Multiparticle scattering amplitudes from lattice QCD

Steve Sharpe
University of Washington

3 (and 2)-particle scattering amplitudes from lattice QCD

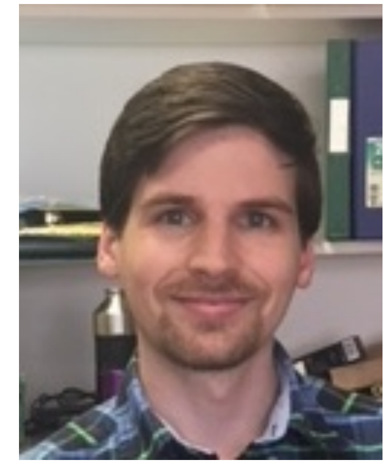
Steve Sharpe
University of Washington

Collaborators


Max Hansen (Edinburgh)


Raúl Briceño (Berkeley)


Fernando Romero-López (MIT/Bern)


Tyler Blanton (UW)


Zach Draper (UW)

Sebastian Dawid (UW)

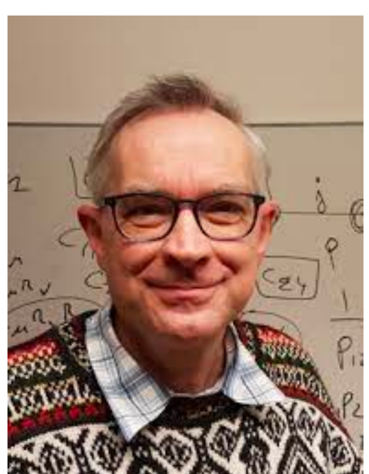

Drew Hanlon (CMU)


Ben Hörz (Intel)

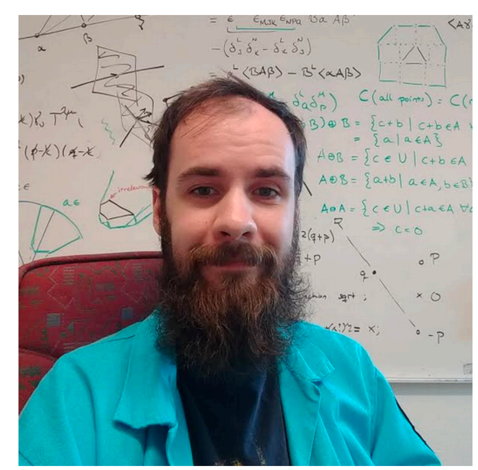

Sarah Skinner (CMU)


Colin Morningstar (CMU)


Jorge Baeza Ballesteros (Valencia/Berlin)


Hans Bijnens (Lund)


Tomas Husek (Birmingham)

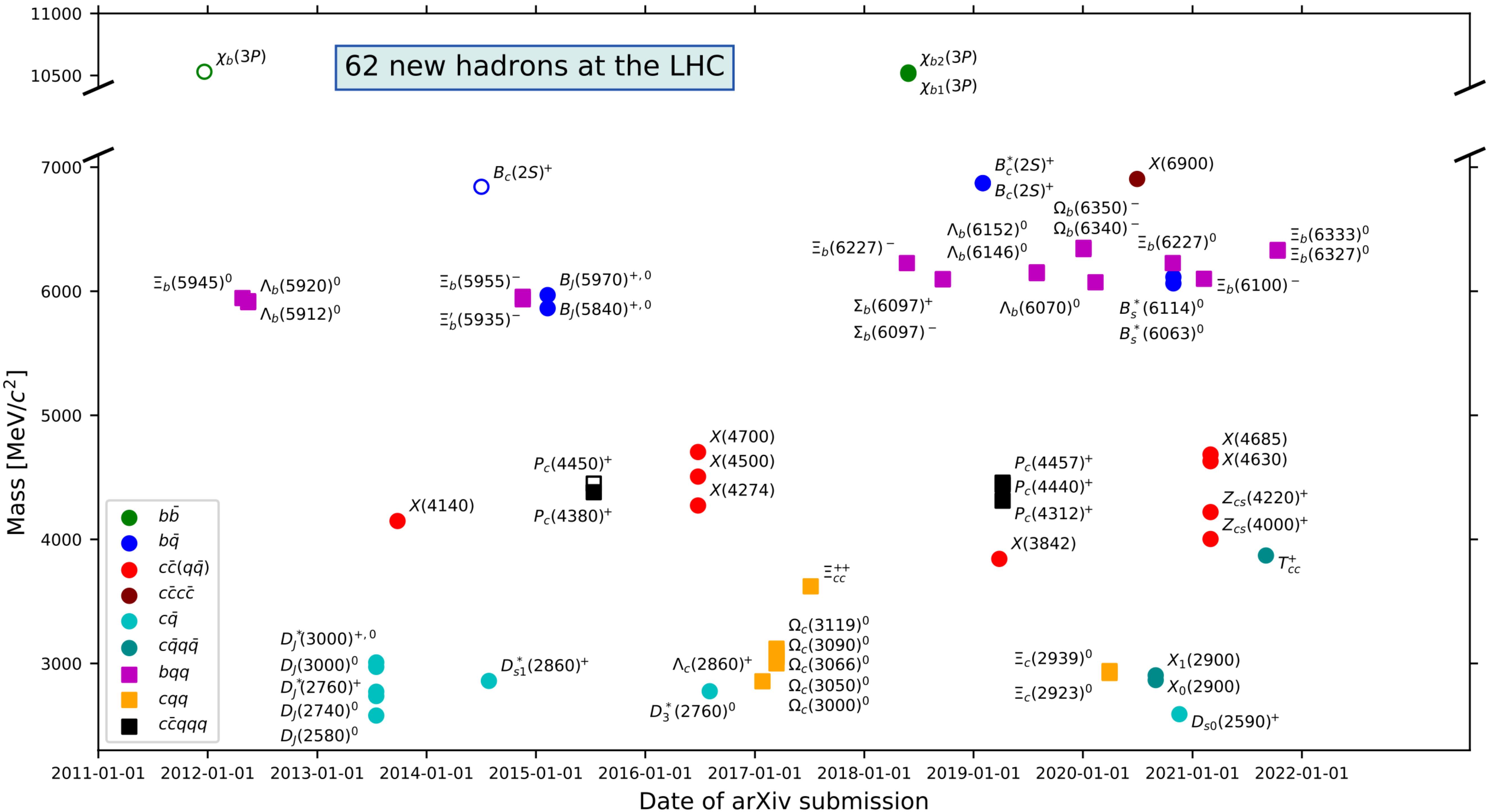

Mattias Sjö (Marseille)


Wilder Schaaf (UW)

Underlying motivations

- Determine properties of strong interaction resonances from QCD
  - E.g. exotics such as $T_{cc}(3875)^+ \to DD^* \to DD\pi$

Cornucopia of exotics



[I. Danilkin, talk at INT workshop, March 23]

+ data from Babar, Belle, COMPASS, …

Underlying motivations

- Determine properties of strong interaction resonances from QCD
  - E.g. exotics such as $T_{cc}(3875)^+ \to DD^* \to DD\pi$
- Determine three particle "forces" for $3n$, 3π, $3K$, …
  - Needed to understand neutron star EoS, …

Underlying motivations

- Determine properties of strong interaction resonances from QCD
    - E.g. exotics such as $T_{cc}(3875)^+ \to DD^* \to DD\pi$
- Determine three particle "forces" for $3n,\ 3\pi,\ 3K, \ldots$
    - Needed to understand neutron star EoS, …
- Calculate weak decay amplitudes within the Standard Model, in order to search for new physics
    - E.g. $K \to 2\pi$ (essentially done), $K \to 3\pi$ (method known), & $D \to \pi^+\pi^-,\ K^+K^-$ (open question)

Underlying motivations

- Determine properties of strong interaction resonances from QCD
  - E.g. exotics such as $T_{cc}(3875)^+ \to DD^* \to DD\pi$
- Determine three particle "forces" for $3n,\ 3\pi,\ 3K, \ldots$
  - Needed to understand neutron star EoS, …
- Calculate weak decay amplitudes within the Standard Model, in order to search for new physics
  - E.g. $K \to 2\pi$ (essentially done), $K \to 3\pi$ (method known), & $D \to \pi^+\pi^-,\ K^+K^-$ (open question)

Will focus most of the discussion on $\mathcal{M}(3\pi \to 3\pi)$

Outline

- The fundamental issue: relating finite and infinite-volume quantities
  - Resolution uses two-step method involving intermediate K matrices ($\mathcal{K}_2, \mathcal{K}_{\rm df,3}$)
- Formalism for $2 \to 2$ scattering
  - Example application: $\pi\pi \to \sigma/f_0(500) \to \pi\pi$
- Sketch derivation of the three-particle formalism for $3\pi^+$
  - Tests of formalism, and generalizations
- Status of applications of the three-particle formalism
  - Fitting $\mathcal{K}_2, \mathcal{K}_{\rm df,3}$ to $\pi^+\pi^+K^+$ spectra from LQCD
  - Comparing $\mathcal{K}_{\rm df,3}(3\pi \to 3\pi)$ to ChPT (Chiral Perturbation Theory)
  - Preliminary results for $\mathcal{M}(3\pi \to 3\pi)$ at nearly physical quark masses from LQCD
  - (Results for $DD\pi$ scattering, relevant for T_{cc}^+)
- Outlook

The fundamental issue

On the one hand…

- LQCD determines energies and properties of finite-volume eigenstates
  - Obtained by fits to (numerically-evaluated) Euclidean correlation functions:

$$\int_L d^3x\, e^{-i\vec{P}\cdot\vec{x}}\, {}_L\langle \Omega | \sigma_{3\pi}(\tau,\vec{x})\sigma^\dagger_{3\pi}(0) | \Omega\rangle_L \propto \sum_n \left| {}_L\langle 0 | \sigma^\dagger_{3\pi}(0) | 3\pi, \vec{P}, n\rangle_L \right|^2 e^{-E_n\tau}; \qquad (\tau > 0)$$

$\vec{P} = 2\pi\vec{n}/L$
Assuming L^3 box with PBC

$\sigma_{3\pi} \sim 3\pi^+$
Lives on timeslice

Tower of finite-volume states with quantum numbers of $3\pi^+$, with momentum $\vec{P}$, and living in irreps of cubic group

Energies of said states

On the one hand…

- LQCD determines energies and properties of finite-volume eigenstates
  - Obtained by fits to (numerically-evaluated) Euclidean correlation functions:

$$\int_L d^3x\, e^{-i\vec{P}\cdot\vec{x}}\, {}_L\langle\Omega|\sigma_{3\pi}(\tau,\vec{x})\sigma_{3\pi}^\dagger(0)|\Omega\rangle_L \propto \sum_n \left| {}_L\langle 0|\sigma_{3\pi}^\dagger(0)|3\pi,\vec{P},n\rangle_L \right|^2 e^{-E_n\tau}; \qquad (\tau > 0)$$

$\vec{P} = 2\pi\vec{n}/L$
Assuming L^3 box with PBC

$\sigma_{3\pi} \sim 3\pi^+$
Lives on timeslice

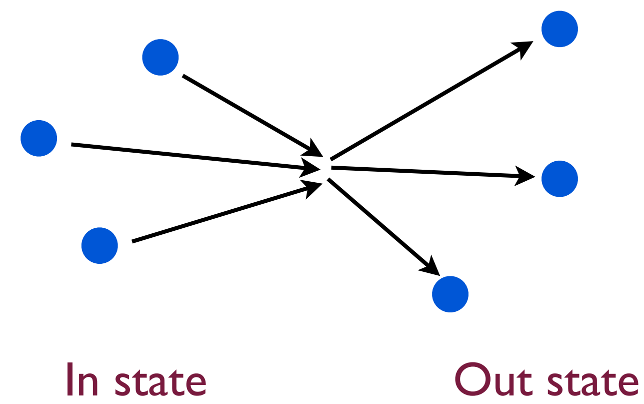
Tower of finite-volume states with quantum numbers of $3\pi^+$, with momentum $\vec{P}$, and living in irreps of cubic group

Energies of said states

- E_n are physical quantities!
  - Can determine 5-10 levels for each choice of quantum numbers ($\vec{P}$, irrep, …)
  - Can now begin to calculate with physical quark masses
  - Results come with statistical & systematic errors (e.g. need $a \to 0$)
  - Mostly, we just assume here that the E_n are provided to us

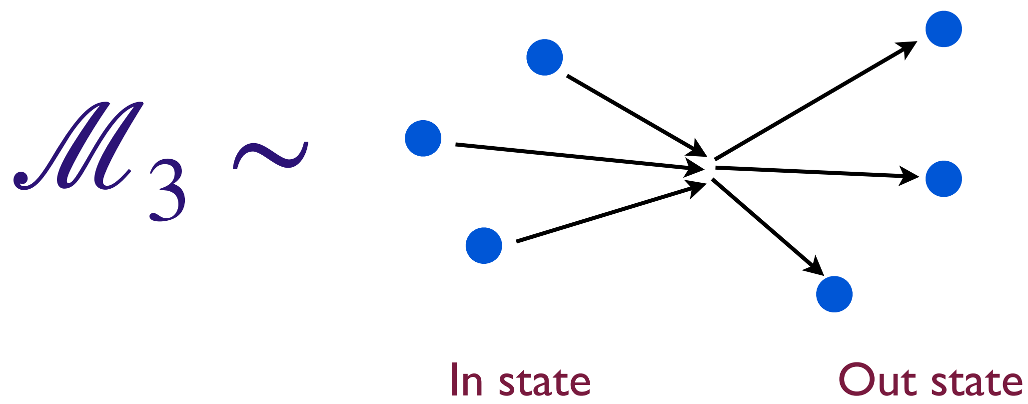
…while on the other

- We want infinite-volume scattering amplitudes

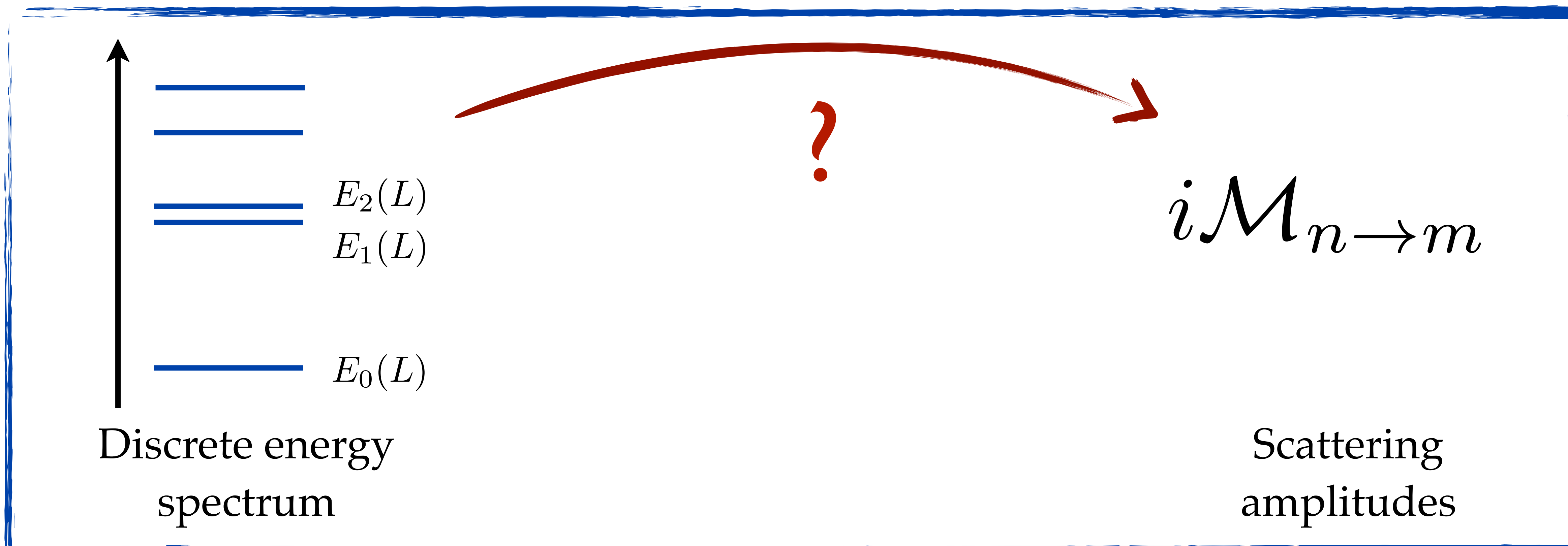
$\mathcal{M}_3 \sim$ 

...while on the other

- We want infinite-volume scattering amplitudes



- How do we relate these? A finite-volume QFT problem.



A related question:

- LQCD can also calculate matrix elements between finite-volume states

$$ {}_L\langle \Omega | \sigma_{3\pi}(\tau_f, \vec{P}) [\int_L d^3x \mathscr{H}_W(0, \vec{x})] K^\dagger(\tau_i, \vec{P}) | \Omega \rangle_L \propto \sum_{n',n} c_{n',n} e^{-E_n \tau_f} {}_L\langle 3\pi, \vec{P}, n' | \mathscr{H}_W(0) | K, n \rangle_L e^{E_{K'_n} \tau_i} $$

$\tau_f > 0$

$\tau_i < 0$

A physical quantity if $E_{n'} = E_n$

A related question:

- LQCD can also calculate matrix elements between finite-volume states

$$ {}_L\langle \Omega | \sigma_{3\pi}(\tau_f, \overrightarrow{P}) [\int_L d^3x \mathscr{H}_W(0, \vec{x})] K^\dagger(\tau_i, \overrightarrow{P}) | \Omega \rangle_L \propto \sum_{n',n} c_{n',n} e^{-E_n \tau_f} {}_L\langle 3\pi, \overrightarrow{P}, n' | \mathscr{H}_W(0) | K, n \rangle_L e^{E_{K'_n} \tau_i} $$

$\tau_f > 0$

$\tau_i < 0$

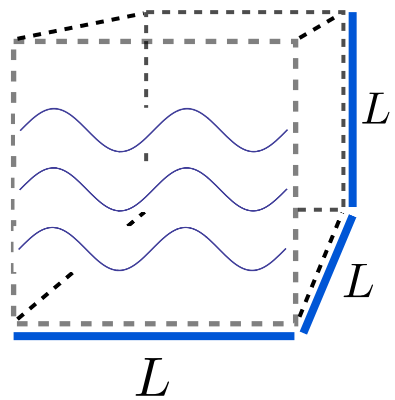
A physical quantity if $E_{n'} = E_n$

- How are these related to decay amplitudes?

$$ \mathscr{A}(K \to 3\pi) = {}_{\text{out}}\langle 3\pi | \mathscr{H}_W(0) | K \rangle $$

Two-step method

2 & 3 particle
Spectra from LQCD



Quantization conditions

QC2: $\det\left[F^{-1} + \mathcal{K}_2\right] = 0$

QC3: $\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$

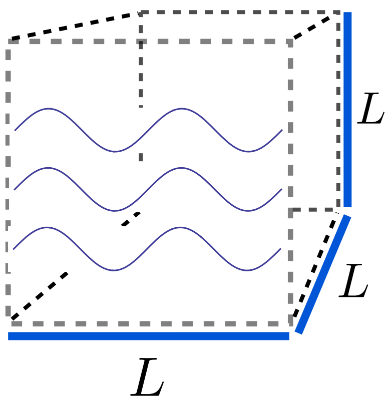
[These are the RFT forms, and assume $\mathbb{Z}_2$ symmetry]

Integral equations in infinite volume

Scattering amplitude $\mathcal{M}_3$

Two-step method

2 & 3 particle
Spectra from LQCD



Infinite-volume K matrix:
Obtained from Feynman diagrams
using PV prescription for poles;
Real, free of unitary cuts

Quantization conditions

QC2: $\det\left[F^{-1} + \mathscr{K}_2\right] = 0$

QC3: $\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$

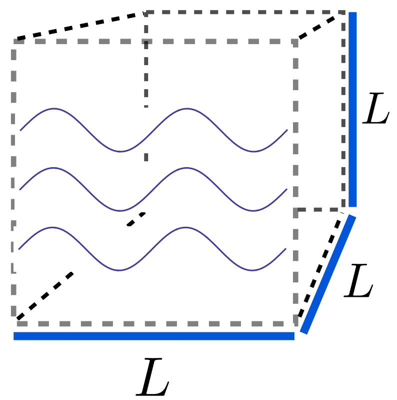
[These are the RFT forms, and assume $\mathbb{Z}_2$ symmetry]

Integral equations in infinite volume

Scattering amplitude $\mathscr{M}_3$

Two-step method

2 & 3 particle
Spectra from LQCD



Infinite-volume K matrix:
Obtained from Feynman diagrams
using PV prescription for poles;
Real, free of unitary cuts

Quantization conditions

QC2: $\det\left[F^{-1} + \mathscr{K}_2\right] = 0$

QC3: $\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$

[These are the RFT forms, and assume $\mathbb{Z}_2$ symmetry]

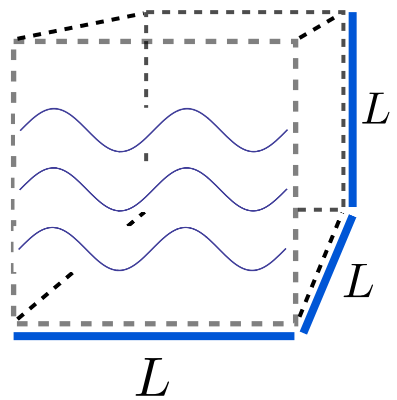
Intermediate infinite-volume K matrix:
A short-distance, real, three-particle
interaction free of unitary cuts, and
with physical divergences subtracted;
unphysical since depends on cutoff

Integral equations in infinite volume

Scattering amplitude $\mathscr{M}_3$

Two-step method

2 & 3 particle
Spectra from LQCD



Infinite-volume K matrix:
Obtained from Feynman diagrams
using PV prescription for poles;
Real, free of unitary cuts

Quantization conditions

QC2: $\det\left[F^{-1} + \mathscr{K}_2\right] = 0$

QC3: $\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$

[These are the RFT forms, and assume $\mathbb{Z}_2$ symmetry]

Intermediate infinite-volume K matrix:
A short-distance, real, three-particle
interaction free of unitary cuts, and
with physical divergences subtracted;
unphysical since depends on cutoff

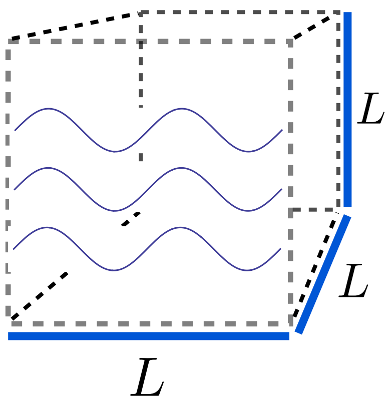
Integral equations in infinite volume

Incorporates initial- and
final-state interactions, and
ensures unitarity

Scattering amplitude $\mathscr{M}_3$

Two-step method

2 & 3 particle
Spectra from LQCD



Infinite-volume K matrix:
Obtained from Feynman diagrams
using PV prescription for poles;
Real, free of unitary cuts

Quantization conditions

QC2: $\det\left[F^{-1}+\mathscr{K}_2\right]=0$

QC3: $\det\left[F_3^{-1}+\mathscr{K}_{\mathrm{df},3}\right]=0$

[These are the RFT
forms, and assume
$\mathbb{Z}_2$ symmetry]

Intermediate infinite-volume K matrix:
A short-distance, real, three-particle
interaction free of unitary cuts, and
with physical divergences subtracted;
unphysical since depends on cutoff

Integral equations in infinite volume

Scattering amplitude $\mathscr{M}_3$

Incorporates initial- and
final-state interactions, and
ensures unitarity

If parametrize K matrices, can
continue $\mathscr{M}_3$ into the complex
plane & look for resonances, etc.

Two-particle formalism

[Lüscher, 1986-91 + many subsequent works by many authors]

I will follow approach of [Kim, Sachrajda, & SS, 2005], generalized to use time-ordered PT following [Blanton & SS, 2020]

Generic relativistic FT (RFT) approach

- Study Minkowski time, finite-volume correlator

$$C_L(E, \overrightarrow{P}) \equiv \int_L d^4x\, e^{iEt - i\overrightarrow{P}\cdot\vec{x}} \langle \Omega \,|\, T\left\{ \sigma_{2\pi}(x)\sigma_{2\pi}^\dagger(0) \right\} |\, \Omega \rangle_L$$

  - For fixed $\overrightarrow{P}$, poles in C_L occur when $E = E_n$

- Analyze in generic EFT for pions, (kaons, ...) working to all orders in (TO)PT
  - For simplicity, assume exact isospin symmetry
  - Restrict kinematic range to $0 < E^* = \sqrt{E^2 - P^2} < 4M_\pi$

Generic relativistic FT (RFT) approach

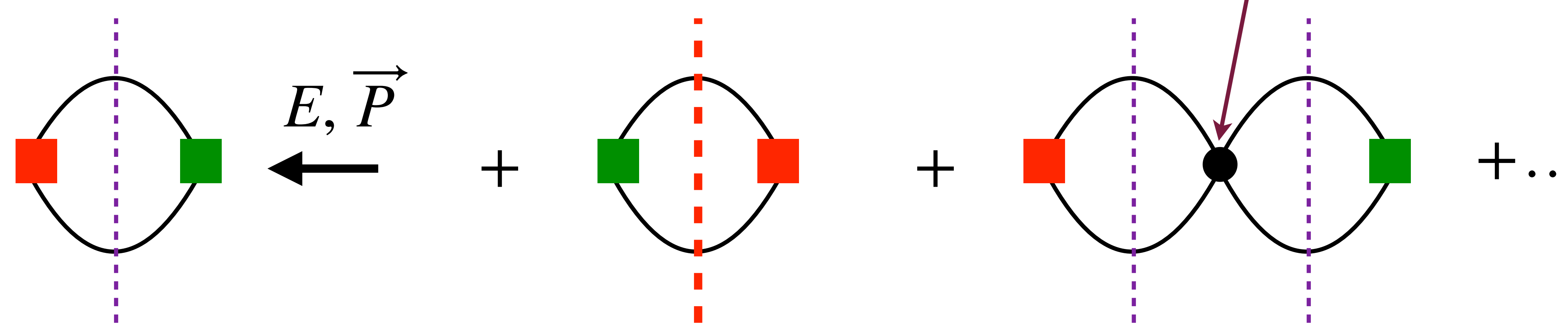
- Study Minkowski time, finite-volume correlator

$$C_L(E, \vec{P}) \equiv \int_L d^4x \, e^{iEt - i\vec{P}\cdot\vec{x}} \langle \Omega | T \left\{ \sigma_{2\pi}(x) \sigma_{2\pi}^\dagger(0) \right\} | \Omega \rangle_L$$

  - For fixed $\vec{P}$, poles in C_L occur when $E = E_n$

- Analyze in generic EFT for pions, (kaons, …) working to all orders in (TO)PT
  - For simplicity, assume exact isospin symmetry
  - Restrict kinematic range to $0 < E^* = \sqrt{E^2 - P^2} < 4M_\pi$

$C_L(E, \vec{P}) =$



No need for $i\epsilon$ in finite volume

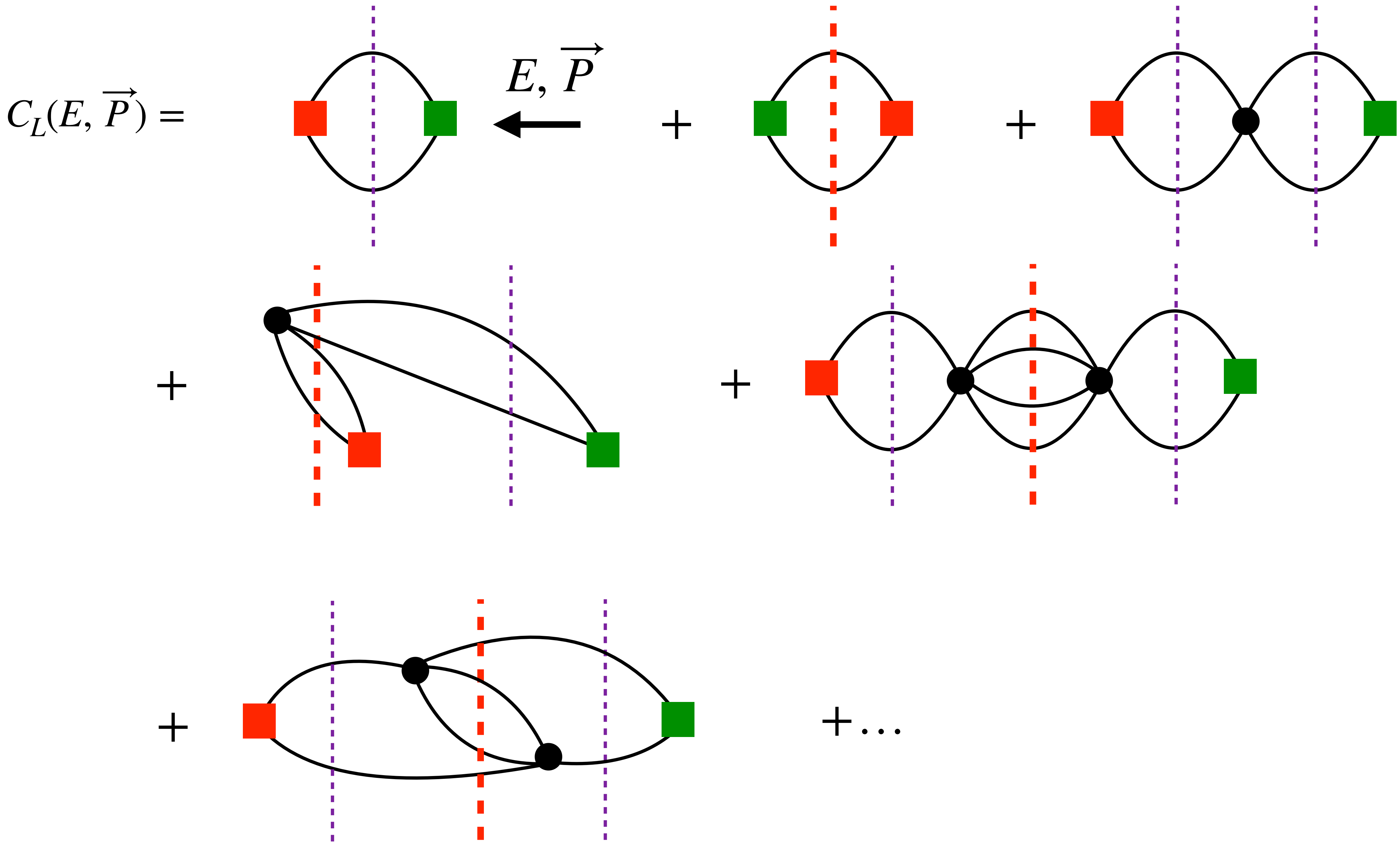
$\frac{1}{E - \omega_1 - \omega_2}$ Can go on shell

$\frac{1}{-E - \omega_1 - \omega_2}$ Cannot go on shell

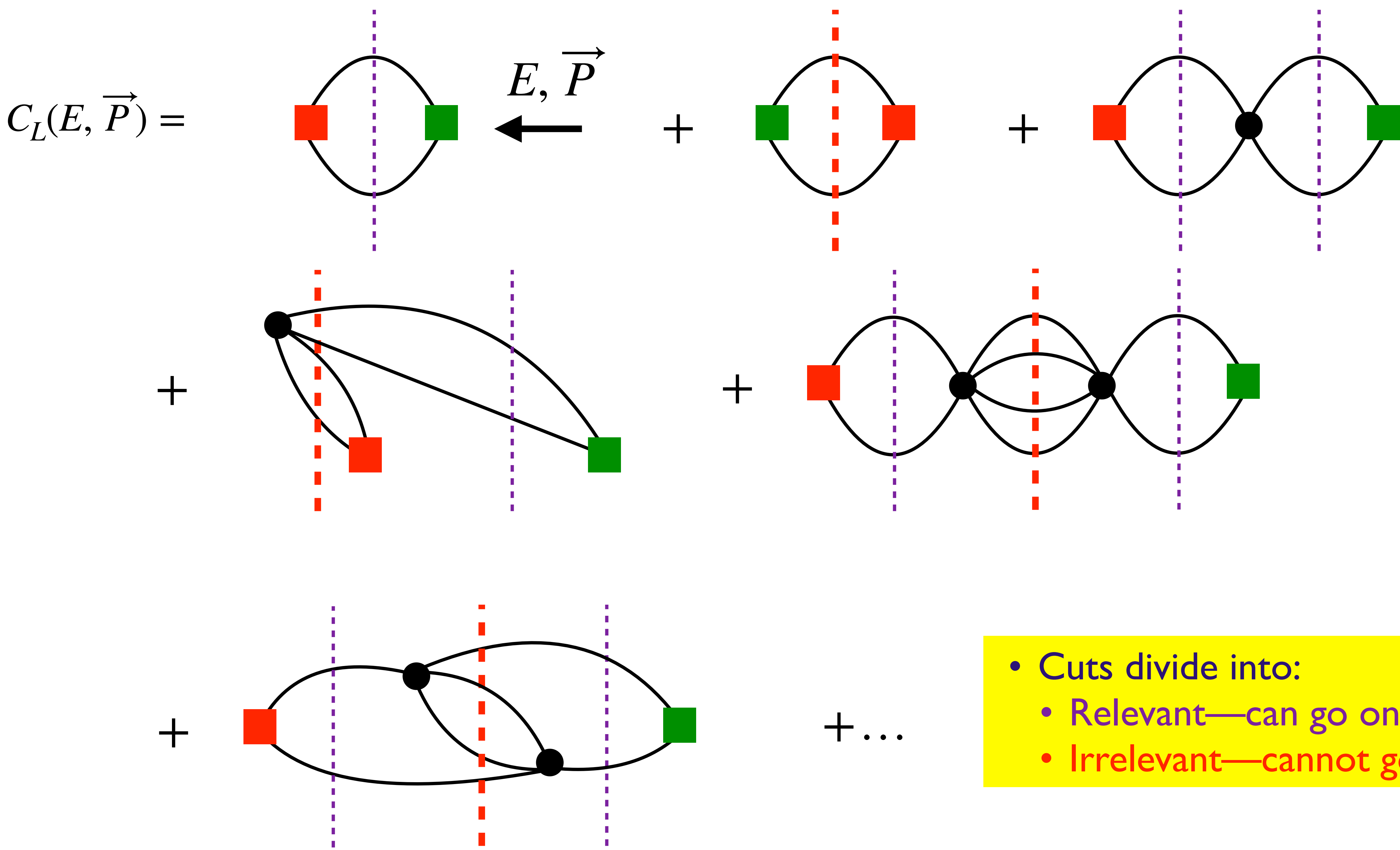
$\frac{1}{E - \omega_1 - \omega_2}$ Can go on shell

$\frac{1}{E - \omega_1 - \omega_2}$ Can go on shell

Generic relativistic FT (RFT) approach

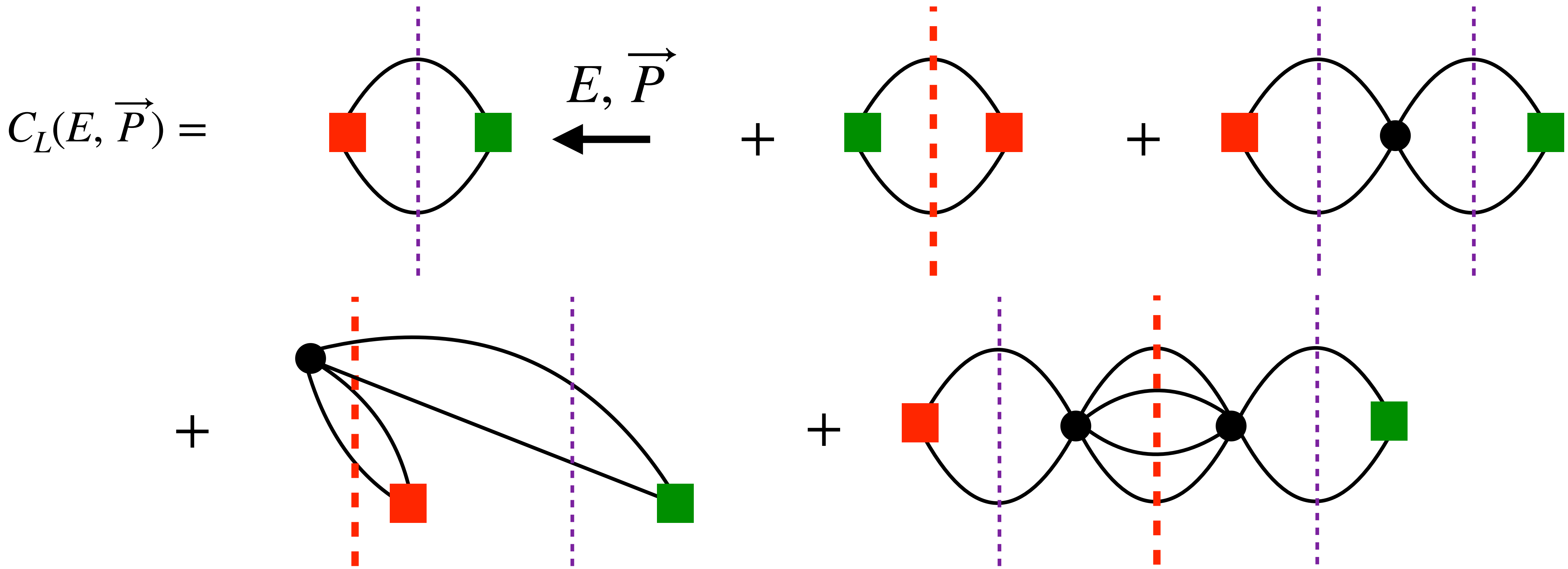


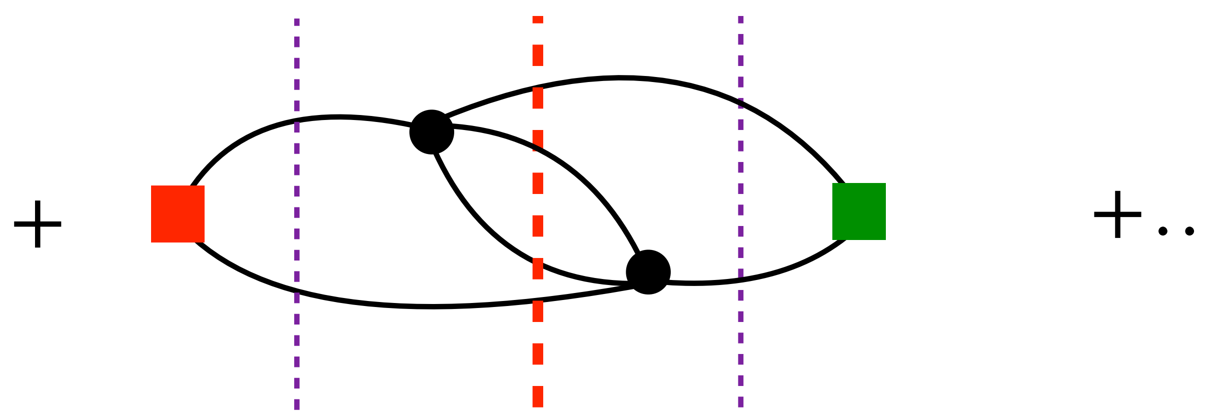
Generic relativistic FT (RFT) approach



- Cuts divide into:
  - Relevant—can go on shell
  - Irrelevant—cannot go on shell

Generic relativistic FT (RFT) approach





- Cuts divide into:
  - Relevant—can go on shell
  - Irrelevant—cannot go on shell

- Three-momenta in loops are summed over finite-volume set

Use Poisson summation formula

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l}\neq\vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})$$

Use Poisson summation formula

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l}\neq\vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})$$

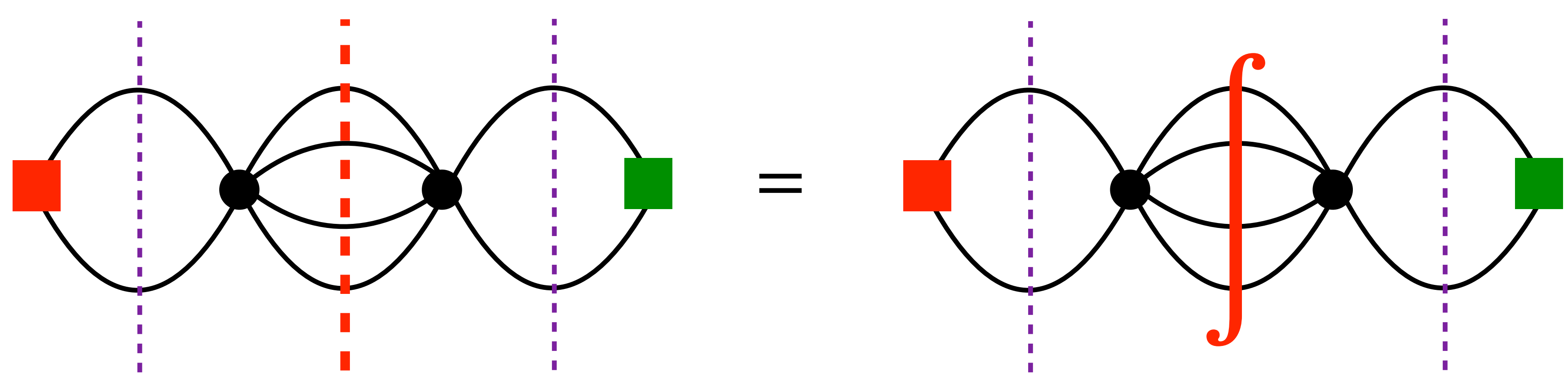
Exp. suppressed if $g(\vec{k})$ is smooth and $g' \sim g/M_\pi$

Use Poisson summation formula

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l}\neq\vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})$$

Exp. suppressed if $g(\vec{k})$ is smooth and $g' \sim g/M_\pi$

- Replace loop sums with integrals where summand/integrand is nonsingular
  - Drop exponentially suppressed terms ($e^{-M_\pi L}, e^{-(M_\pi L)^2}$, etc.) while keeping power-law dependence



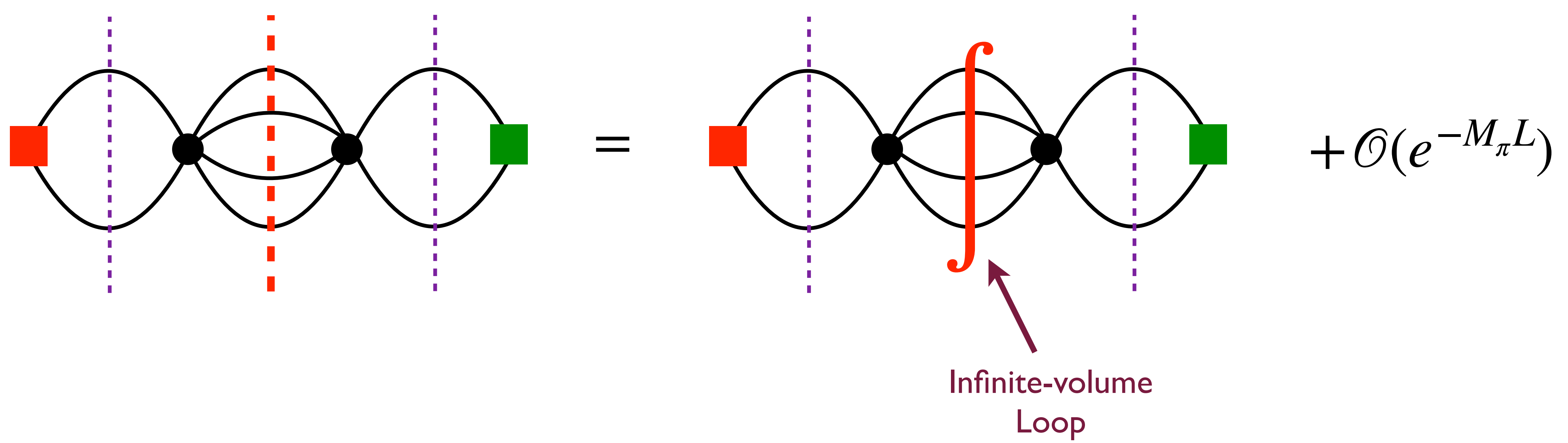
$+\mathcal{O}(e^{-M_\pi L})$

Use Poisson summation formula

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l}\neq\vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})$$

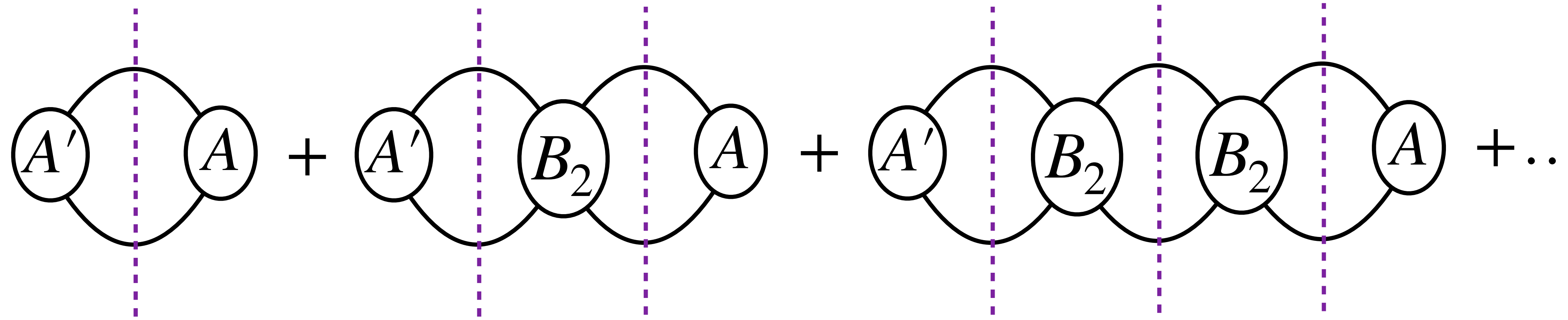
Exp. suppressed if $g(\vec{k})$ is smooth and $g' \sim g/M_\pi$

- Replace loop sums with integrals where summand/integrand is nonsingular
  - Drop exponentially suppressed terms ($e^{-M_\pi L}, e^{-(M_\pi L)^2}$, etc.) while keeping power-law dependence

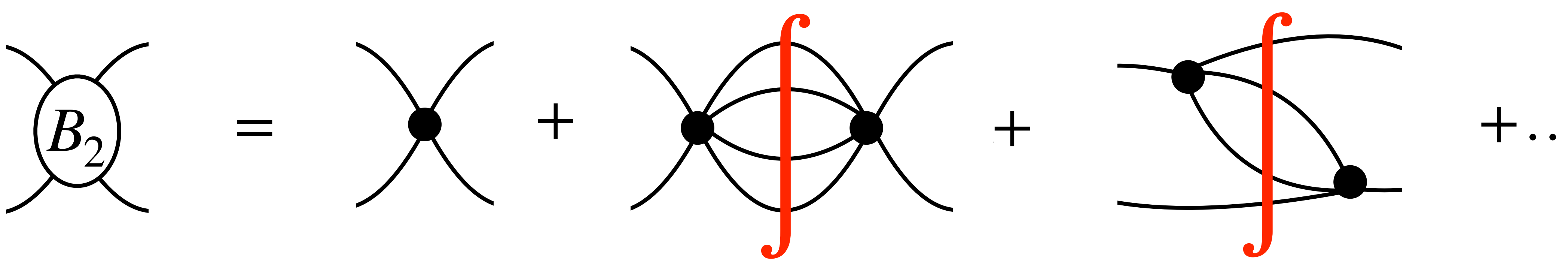


Expansion in relevant cuts

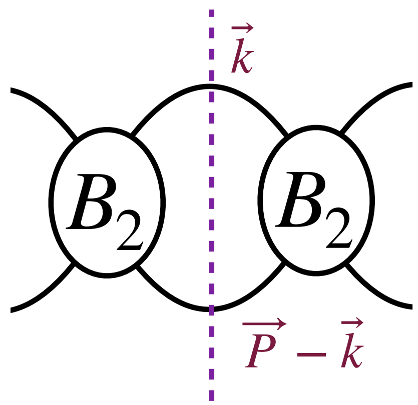
$$C_L(E, \vec{P}) = C_L^{(0)}(E, \vec{P}) + \ldots$$



- B_2 is the TOPT version of a Bethe-Salpeter kernel (2PI in s-channel)
  - A' and A are corresponding "endcaps"



Dealing with relevant cuts



$$\frac{1}{L^3}\sum_{\vec{k}} f(E,\vec{P},\vec{k})\frac{1}{2}\frac{1}{4\omega_k\omega_{P-k}}\frac{1}{E-\omega_k-\omega_{P-k}}g(E,\vec{P},\vec{k})$$

$$= \mathrm{PV}\int \frac{d^3k}{(2\pi)^3} f(E,\vec{P},\vec{k})\frac{1}{2}\frac{1}{4\omega_k\omega_{P-k}}\frac{1}{E-\omega_k-\omega_{P-k}}g(E,\vec{P},\vec{k})$$

$$+\sum_{\ell',m';\ell,m} f^{\rm on}_{\ell' m'}(E^*)\; F_{\ell' m';\ell m}(E,\vec{P},L)\; g^{\rm on}_{\ell m}(E^*)$$

On-shell projected in pair CM frame, and decomposed into harmonics

- F is a known, calculable kinematic finite-volume function

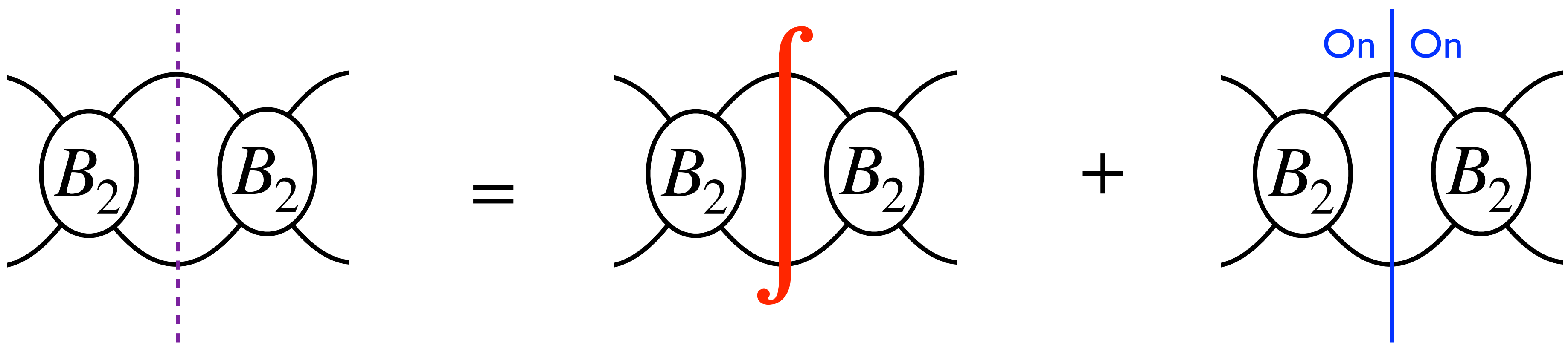
CM frame relative momentum

Harmonic polynomial

$$F_{\mathrm{PV};\ell' m';\ell m}(E,\vec{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \mathrm{PV}\int\frac{d^3k}{(2\pi)^3}\right)\frac{\mathcal{Y}_{\ell' m'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k})}$$

UV cutoff

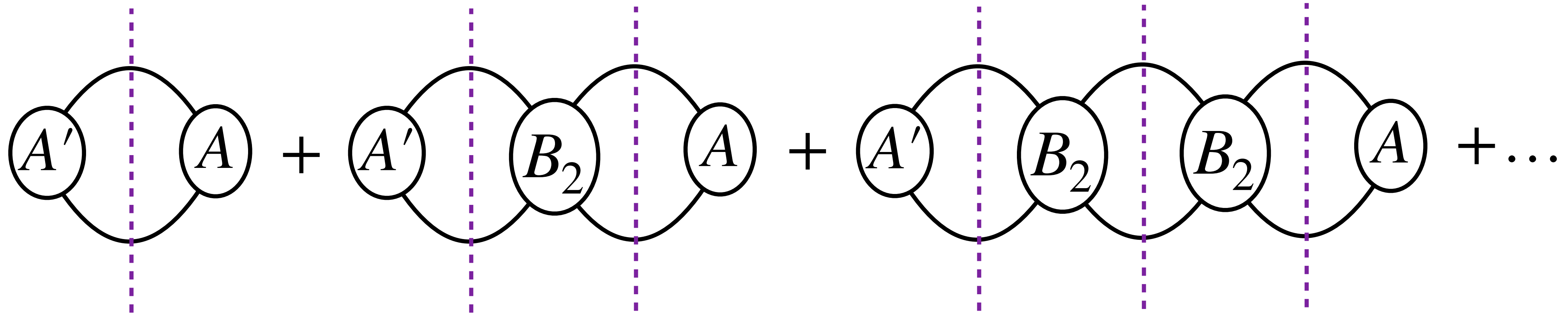
Key move



$$\Sigma = \int + F$$

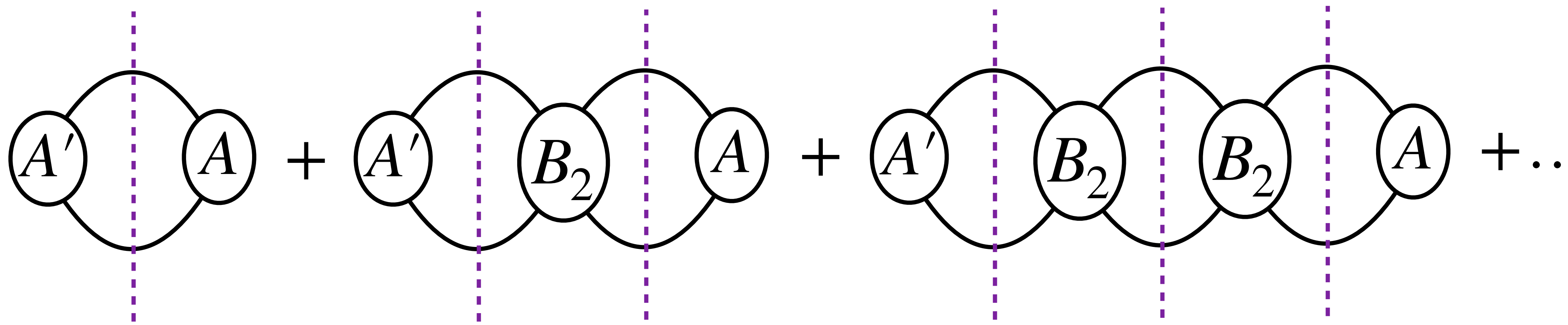
Resummations

$$C_L(E, \vec{P}) = C_L^{(0)}(E, \vec{P}) + \ldots$$

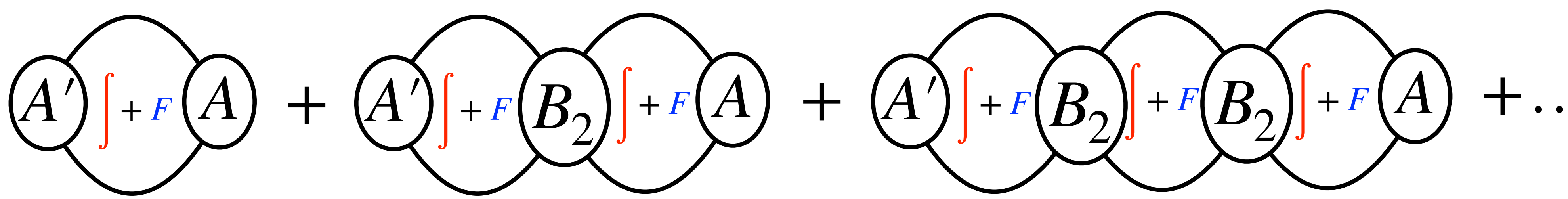


Resummations

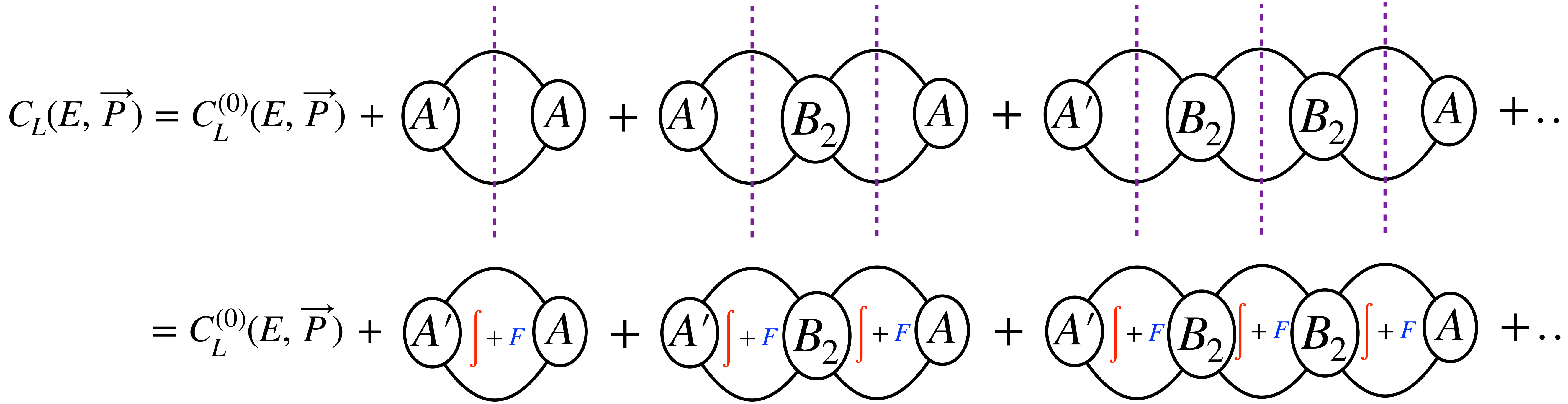
$$C_L(E, \vec{P}) = C_L^{(0)}(E, \vec{P}) +$$



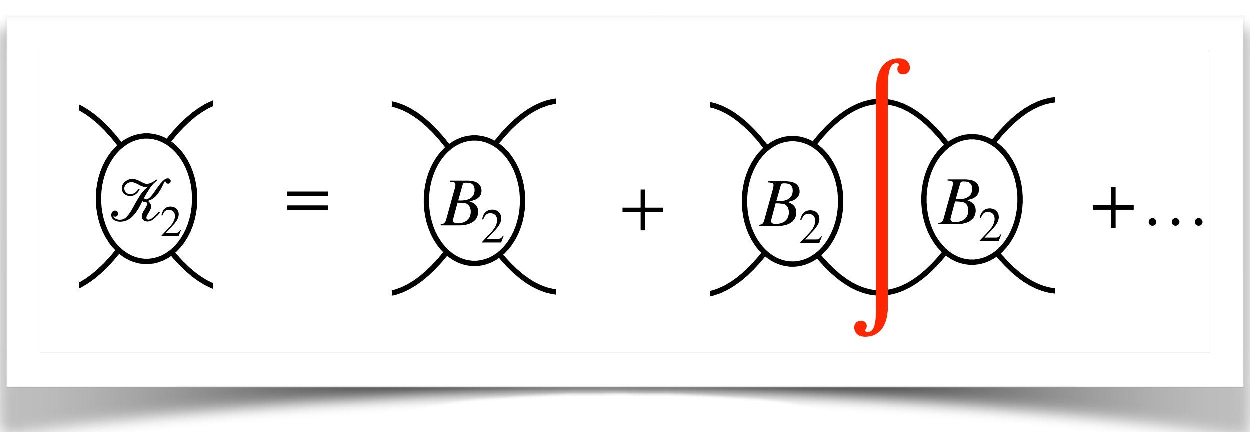
$$= C_L^{(0)}(E, \vec{P}) +$$

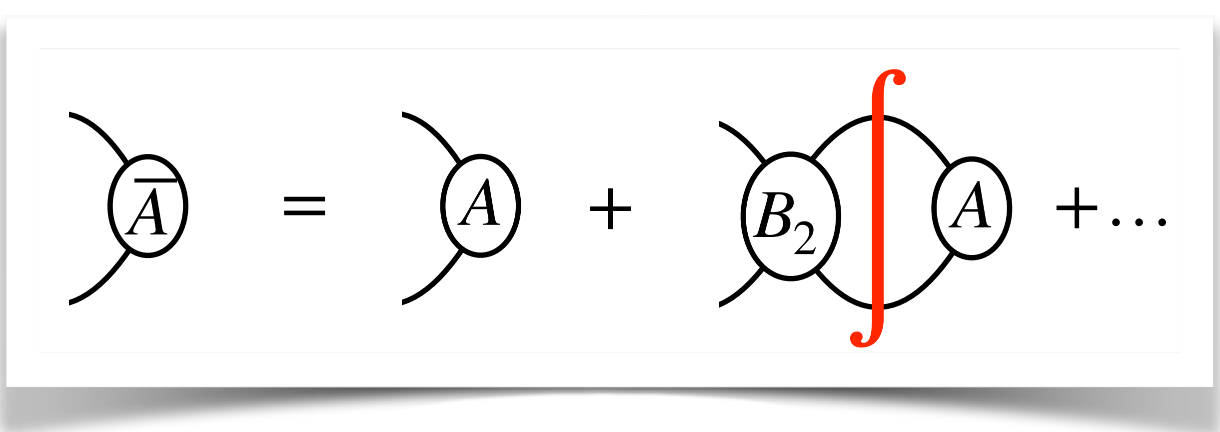


Resummations



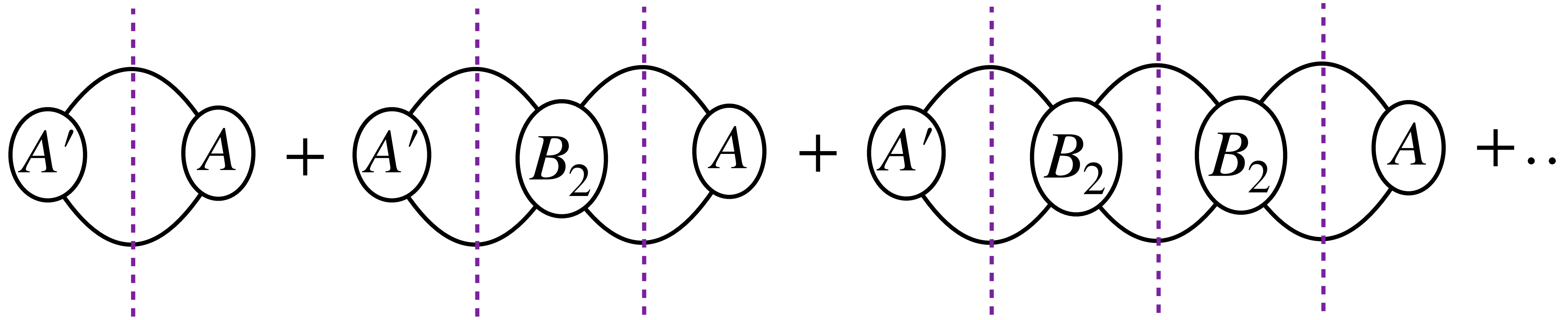
$$= C_\infty(E, \vec{P}) \ + \overline{A}' \cdot iF \cdot \overline{A} + \overline{A}' \cdot iF \cdot i\mathcal{K}_2 \cdot iF \cdot \overline{A} + \overline{A}' \cdot iF \cdot i\mathcal{K}_2 \cdot iF \cdot i\mathcal{K}_2 \cdot iF \cdot \overline{A} + \ldots$$



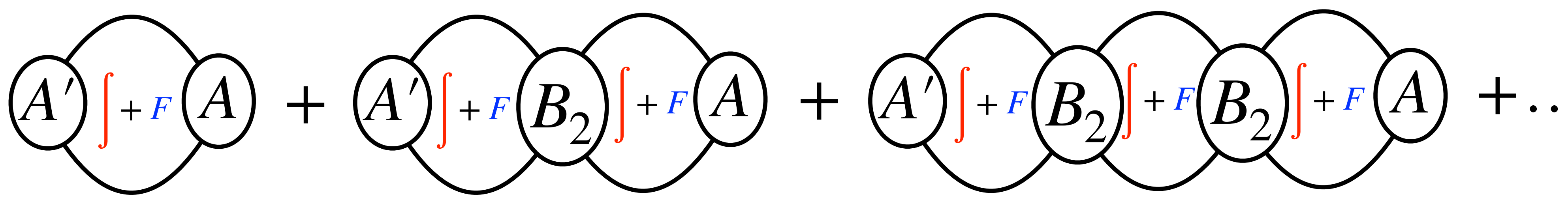


Resummations

$C_L(E, \vec{P}) = C_L^{(0)}(E, \vec{P}) +$

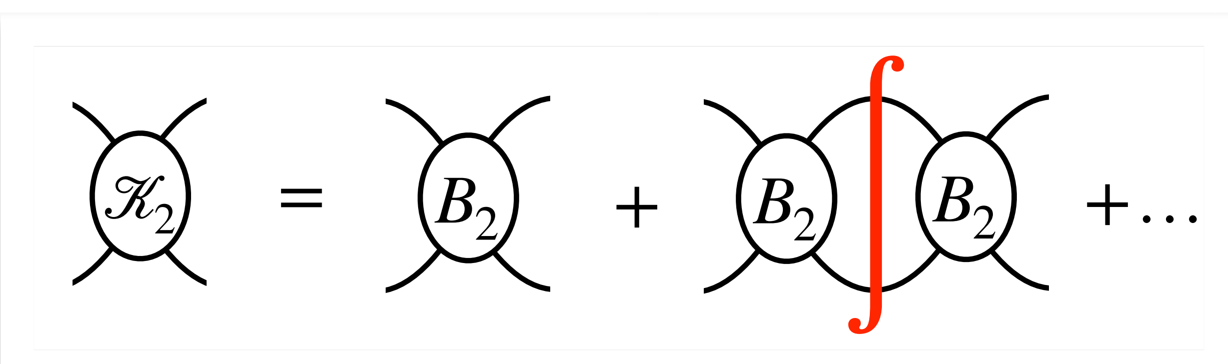


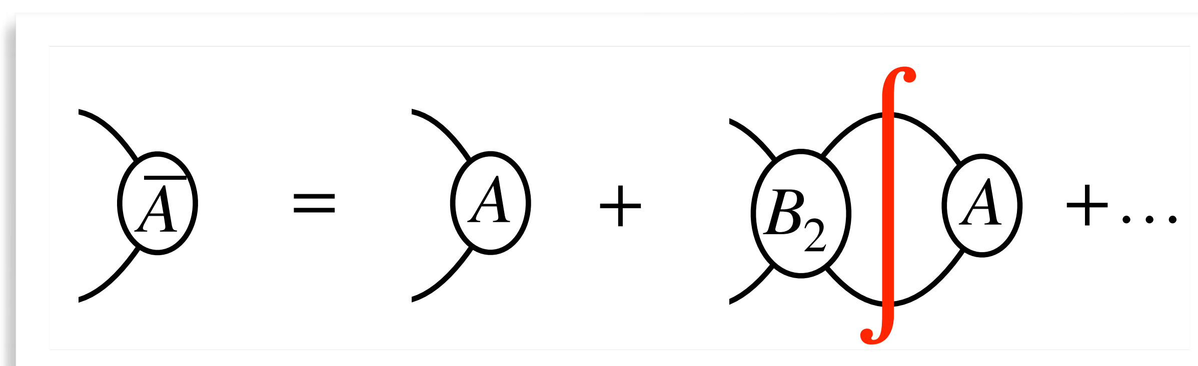
$= C_L^{(0)}(E, \vec{P}) +$



$$= C_\infty(E, \vec{P}) \; + \bar{A}' \cdot iF \cdot \bar{A} + \bar{A}' \cdot iF \cdot i\mathscr{K}_2 \cdot iF \cdot \bar{A} + \bar{A}' \cdot iF \cdot i\mathscr{K}_2 \cdot iF \cdot i\mathscr{K}_2 \cdot iF \cdot \bar{A} + \ldots$$

$$= C_\infty(E, \vec{P}) \; + \; \bar{A}' \cdot iF \cdot \frac{1}{1 + \mathscr{K}_2 \cdot F} \cdot \bar{A}$$





Quantization condition (QC2)

$$C_L(E, \overrightarrow{P}) = C_\infty(E, \overrightarrow{P}) \; + \; \overline{A}' \cdot iF \cdot \frac{1}{1 + \mathcal{K}_2 \cdot F} \cdot \overline{A}$$

Has no L-dependent poles

Only source of L-dependent poles

Quantization condition (QC2)

$$C_L(E,\overrightarrow{P}) = C_\infty(E,\overrightarrow{P}) \;+\; \overline{A}'\cdot iF\cdot\frac{1}{1+\mathscr{K}_2\cdot F}\cdot\overline{A}$$

Has no L-dependent poles

Only source of L-dependent poles

- QC2: finite-volume energies occur when

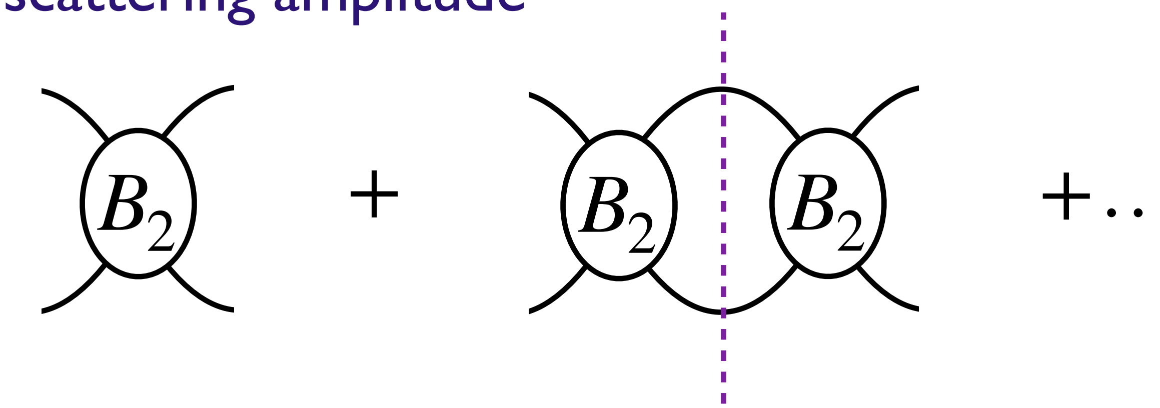
$$\det(F^{-1}+\mathscr{K}_2)=0$$

  - Matrix indices are CM-frame ℓ, m
  - $\mathscr{K}_2$ is an infinite-volume quantity: diagonal in ℓ, m
  - F depends on finite-volume size & geometry, mixes ℓ, m
  - In practical applications, must truncate in ℓ

Step 2: relating $\mathscr{K}_2$ to $\mathscr{M}_2$

- Consider "finite-volume scattering amplitude"

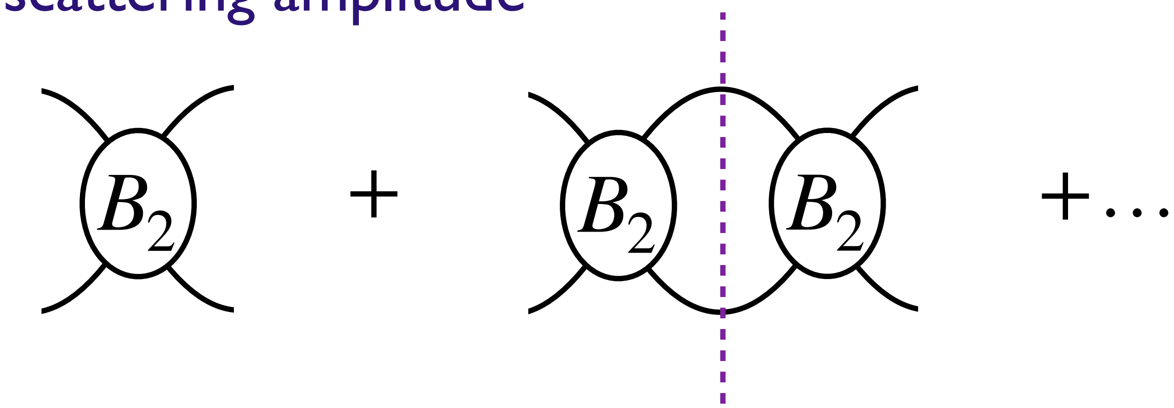
$$\mathscr{M}_{2,L}^{(\mathrm{off})} =$$



Step 2: relating $\mathcal{K}_2$ to $\mathcal{M}_2$

- Consider "finite-volume scattering amplitude"

$$\mathcal{M}_{2,L}^{(\text{off})} =$$



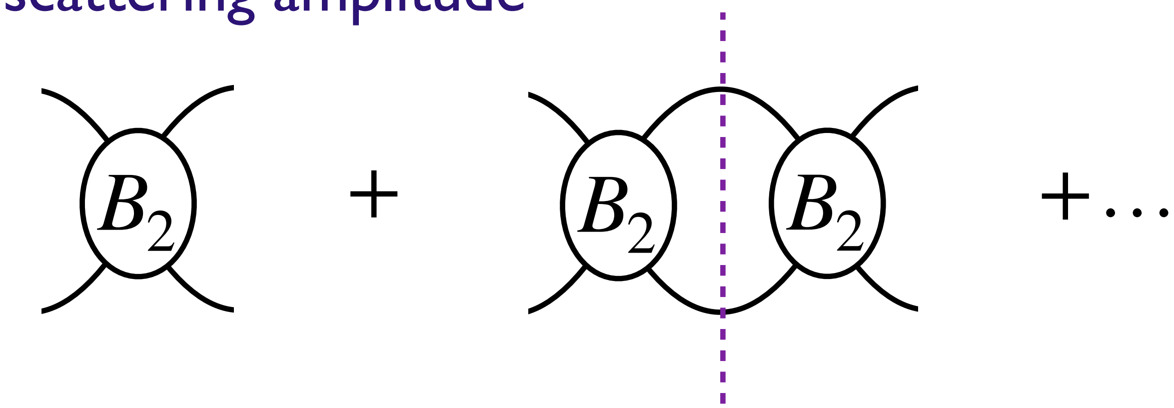
  - Use similar steps as for $C_{2,L}$: project on ℓ, m, project on shell, use "key move"

$$i\mathcal{M}_{2,L} = i\mathcal{K}_2 + i\mathcal{K}_2 \cdot iF \cdot i\mathcal{K}_2 + \ldots = i\mathcal{K}_2 \frac{1}{1 + F\mathcal{K}_2}$$

Step 2: relating $\mathscr{K}_2$ to $\mathscr{M}_2$

- Consider "finite-volume scattering amplitude"

$$\mathscr{M}^{(\mathrm{off})}_{2,L} =$$



  - Use similar steps as for $C_{2,L}$: project on ℓ, m, project on shell, use "key move"

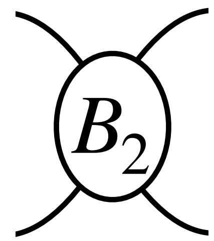
$$i\mathscr{M}_{2,L} = i\mathscr{K}_2 + i\mathscr{K}_2 \cdot iF \cdot i\mathscr{K}_2 + \ldots = i\mathscr{K}_2 \frac{1}{1 + F\mathscr{K}_2}$$

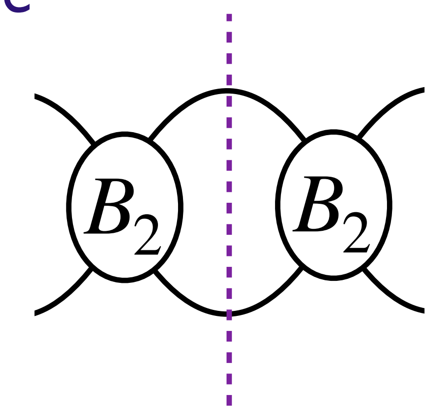
- Take $L \to \infty$ limit, regularizing integrals with $i\epsilon$ prescription

$$\mathscr{M}_{2,L} \to \mathscr{M}_2, \qquad F_{\ell',m';\ell,m} \to -i\delta_{\ell'\ell}\delta_{m'm}\rho, \qquad \rho = -i\sqrt{q^{*2}}/16\pi E^*$$

Step 2: relating $\mathcal{K}_2$ to $\mathcal{M}_2$

- Consider "finite-volume scattering amplitude"

$\mathcal{M}_{2,L}^{(\text{off})} =$ 

$+$ 

$+ \ldots$

  - Use similar steps as for $C_{2,L}$: project on ℓ, m, project on shell, use "key move"

$$i\mathcal{M}_{2,L} = i\mathcal{K}_2 + i\mathcal{K}_2 \cdot iF \cdot i\mathcal{K}_2 + \ldots = i\mathcal{K}_2 \frac{1}{1 + F\mathcal{K}_2}$$

- Take $L \to \infty$ limit, regularizing integrals with $i\epsilon$ prescription

$$\mathcal{M}_{2,L} \to \mathcal{M}_2, \qquad F_{\ell',m';\ell,m} \to -i\delta_{\ell'\ell}\delta_{m'm}\rho, \qquad \rho = -i\sqrt{q^{*2}}/16\pi E^*$$

  - Leads to standard relation between $\mathcal{M}_2$ & $\mathcal{K}_2$, showing that $\mathcal{K}_2$ is the standard, relativistic two-particle K matrix

$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho\mathcal{K}_2}$$

Applications of QC2 are well developed

- LQCD gives spectrum, fit to QC2 with parametrized, truncated $\mathscr{K}_2$, determine $\mathscr{M}_2$, look for poles in complex plane

- State-of-the-art involves multiple channels, particles with spin, as well as decay and transition amplitudes

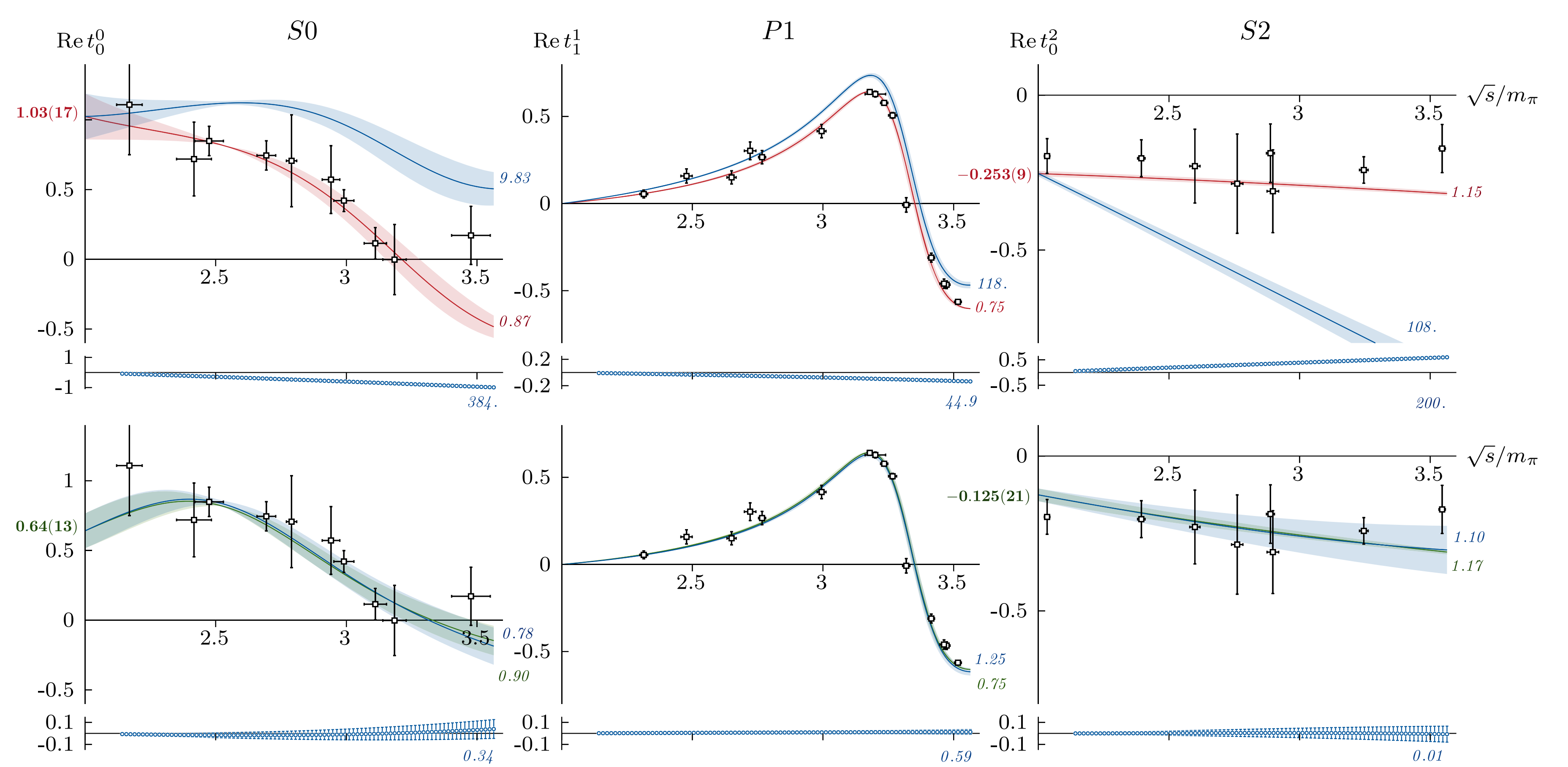
- Nice recent example [Rodas et al., 2304.03762 (PRD)] for $\pi\pi \to \sigma/f_0(500) \to \pi\pi$ where crossing symmetry/dispersion relations restrict parametrizations of $\mathscr{K}_2$

Applications of QC2 are well developed

- LQCD gives spectrum, fit to QC2 with parametrized, truncated $\mathscr{K}_2$, determine $\mathscr{M}_2$, look for poles in complex plane
- State-of-the-art involves multiple channels, particles with spin, as well as decay and transition amplitudes
- Nice recent example [Rodas et al., 2304.03762 (PRD)] for $\pi\pi \to \sigma/f_0(500) \to \pi\pi$ where crossing symmetry/dispersion relations restrict parametrizations of $\mathscr{K}_2$

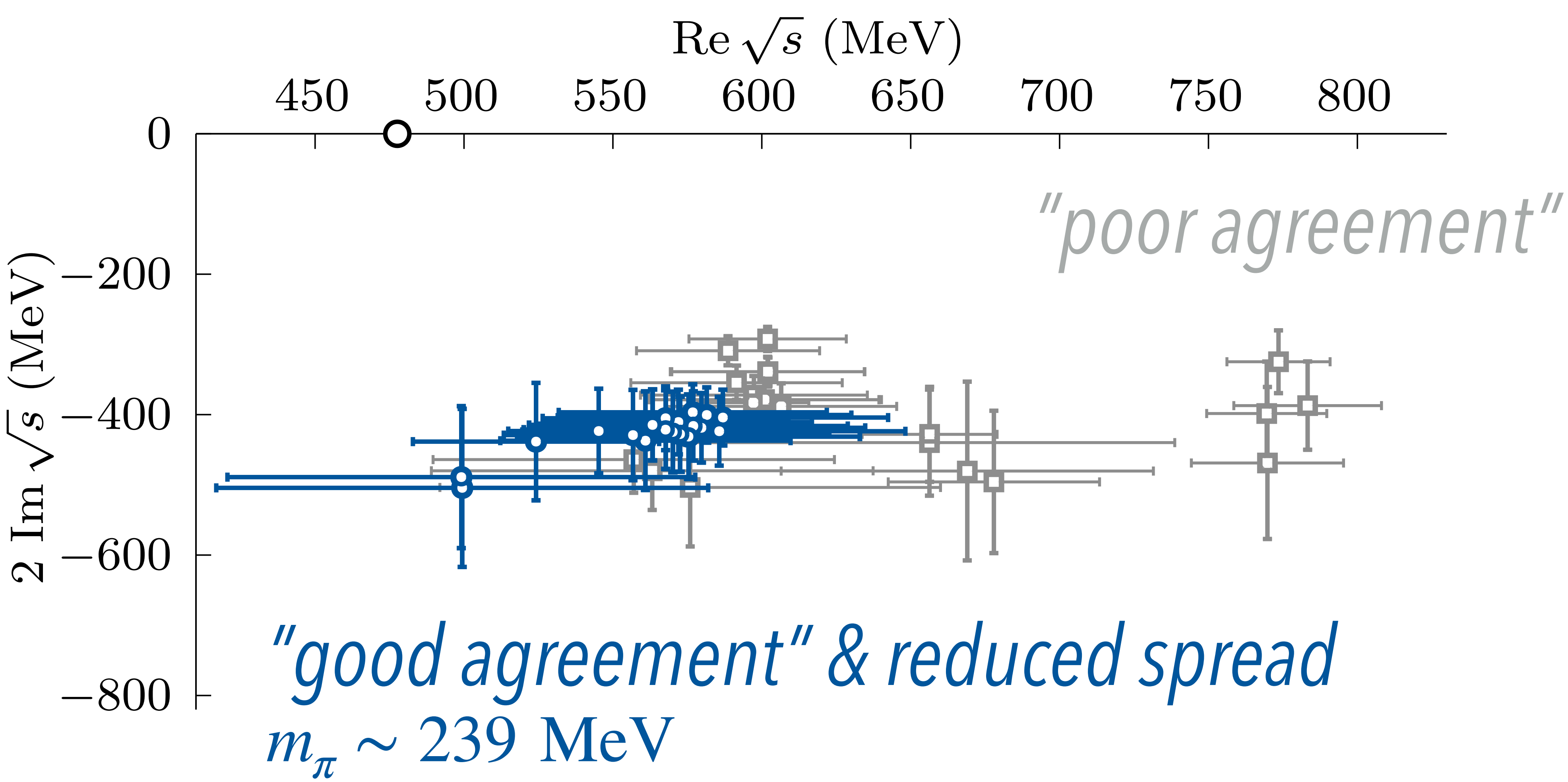
DETERMINATION OF CROSSING-SYMMETRIC $\pi\pi$ … PHYS. REV. D **109,** 034513 (2024)

91 $\pi\pi$ levels
$M_\pi \approx 240$ MeV



Applications of QC2 are well developed

- LQCD gives spectrum, fit to QC2 with parametrized, truncated $\mathcal{K}_2$, determine $\mathcal{M}_2$, look for poles in complex plane
- State-of-the-art involves multiple channels, particles with spin, as well as decay and transition amplitudes
- Nice recent example [Rodas et al., 2304.03762 (PRD)] for $\pi\pi \to \sigma/f_0(500) \to \pi\pi$



Three-particle formalism

[Hansen & SS, 2014 & 2015]

[Blanton & SS, 2020]

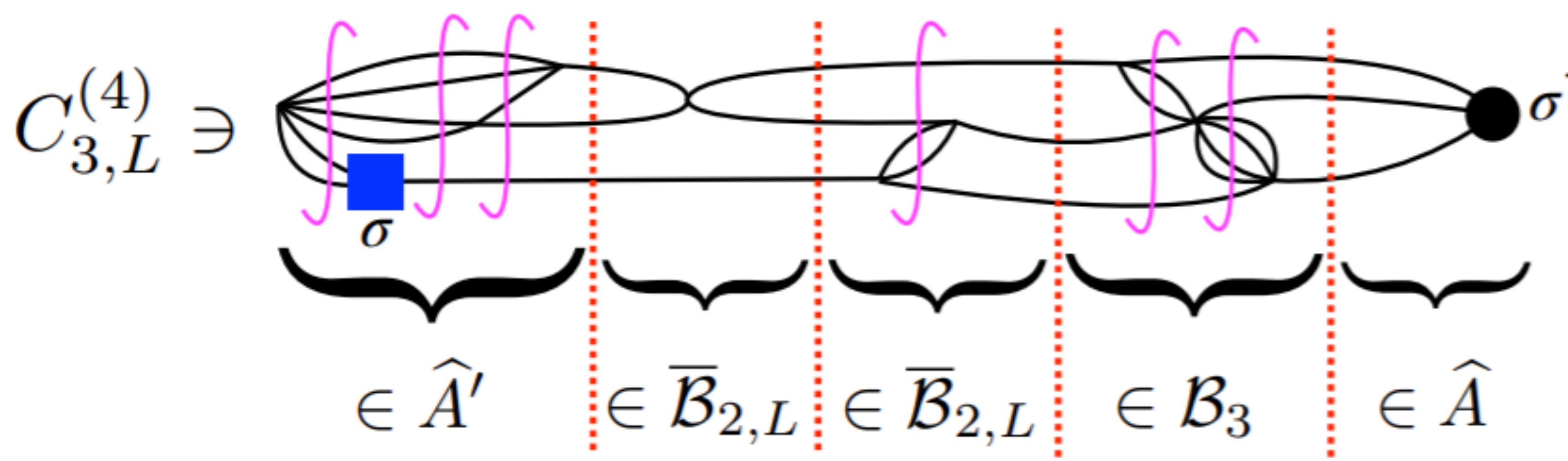
RFT approach

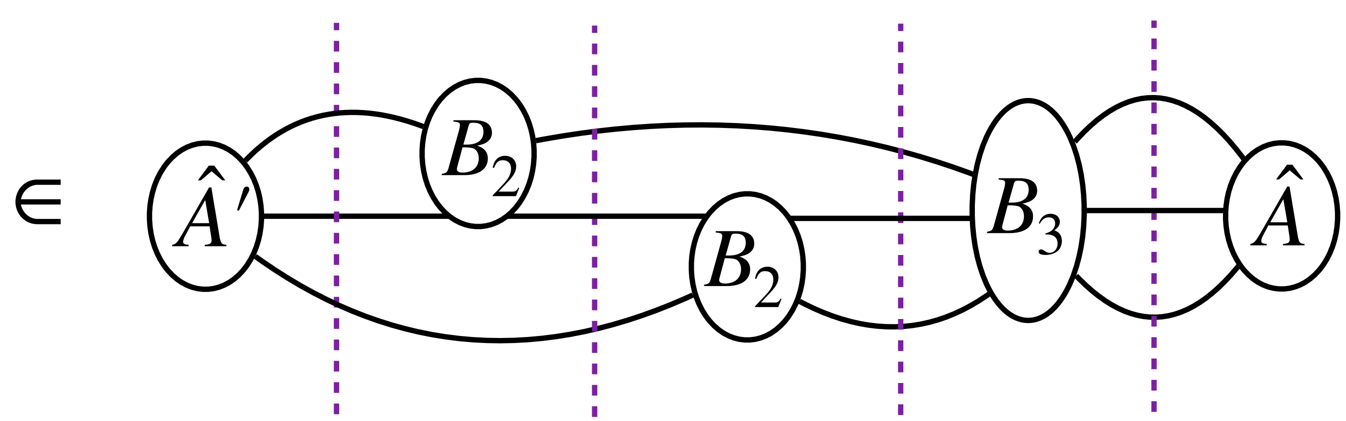
- Study Minkowski time, finite-volume correlator, and look for poles

$$C_L(E, \vec{P}) \equiv \int_L d^4x\, e^{iEt - i\vec{P}\cdot\vec{x}} \langle \Omega | T\left\{ \sigma_{3\pi}(x)\sigma_{3\pi}^\dagger(0) \right\} | \Omega \rangle_L$$

  - Restrict kinematic range to $M_\pi < E^* = \sqrt{E^2 - P^2} < 5M_\pi$

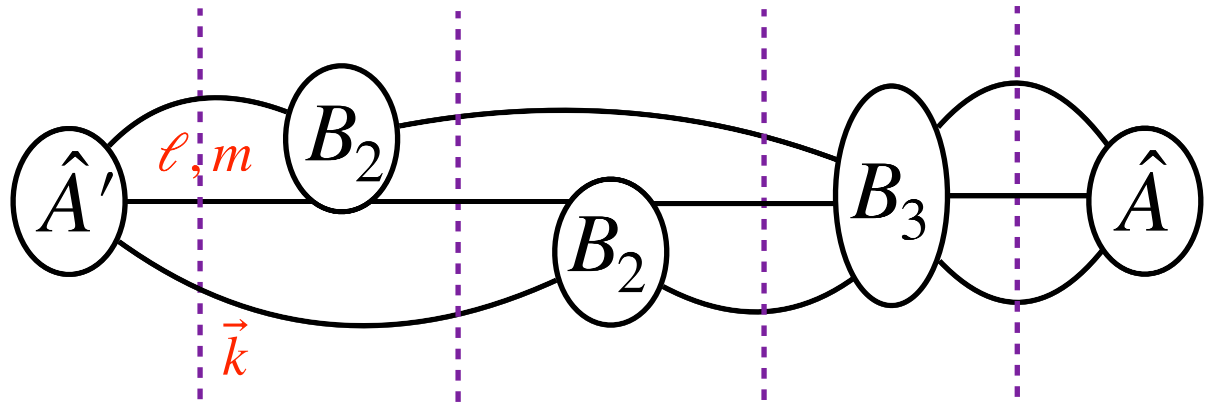
- Use TOPT, and decompose into kernels, separated by relevant (3 particle) cuts





New features

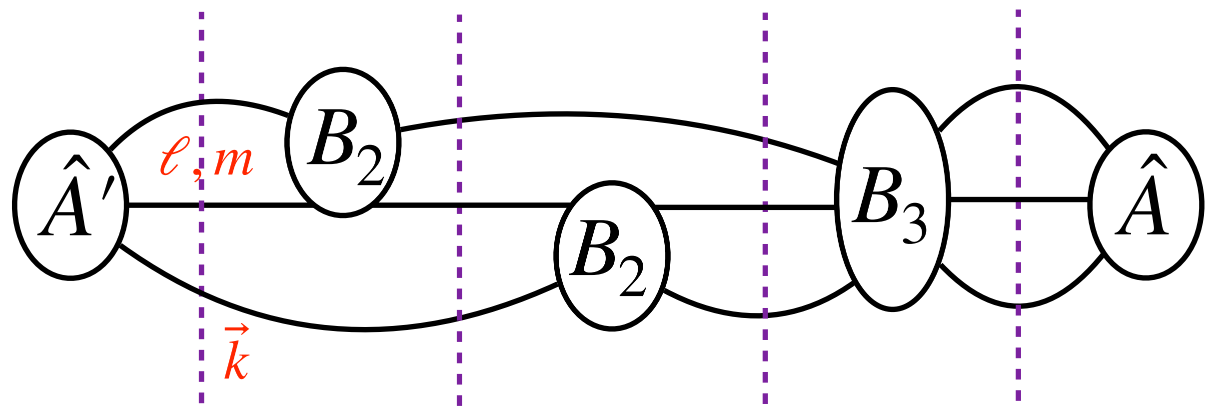
$$C_L(E, \vec{P}) = \ldots +$$



$$+ \ldots$$

- Sum over 3 momenta: when project a pair on shell, have additional finite-volume spectator momentum ⇒ Indices are $\vec{k}, \ell, m$

New features

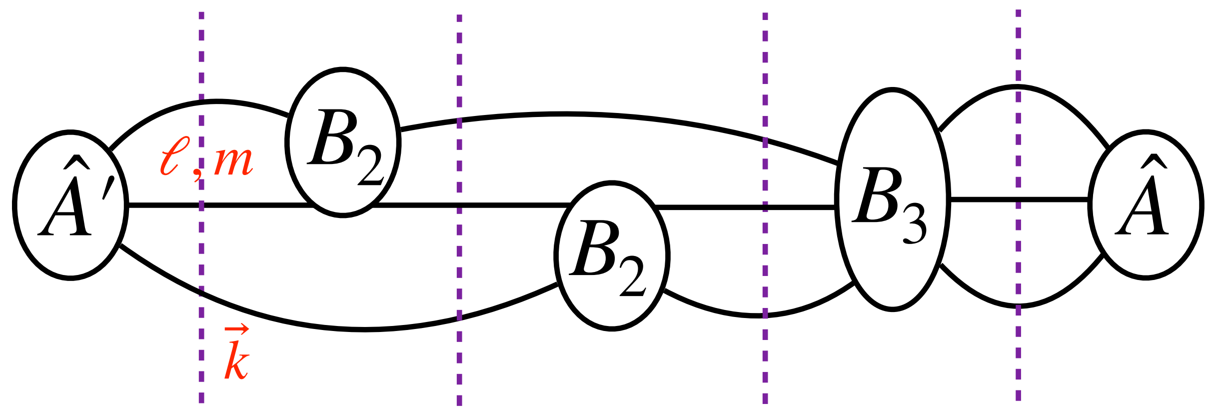
$C_L(E, \vec{P}) = \ldots +$  $+ \ldots$

- Sum over 3 momenta: when project a pair on shell, have additional finite-volume spectator momentum ⇒ Indices are $\vec{k}, \ell, m$

- Sum over spectator momentum leads to subthreshold pair when project on shell
  - Introduce smooth cutoff function so pair cannot go too far below threshold
  - Truncates sum over $\vec{k}$, and avoids left-hand cut in B_2

New features

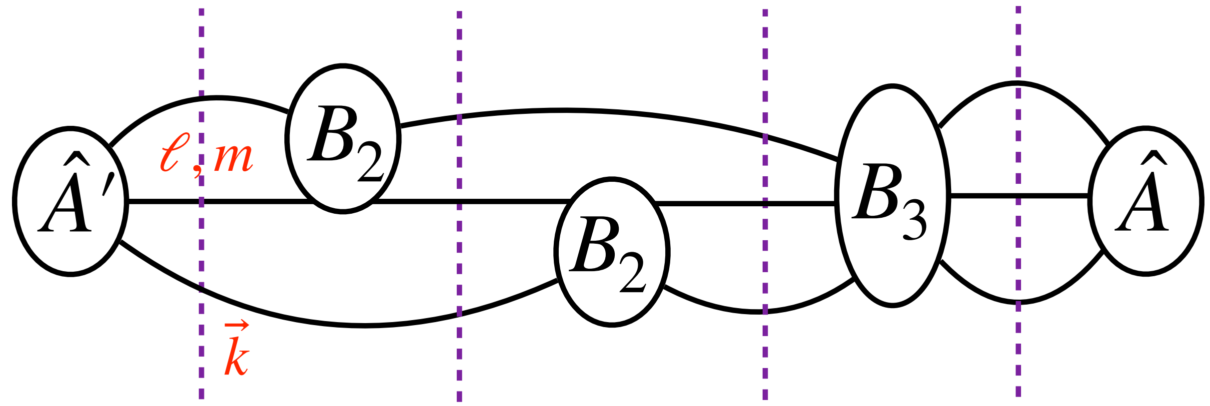
$C_L(E, \vec{P}) = \ldots +$



$+ \ldots$

- Sum over 3 momenta: when project a pair on shell, have additional finite-volume spectator momentum ⇒ Indices are $\vec{k}, \ell, m$
- Sum over spectator momentum leads to subthreshold pair when project on shell
  - Introduce smooth cutoff function so pair cannot go too far below threshold
  - Truncates sum over $\vec{k}$, and avoids left-hand cut in B_2
- Switches between spectators: leads to two types of finite-volume kinematic function, F and G

New features

$C_L(E, \vec{P}) = \ldots +$  $+\ldots$

- Sum over 3 momenta: when project a pair on shell, have additional finite-volume spectator momentum ⇒ Indices are $\vec{k}, \ell, m$

- Sum over spectator momentum leads to subthreshold pair when project on shell
  - Introduce smooth cutoff function so pair cannot go too far below threshold
  - Truncates sum over $\vec{k}$, and avoids left-hand cut in B_2

- Switches between spectators: leads to two types of finite-volume kinematic function, F and G

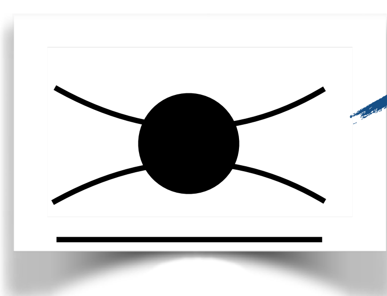
- Tree particle Bethe-Salpeter kernel B_3: once "dressed" it will become $\mathscr{K}_{\mathrm{df},3}$

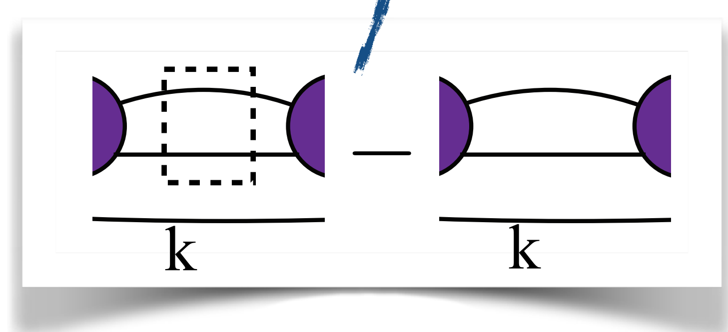
…skipping over details…

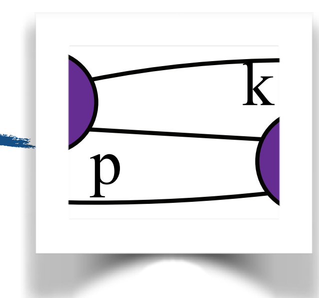
- Can reorganize into geometric series and sum to find poles
  - Involves $\mathcal{K}_3$ that is neither Lorentz invariant nor symmetric under particle exchange
- Nasty algebraic reorganization brings $\mathcal{K}_3$ into symmetric, Lorentz-invariant form

QC3: $\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$ [cf. QC2: $\det(F^{-1} + \mathcal{K}_2) = 0$]

$$F_3 = \frac{1}{2\omega L^3}\left[\frac{\widetilde{F}}{3} - \widetilde{F}\,\frac{1}{1/\widetilde{\mathcal{K}}_{2,L} + \widetilde{F} + \widetilde{G}}\,\widetilde{F}\right]$$







Explicit forms

- F & G are known geometrical functions, containing cutoff function $H(k)$

$$\widetilde{F}_{p\ell'm';k\ell m} = \delta_{pk}\ H(\vec{k})\ F_{\ell'm';\ell m}(E-\omega_k, \overrightarrow{P}-\vec{k}, L)$$

$$F_{\ell'm';\ell m}(E,\overrightarrow{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \mathrm{PV}\int\frac{d^3k}{(2\pi)^3}\right)\frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k})}$$

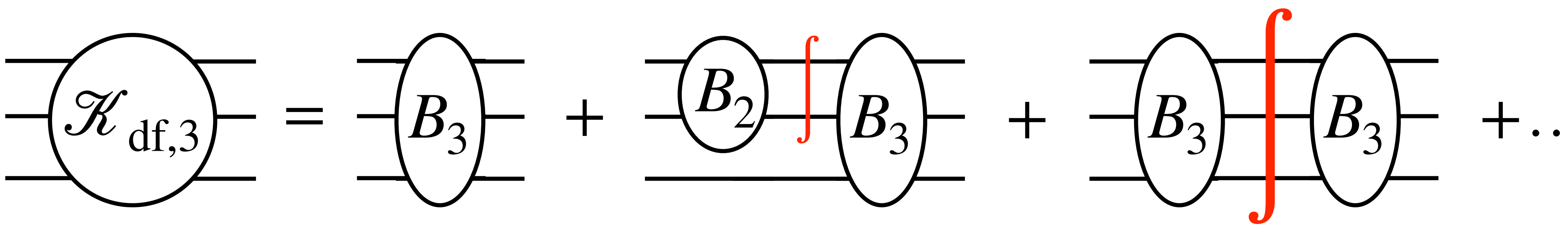
$$\mathcal{Y}_{\ell m}(\vec{k}^*) = \sqrt{4\pi}\left(\frac{k^*}{q^*}\right)^{\ell} Y_{\ell m}(\hat{k}^*)$$

$$G_{p\ell'm';k\ell m} = \left(\frac{k^*}{q_p^*}\right)^{\ell'}\frac{4\pi Y_{\ell'm'}(\hat{k}^*)H(\vec{p})H(\vec{k})Y^*_{\ell m}(\hat{p}^*)}{(P-k-p)^2-m^2}\left(\frac{p^*}{q_k^*}\right)^{\ell}\frac{1}{2\omega_k L^3}$$

Relativistic form introduced in [BHS17]

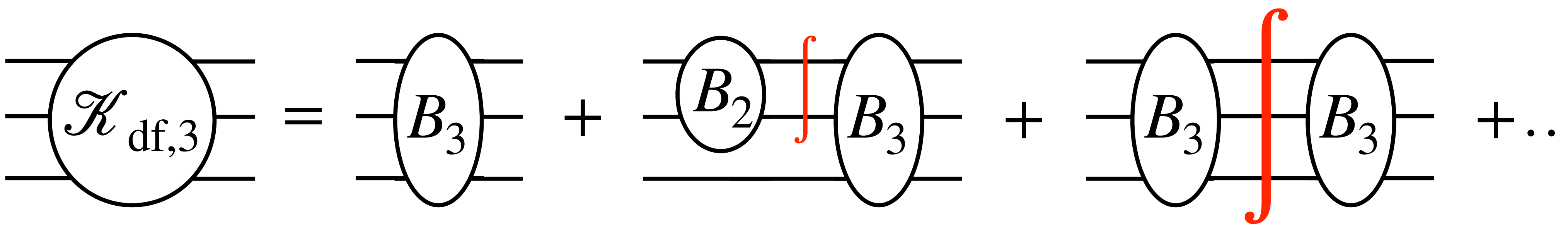
$\mathcal{K}_{\mathrm{df},3}$

- $\mathcal{K}_{\mathrm{df},3}$ has known, complicated expression; can crudely represent as



$\mathcal{K}_{\mathrm{df},3}$

- $\mathcal{K}_{\mathrm{df},3}$ has known, complicated expression; can crudely represent as



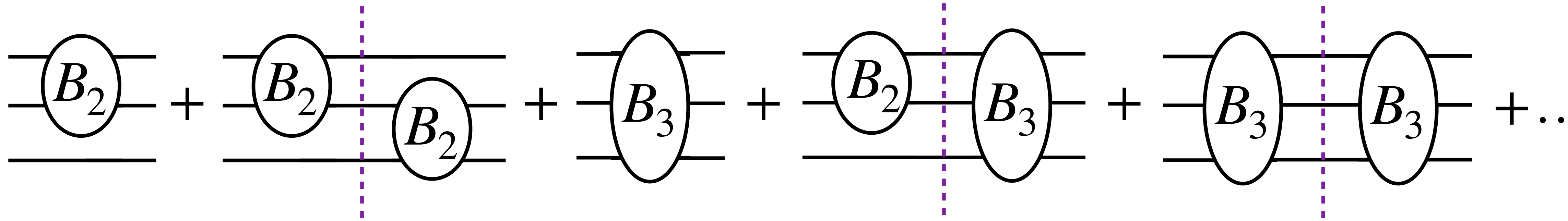
- Key properties:
  - Infinite-volume quantity, with same symmetries as $\mathcal{M}_3$
  - Unlike $\mathcal{M}_3$, does not contain one-particle exchange singularities
  - Real, smooth function of momenta, aside from possible three-particle poles
  - Relativistically invariant, so can expand about threshold in "effective-range exp."
  - Unphysical since depends on cutoff function $H(\vec{k})$

- Can think of $\mathcal{K}_{\mathrm{df},3}$ as a quasi-local three-particle interaction

Step 2: relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

- Consider "finite-volume scattering amplitude" in TOPT

$\mathcal{M}^{(\mathrm{off})}_{23,L} =$



- Resum geometric series; project onto $\vec{k}, \ell, m$; project on shell; use "key move"; algebraic reorganization; take $L \to \infty$ $(i\epsilon)$ limit

- Result is set of integral equations relating $\mathcal{M}_3$ to $\mathcal{M}_2$ and $\mathcal{K}_{\mathrm{df},3}$ (all on shell)

$$\mathcal{M}_3 = \lim_{L\to\infty} \mathcal{S}\left\{\mathcal{D}^{(u,u)}_L + \mathcal{M}^{(u,u)}_{\mathrm{df,3,L}}\right\}, \qquad \mathcal{S} \Rightarrow \text{symmetrization}$$

$$i\mathcal{D}^{(u,u)}_L = i\mathcal{M}_{2,L} i\tilde{G} i\mathcal{M}_{2,L} \frac{1}{1 - i\tilde{G} i\mathcal{M}_{2,L}}, \qquad \mathcal{M}_{2,L} = 2\omega L^3 \mathcal{M}_2$$

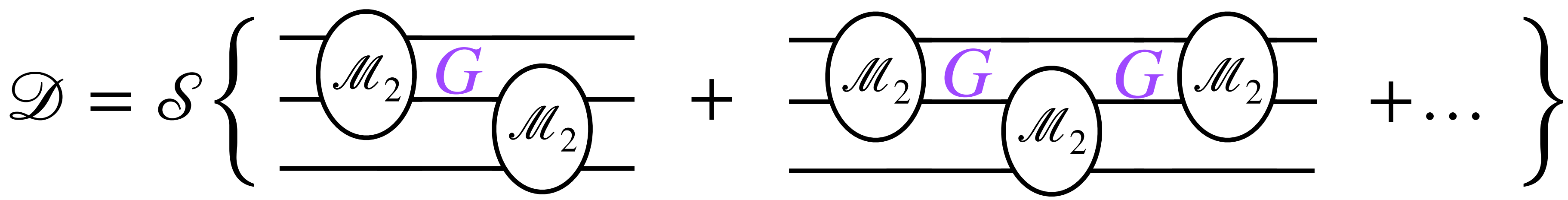
$$i\mathcal{M}^{(u,u)}_{\mathrm{df,3,L}} = \mathcal{L}^{(u)}_L i\mathcal{K}_{\mathrm{df},3} \frac{1}{1 - iF_3 i\mathcal{K}_{\mathrm{df},3}} \mathcal{L}^{(u)\dagger}_L$$

$$\mathcal{L}^{(u)}_L = \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{2,L} i\tilde{G}} i\mathcal{M}_{2,L} i\tilde{F}$$

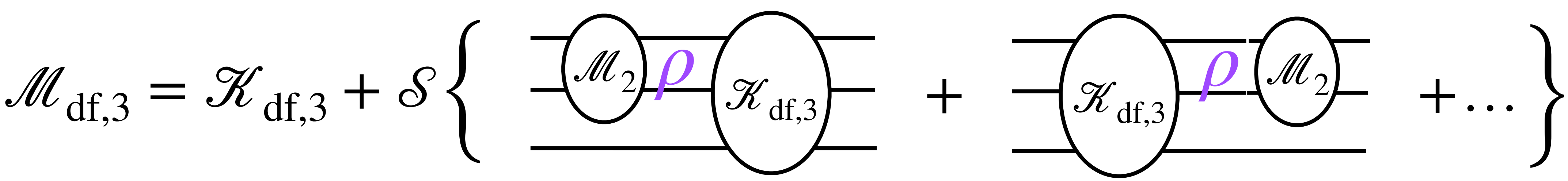
Step 2: relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

$$\mathcal{M}_3 = \lim_{L\to\infty} \mathcal{S}\left\{ \mathcal{D}_L^{(u,u)} + \mathcal{M}_{\mathrm{df,3,L}}^{(u,u)} \right\} = \mathcal{D} + \mathcal{M}_{\mathrm{df},3}$$

- $\mathcal{D}$ contains all divergent contributions to $\mathcal{M}_3$, but depends on cutoff function $H(\vec{k})$



- $\mathcal{M}_{\mathrm{df},3}$ is divergence-free, equals $\mathcal{K}_{\mathrm{df},3}$ at leading order, and is also cutoff-dependent



- "Decorations" ensure that $\mathcal{M}_3$ is unitary
- Methods for solving integral equations, and analytically continuing to complex momenta, are now well established [Briceño, Dawid, Hansen, Islam, Jackura, 2020-23]
  - In practice, project on definite overall J^P

Tests of formalism [Refs. at end]

Tests of formalism [Refs. at end]

- Expansion in L^{-1} of ground-state 3-particle energy agrees with NRQM through L^{-5}
  - Agreement extended to L^{-6} in relativistic ϕ^4 theory at 3-loop order
- Volume dependence of energy and form factor of Efimov "trimer" matches NRQM
- s-channel unitarity of $\mathscr{M}_3$
- Decomposition into $\mathscr{D} + \mathscr{M}_{\rm df,3}$ checked at NLO in ChPT for 3π
  - Leads to NLO ChPT prediction for $\mathscr{K}_{\rm df,3}$
- Three approaches to deriving formalism lead to equivalent results

Status: formalism

- 3 identical spinless particles [Hansen & SRS 14,15 (RFT); Hammer, Pang, Rusetsky 17 (NREFT); Mai, Döring 17 (FVU)]
  - Applications: $3\pi^+,\ 3K^+$, as well as ϕ^4 theory
- Mixing of two- and three-particle channels for identical spinless particles [Briceño, Hansen, SRS 17]
  - Step on the way to $N(1440) \to N\pi,\ N\pi\pi$, etc.
- 3 degenerate but distinguishable spinless particles, e.g 3π with isospin 0, 1, 2, 3 [Hansen, Romero-López, SRS 20]; $I = 1$ case in FVU approach [Mai et al., 21]
  - Potential applications: $\omega(782),\ a_1(1260),\ h_1(1170),\ \pi(1300), \ldots$
- 3 nondegenerate spinless particles [Blanton, SRS 20]
  - Potential applications: $D_s^+ D^0 \pi^-$
- 2 identical +1 different spinless particles [Blanton, SRS 21]
  - Applications: $\pi^+\pi^+K^+$, $K^+K^+\pi^+$
- 3 identical spin-½ particles [Draper, Hansen, Romero-López, SRS 23]
  - Potential applications: $3n,\ 3p,\ 3\Lambda$
- $DD\pi$ for all isospins (also $BB\pi,\ KK\pi$) [Draper, Hansen, Romero-López, SRS 23]
  - Potential applications: $T_{cc} \to D^*D$ incorporating LH cut
- Multiple three-particle channels: $\eta\pi\pi + K\overline{K}\pi$ [Draper & SRS 24]
  - Potential applications: $b_1(1235)$, $\eta(1295)$

Applications of 3-particle formalism:

Fitting $\mathcal{K}_2, \mathcal{K}_{\text{df},3}$ to $\pi^+\pi^+K^+$ spectra from LQCD

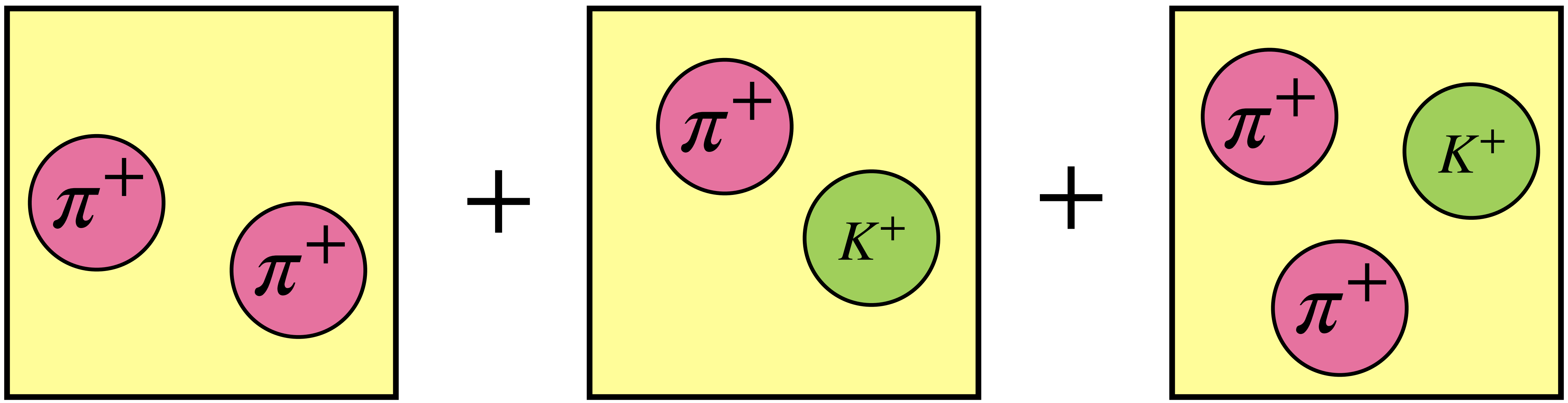
[Draper, Hanlon, Hörz, Morningstar, Romero-López & SRS, 2302.13587 (JHEP)]





$\pi^+\pi^+K^+$ interactions

- System with weakly repulsive interactions
  - No resonances in two-particle subchannels or in three-particle system
- Simultaneously fit to several spectra to QC2/QC3 to obtain $\mathcal{K}_2$ and $\mathcal{K}_{\rm df,3}$



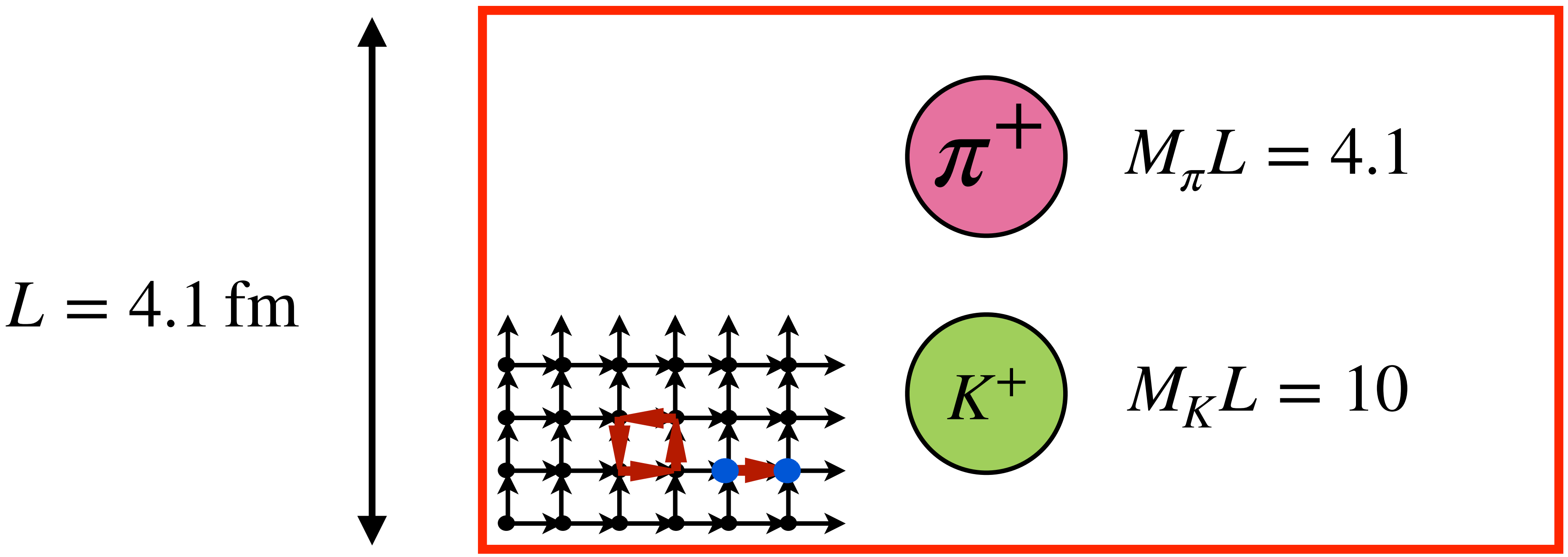
- Parametrize $\mathcal{K}_{\rm df,3}$ (and $\mathcal{K}_2$) as the most general smooth functions consistent with particle interchange, time-reversal and parity symmetries, using an expansion about threshold
  - s-wave interactions in $\pi^+\pi^+$ (sub)channel, s- and p-wave in π^+K^+; 9 or 10 parameters in all

Lattices used in pilot calculation

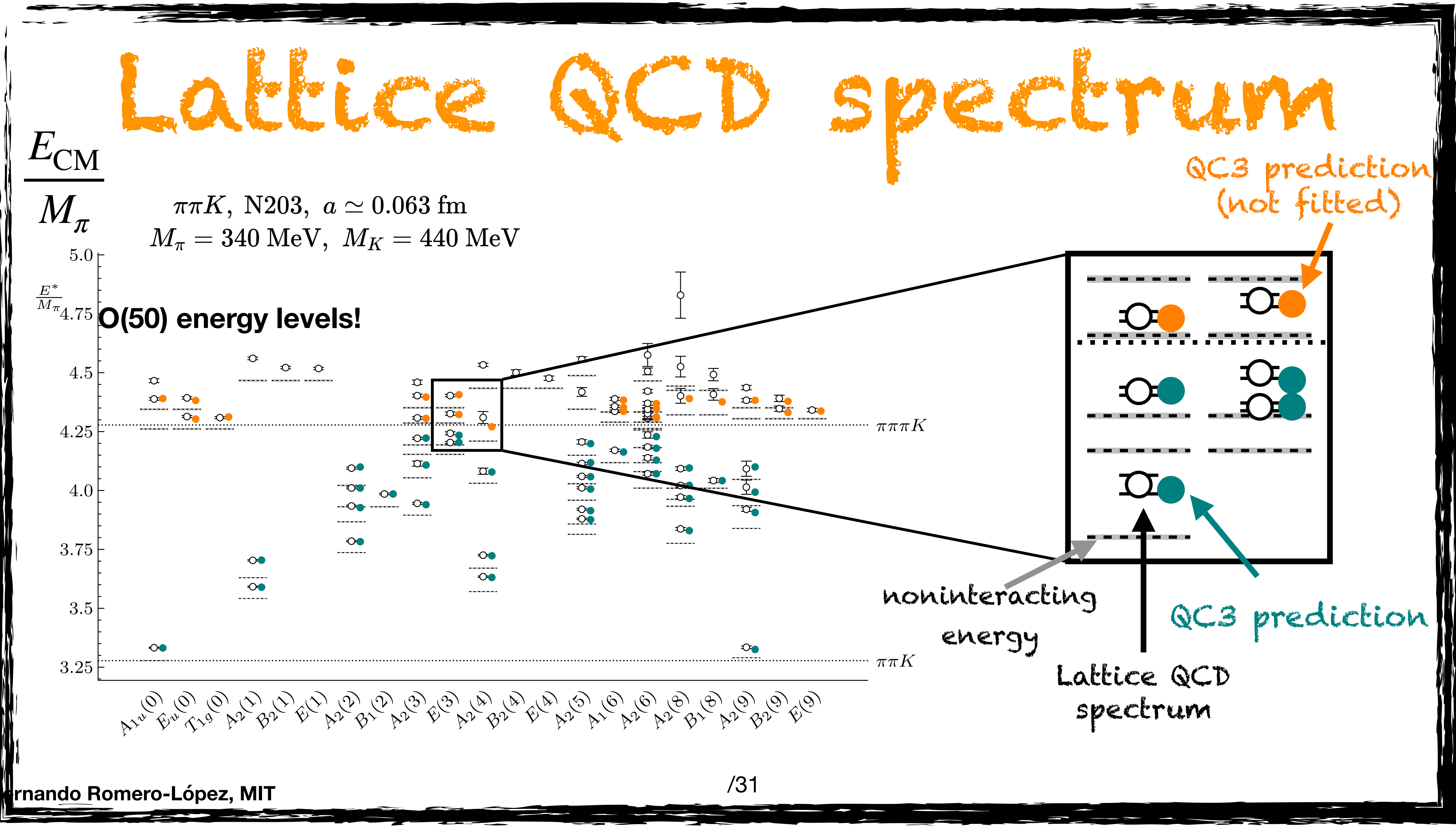
- Improved Wilson fermions at $a = 0.064\,\mathrm{fm}$ (CLS lattices)

	$(L/a)^3 \times (T/a)$	M_π [MeV]	M_K [MeV]	$N_{\rm cfg}$	$t_{\rm src}/a$	$N_{\rm ev}$	dilution	$N_r(\ell/s)$
N203	$48^3 \times 128$	340	440	771	32, 52	192	(LI12,SF)	6/3
D200	$64^3 \times 128$	200	480	2000	35, 92	448	(LI16,SF)	6/3

D200 configurations



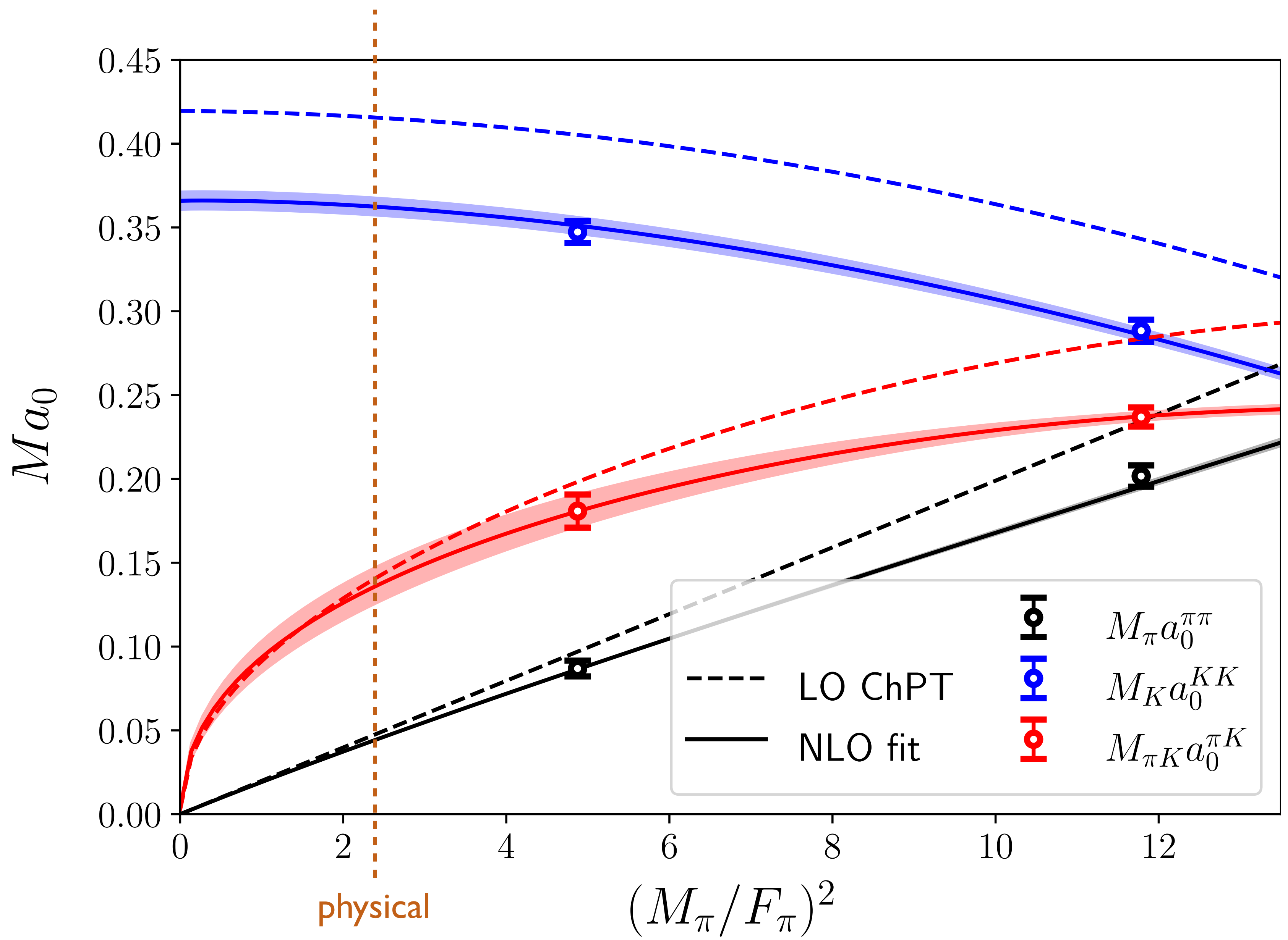
Example of fit



Simultaneous fit to 27 $\pi^+\pi^+$, 19 π^+K^+, & 36 $\pi^+\pi^+K^+$ $\Rightarrow$ 82 levels with 9 parameters

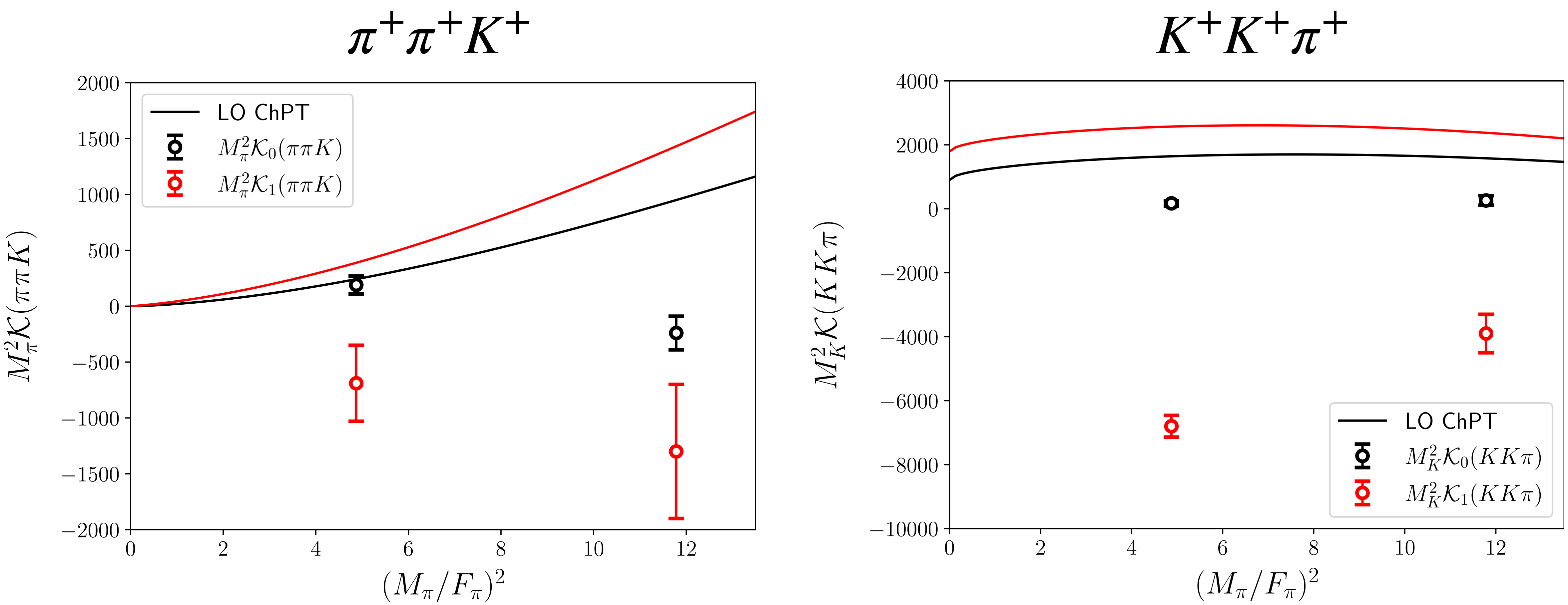
$\chi^2/\mathrm{DOF} = 119/(82-9)$

Results: scattering lengths



- 2-particle s-wave scattering lengths are well determined
- All are repulsive and consistent with ChPT
  - Evidence for small discretization errors

s-wave contributions to $\mathcal{K}_{\text{df},3}$

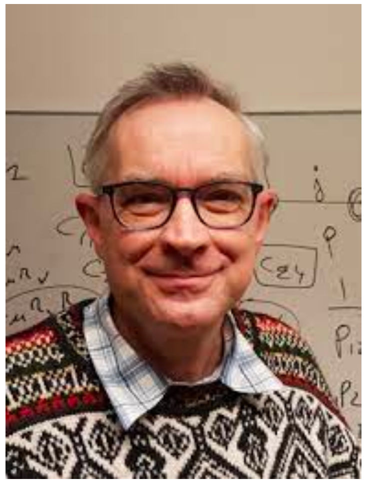


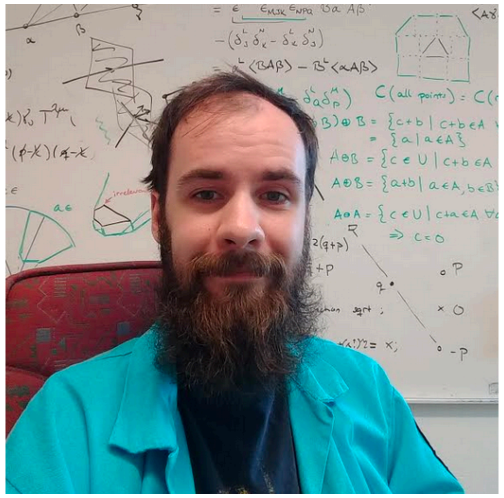
- Evidence for nonzero values ($2-5\sigma$)
- Overall effect of $\mathcal{K}_{\text{df},3}$ is repulsive
- LO ChPT predicts opposite sign (but see later)

Applications of 3-particle formalism:

Calculating $\mathscr{K}_{\rm df,3}$ for $3\pi \to 3\pi$ in ChPT

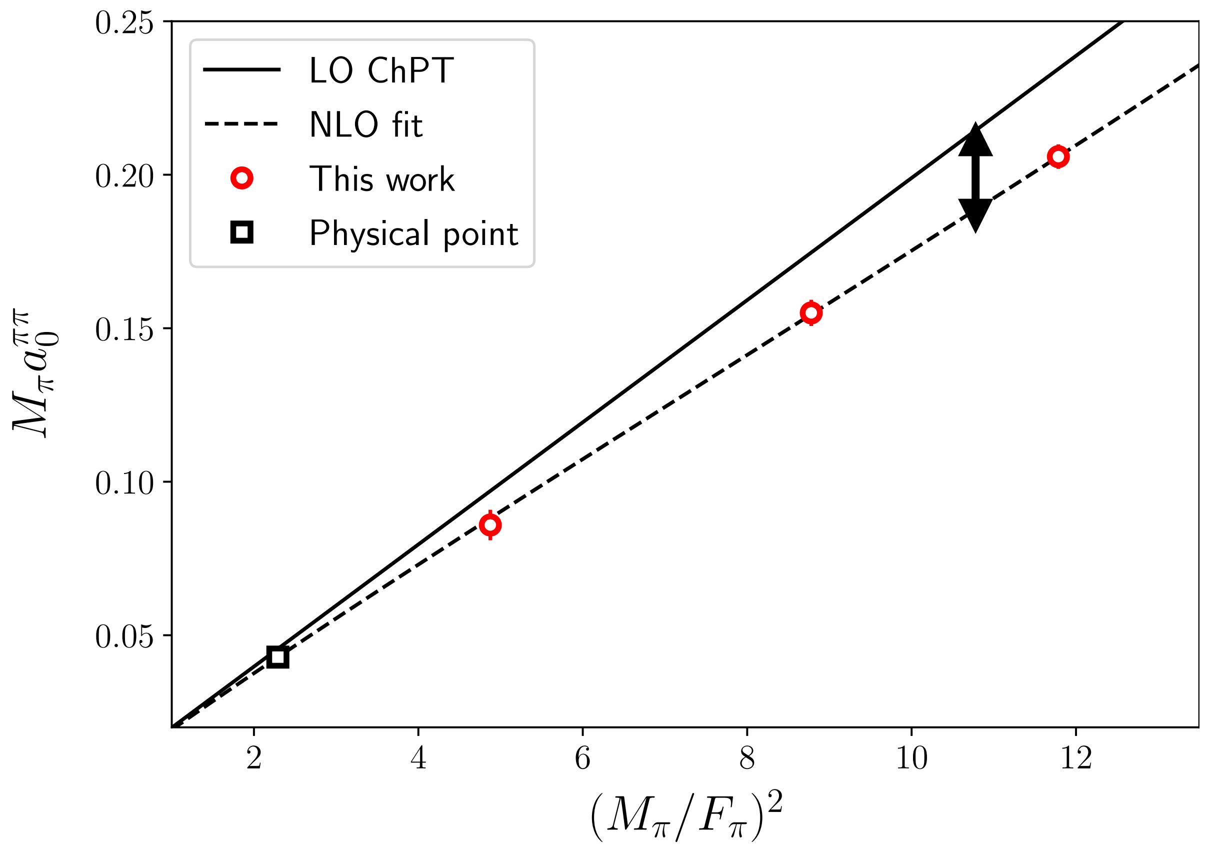
[Baeza-Ballesteros, Bijnens, Husek, Romero-López, SRS, Sjö, 2303.13206 (JHEP) & 2401.14293 (JHEP)]

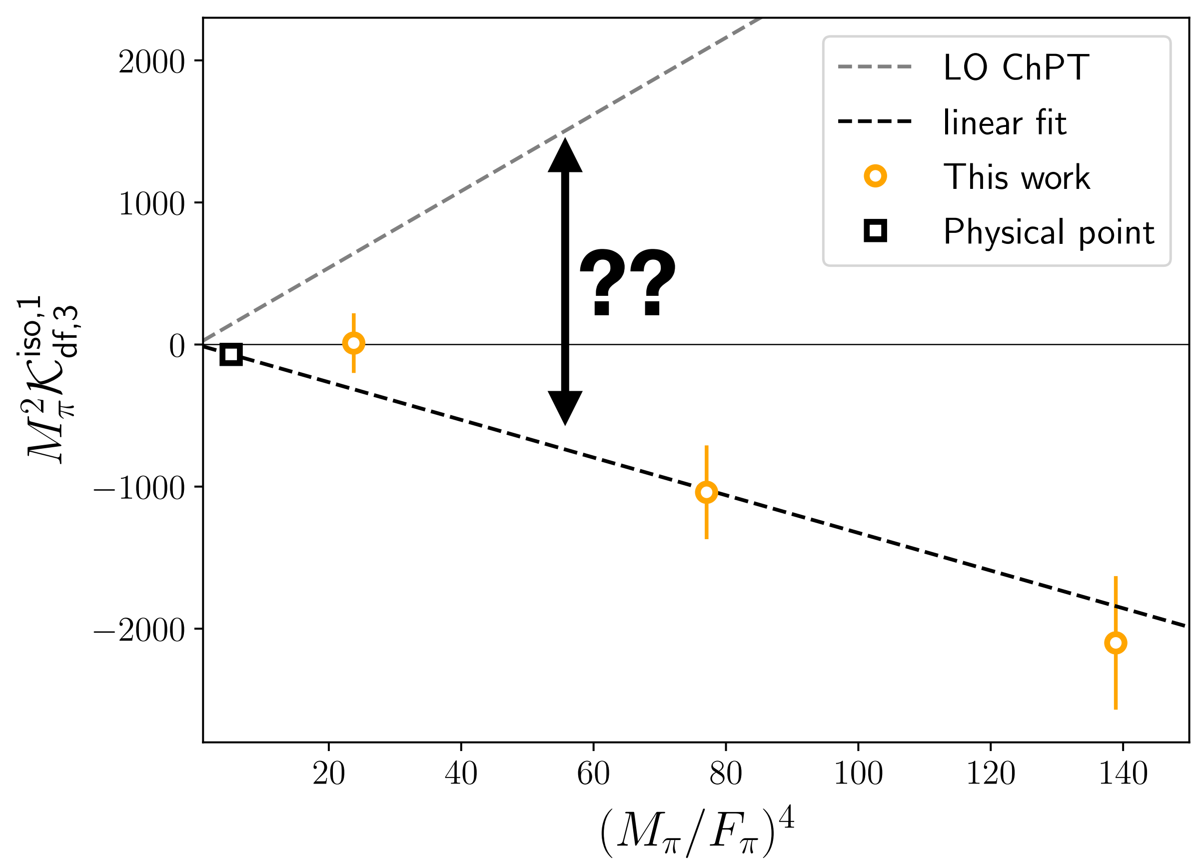



$2\pi/3\pi$ K matrices vs ChPT

$2\pi^+$ scattering length



$3\pi^+$ K matrix

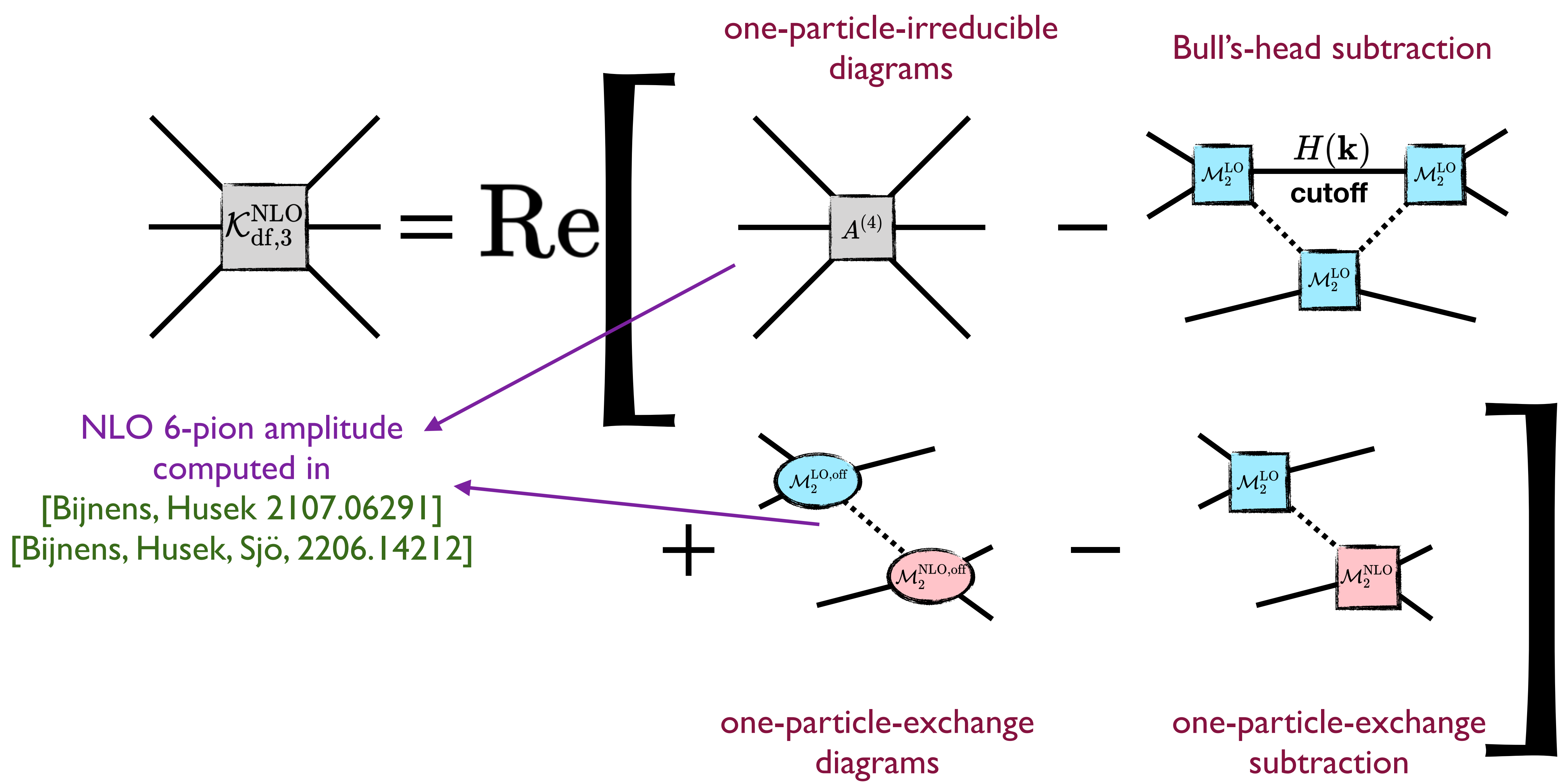


[Results from Blanton, Hanlon, Hörz, Morningstar, Romero-López, SRS, 2106.05590 (JHEP)]

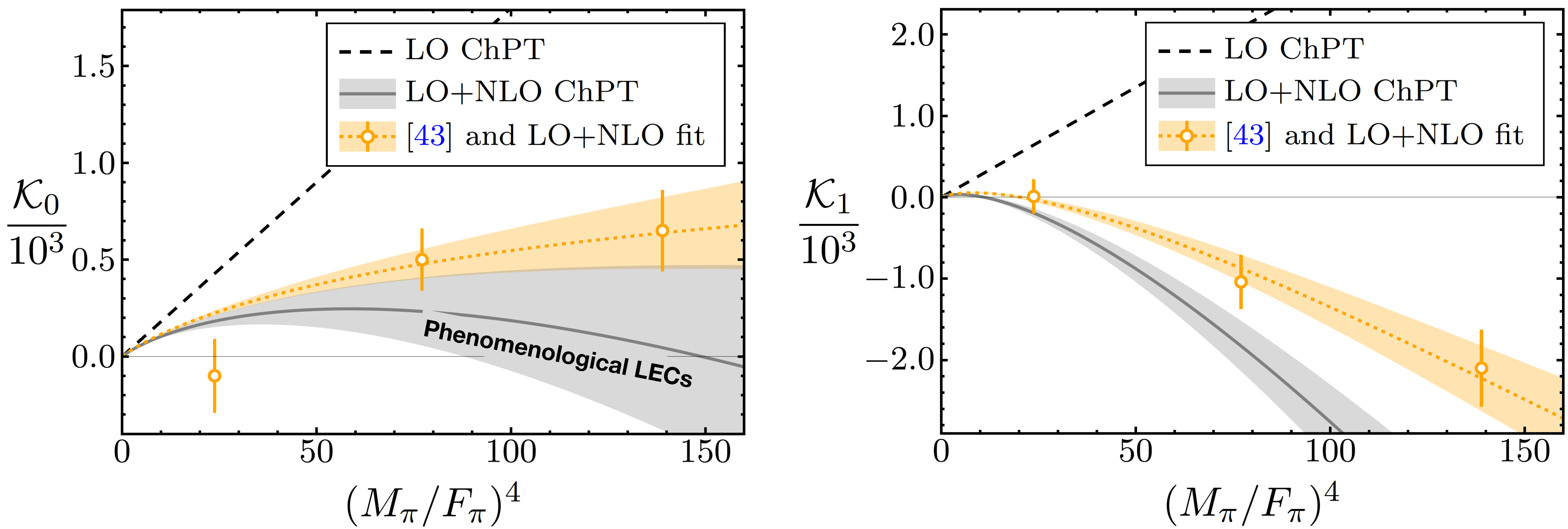
- LO ChPT describes 2-pion sector well
- Large discrepancy in 3-pion sector!

NLO ChPT for $\mathcal{K}_{\rm df,3}$

- Integral equations simplify to: $\mathcal{K}_{\rm df,3}^{\rm NLO} = \text{Re}\, \mathcal{M}_{\rm df,3}^{\rm NLO}$



Comparison to LQCD



- (Very) large NLO corrections
  - Discrepancy with LO ChPT resolved!
  - ChPT not trustworthy for $\mathscr{K}_1$

Applications of 3-particle formalism:

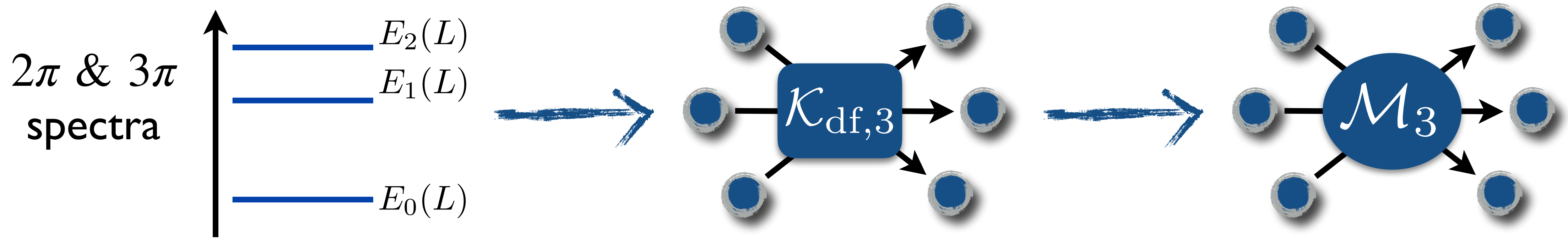
Results for $\mathcal{M}(3\pi \to 3\pi)$ at nearly physical quark masses

[Dawid, Draper, Hanlon, Hörz, Skinner, Morningstar, Romero-López & SRS, in progress]

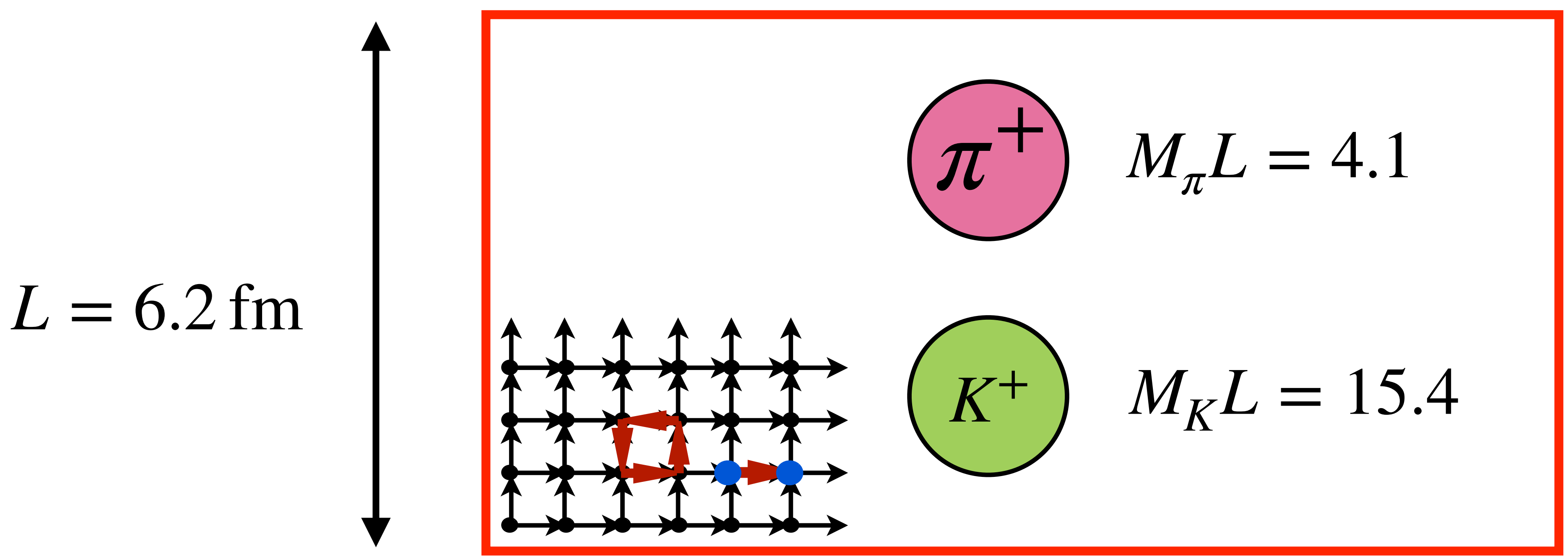




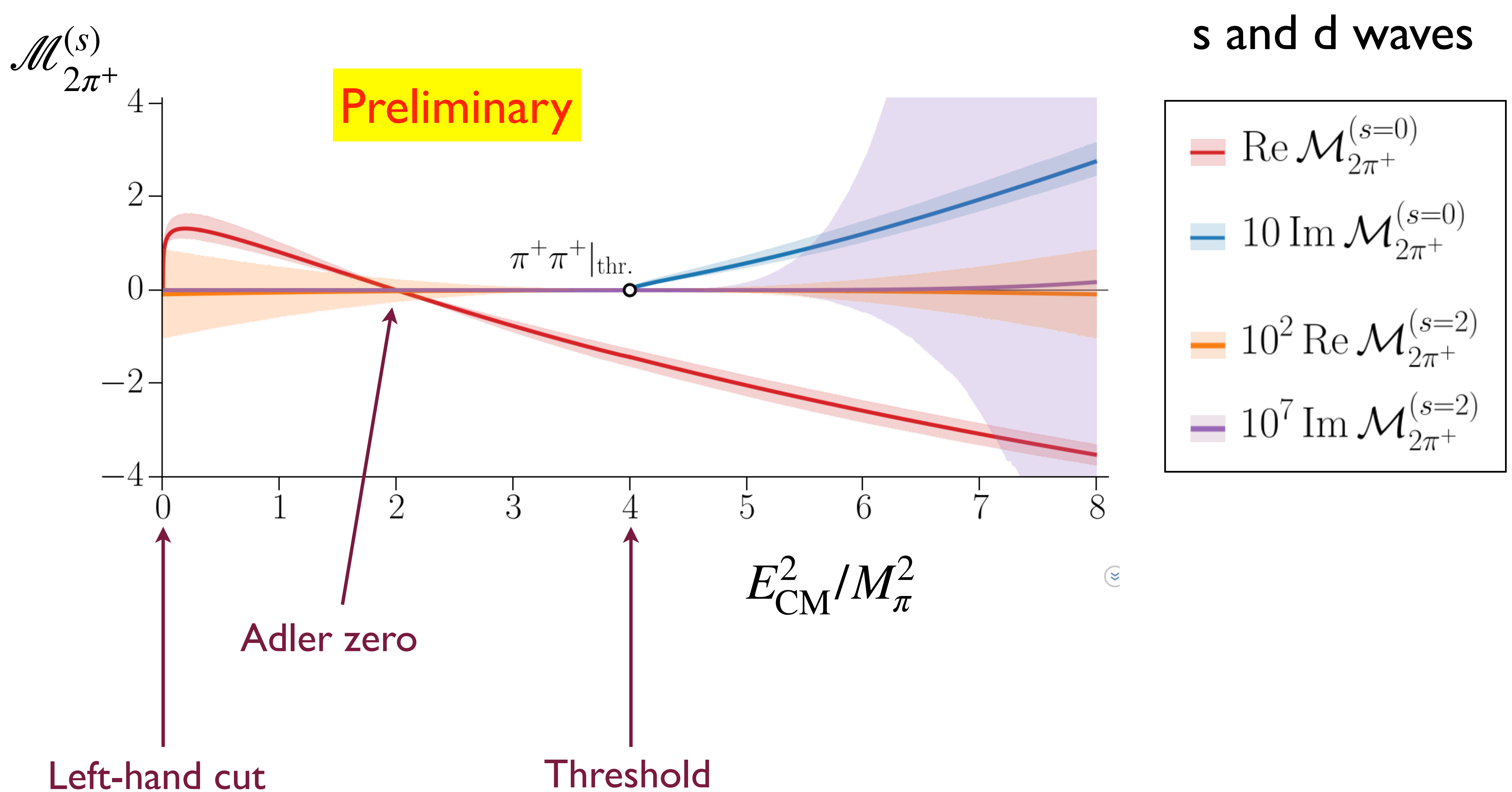

Example of complete application



- First calculation used $M_\pi \approx 390$ MeV, $a \approx 0.12$ fm, $L \approx 2.5$ & 2.9 fm
    - [Hansen, Briceño, Dudek, Edwards, Wilson (HADSPEC collaboration), 2009.04931, PRL 21]
- We use almost physical quark masses (E250 CLS ensemble, 500 configurations)
    - $96^3 \times 192, a = 0.064$ fm, $M_\pi = 130(1)$ MeV, $M_K = 488(5)$ MeV (isosymmetric)

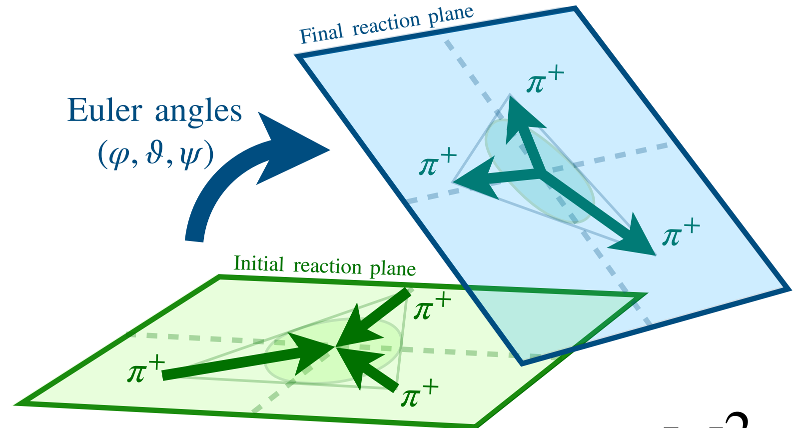


$2\pi^+$ amplitudes



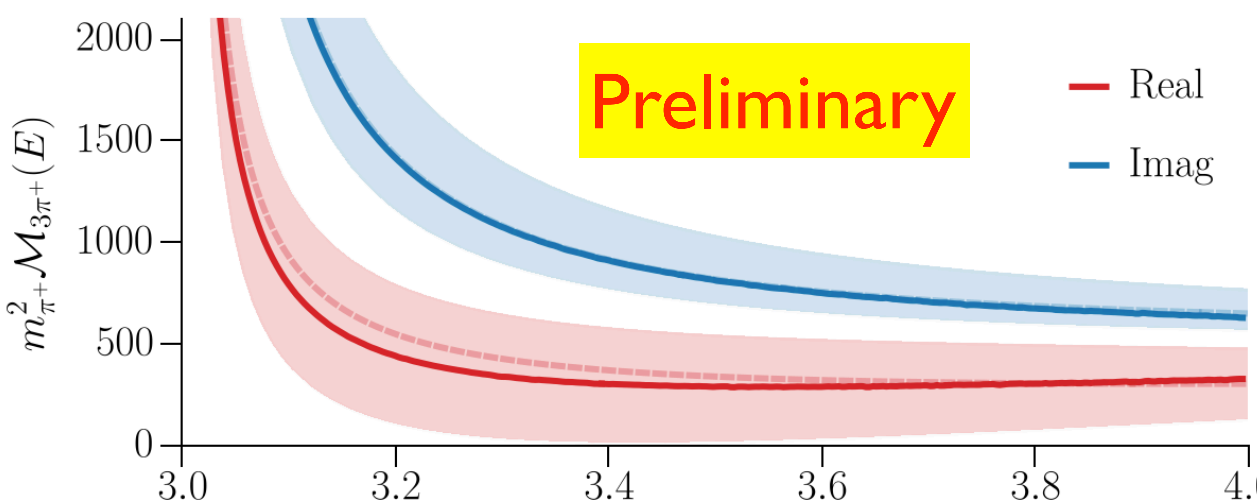
$3\pi^+$ amplitude ($J^P = 0^-$)

Equilateral triangle configuration



s & d waves in $\mathcal{K}_2$ & $\mathcal{K}_{\mathrm{df},3}$

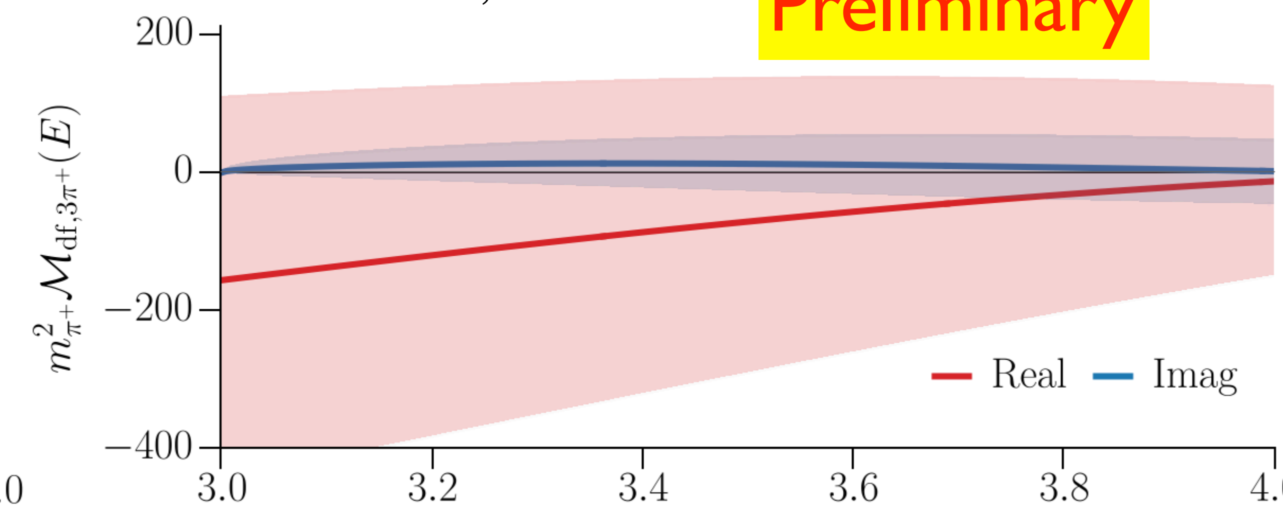
$M_\pi^2 \mathcal{M}_{3\pi^+}$



Threshold

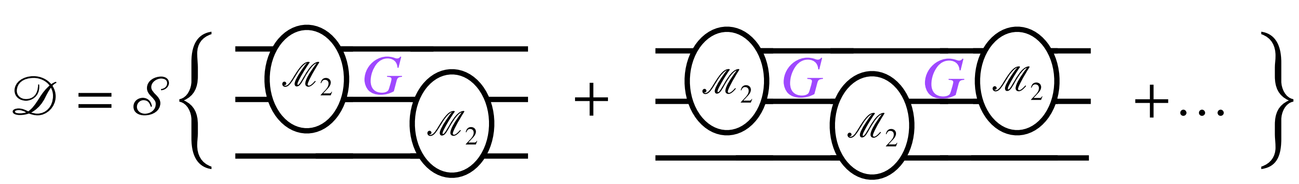
E_{CM}/M_π

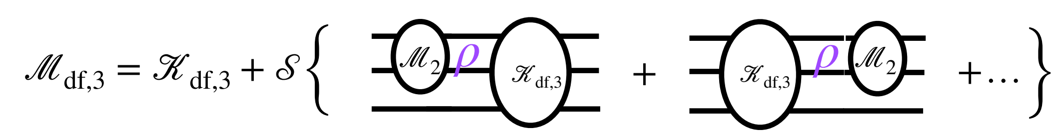
$M_\pi^2 \mathcal{M}_{\mathrm{df},3}$



E_{CM}/M_π

$$\mathcal{M}_3 = \mathcal{D} + \mathcal{M}_{\mathrm{df},3}$$





Summary & outlook

Summary & Outlook

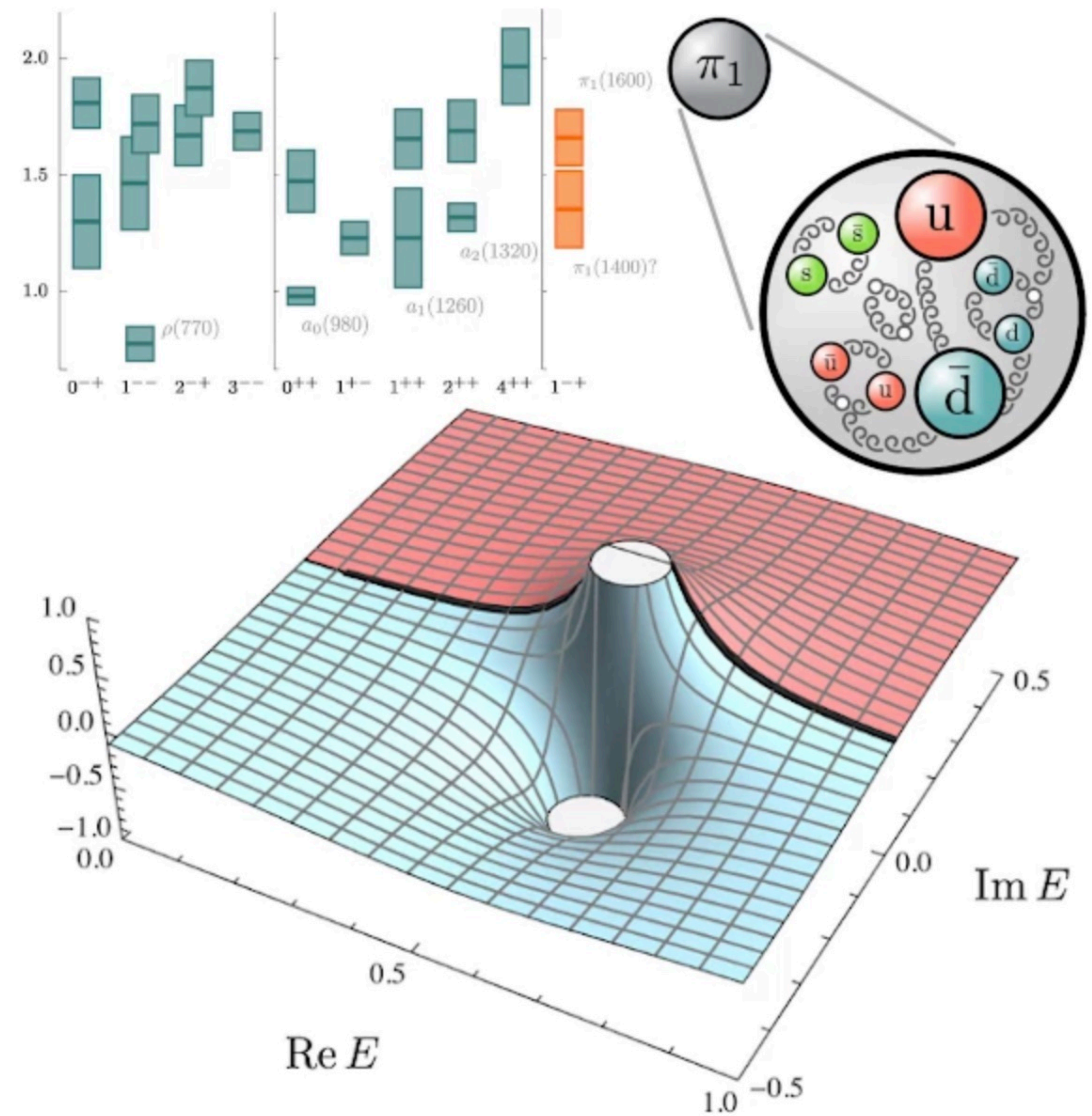
- Two-particle sector is entering precision phase
  - Frontier is two nucleons, and form factors of mesonic resonances
- Major steps have been taken in the three-particle sector
  - Formalism well established & cross checked, and almost complete
  - Several applications to three-particle spectra from LQCD
  - Initial discrepancy with LO ChPT explained by large NLO contributions
  - Path to a calculation of $K \to 3\pi$ decay amplitudes is now open
- Next steps in implementation
  - $T_{cc}^{+} \to D^{*}D \to DD\pi$
  - $3\pi(I=2) \leftrightarrow \rho\pi\,;\; 3\pi(I=0) \leftrightarrow \omega(782) \leftrightarrow K\overline{K}$(I=0) (WZW term)
  - $N\pi\pi \leftrightarrow \Delta\pi\,;\; N\pi\pi + N\pi$ [Roper]
- Next steps in formalism
  - $NNN(I=\frac{1}{2}), N\pi\pi + N\pi$ [for Roper] & $NN\pi + NN$ (all underway)
  - Four particles!

ExoHad collaboration

ExoHad Collaboration exohad.org People Events Talks Publications

EXOHAD
EXOTIC HADRONS TOPICAL COLLABORATION

The Exo(tic) Had(ron) Collaboration started in 2023 to explore all aspects of exotic hadron physics, from predictions within lattice QCD, through reliable extraction of their existence and properties from experimental data, to descriptions of their structure within phenomenological models.



Thank you!
Questions?

References

(Highly-)selected 2-particle refs

★ **Original papers**

- M. Lüscher, Commun.Math.Phys. 105 (1986) 153-188; Nucl.Phys.B 364 (1991) 237-251 [Derived QC2 using NRQM and proving relation to QFT]

- L. Lellouch & M. Lüscher, Commun.Math.Phys. 219 (2001) 31-44; arXiv:hep-lat/0003023 [Determined LL factors relating finite- and infinite-volume matrix elements]

★ **Generalizations**

- C. Kim, C. Sachrajda, & SRS, Nucl.Phys.B 727 (2005) 218-243; arXiv:hep-lat/0507006 [QFT-based approach; LL factors in moving frames]

- R. Briceño, Phys.Rev.D 89 (2014) 7, 074507; arXiv:1401.3312 [QC2 for arbitrary spin]

RFT 3-particle papers

Max Hansen & SRS:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]

Raúl Briceño, Max Hansen & SRS:

"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]

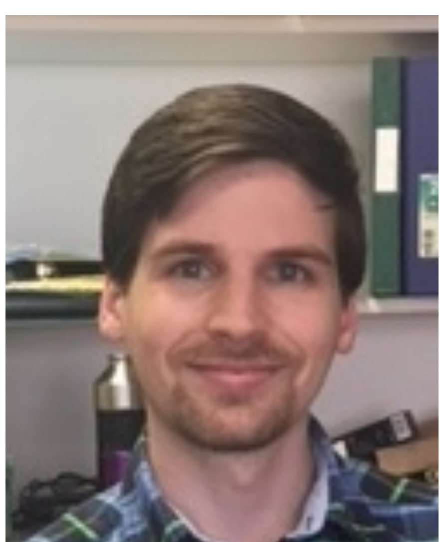
"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

SRS

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory,"

arXiv:1707.04279 (PRD) [SPT17]



Tyler Blanton, Fernando Romero-López & SRS:

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]

"I=3 three-pion scattering amplitude from lattice QCD," arXiv:1909.02973 (PRL) [BRS-PRL19]

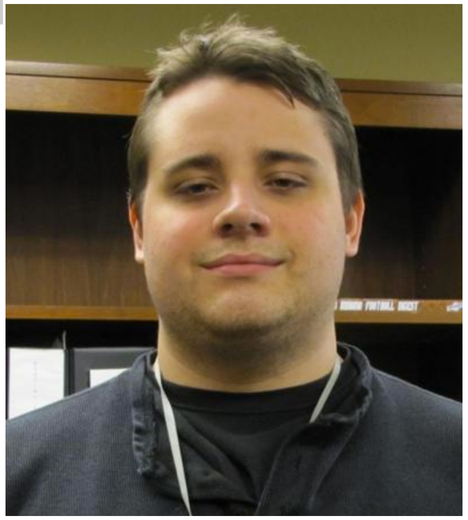
"Implementing the three-particle quantization condition for $\pi^+\pi^+K^+$ and related systems" 2111.12734 (JHEP)

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP) [BBHRS19]

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)



Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

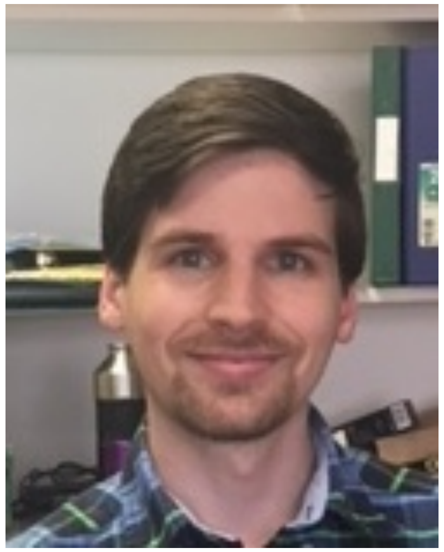
"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

"Generalizing the relativistic quantization condition to include all three-pion isospin channels", arXiv:2003.10974 (JHEP) [HRS20]

"Decay amplitudes to three particles from finite-volume matrix elements," arXiv: 2101.10246 (JHEP)

Tyler Blanton & SRS:



"Alternative derivation of the relativistic three-particle quantization condition," arXiv:2007.16188 (PRD) [BS20a]

"Equivalence of relativistic three-particle quantization conditions," arXiv:2007.16190 (PRD) [BS20b]

"Relativistic three-particle quantization condition for nondegenerate scalars," arXiv:2011.05520 (PRD)

"Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ & related systems," arXiv:2105.12904 (PRD)

Tyler Blanton, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS

"$3\pi^+$ & $3K^+$ interactions beyond leading order from lattice QCD," arXiv:2106.05590 (JHEP)

Zack Draper, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS

"Interactions of πK, $\pi\pi K$ and $KK\pi$ systems at maximal isospin from lattice QCD," arXiv:2302.13587 (JHEP)





<u>**Zach Draper, Max Hansen, Fernando Romero-López & SRS:**</u>

"Three relativistic neutrons in a finite volume," arXiv:2303.10219 (JHEP)

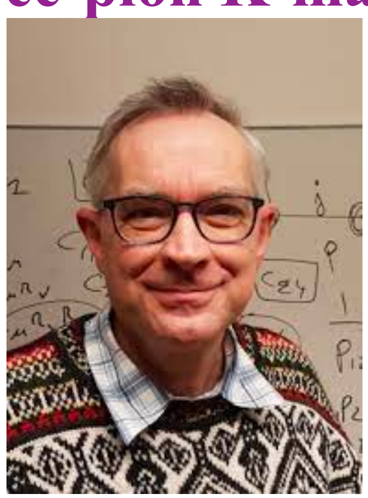
<u>**Zach Draper & SRS:**</u>

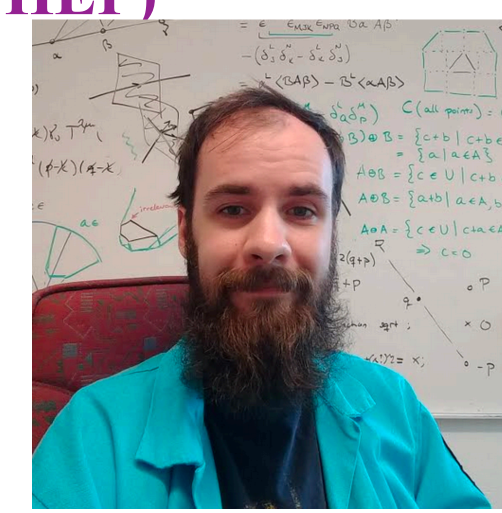
"Three-particle formalism for multiple channels: the $\eta\pi\pi + K\overline{K}\pi$ system in isosymmetric QCD," arXiv:2403.20064 (JHEP)

<u>**Max Hansen, Fernando Romero-López & SRS:**</u>

"Incorporating $DD\pi$ effects and left-hand cuts in lattice studies of the $T_{cc}(3875)^+$," arXiv:2401.06609 (JHEP)

<u>**Jorge Baeza-Ballesteros, Johan Bijnens, Tomas Husek, Fernando Romero-López, SRS & Mattias Sjö:**</u> **"The isospin-3 three-particle K-matrix at NLO in ChPT," arXiv:2303.13206 (JHEP) & "The three-pion K-matrix at NLO in ChPT," arXiv:2401.14293 (JHEP)**





Other work

★ Implementing RFT integral equations

- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating $3\pi^+$ spectrum and using to determine three-particle scattering amplitude]
- A. Jackura et al., 2010.09820 , PRD [Solving s-wave RFT integral equations in presence of bound states]
- S. Dawid, Md. Islam and R. Briceño, 2303.04394 [Analytic continuation of 3-particle amplitudes]
- A. Jackura, 2208.10587, PRD [3-body scattering and quantization conditions from S-matrix unitarity]

★ Reviews

- A. Rusetsky, 1911.01253 [LATTICE 2019 plenary]
- M. Mai, M. Döring and A. Rusetsky, 2103.00577 [Review of formalisms and chiral extrapolations]
- F. Romero-López, 2112.05170, [Three-particle scattering amplitudes from lattice QCD]

★ Other numerical simulations

- F. Romero-López, A. Rusetsky, C. Urbach, 1806.02367, JHEP [2- & 3-body interactions in φ^4 theory]
- M. Fischer et al., 2008.03035, Eur.Phys.J.C [$2\pi^+$ & $3\pi^+$ at physical masses]
- M. Garofolo et al., 2211.05605, JHEP [3-body resonances in φ^4 theory]

Other work

★ **Other RFT (and related) derivations**

- A. Jackura, 2208.10587, PRD [3-body scattering and quantization conditions from S-matrix unitarity]
- R. Briceño, A. Jackura, D. Pefkou & F. Romero-López, 2402.12167, JHEP [Electroweak three-body decays in the presence of two- and three-body bound states]

★ **Implementing RFT integral equations**

- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating $3\pi^+$ spectrum and using to determine three-particle scattering amplitude]
- A. Jackura et al., 2010.09820, PRD [Solving s-wave RFT integral equations in presence of bound states]
- S. Dawid, Md. Islam & R. Briceño, 2303.04394, PRD [Analytic continuation of 3-particle amplitudes]
- S. Dawid, Md. Islam, R. Briceño, & A. Jackura, 2309.01732 [Evolution of Efimov States]
- A. Jackura & R. Briceño, 2312.00625 [Partial-wave projection of the one-particle exchange in three-body scattering amplitudes]

Other work

★ **NREFT approach**

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, 1706.07700, JHEP & 1707.02176, JHEP [Formalism & examples]
- M. Döring et al., 1802.03362, PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111, PRD [large volume expansion for excited levels]
- F. Müller, T. Yu & A. Rusetsky, 2011.14178, PRD [large volume expansion for I=1 three pion ground state]
- F. Romero-López, A. Rusetsky, N. Schlage & C. Urbach, 2010.11715, JHEP [generalized large-volume exps]
- F. Müller & A. Rusetsky, 2012.13957, JHEP [Three-particle analog of Lellouch-Lüscher formula]
- J-Y. Pang, M. Ebert, H-W. Hammer, F. Müller, A. Rusetsky, 2204.04807, JHEP, [Spurious poles in a finite volume]
- F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2110.09351, JHEP [Relativistic-invariant formulation of the NREFT three-particle quantization condition]
- J. Lozano, U. Meißner, F. Romero-López, A. Rusetsky & G. Schierholz, 2205.11316, JHEP [Resonance form factors from finite-volume correlation functions with the external field method]
- F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2211.10126, JHEP [3-particle Lellouch-Lüscher formalism in moving frames]
- R. Bubna, F. Müller, A. Rusetsky, 2304.13635 [Finite-volume energy shift of the three-nucleon ground state]
- J-Y. Pang, R. Bubna, F. Müller, A. Rusetsky, J-J. Wu, 2312.04391 [Lellouch-Lüscher factor for $K \to 3\pi$ decays]
- R. Bubna, H-W. Hammer, F. Müller, J-Y. Pang, A. Rusetsky, 2402.12985 [Lüscher equation with long range forces]

Alternate 3-particle approaches

★ **Finite-volume unitarity (FVU) approach**

- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of M_3 involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749, PRD [applying FVU approach to $3\pi^+$ spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]
- A. Alexandru et al., 2009.12358, PRD [calculating $3K^-$ spectrum and comparing with FVU predictions]
- R. Brett et al., 2101.06144, PRD [determining $3\pi^+$ interaction from LQCD spectrum]
- M. Mai et al., 2107.03973, PRL [three-body dynamics of the $a_1(1260)$ from LQCD]
- D. Dasadivan et al., 2112.03355, PRD [pole position of $a_1(1260)$ in a unitary framework]
- D. Seivert, M. Mai, U-G. Meißner, 2212.02171, JHEP [Particle-dimer approach for the Roper resonance]

★ **HALQCD approach**

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]

Backup slides

Matrix structure in QC3

- All quantities are infinite-dimensional matrices with indices $\mathbf{k}\ell mi$ describing 3 on-shell particles

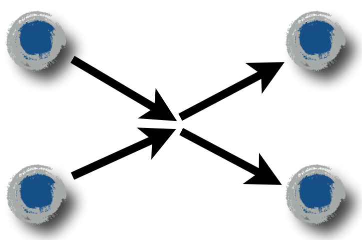
[finite volume "spectator" momentum: $\boldsymbol{k} = (2\pi/L)\boldsymbol{n}$] x [2-particle CM angular momentum: ℓ, m] x [spectator flavor: i]

- For large $\boldsymbol{k}$ (at fixed E, L), the other two particles are below threshold
- Must include such configurations, by analytic continuation, up to a cut-off at $k \approx m$ [Polejaeva & Rusetsky, `12]

Divergence-free K matrix

- $K_{df,3}$ has the same symmetries as M_3: relativistic invariance, particle interchange, T-reversal

M_2, K_2



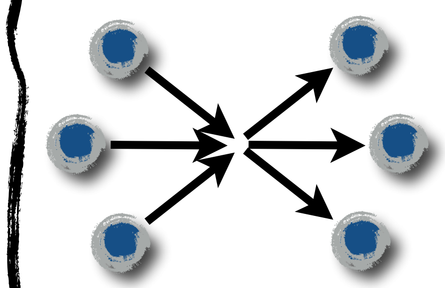
12 momentum components

-10 Poincaré generators

2 degrees of freedom

$s=E^{*2}$ + θ

M_3, $K_{df,3}$



18 momentum components

-10 Poincaré generators

8 degrees of freedom

$s=E^{*2}$ + 7 "angles"

- Need more parameters to describe $\mathcal{K}_{df,3}$ than $\mathcal{K}_2$ (will be discussed in lecture 3)
- Why $\mathcal{K}_2$ and $\mathcal{K}_{df,3}$ appear in QC3, rather than $\mathcal{M}_2$ and $\mathcal{M}_{df,3}$, will be explained shortly

Threshold expansion for $\mathscr{K}_{\mathrm{df},3}$

- $\mathscr{K}_{\mathrm{df},3}$ is a real, smooth function which is Lorentz, P and T invariant
- Expand about threshold in powers of $\Delta = (s - 9M_\pi^2)/9M_\pi^2$, $\tilde{t}_{ij} = (p_i' - p_j)^2/9M_\pi^2, \ldots$

Depend on CM energy | Angular dependence

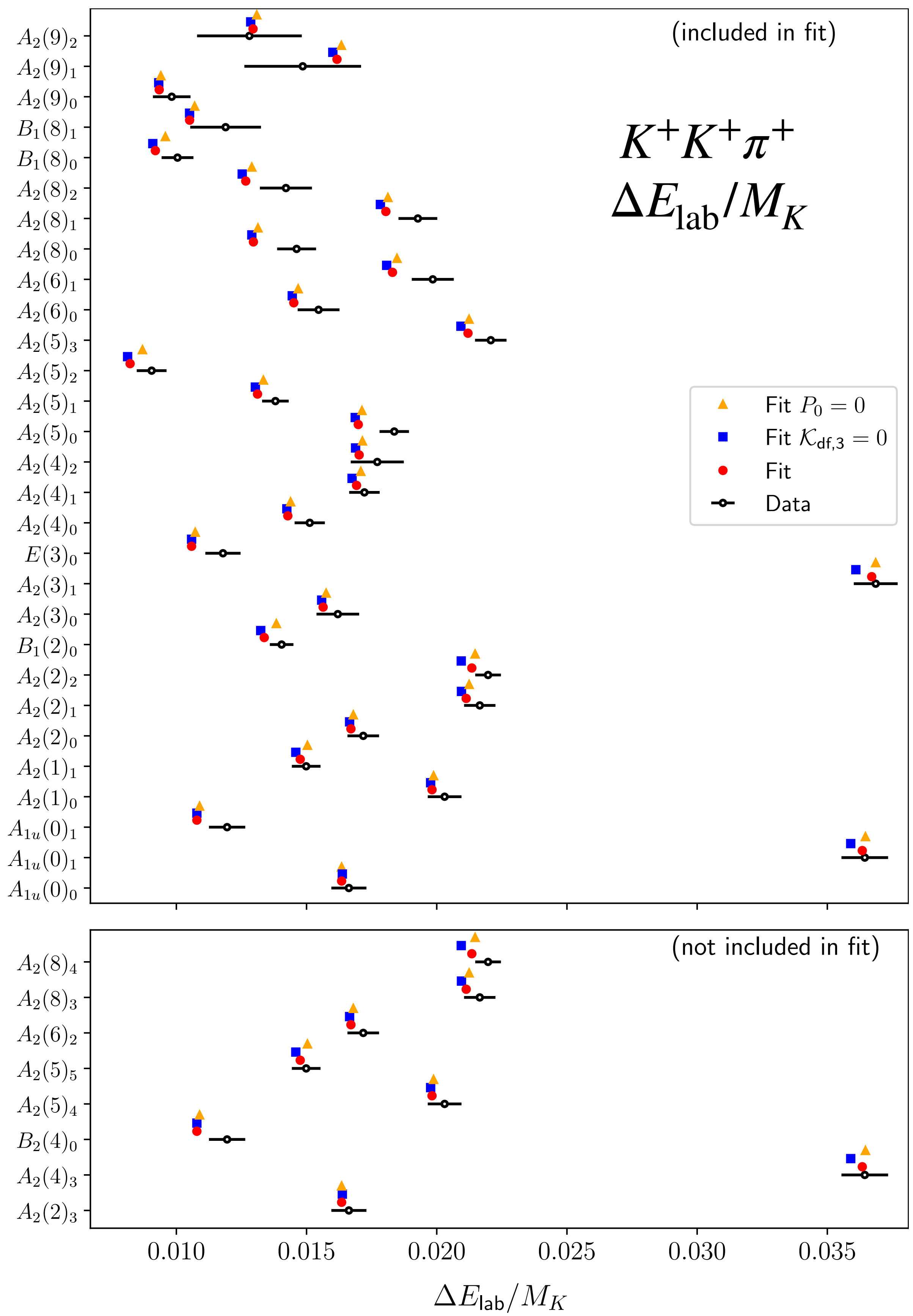
$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},\,1}\Delta + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},\,2}\Delta^2 + \mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B + \mathcal{O}(\Delta^3)$$

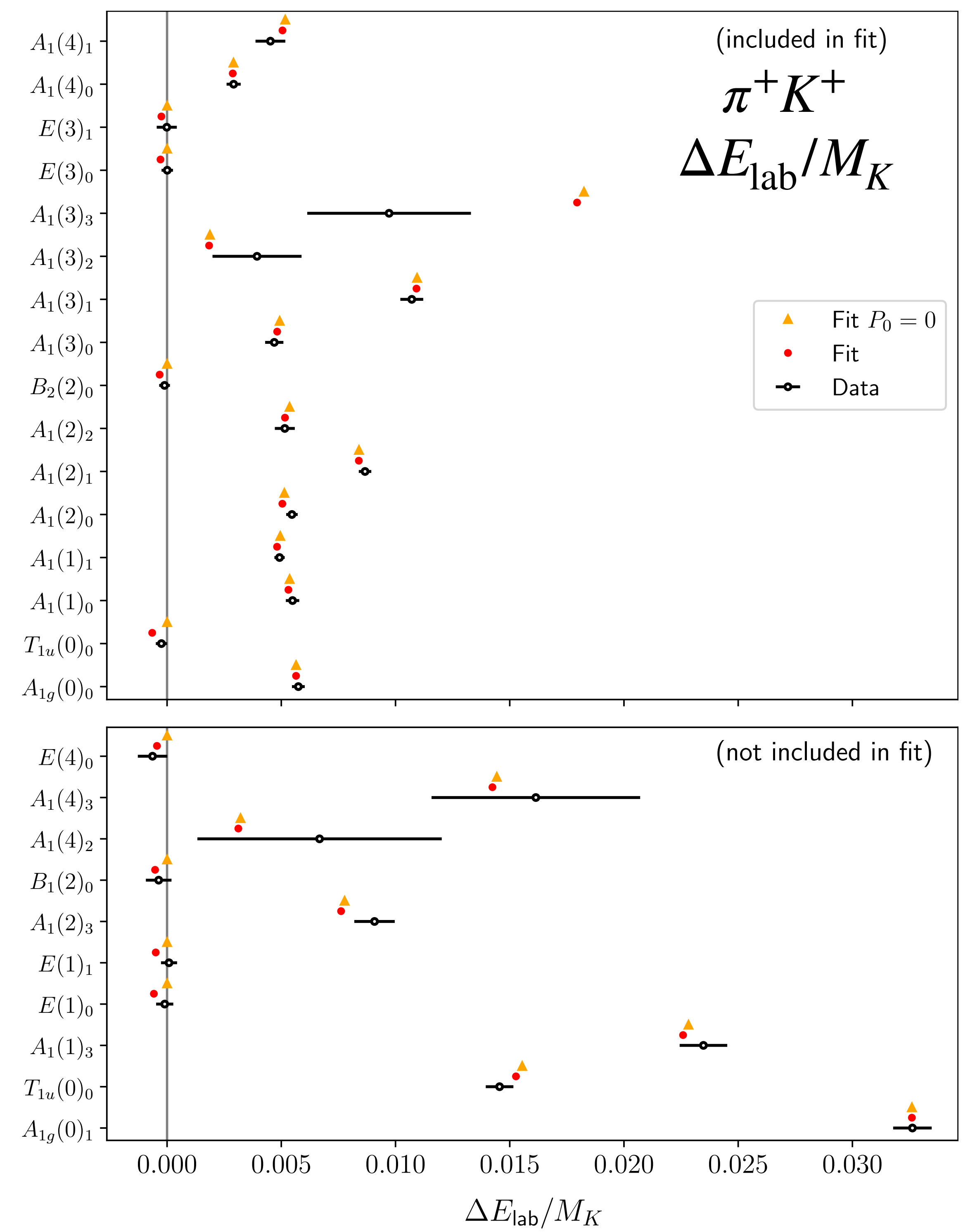
$$\Delta_B = \sum_{i,j=1}^{3} \tilde{t}_{ij}^2 - \Delta^2,$$

- Can separate terms in fit based on dependence on energy and rotational properties
  - E.g. only $\mathscr{K}_B$ contributes to nontrivial irreps

Sensitivity to $\mathcal{K}_{\mathrm{df},3}$

Simultaneous fit to 28 K^+K^+, 16 π^+K^+, & 29 $K^+K^+\pi^+$ levels with 10 parameters on D200: $\chi^2/\mathrm{DOF} = 162/(73-10)$





NLO ChPT results for $\mathscr{K}_{\mathrm{df},3}$

$\kappa = 1/(16\pi^2)$

$$\mathcal{K}_0 = \left(\frac{M_\pi}{F_\pi}\right)^4 18 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-3\kappa(35 + 12\log 3) - \mathcal{D}_0 + 111L + \ell^{\mathrm{r}}_{(0)}\right],$$

$$\mathcal{K}_1 = \left(\frac{M_\pi}{F_\pi}\right)^4 27 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-\frac{\kappa}{20}\left(1999 + 1920\log 3\right) - \mathcal{D}_1 + 384L + \ell^{\mathrm{r}}_{(1)}\right],$$

$$\mathcal{K}_2 = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{207\kappa}{1400}(2923 - 420\log 3) - \mathcal{D}_2 + 360L + \ell^{\mathrm{r}}_{(2)}\right],$$

$$\mathcal{K}_{\mathrm{A}} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{9\kappa}{560}(21809 - 1050\log 3) - \mathcal{D}_{\mathrm{A}} - 9L + \ell^{\mathrm{r}}_{(\mathrm{A})}\right],$$

$$\mathcal{K}_{\mathrm{B}} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{27\kappa}{1400}(6698 - 245\log 3) - \mathcal{D}_{\mathrm{B}} + 54L + \ell^{\mathrm{r}}_{(\mathrm{B})}\right].$$

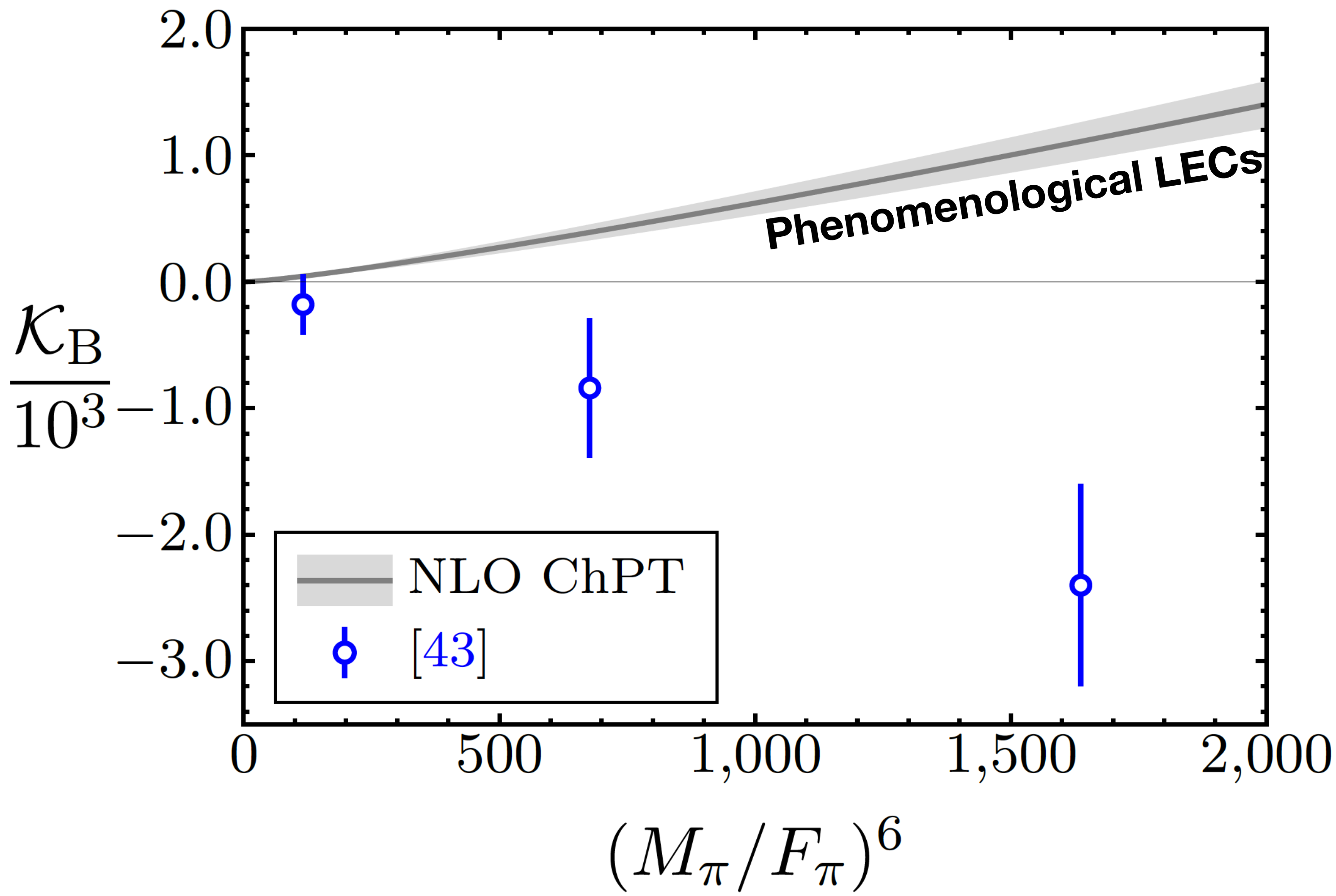
$L \equiv \kappa \log(M_\pi^2/\mu^2)$

LECs

Numerical coefficients
Depend on cutoff H(**k**)

μ-dependence cancels

Comparison to LQCD



- $\mathscr{K}_B$ first appears at NLO in ChPT
- Discrepancy may be resolved by NNLO terms?

Finite- and infinite-volume analysis of the tetraquark $T_{cc}^{+}(3875)$

Sebastian M. Dawid

with the honorable
F. Romero-López & S. Sharpe

SUMMARY

1) We lay out a strategy for a rigorous determination
of T_{cc} and related systems from Lattice QCD
2) We propose resolution of the so-called "left-hand cut problem"

UNIVERSITY of WASHINGTON

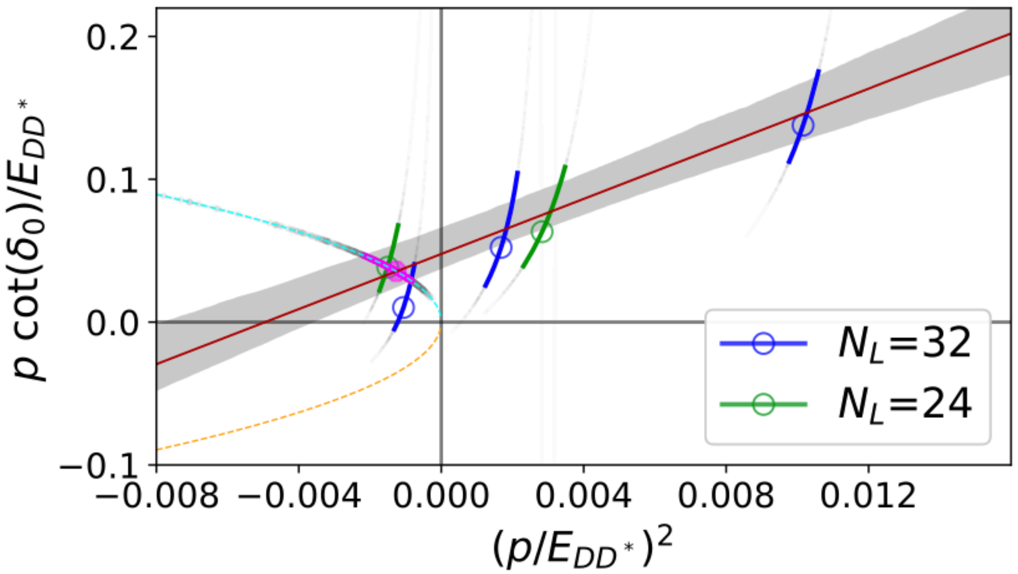
PWA13/ATHOS8, William & Mary, 30.05.2024

Available lattice results

Signature of a doubly charm tetraquark pole in DD scattering on the lattice*
Padmanath, Prelovsek, PRL 129, 032002 (2022)

Towards the quark mass dependence of from lattice QCD
Collins, Nefediev, Padmanath, Prelovsek, PRD 109 (2024) 9, 094509

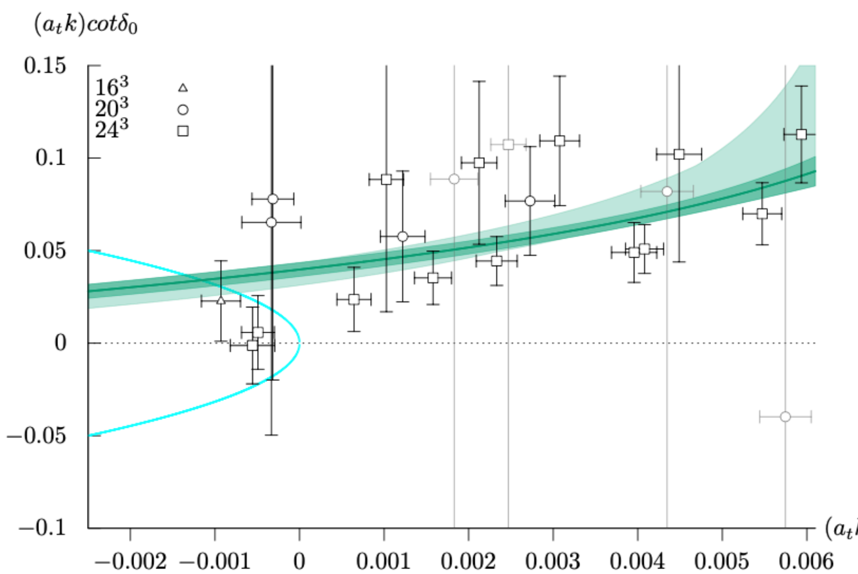
$m_\pi \approx 280$ MeV



Lyu et al., PRL 131, 161901 (2023) $m_\pi \approx 146$ MeV

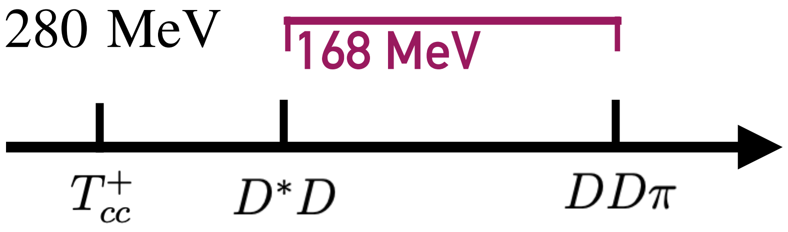
Chen et al. PLB 833, 137391 (2022) $m_\pi \approx 350$ MeV

Whyte, Wilson, Thomas, $m_\pi \approx 391$ MeV



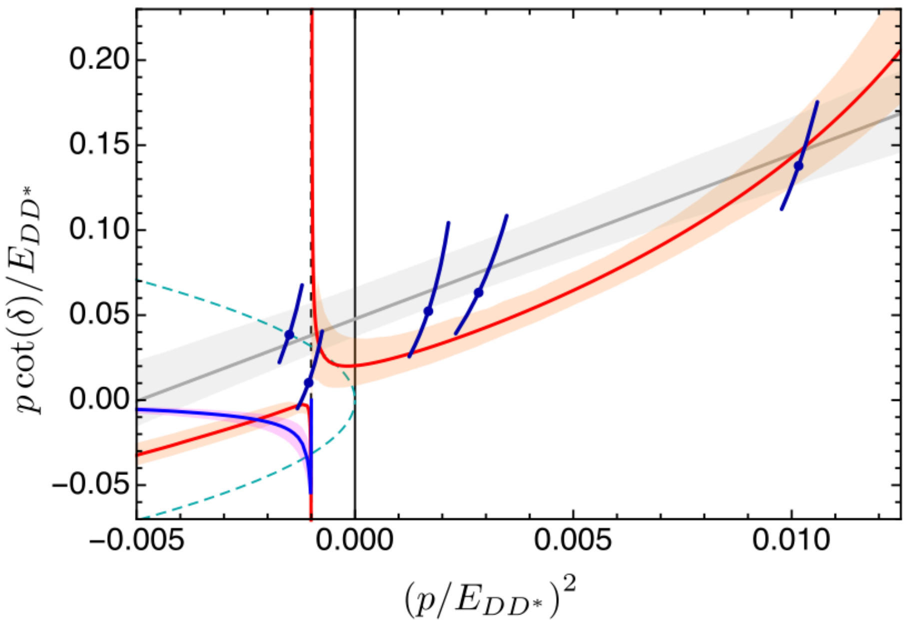
Thresholds are inverted but the three-body effects
still play an important role in the analysis

$m_\pi \approx 280$ MeV



The left-hand cut problem

Role of the left-hand cut contributions on pole extractions

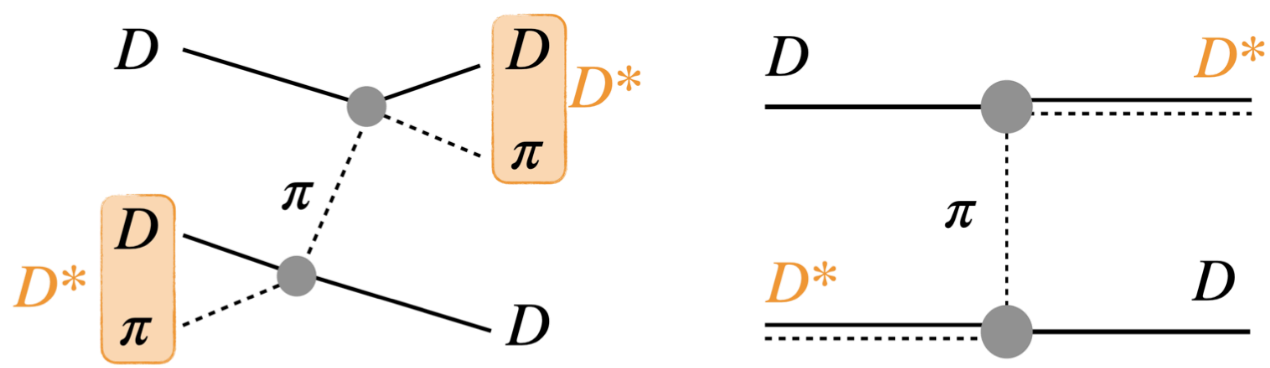


$$s_{\text{lhc}} = s_{\text{thr}} - m_\pi^2 + (m_{D^*} - m_D)^2$$

$$\sqrt{s_{\text{lhc}}} \approx 3966 \text{ MeV} \quad \sqrt{s_{\text{thr}}} \approx 3975 \text{ MeV}$$

Presence of the left-hand cut:
a) invalidates the Lüscher

Incorporating DDπ effects and left-hand cuts in lattice QCD studies of T_{cc}^+
Hansen, Romero-López, Sharpe, arXiv:2401.06609



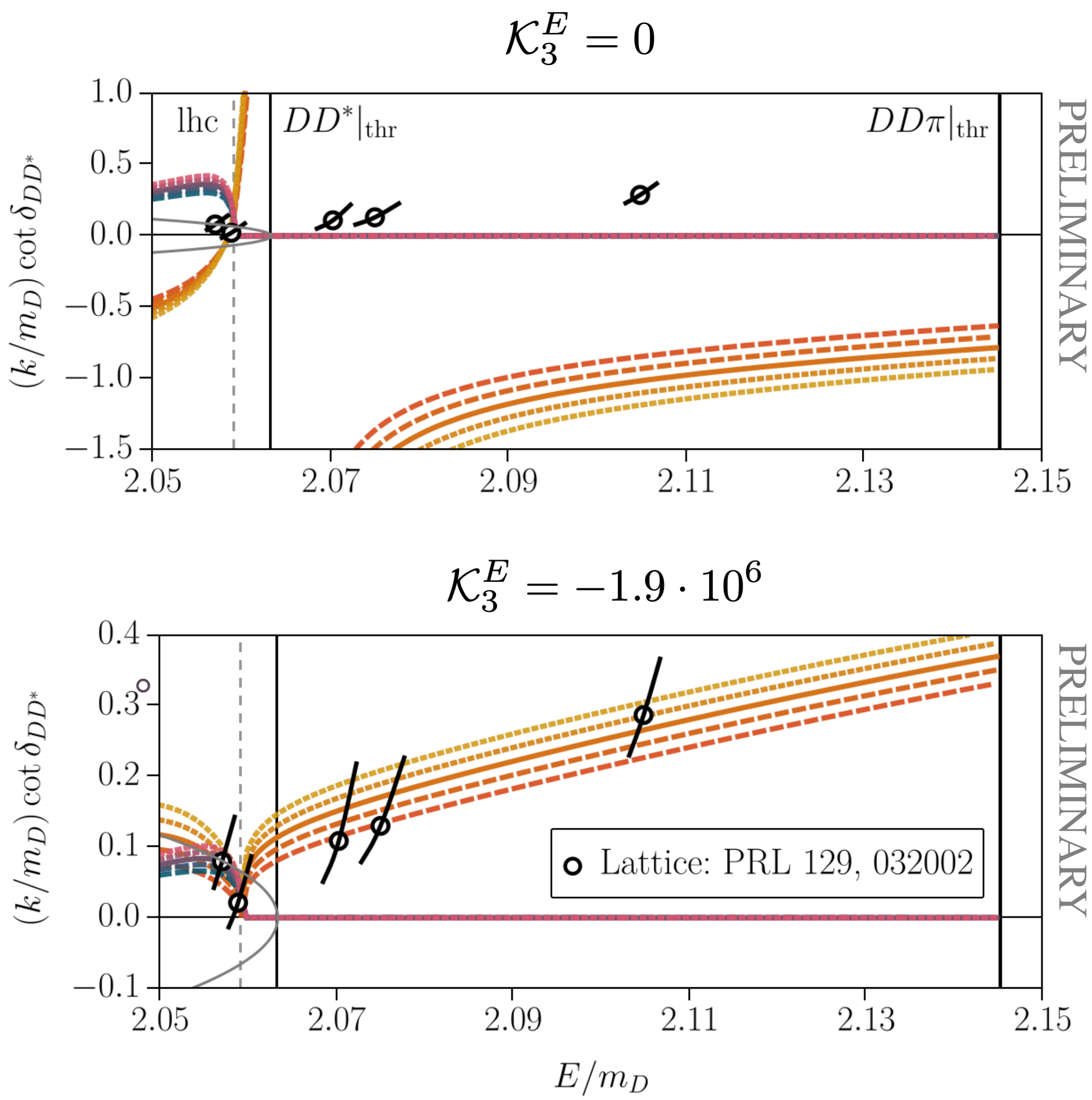
$$s_{\text{lhc},2} = s_{\text{thr}} - 4m_\pi^2 + (m_{D^*} - m_D)^2$$

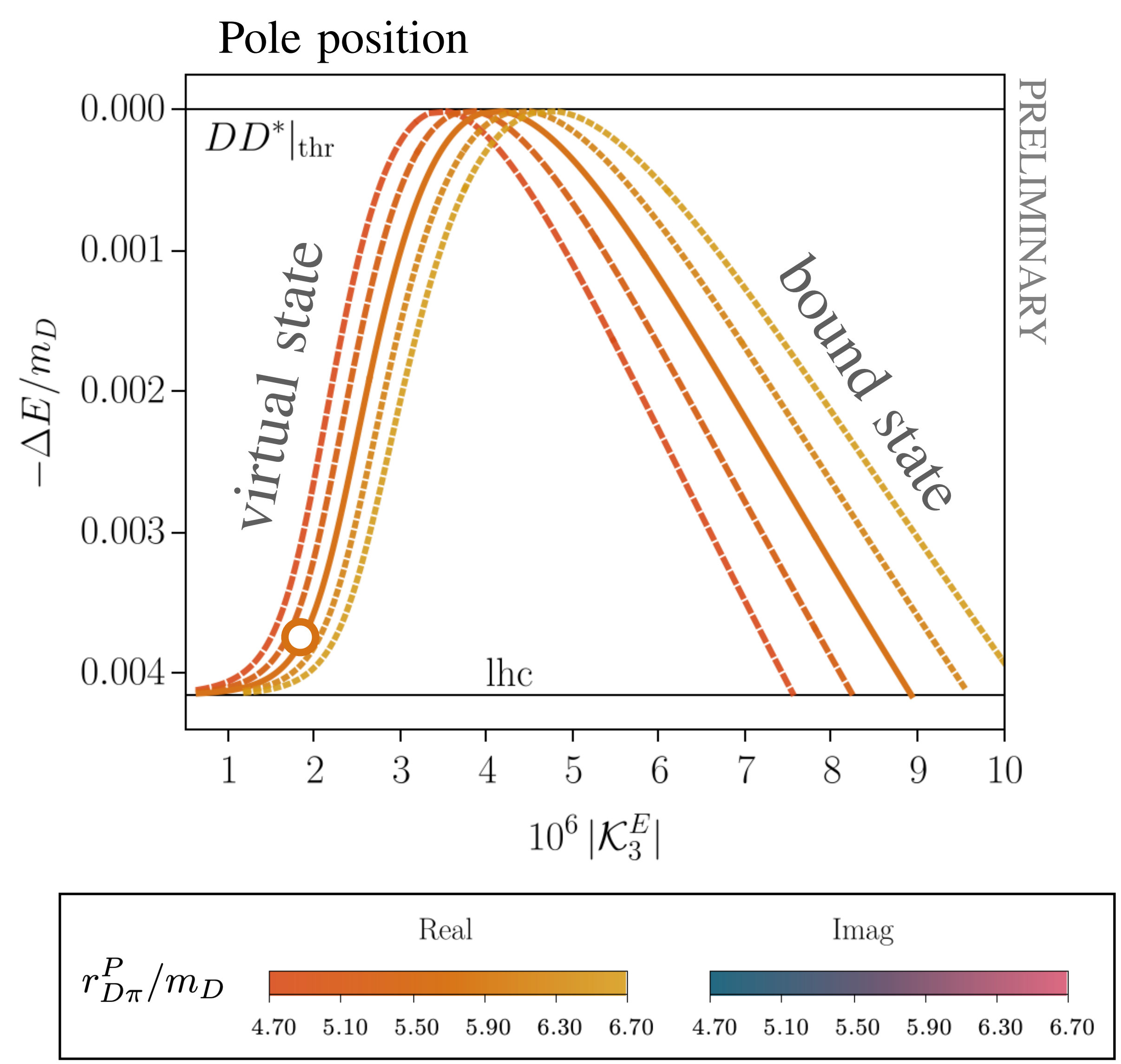
$$\sqrt{s_{\text{lhc},2}} \approx 3937 \text{ MeV} \quad \sqrt{s_{\text{thr}}} \approx 3975 \text{ MeV}$$

Single-channel approximation

$[\mathcal{M}_3({}^3S_1|{}^3S_1)] \quad (\kappa = m_\pi/m_D = 0.145)$

$J^P = 1^+$





73