Scattering amplitudes from lattice QCD

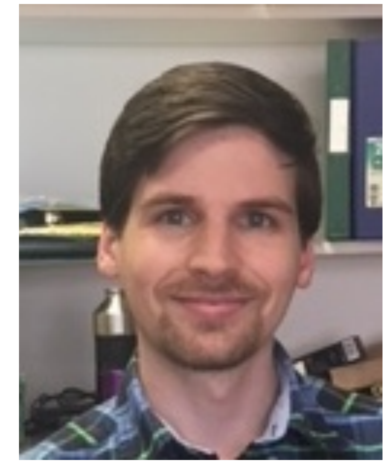
Steve Sharpe
University of Washington

Collaborators

Max Hansen (Edinburgh)

Raúl Briceño (Berkeley)

Fernando Romero-López (MIT/Bern)



Tyler Blanton (UW)

Zach Draper (UW)

Sebastian Dawid (UW)

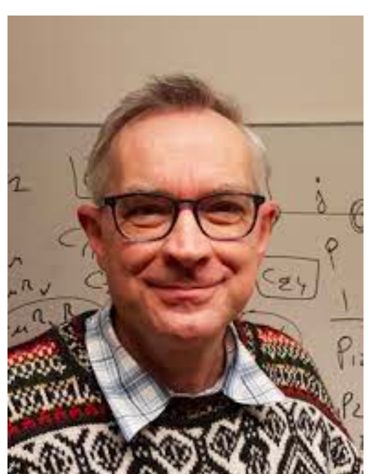
Drew Hanlon (CMU)

Ben Hörz (Intel)

Sarah Skinner (CMU)

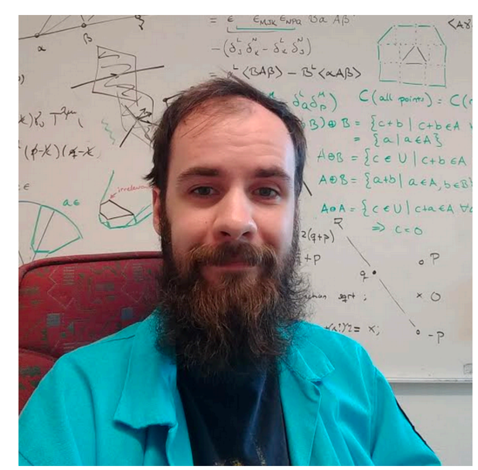
Colin Morningstar (CMU)

Jorge Baeza Ballesteros (Valencia/Berlin)



Hans Bijnens (Lund)

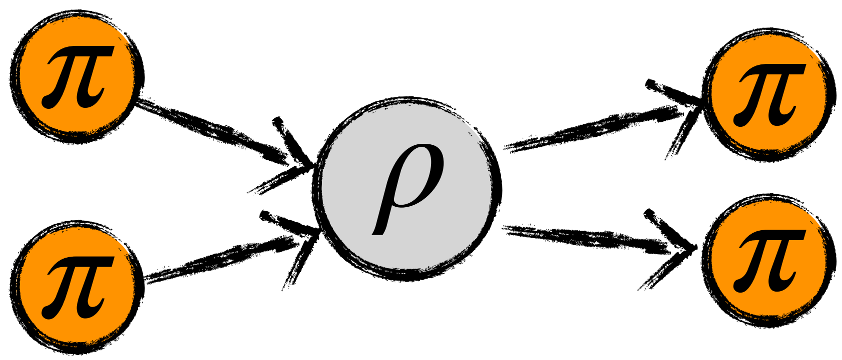
Tomas Husek (Birmingham)



Mattias Sjö (Marseille)

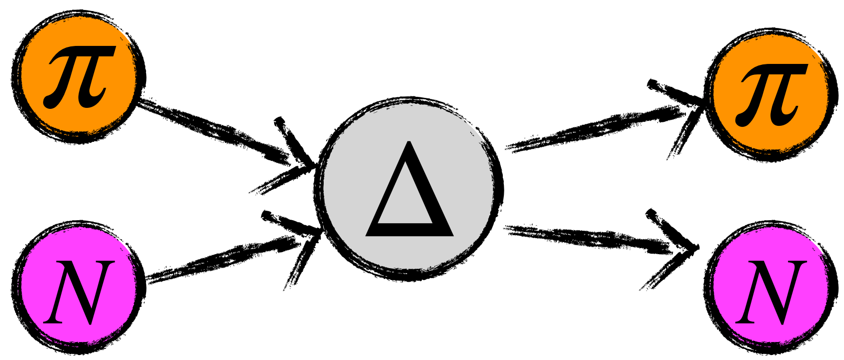
Wilder Schaaf (UW)

Overview: present status



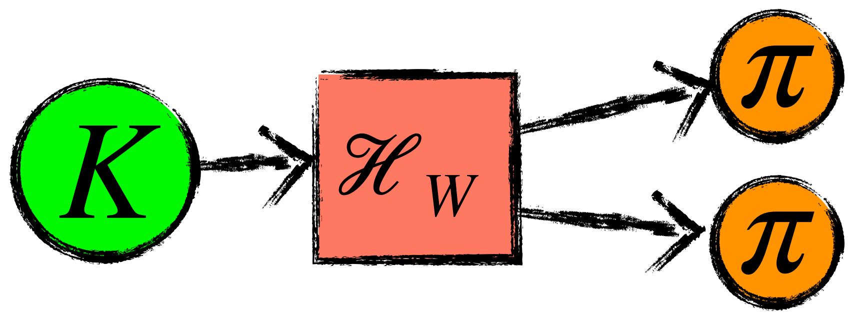
Physical quark masses

E.g.: Wang, Leinweber, Liu, Liu, Sun, Thomas, Wu, Xing, Yu, 2502.03700



$M_\pi = 200$ MeV

Bulava, Hanlon, Hörz, Morningstar, Nicholson, Romero-López Skinner, Varnas, Walker-Loud, 2208.03867

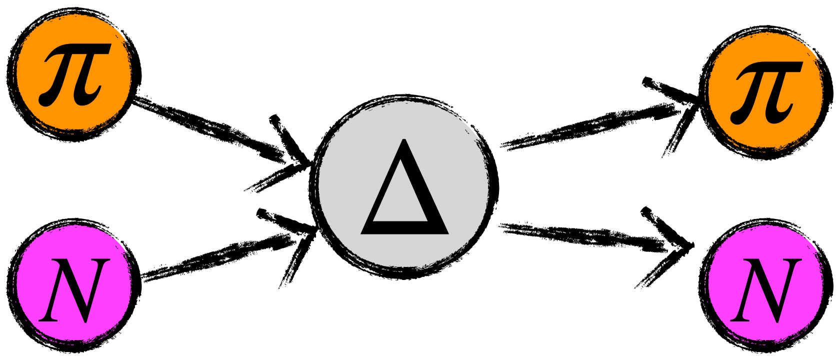


Almost "FLAG-ready", i.e. fully controlled

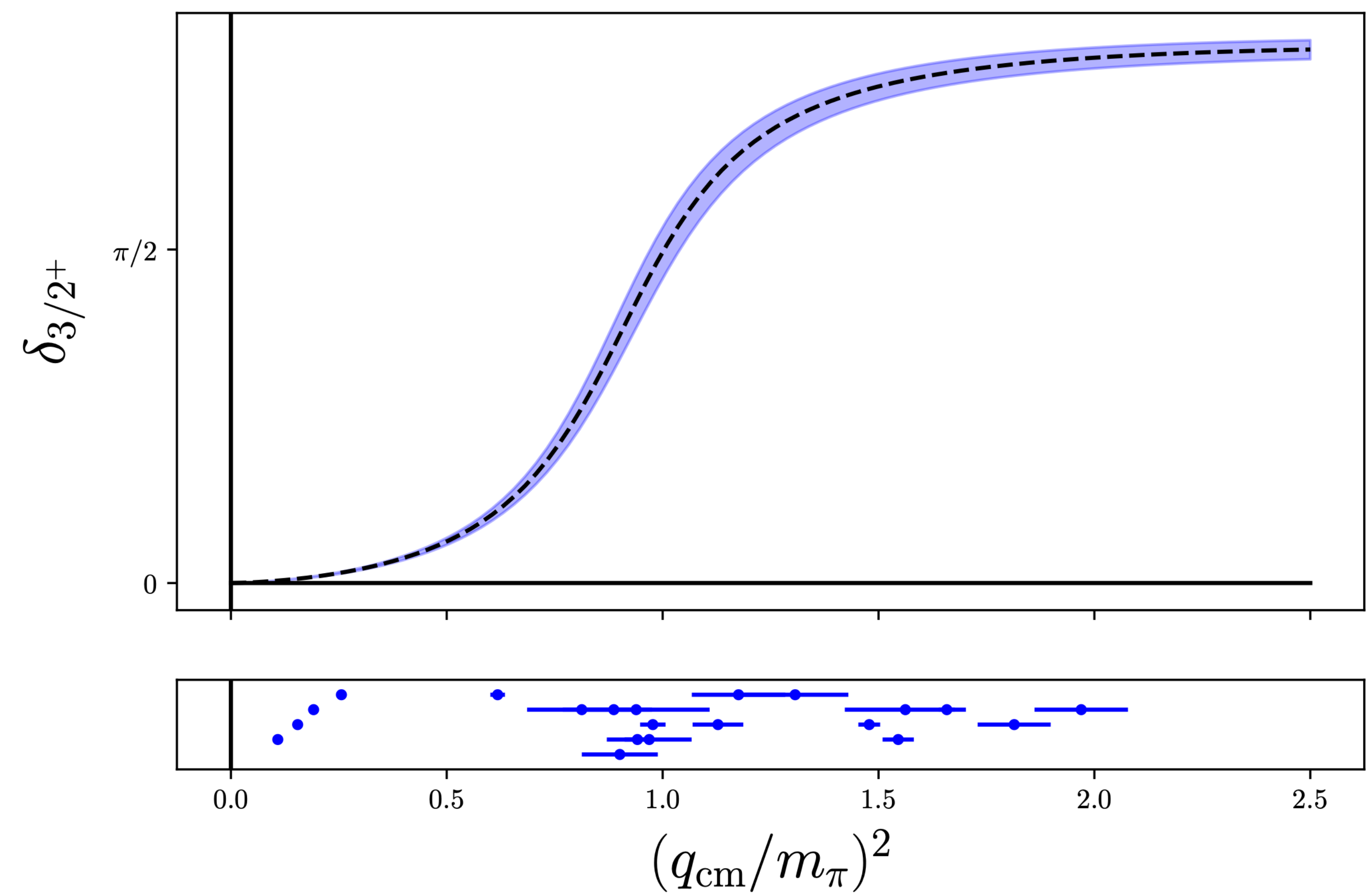
Abbott et al. [RBC-UKQCD collaboration], 2004.09440

Δ resonance

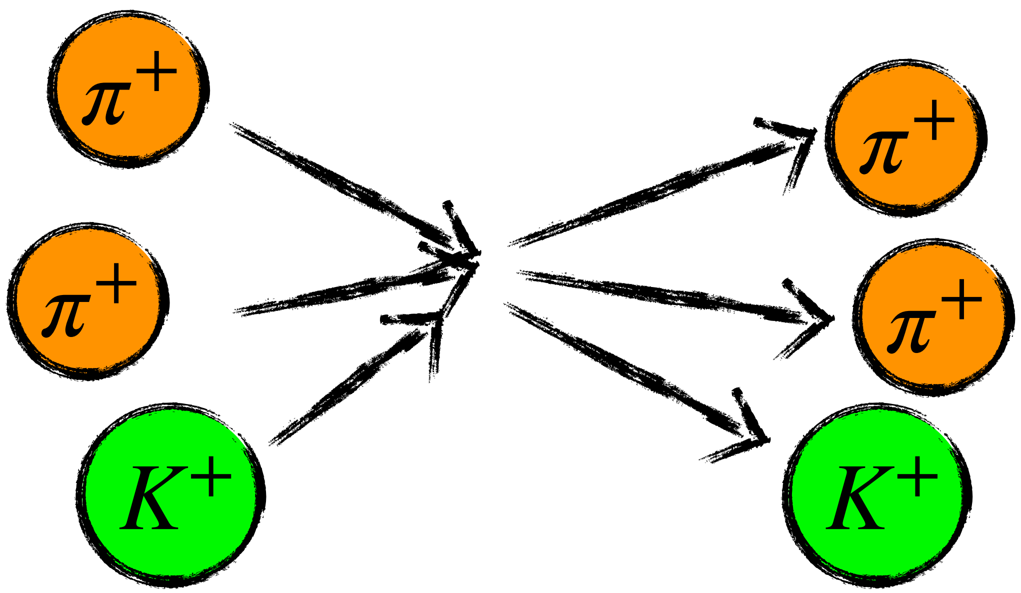
Bulava, Hanlon, Hörz, Morningstar, Nicholson, Romero-López, Skinner, Varnas, Walker-Loud, 2208.03867



$M_\pi \approx 200$ MeV
$M_N \approx 950$ MeV
$a = 0.063$ fm
$L^3 \times T = 64^3 \times 128$
Clover fermions (CLS)

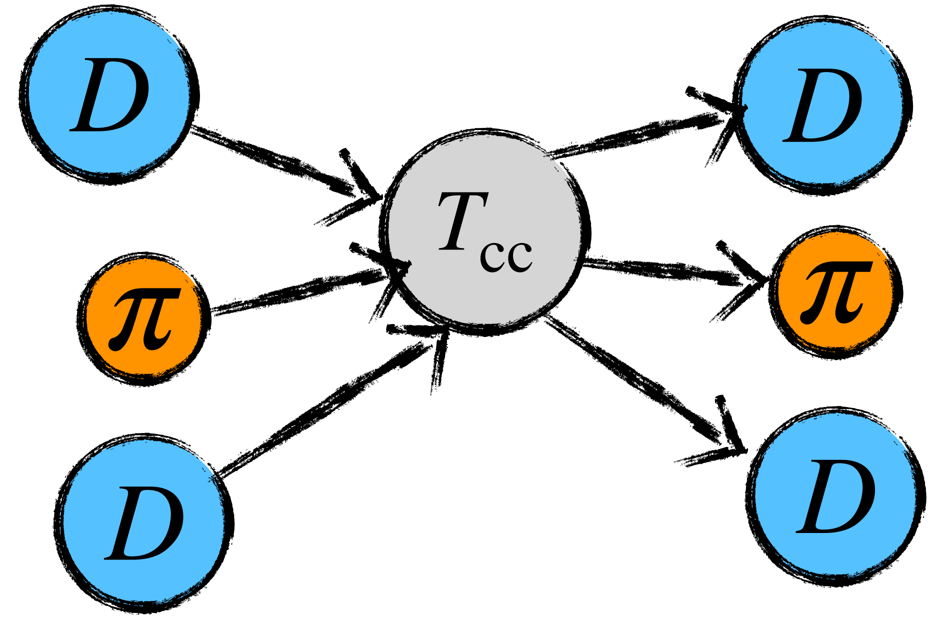


Overview: present frontier



Physical quark masses

Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, SRS, Skinner, 2502.14348 & 2502.17976

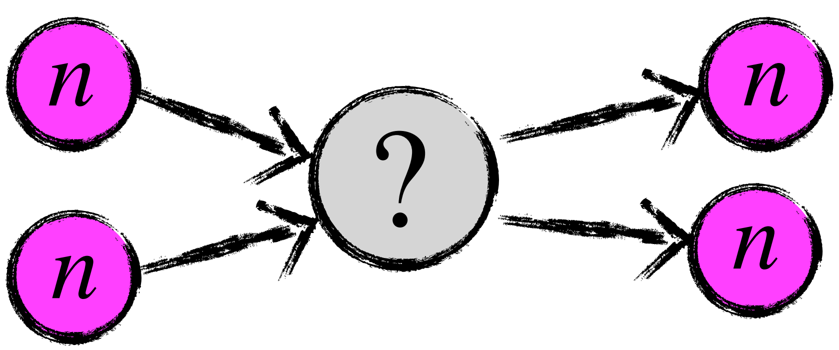


$M_\pi = 200$ MeV

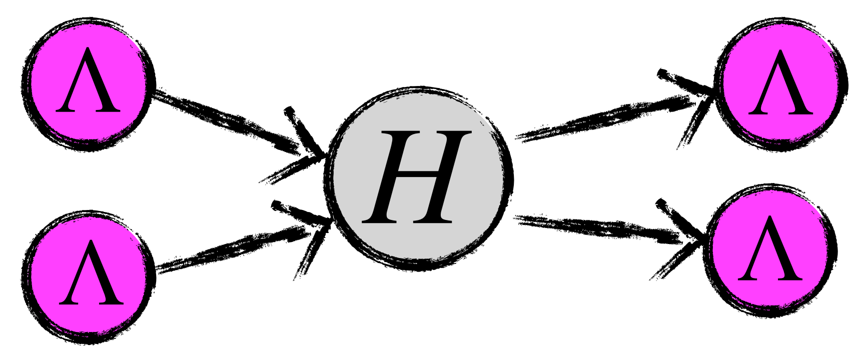
Padmanath, Prelovsek, 2202.10110

…

Dawid, Romero-López, SRS, 2409.17059

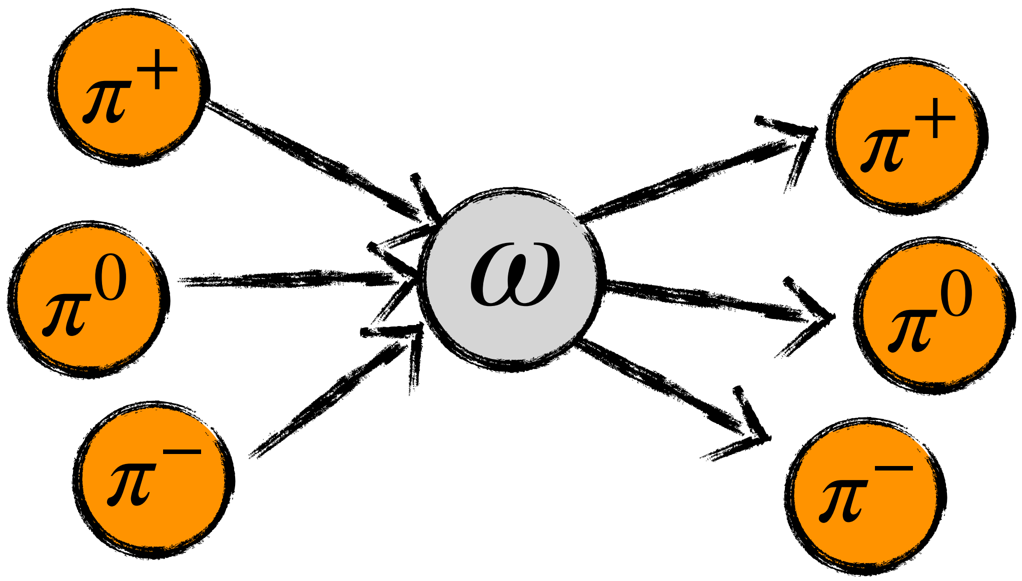


Di-neutron bound state?

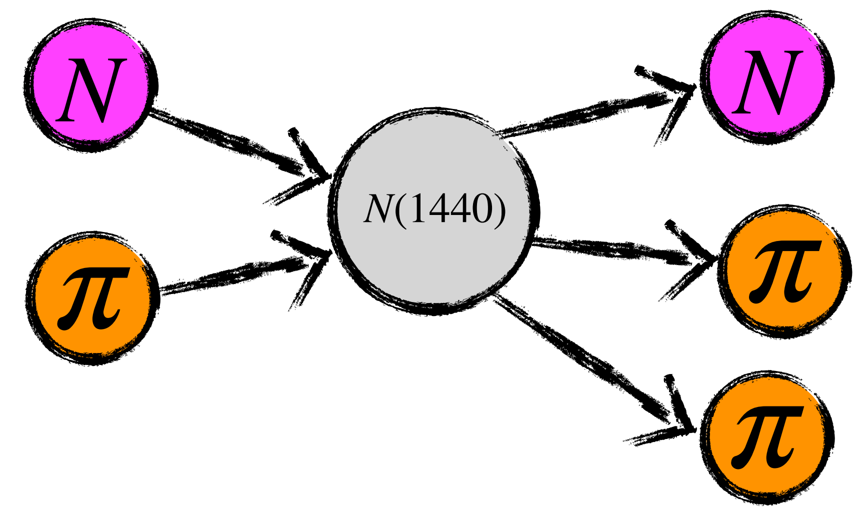


Is H dibaryon bound?

Overview: “near” future

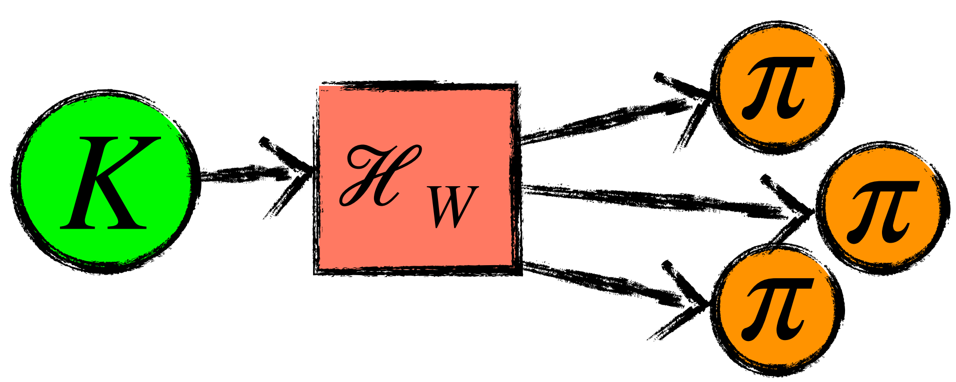


Formalism: Hansen, Romero-López, SRS, 2003.10974
First LQCD results and alternative formalism:
Yan, Mai, Garofalo, Meißner, Liu, Liu, Urbach, 2407.16659



Roper resonance

Formalism: Hansen, Romero-López, SRS, … in prep.



Formalism: Hansen, Romero-López, SRS, 2101.10246

Outline

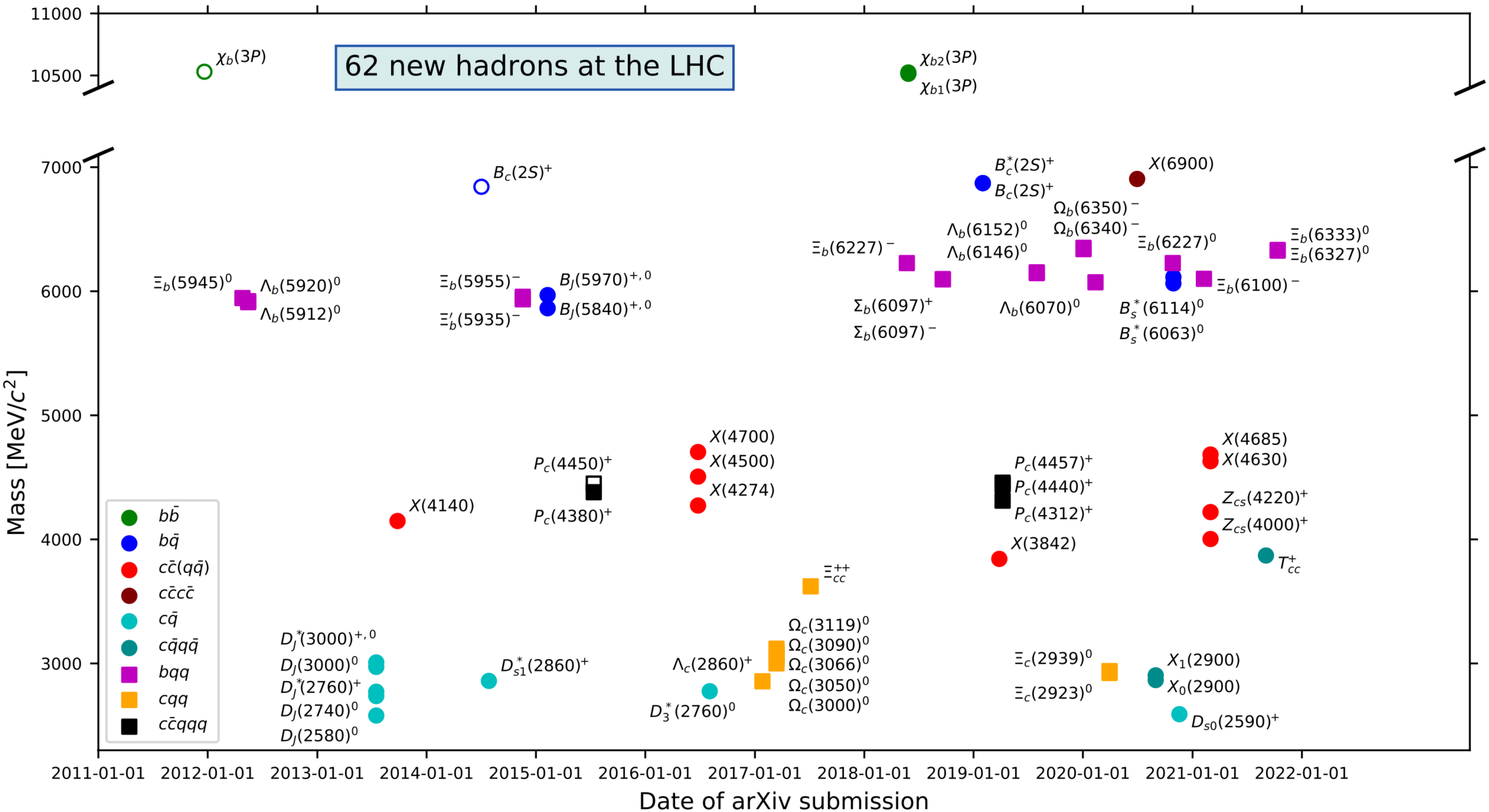
- Motivation
- The fundamental issue
- Summary of formalism
- Applications of three-particle formalism
  - Three-particle amplitudes involving pions & kaons at physical quark masses
  - $DD\pi$ scattering, relevant for T_{cc}^{+}
- New issue: unexpected subthreshold singularities
- Outlook

Motivation

Underlying motivations

- Determine properties of strong interaction resonances from QCD
  - E.g. exotics such as $T_{cc}(3875)^+ \to DD^* \to DD\pi$

Cornucopia of exotics



[I. Danilkin, talk at INT workshop, March 23]

+ data from Babar, Belle, COMPASS, …

Underlying motivations

- Determine properties of strong interaction resonances from QCD
  - E.g. exotics such as $T_{cc}(3875)^+ \to DD^* \to DD\pi$
- Determine three particle "forces" for $3n,\ 3\pi,\ 3K, \ldots$
  - Needed to understand neutron star EoS, properties of large nuclei, …
- Calculate weak decay amplitudes within the Standard Model, in order to search for new physics
  - E.g. $K \to 2\pi$ (essentially done), $K \to 3\pi$ (method known), & $D \to \pi^+\pi^-,\ K^+K^-$ (open question)

The fundamental issue

On the one hand…

- LQCD determines energies and properties of finite-volume eigenstates
  - Obtained by fits to (numerically-evaluated) Euclidean correlation functions:

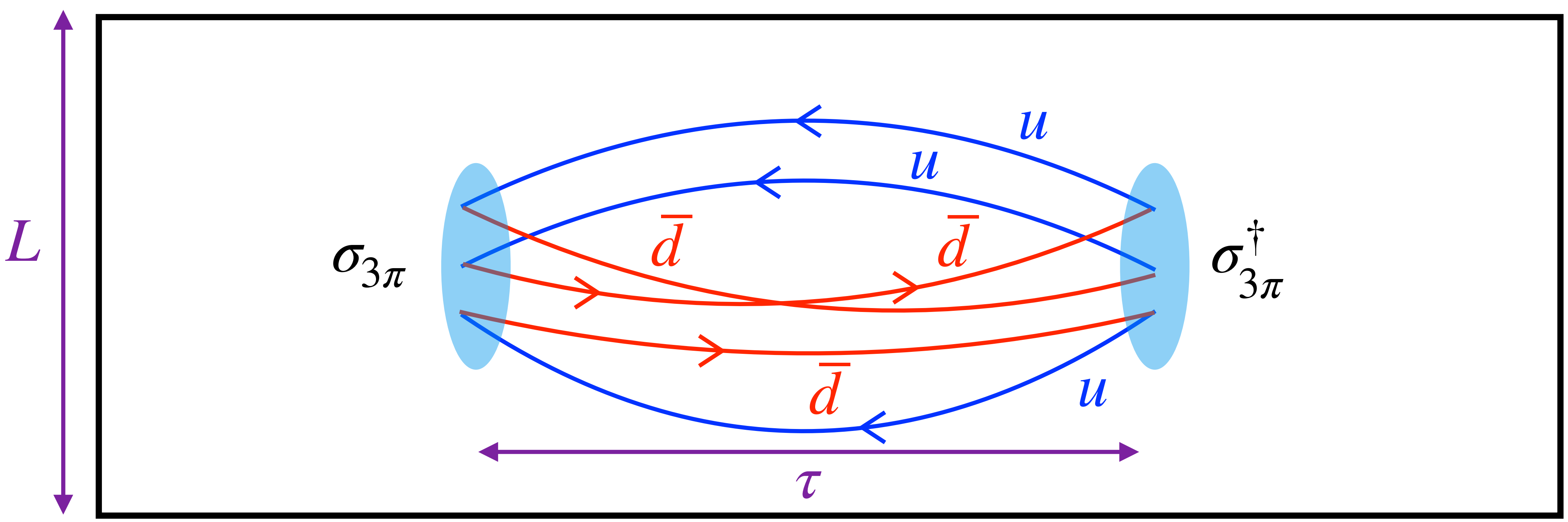
$$\int_L d^3x\, e^{-i\vec{P}\cdot\vec{x}}\, {}_L\langle\Omega\,|\,\sigma_{3\pi}(\tau,\vec{x})\sigma_{3\pi}^\dagger(0)\,|\,\Omega\rangle_L \propto \sum_n \left|{}_L\langle 0\,|\,\sigma_{3\pi}^\dagger(0)\,|\,3\pi,\vec{P},n\rangle_L\right|^2 e^{-E_n\tau}; \qquad (\tau>0)$$

$\vec{P} = 2\pi\vec{n}/L$
Assuming L^3 box with PBC

$\sigma_{3\pi} \sim 3\pi^+$
Lives on timeslice

Tower of finite-volume states with quantum numbers of $3\pi^+$, with momentum $\vec{P}$, and living in irreps of cubic group

Energies of said states



On the one hand…

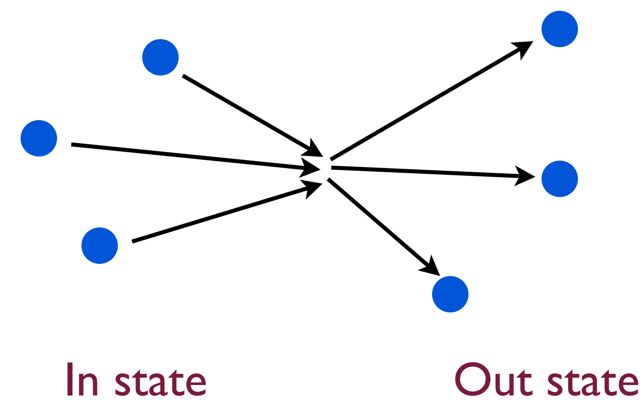
- LQCD determines energies and properties of finite-volume eigenstates
  - Obtained by fits to (numerically-evaluated) Euclidean correlation functions:

$$\int_L d^3x\, e^{-i\vec{P}\cdot\vec{x}}\, {}_L\langle\Omega|\sigma_{3\pi}(\tau,\vec{x})\sigma^\dagger_{3\pi}(0)|\Omega\rangle_L \propto \sum_n \left| {}_L\langle 0|\sigma^\dagger_{3\pi}(0)|3\pi, \vec{P}, n\rangle_L \right|^2 e^{-E_n\tau}; \qquad (\tau > 0)$$

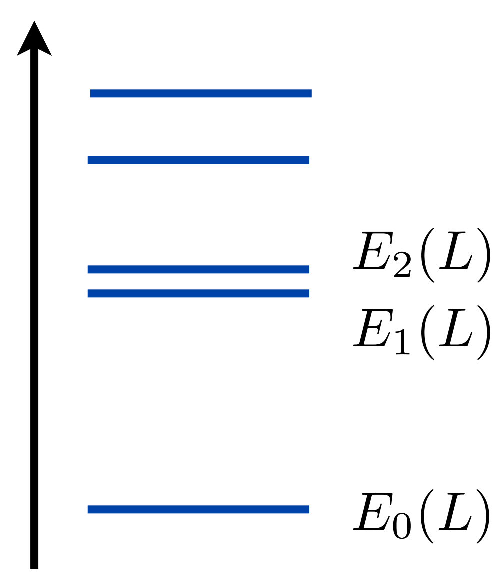
- E_n are physical quantities!
  - Can determine 5-10 levels for each choice of quantum numbers ($\vec{P}$, irrep, …)
  - Can now begin to calculate with physical quark masses
  - Results come with statistical & systematic errors (e.g. need $a \to 0$)
  - Today, I assume that the physical E_n are provided by LQCD simulations

…while on the other

- We want infinite-volume scattering amplitudes, e.g.

$\mathscr{M}_3 \sim$ 

- How do we relate these? A finite-volume QFT problem.



Discrete energy spectrum

?

$$i\mathcal{M}_{n\to m}$$

Scattering amplitudes

A related question:

- LQCD can also calculate matrix elements between finite-volume states

$$ {}_L\langle \Omega | \sigma_{3\pi}(\tau_f, \overrightarrow{P})[\int_L d^3x \mathscr{H}_W(0, \vec{x})] K^\dagger(\tau_i, \overrightarrow{P}) | \Omega \rangle_L \propto \sum_{n',n} c_{n',n} e^{-E_n \tau_f} \underbrace{{}_L\langle 3\pi, \overrightarrow{P}, n' | \mathscr{H}_W(0) | K, n \rangle_L}_{\text{A physical quantity if } E_{n'} = E_n} e^{E_{K'_n} \tau_i} $$

$\tau_f > 0$ $\tau_i < 0$

- How are these related to decay amplitudes?

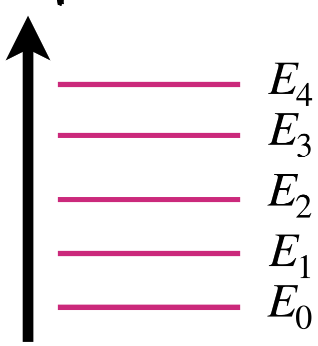
$$ \mathscr{A}(K \to 3\pi) = {}_{\text{out}}\langle 3\pi | \mathscr{H}_W(0) | K \rangle $$

Summary of formalism

2-particle formalism

[Lüscher, 1986-91 + many subsequent works]

Two-particle spectrum



Quantization condition: QC2

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

Fit

K-matrix

$$\mathcal{K}_2$$

Parametrize:

$$\mathcal{K}_2 = c_0 + c_1 k^2 + \ldots$$

Unitarity relation

Scattering amplitude

$$\mathcal{M}_2$$

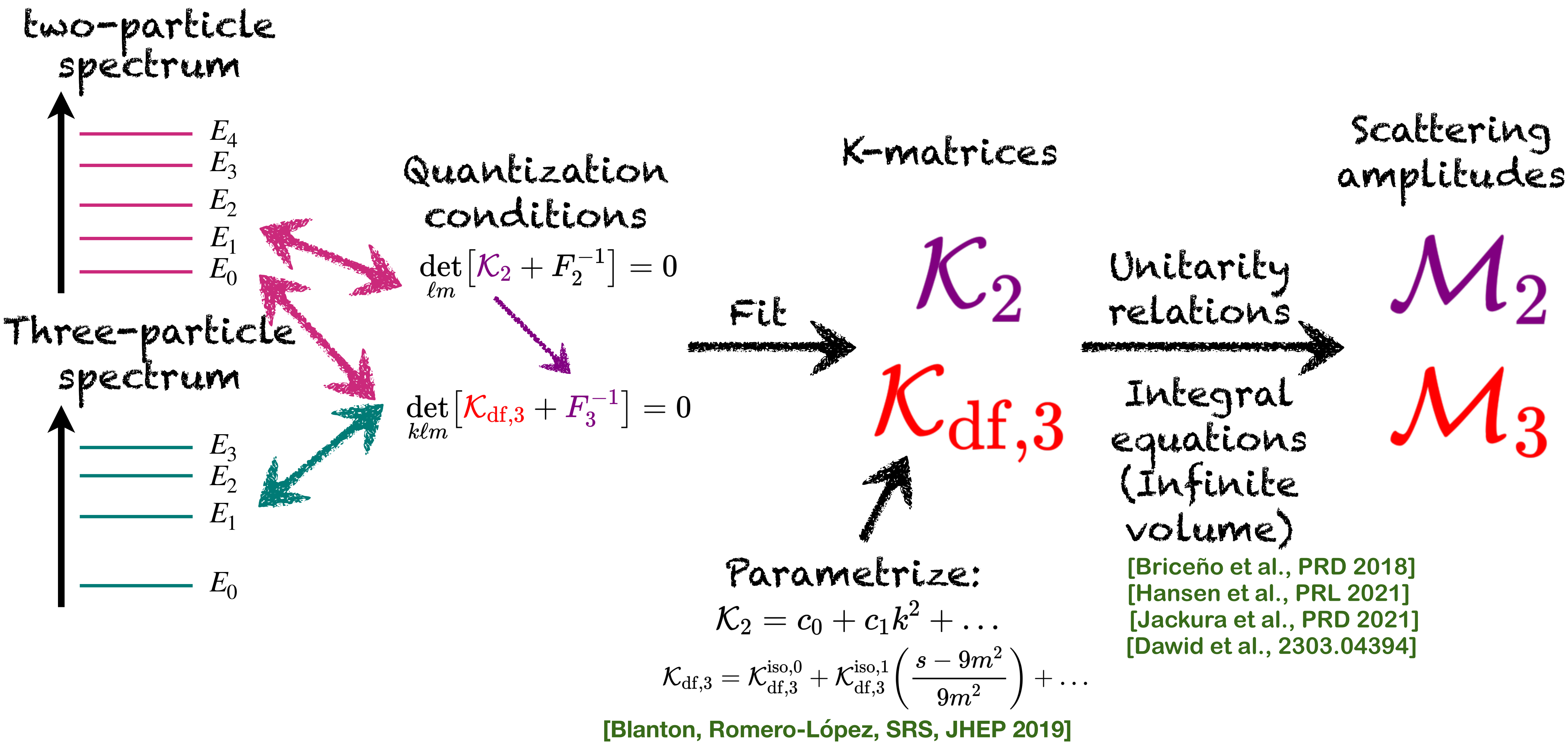
$$\mathscr{M}_2 = \mathscr{K}_2 \frac{1}{1 - i\rho\mathscr{K}_2}$$

$$\rho = \frac{\eta\sqrt{q^{*2}}}{8\pi E^*}$$

- QC2 valid up to corrections $\propto \exp(-M_\pi L)$
- QC2 valid up to inelastic threshold
- Matrix indices are CM-frame ℓ, m
- $\mathscr{K}_2$ is an infinite-volume quantity; diagonal in ℓ, m
- F depends on $E, \boldsymbol{P}, L$; mixes ℓ, m
- In practical applications, must truncate in ℓ

3-particle formalism

[Hansen & SRS, 2014 & 2015 + many subsequent works]

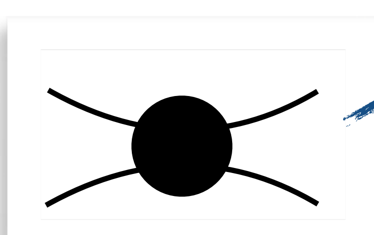


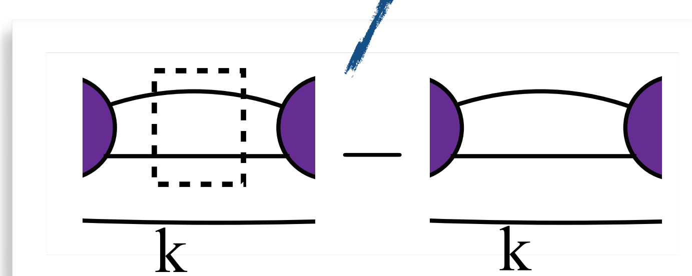
QC3

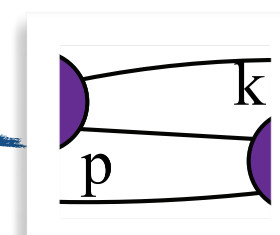
$$\det_{k\ell m}\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$

- QC3 valid up to corrections $\propto \exp(-M_\pi L)$
- QC3 valid up to first inelastic threshold, e.g. $E^* = 5M_\pi$ for 3 pion system
- Matrix indices are spectator finite-volume momentum $\boldsymbol{k}$, and pair CM-frame ℓ, m
- Formalism includes smooth cutoff in $\boldsymbol{k}$; must truncate "by hand" in ℓ

$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\frac{1}{1/\mathscr{K}_{2,L} + F + G}F\right]$$



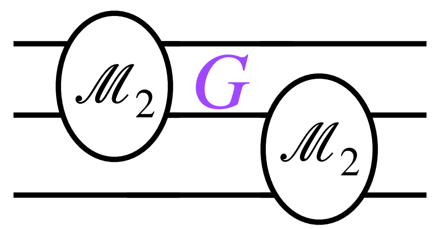


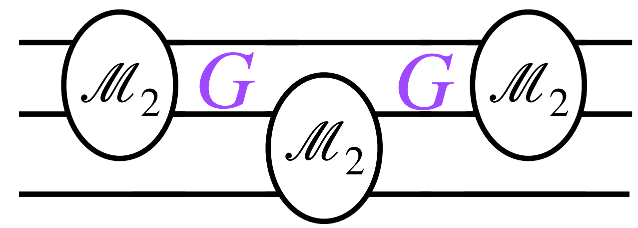


Relating $\mathscr{K}_{\mathrm{df},3}$ to $\mathscr{M}_3$

$$\mathscr{M}_3 = \lim_{L\to\infty} \mathcal{S}\left\{ \mathscr{D}_L^{(u,u)} + \mathscr{M}_{\mathrm{df,3,L}}^{(u,u)} \right\} = \mathscr{D} + \mathscr{M}_{\mathrm{df},3}$$

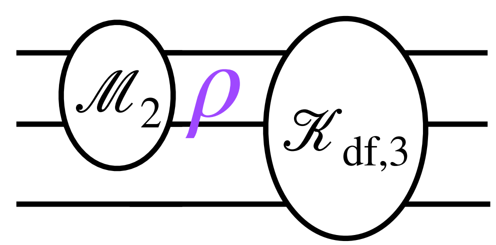
- $\mathscr{D}$ contains all divergent contributions to $\mathscr{M}_3$, but depends on cutoff function $H(\vec{k})$

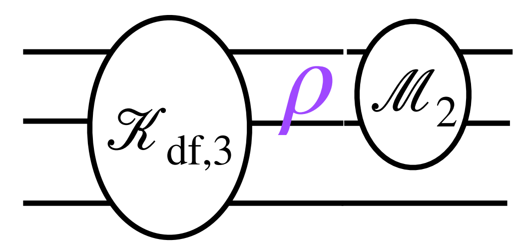
$\mathscr{D} = \mathcal{S}\Big\{$ 

$+$ 

$+ \dots \Big\}$

- $\mathscr{M}_{\mathrm{df},3}$ is divergence-free, equals $\mathscr{K}_{\mathrm{df},3}$ at leading order, and is also cutoff-dependent

$\mathscr{M}_{\mathrm{df},3} = \mathscr{K}_{\mathrm{df},3} + \mathcal{S}\Big\{$ 

$+$ 

$+ \dots \Big\}$

- "Decorations" ensure that $\mathscr{M}_3$ is unitary

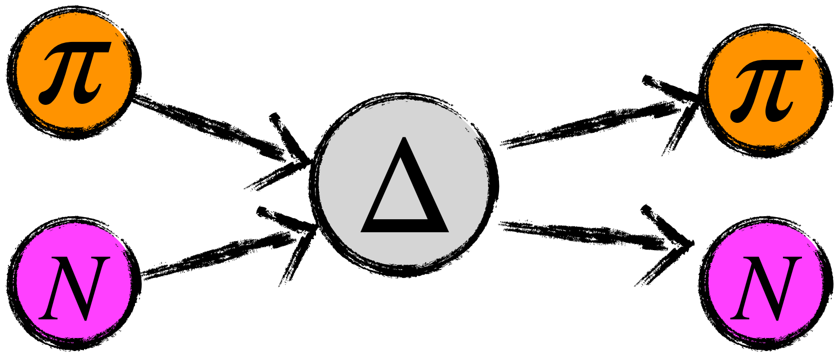
- Methods for solving integral equations, and analytically continuing to complex momenta, are now well established [Briceño, Dawid, Hansen, Islam, Jackura, 2020-23]
  - In practice, project on definite overall J^P

Status: formalism

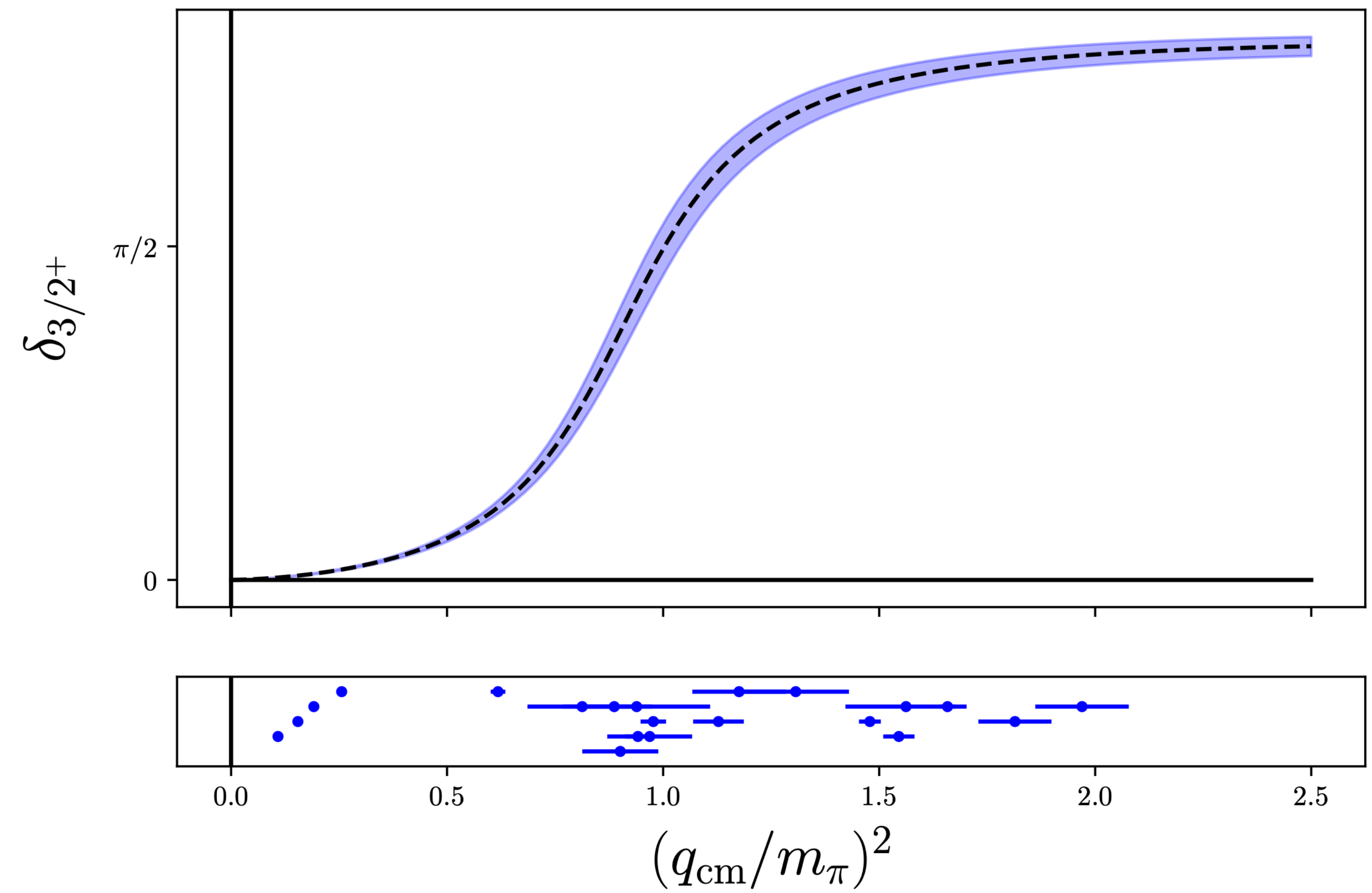
- 3 identical spinless particles [Hansen & SRS 14,15 (RFT); Hammer, Pang, Rusetsky 17 (NREFT); Mai, Döring 17 (FVU)]
  - Applications: $3\pi^+,\ 3K^+$, as well as ϕ^4 theory
- Mixing of two- and three-particle channels for identical spinless particles [Briceño, Hansen, SRS 17]
- 3 degenerate but distinguishable spinless particles, e.g 3π with isospin 0, 1, 2, 3 [Hansen, Romero-López, SRS 20]; $I = 1$ case in FVU approach [Mai et al., 21]
  - Potential applications: $\omega(782),\ a_1(1260),\ h_1(1170),\ \pi(1300), \ldots$
- 3 nondegenerate spinless particles [Blanton, SRS 20]
  - Potential applications: $D_s^+ D^0 \pi^-$
- 2 identical +1 different spinless particles [Blanton, SRS 21]
  - Applications: $\pi^+\pi^+K^+, K^+K^+\pi^+$
- 3 identical spin-½ particles [Draper, Hansen, Romero-López, SRS 23]
  - Potential applications: $3n,\ 3p,\ 3\Lambda$
- $DD\pi$ for all isospins (also $BB\pi,\ KK\pi$) [Draper, Hansen, Romero-López, SRS 23]
  - Potential applications: $T_{cc} \to D^*D$ incorporating LH cut
- Multiple three-particle channels: $\eta\pi\pi + K\overline{K}\pi$ [Draper & SRS 24]
  - Potential applications: $b_1(1235), \eta(1295)$
- $N\pi\pi$ at maximal isospin [Hansen, Romero-López, SRS 23]
  - Step on the way to Roper: $N(1440) \to N\pi + N\pi\pi$

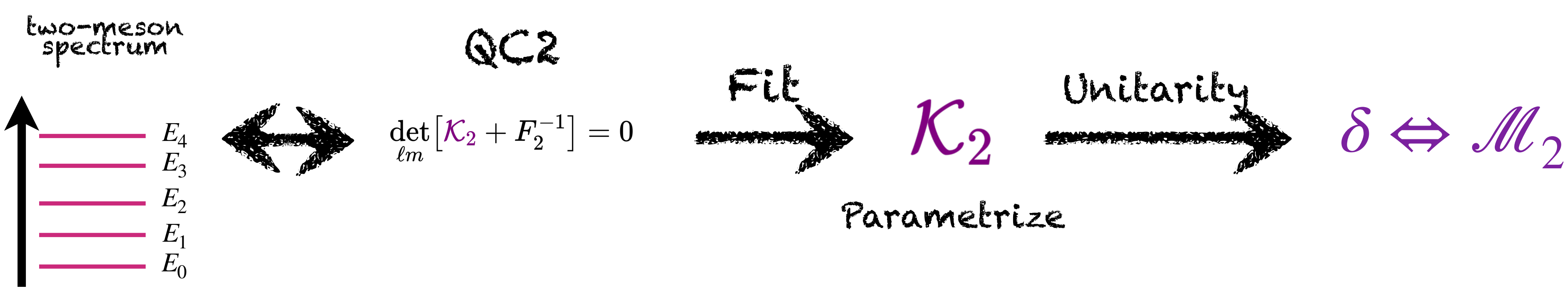
Δ resonance

Bulava, Hanlon, Hörz, Morningstar, Nicholson, Romero-López, Skinner, Varnas, Walker-Loud, 2208.03867



$M_\pi \approx 200$ MeV
$M_N \approx 950$ MeV
$a = 0.063$ fm
$L^3 \times T = 64^3 \times 128$
Clover fermions (CLS)





Application of 3-particle formalism:

QCD predictions for physical multimeson scattering amplitudes

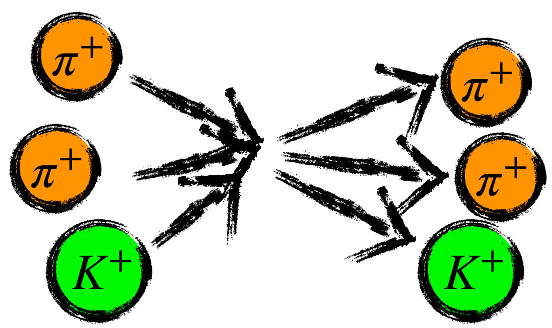
Sebastian M. Dawid,[1] Zachary T. Draper,[1] Andrew D. Hanlon,[2] Ben Hörz,[3] Colin Morningstar,[4] Fernando Romero-López,[5,6] Stephen R. Sharpe,[1] and Sarah Skinner[4]

[2502.14348 (PRL) & 2502.17976 (PRD)]






Three PGBs at maximal isospin



$$\boxed{3\pi^+, \quad 3K^+, \quad \pi^+\pi^+K^+, \quad K^+K^+\pi^+}$$

- Benchmark system, with simple repulsive dynamics
  - First calculation at physical quark masses
  - Study $J^P = 0^-, 1^+, 2^-$, including subchannels with $\ell = 0,1,2$
  - Compare to expectations from ChPT
- Use CLS ensembles ($\mathcal{O}(a)$ improved Wilson) + GEVP

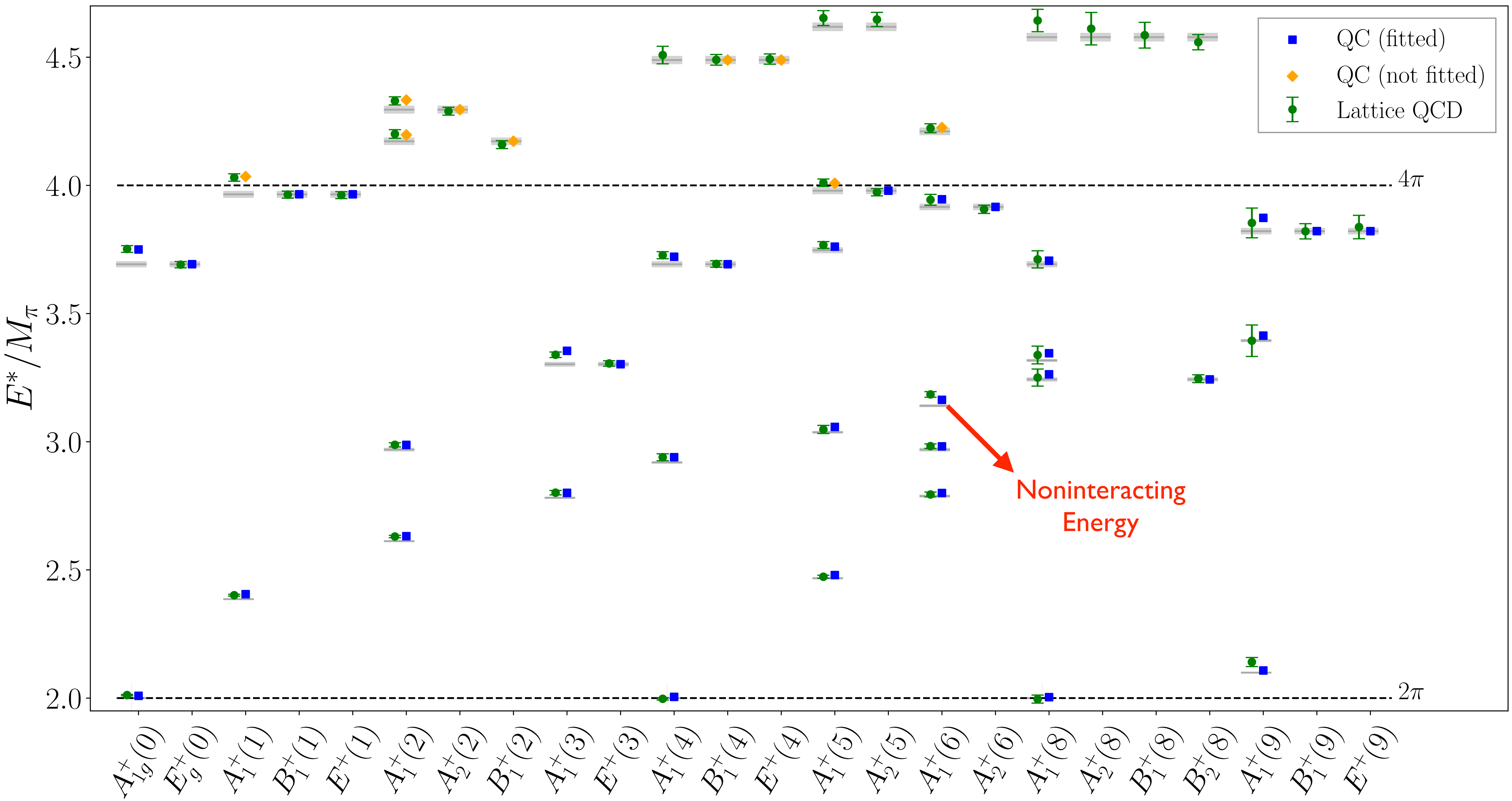
	$(L/a)^3 \times (T/a)$	M_π[MeV]	M_K[MeV]	N_{cfg}	$M_\pi L$
N203	$48^3 \times 128$	340	440	771	5.41
N200	$48^3 \times 128$	280	460	1712	4.42
D200	$64^3 \times 128$	200	480	2000	4.20
E250	$96^3 \times 192$	130	500	505	4.05

[Blanton, SRS, et al., PRL 2020 & JHEP 2021]
[Draper, SRS, et al., JHEP 2023]

Present work

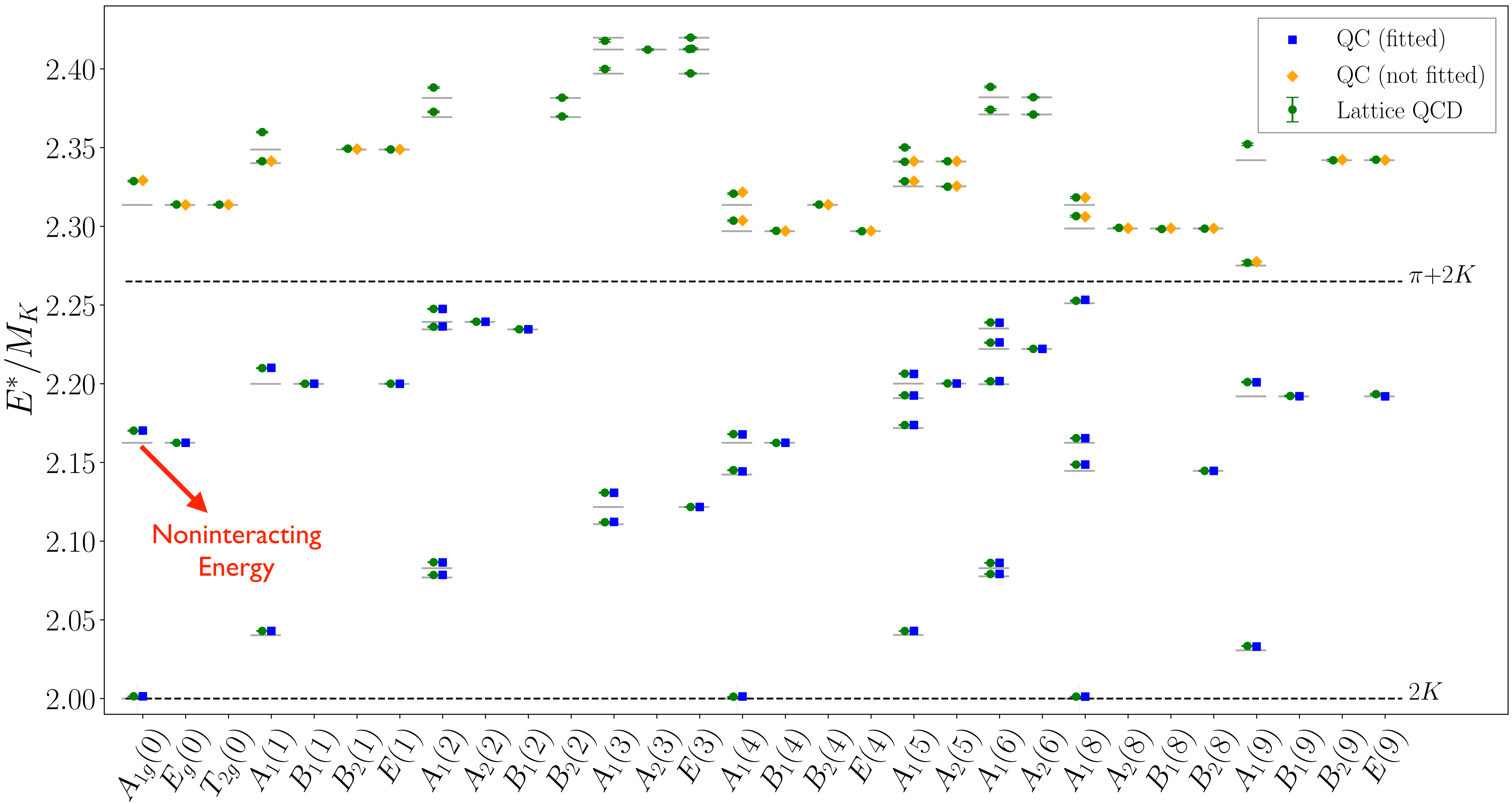
$$a \simeq 0.063 \text{ fm}$$
$$\text{tr } m_q = 2m_{ud} + m_s \simeq \text{const}$$

Example of spectrum



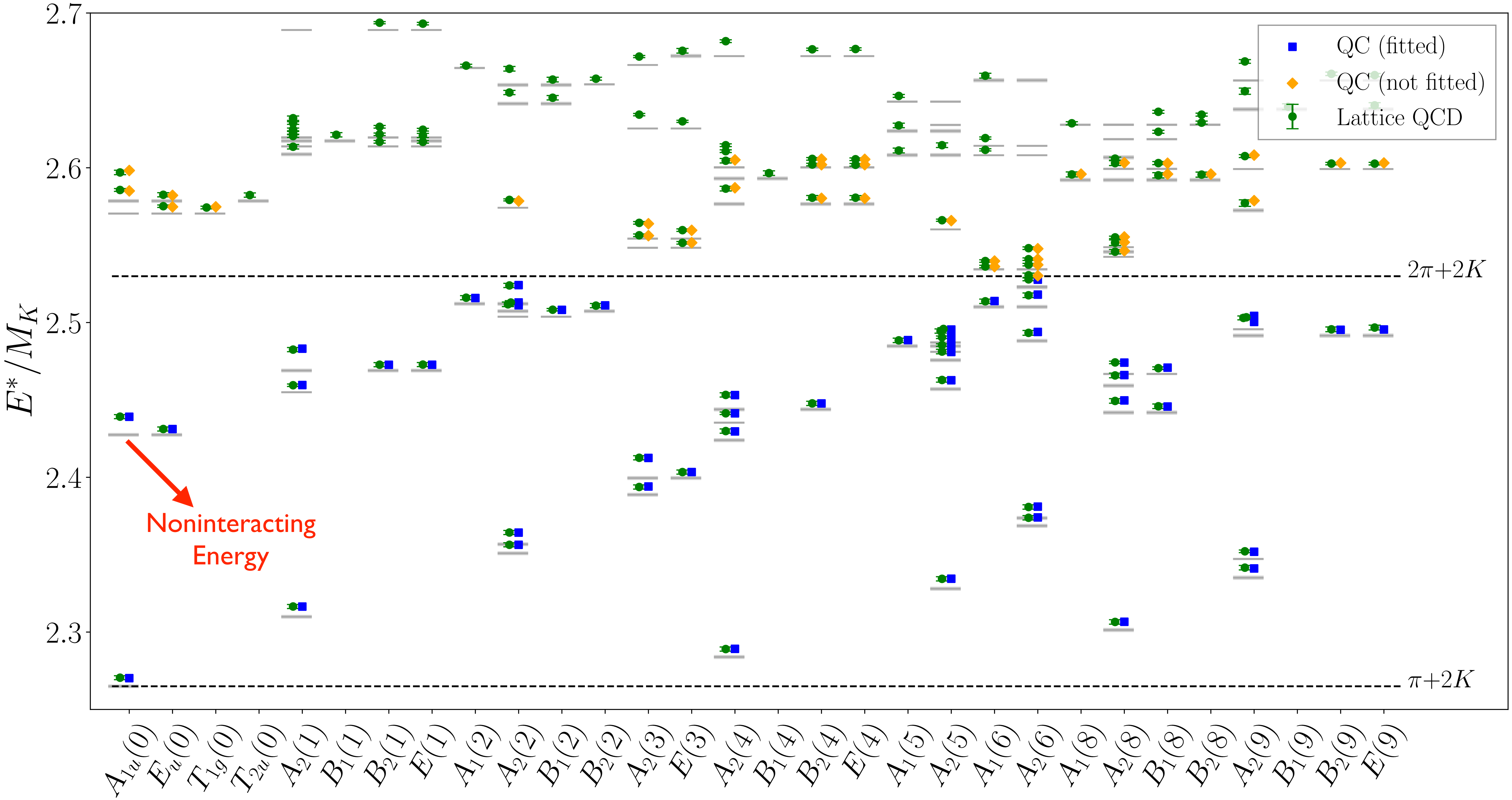
E250: $\pi^+\pi^+$

Example of spectrum



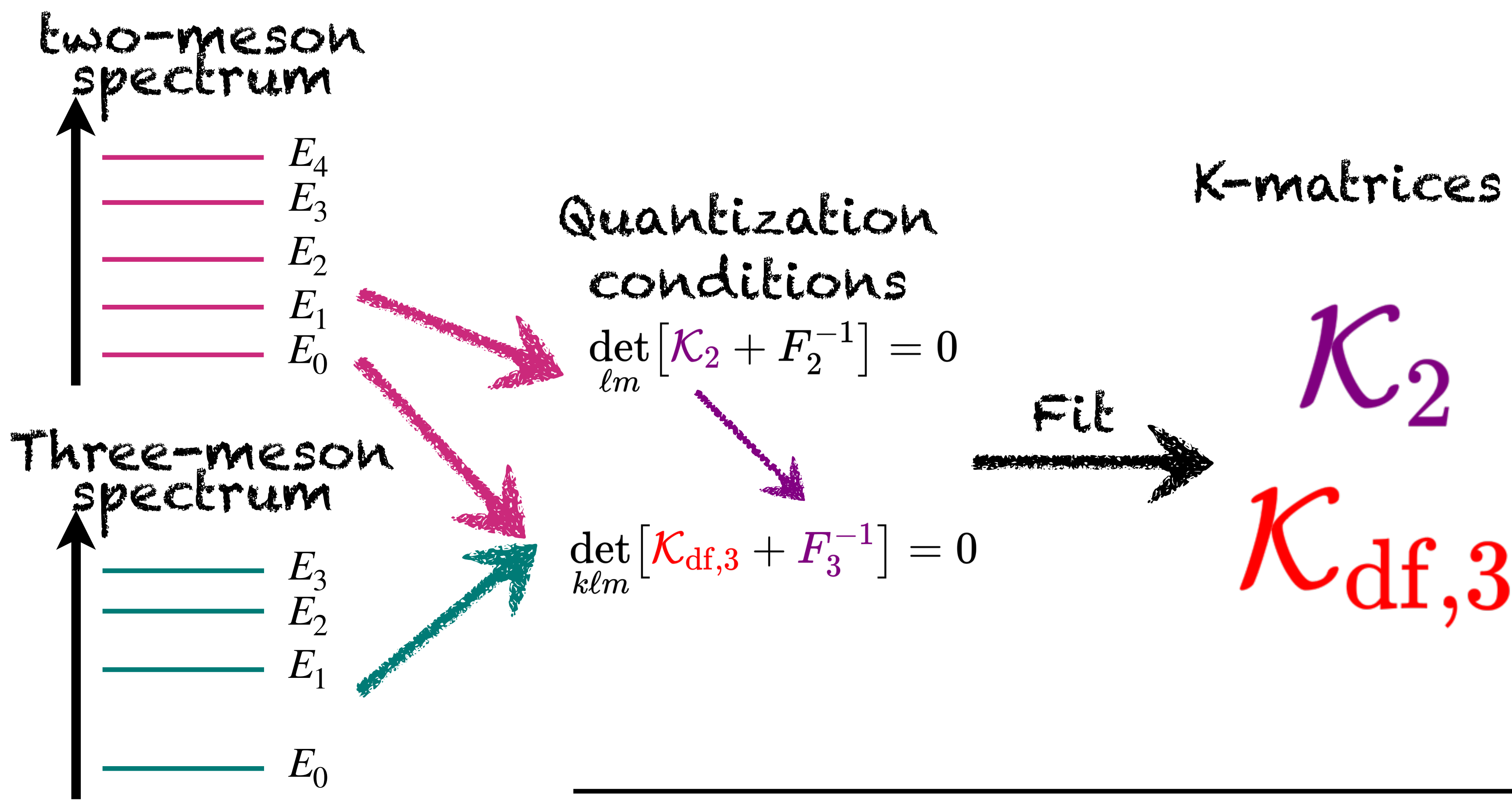
E250: K^+K^+

Example of spectrum



E250: $K^+K^+\pi^+$

Fits to spectra

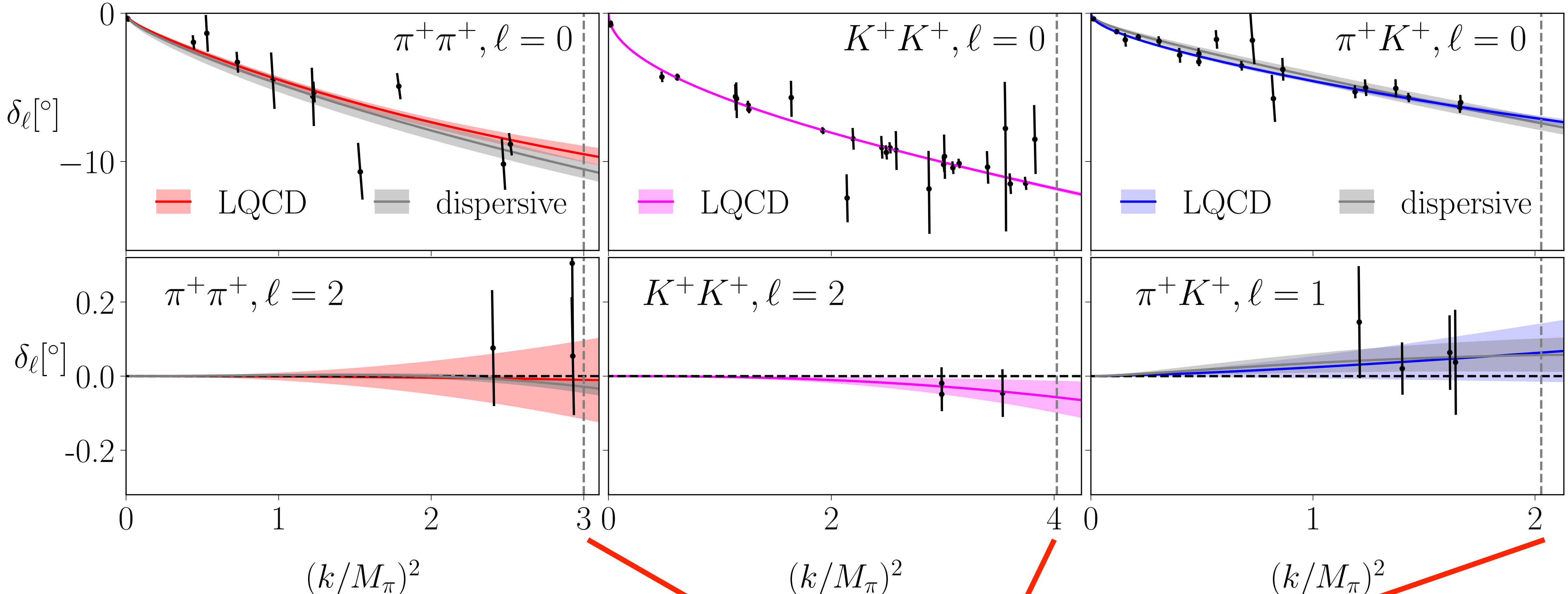


channels	# levels	# parameters	χ^2/DOF
$2\pi/3\pi$	$34+32$	6	1.14
$2\pi/\pi K/2\pi K$	$24+25+23$	9	1.23
$2K/\pi K/2K\pi$	$25+40+50$	10	1.95
$2K/3K$	$40+53$	6	1.49

Parameters in $\mathscr{K}_2$ and $\mathscr{K}_3$: use threshold expansions

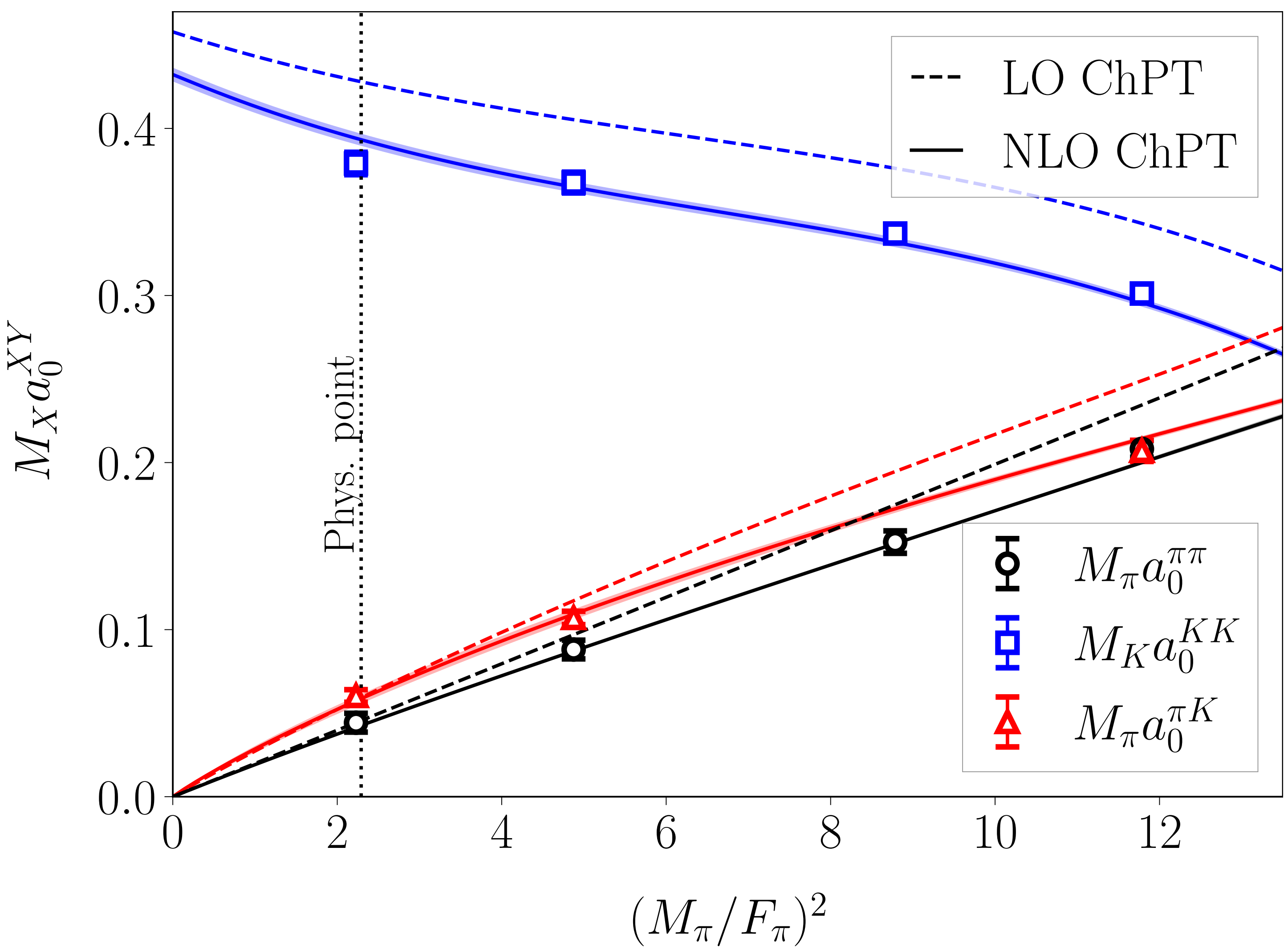
2-particle interactions

E250: ~physical quark masses



Inelastic thresholds

Scattering lengths



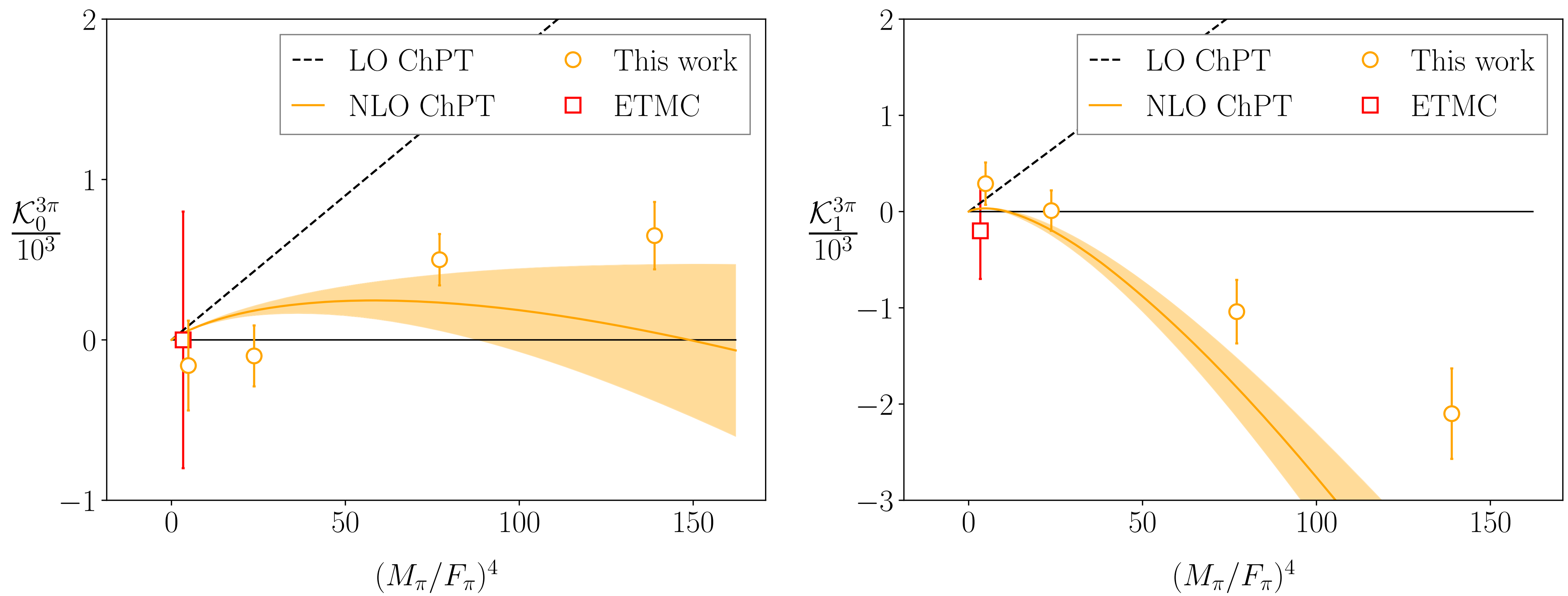
$\chi^2/\text{dof} = 12.4/(11-2)$

- Simultaneous chiral fit to $\pi\pi$, πK, KK scattering lengths
  - Fit involves two LECs, one of which is determined with 2% stat. Errors

Comparing $\mathscr{K}_{\mathrm{df},3}$ to ChPT

- Parametrize $\mathscr{K}_{\mathrm{df},3}$ in a threshold expansion, keeping first two terms
- From fits to spectra, we find non vanishing results for these terms at heavier quark masses
- Compare to NLO Chiral Perturbation theory calculation of $\mathscr{K}_{\mathrm{df},3}$ for $3\pi \to 3\pi$

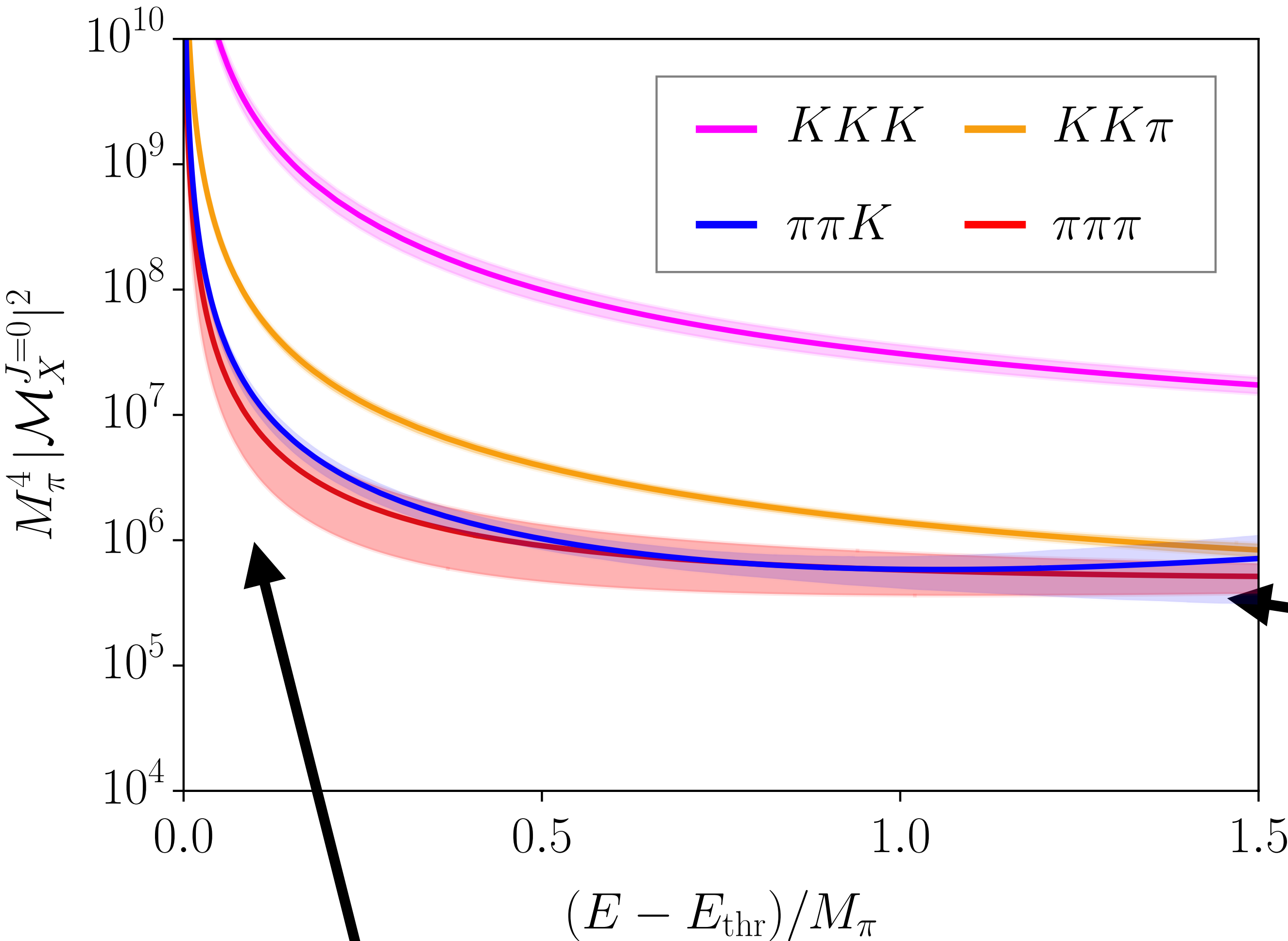
[Baeza-Ballesteros, Bijnens, Husek, Romero-López, SRS, Sjö, 2303.13206 (JHEP) & 2401.14293 (JHEP)]



Large NLO corrections in ChPT resolve LO inconsistency

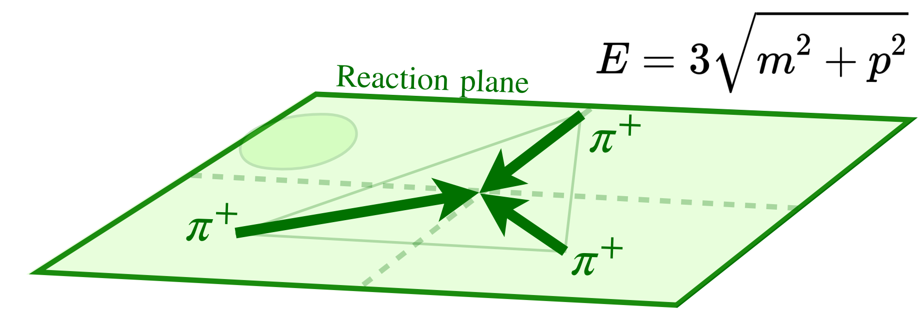
Physical 3-particle amplitudes

$M_\pi^4|\text{Amplitude}|^2$



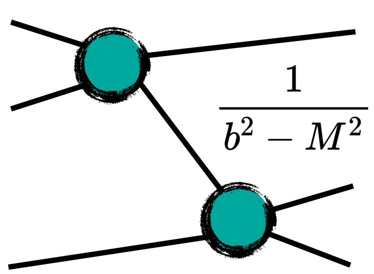
$J^P = 0^-$

Equilateral kinematic configuration

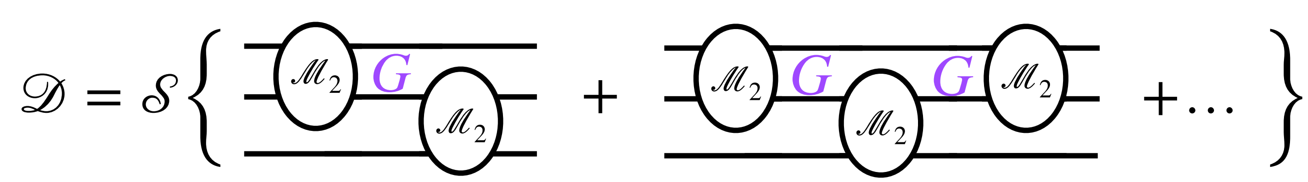


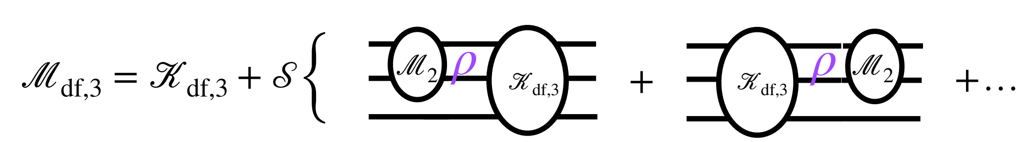
Pion interactions are chirally suppressed

Divergent at threshold



$$\mathcal{M}_3 = \mathcal{D} + \mathcal{M}_{\rm df,3}$$

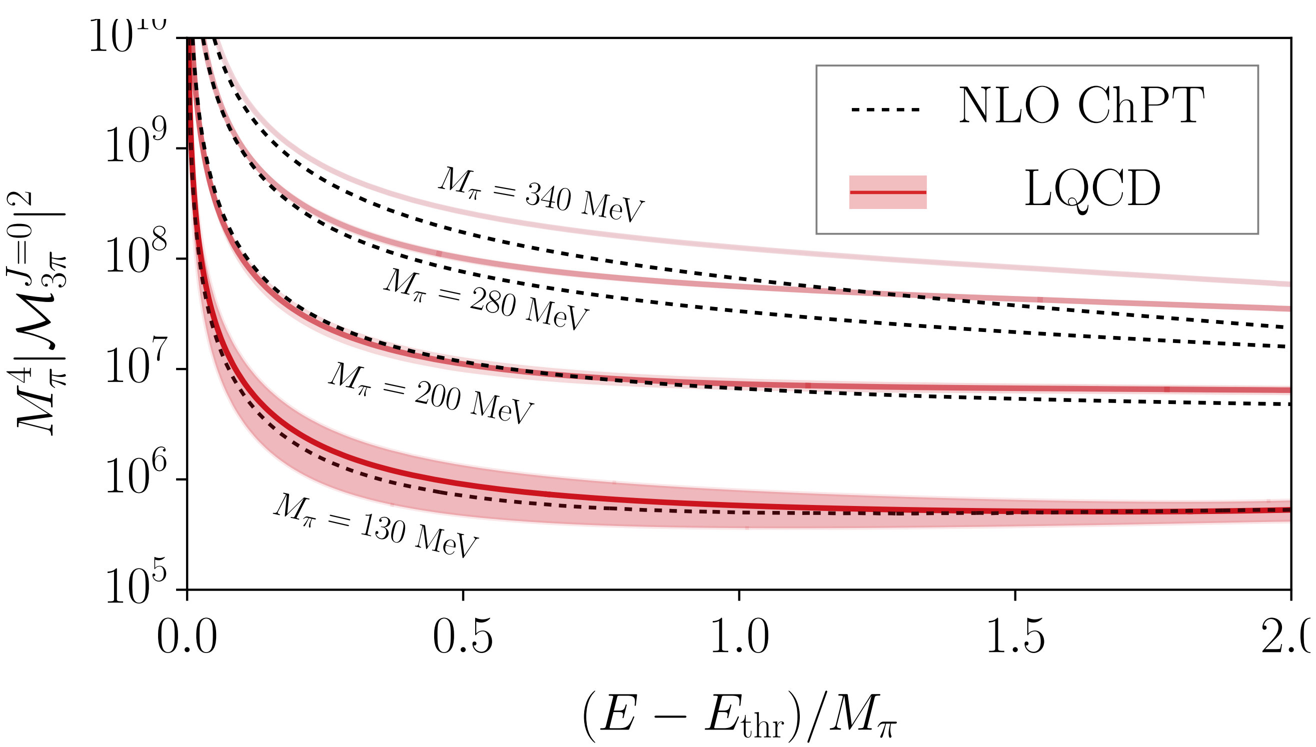


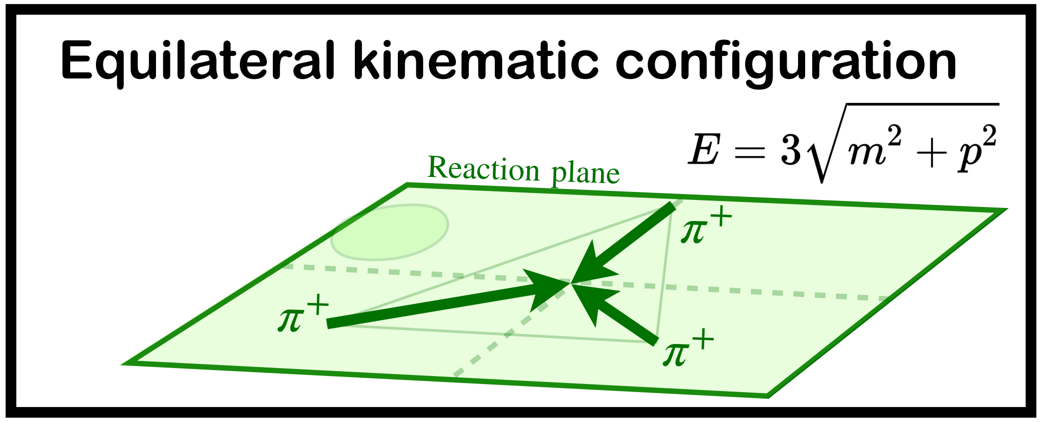


Comparison with ChPT

$M_\pi^4 |\text{Amplitude}|^2$

$J^P = 0^-$





Observe expected breakdown as M_π or p_π increase

Application of 3-particle formalism:

Incorporating DDπ effects and left-hand cuts in lattice QCD studies of the $\mathrm{T_{cc}}(3875)^+$

Maxwell T. Hansen,[a] Fernando Romero-López[b] and Stephen R. Sharpe[c]

[2401.06609 JHEP]

Finite- and infinite-volume study of $DD\pi$ scattering

Sebastian M. Dawid,[a] Fernando Romero-López[b,c] and Stephen R. Sharpe[a]

[2409.17059 JHEP]

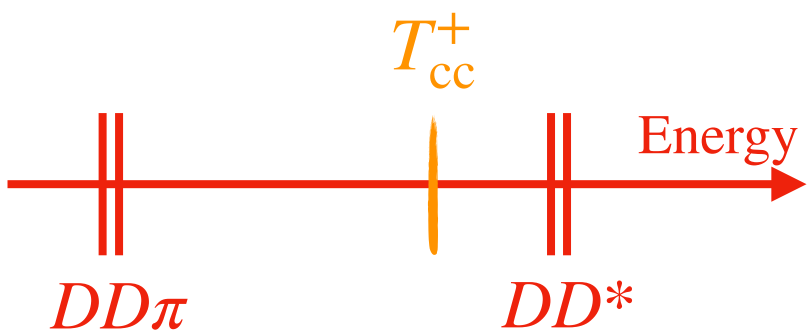
Comparison of integral equations used to study T_{cc}^+ for a stable D^*

Sebastian M. Dawid,[a] Fernando Romero-López[b] and Stephen R. Sharpe[a]

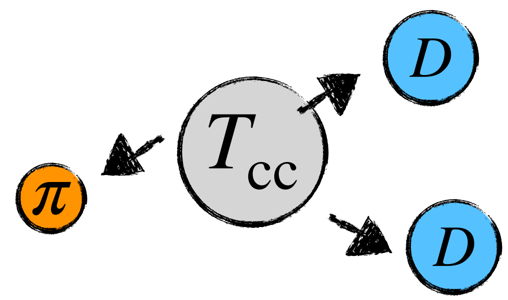
[2505.05466 JHEP]

Doubly-charmed tetraquark

Experiment

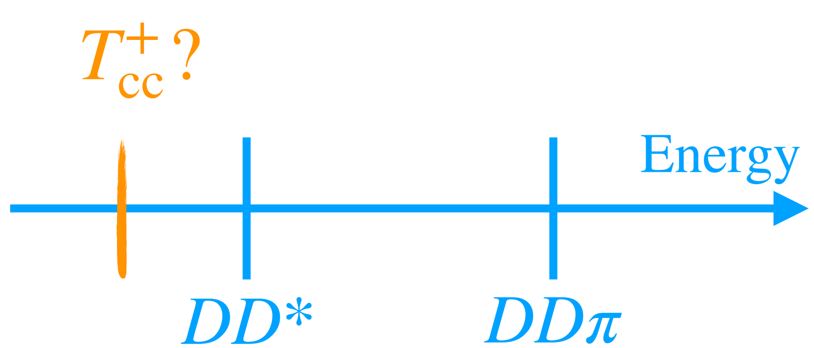


- **For physical quark masses is a three-body resonance**

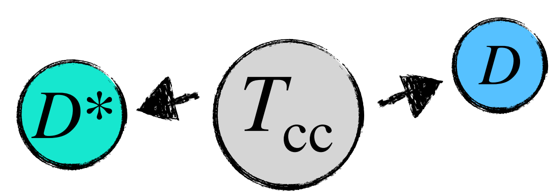


need three-body formalism!

LQCD calculations with heavier-than-physical quarks



- **D* is stable for slightly heavier-than-physical quarks**



suitable for the two-body finite-volume formalism?

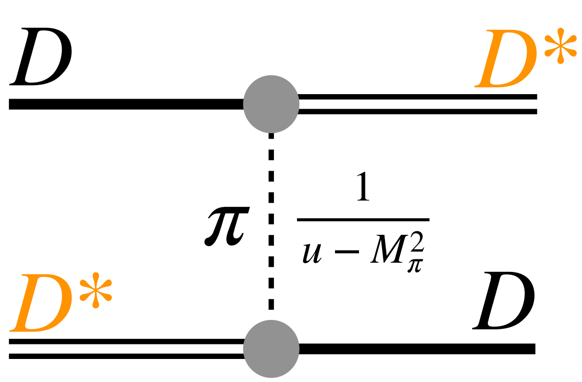
Adapted from: Fernando Romero-López

DD^* scattering

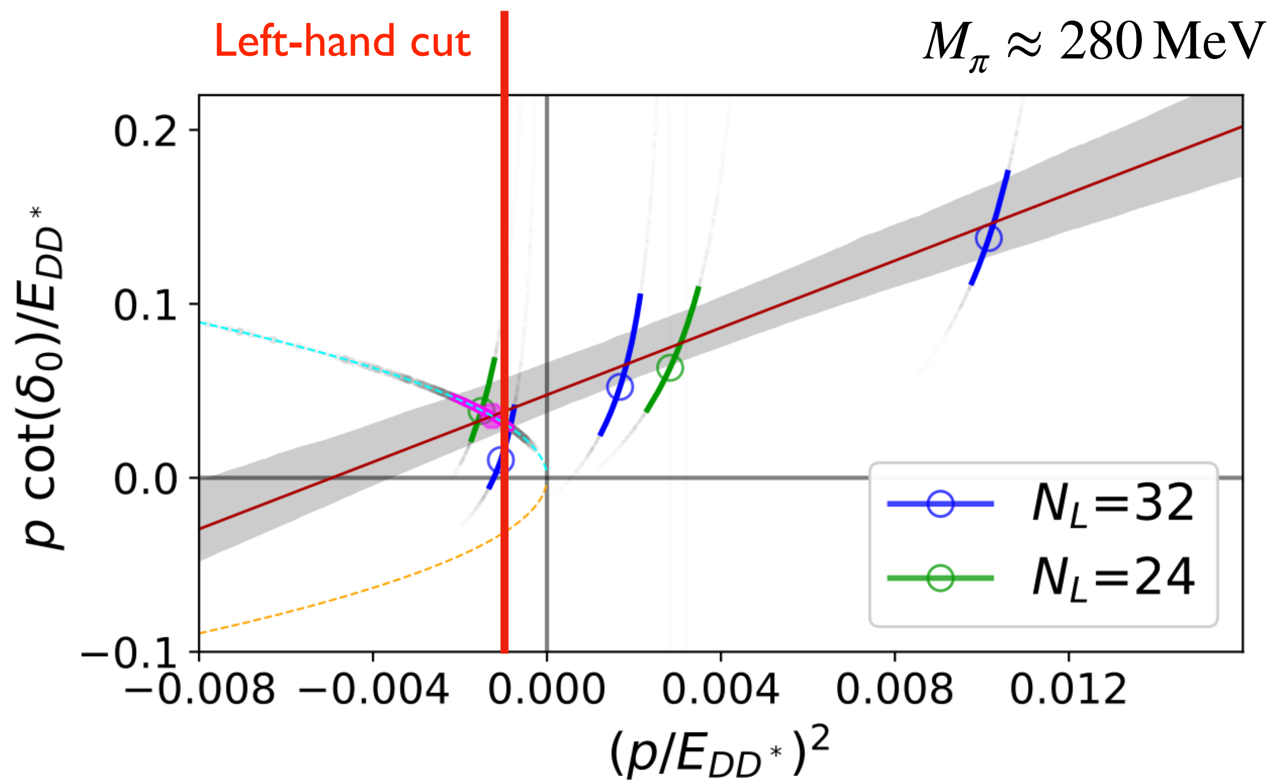
- Several LQCD studies using QC2 in T_{cc} channel

  [Chen et al., 2206.06185]
  [Lyu et al. (HALQCD), 2302.04505]
  [Padmanath & Prelovsek, 2202.10110]
  [Whyte, Thomas, Wilson, 2405.15741]

  - Use heavier-than-physical quarks
  - Find signature of virtual bound state
  - But QC2 fails at left-hand cut, below which phase shift becomes complex



[Padmanath, Prelovsek, arXiv:2202.10110]



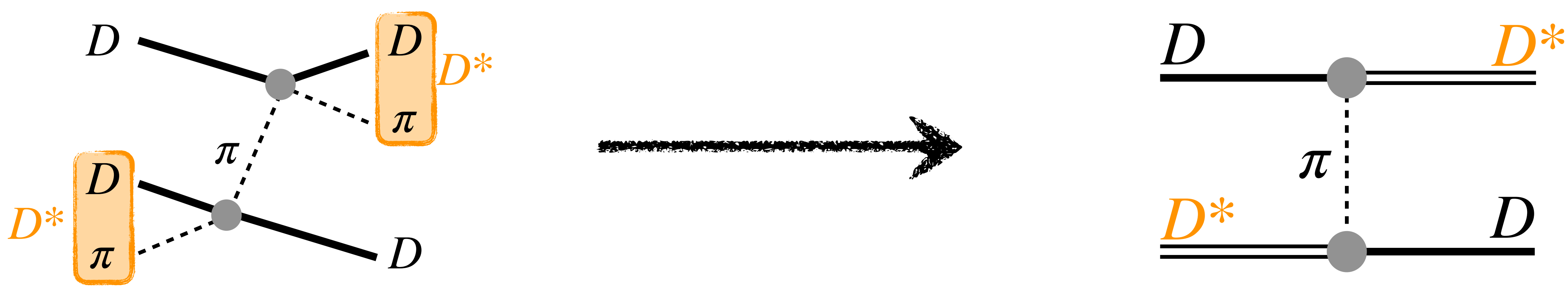
- Several solutions to failure have been proposed, by generalizing the QC2

  [Du et al (2408.09375), Abolnikov et al. (2407.04649), Bubna et al. (2402.12985), Meng et al. (2312.01930), Raposo, Hansen (2311.18793, Raposo, Hansen, Briceño, Jackura (2502.19375)]

3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

- Use QC3, but include D^* as a bound state in p-wave $D\pi$ channel
  - Finite-volume effects from u-channel pion exchange naturally incorporated

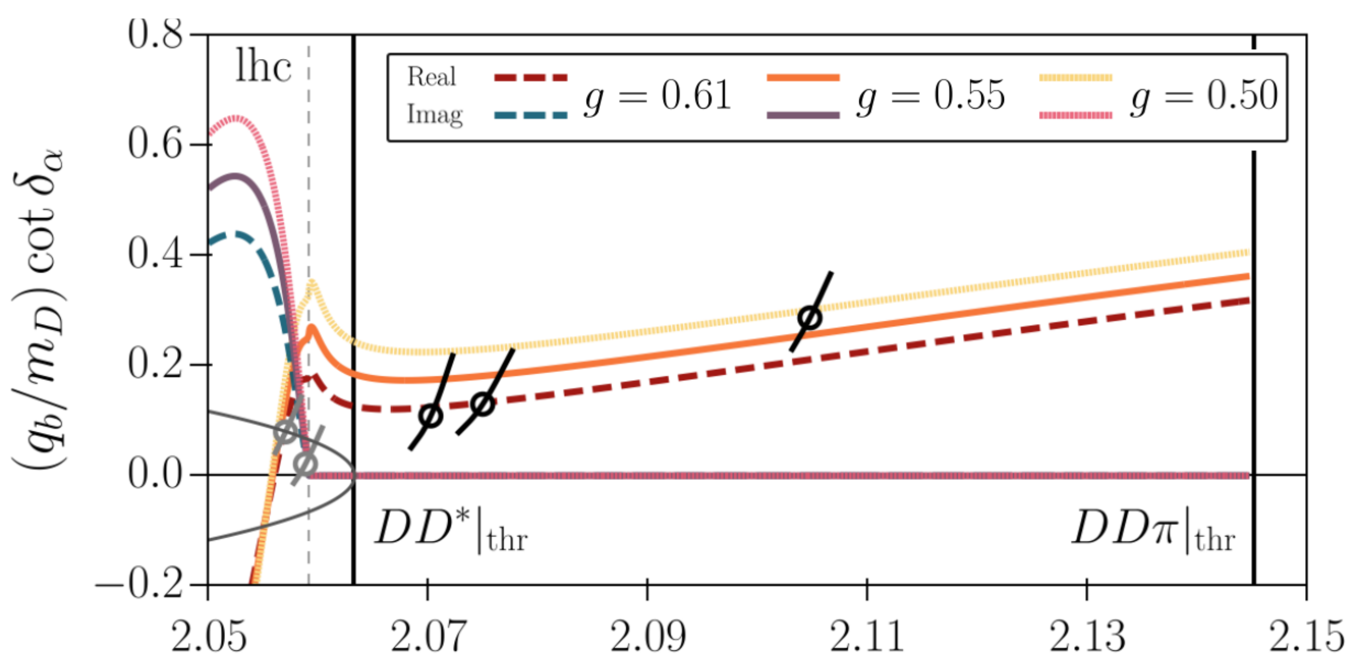


- Important advantage of QC3 approach:
  - Works for all choices of quark masses, including physical case of unbound D^*

Application to LQCD data

[Dawid, Romero-López, SRS, 2409.17059, JHEP]

- Proof of concept study to (limited) existing LQCD data with bound D^*
  - Choose p-wave $D\pi$ scattering amplitude to obtain D^*, for given choice of $g_{D^*D\pi}$
  - Choose reasonable s-wave $D\pi$ and DD amplitudes
  - Eyeball fit to results of [Padmanath, Prelovsek, arXiv:2202.10110] requires inclusion of $\mathcal{K}_{\mathrm{df},3}$
  - Solve integral equations, analytically continue to D^* pole, use LSZ to obtain $\mathcal{M}(DD^*)$
  - Result shows appropriate behavior at and below left-hand cut



- Virtual bound-state pole moves into complex plane!
- Looking forward: complete analysis requires more extensive LQCD spectra
  - DD, $D\pi$, $DD\pi$ & DD^* levels

Unexpected subthreshold singularities

Finite-volume formalism for $N\pi\pi$ at maximal isospin

Maxwell T. Hansen[a], Fernando Romero-López[b], and Stephen R. Sharpe[c]

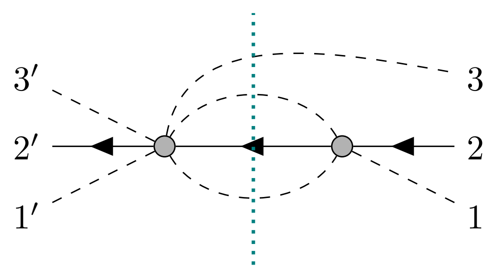
[2509.24778]

Parameter space in QC3

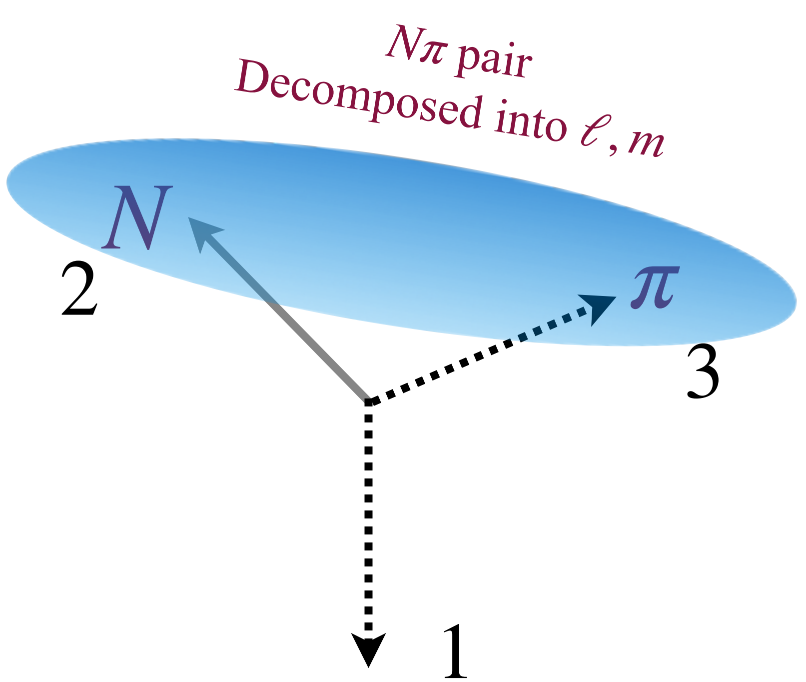
QC3 for $N\pi\pi$: $$\det_{i\boldsymbol{p}\ell m m_s}\left(1+\widehat{\mathcal{K}}_{\mathrm{df},3}(E^\star)\widehat{F}_3(E,\boldsymbol{P},L)\right)=0$$

Matrix indices are:
channel, spectator momentum,
pair angular momentum,
lab frame nucleon spin

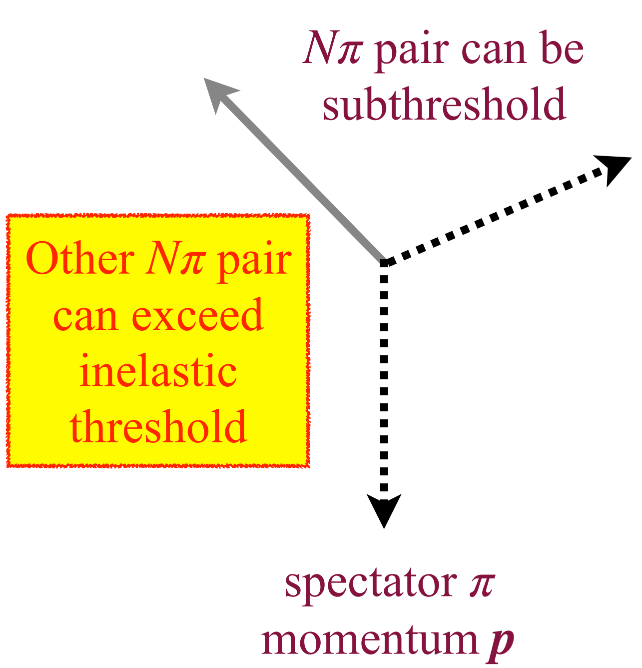
- A key quantity in derivation is B_3, $3\to 3$ Bethe-Salpeter kernel (3 Particle Irreducible)
  - Must be nonsingular in range of kinematic parameters
  - Otherwise introduce uncontrolled L^{-n} finite-volume effects



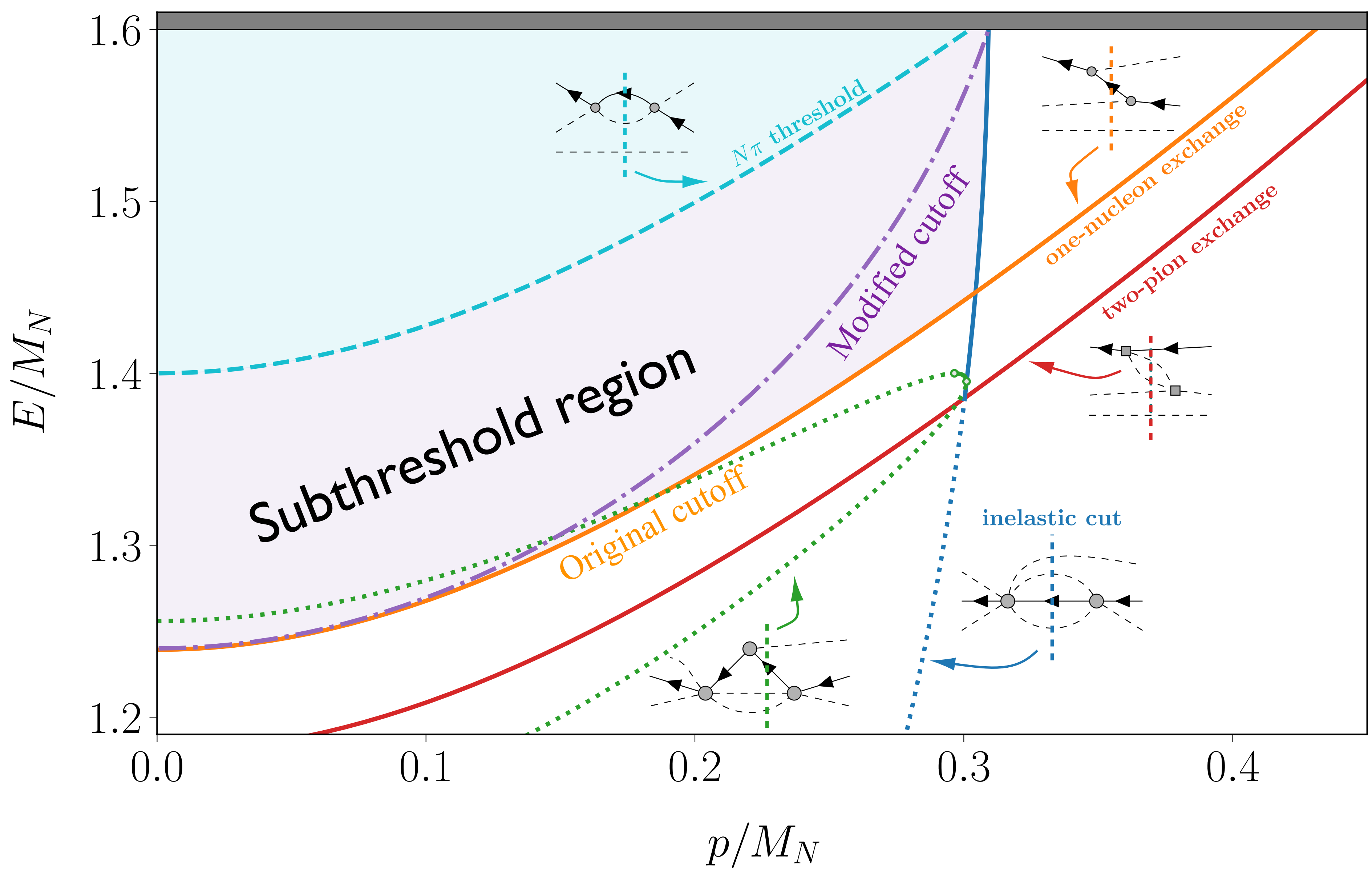
- Parameter space (for pion spectator)
  - Energy E, momentum $\boldsymbol{P}$



Subthreshold singularities



Parameter space for $\boldsymbol{P} = 0$, $M_\pi/M_N = 0.2$



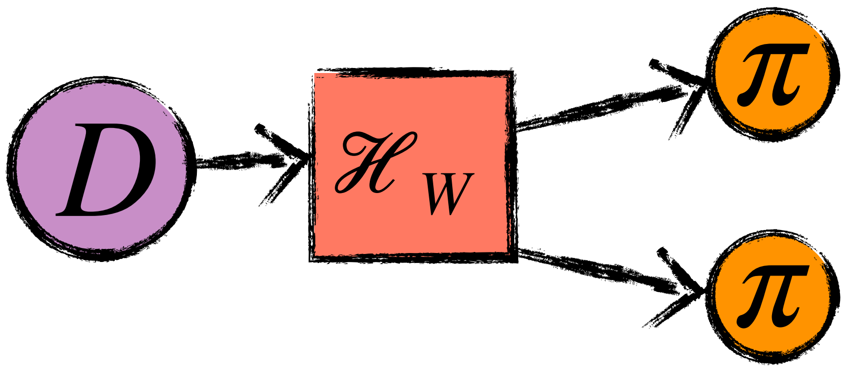
Summary & outlook

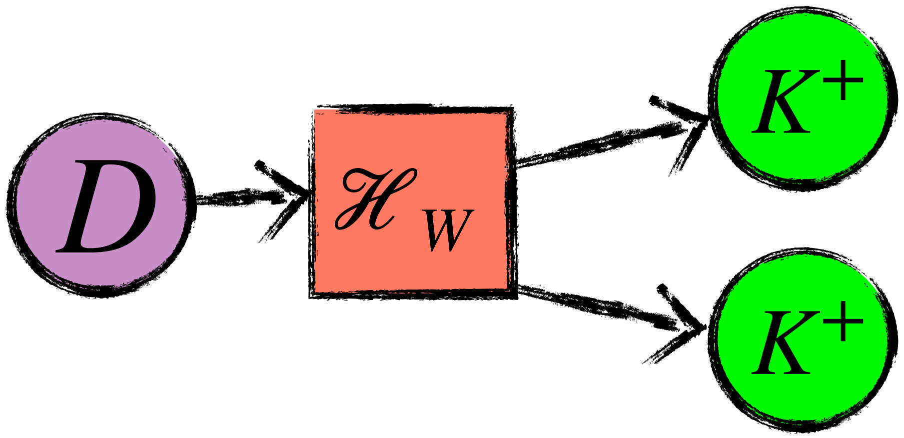
Summary & Outlook

- Two-particle sector is entering precision phase
  - Frontier is two nucleons, and form factors of mesonic resonances
- Major steps have been taken in the three-particle sector
  - Formalism well established & cross checked, and almost complete
  - Several applications to three-particle spectra from LQCD
  - Path to a calculation of $K \to 3\pi$ decay amplitudes is now open
- Next steps in implementation
  - $T_{cc}^{+} \to D^{*}D \to DD\pi$ for lighter quark masses
  - $3\pi(I=2) \leftrightarrow \rho\pi\ ;\ 3\pi(I=0) \leftrightarrow \omega(782) \leftrightarrow K\overline{K}$(I=0) (WZW term)
  - $N\pi\pi \leftrightarrow \Delta\pi\ ;\ N\pi\pi + N\pi$ [Roper]
  - Improved, model-independent fitting & analytic continuation [Salg, Romero-López, Jay, 2506.16161]
- Next steps in formalism
  - $NNN(I=\frac{1}{2}), N\pi\pi + N\pi$ [for Roper] & $NN\pi + NN$ (all underway)
  - Four particles!

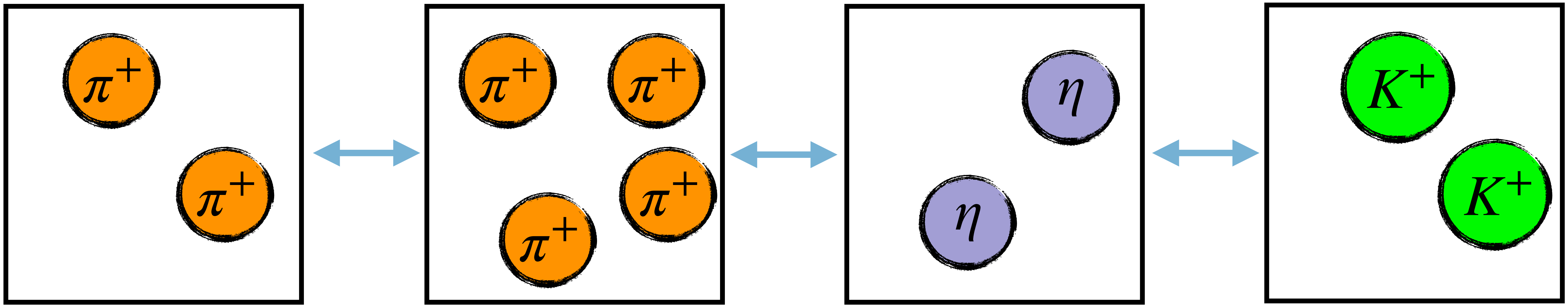
Long-term dream

CP violation in D decays





Challenge: finite-volume mixing with $4\pi, 6\pi, \eta\eta, \ldots$

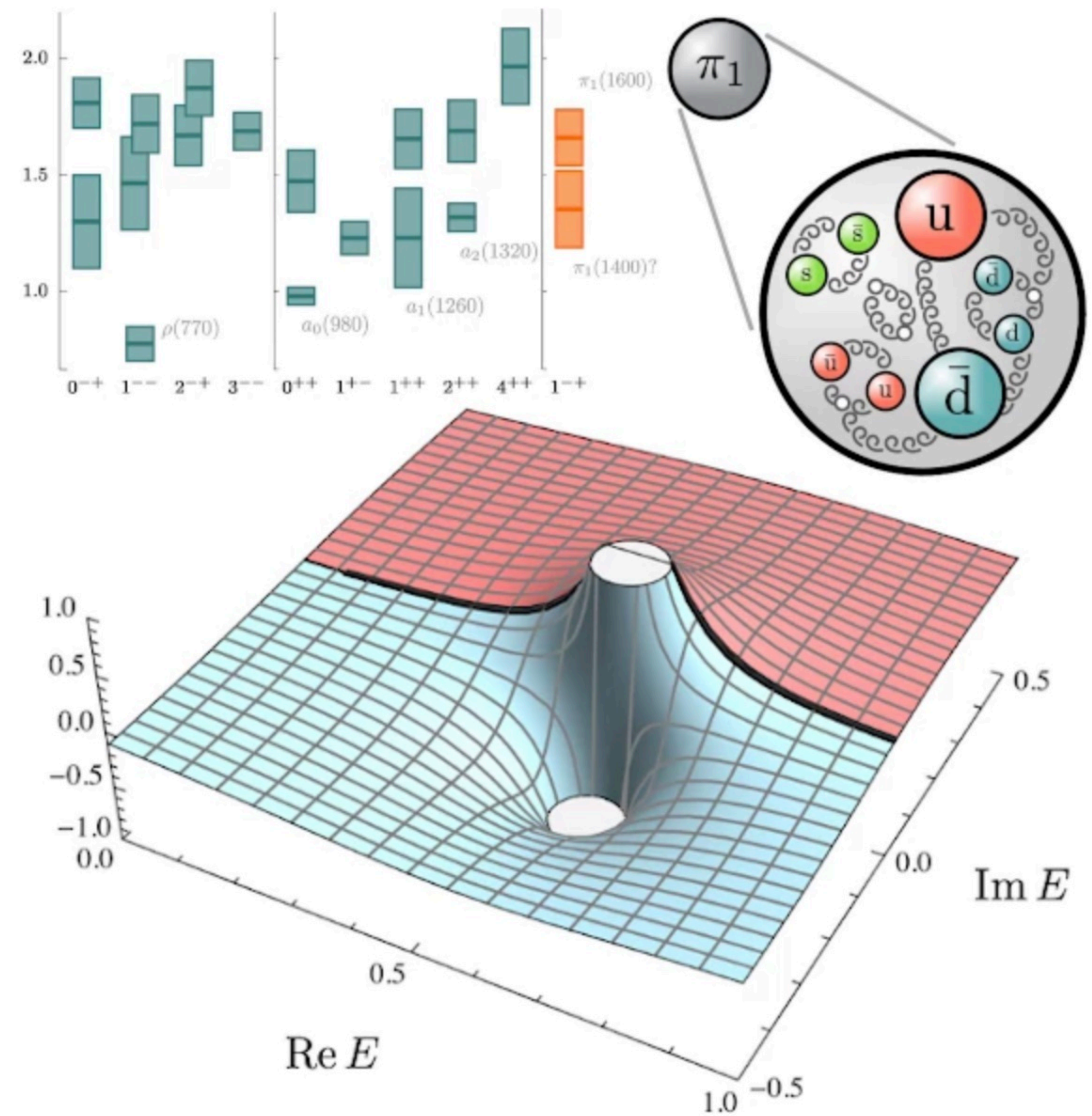


4+-particle formalism not yet developed

ExoHad collaboration

ExoHad Collaboration exohad.org People Events Talks Publications

EXOHAD
EXOTIC HADRONS TOPICAL COLLABORATION



The Exo(tic) Had(ron) Collaboration started in 2023 to explore all aspects of exotic hadron physics, from predictions within lattice QCD, through reliable extraction of their existence and properties from experimental data, to descriptions of their structure within phenomenological models.

Thank you!
Questions?

Backup slides

Explicit forms of F & G

- F & G are known geometrical functions, containing cutoff function $H(k)$

$$\widetilde{F}_{p\ell'm';k\ell m} = \delta_{pk}\ H(\vec{k})\ F_{\ell'm';\ell m}(E-\omega_k, \overrightarrow{P}-\vec{k}, L)$$

$$F_{\ell'm';\ell m}(E,\overrightarrow{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \mathrm{PV}\int\frac{d^3k}{(2\pi)^3}\right)\frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k})}$$

$$\mathcal{Y}_{\ell m}(\vec{k}^*) = \sqrt{4\pi}\left(\frac{k^*}{q^*}\right)^{\ell} Y_{\ell m}(\hat{k}^*)$$

$$G_{p\ell'm';k\ell m} = \left(\frac{k^*}{q_p^*}\right)^{\ell'}\frac{4\pi Y_{\ell'm'}(\hat{k}^*)H(\vec{p})H(\vec{k})Y^*_{\ell m}(\hat{p}^*)}{(P-k-p)^2-m^2}\left(\frac{p^*}{q_k^*}\right)^{\ell}\frac{1}{2\omega_k L^3}$$

Relativistic form introduced in [BHS17]

Two-particle formalism

[Lüscher, 1986-91 + many subsequent works by many authors]

I will follow approach of [Kim, Sachrajda, & SS, 2005], generalized to use time-ordered PT following [Blanton & SS, 2020]

Generic relativistic FT (RFT) approach

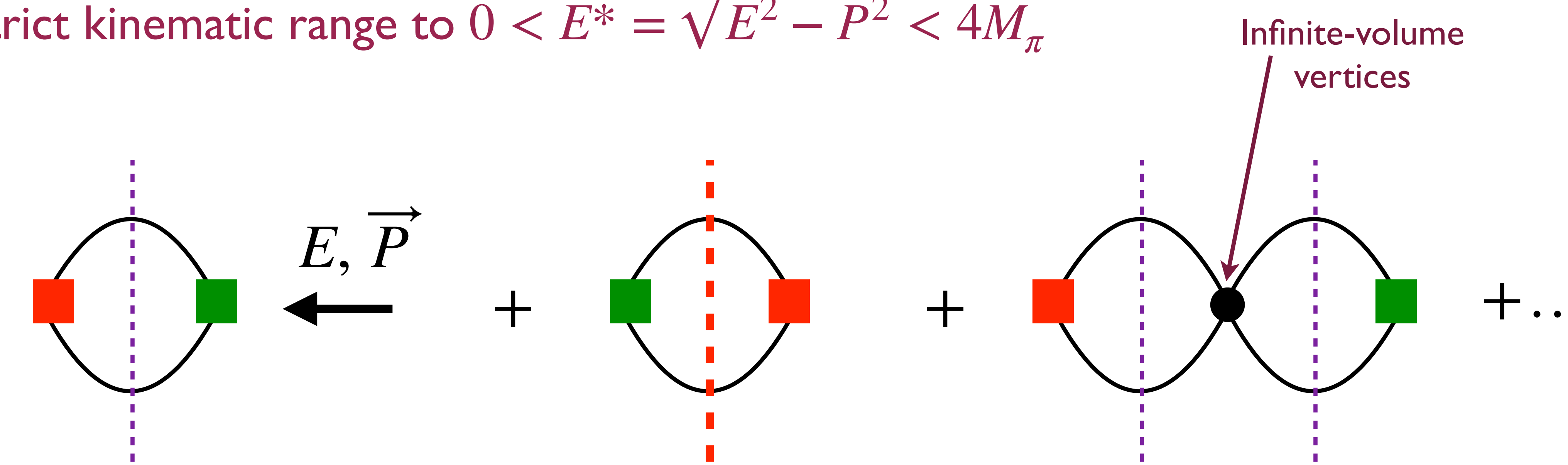
- Study Minkowski time, finite-volume correlator

$$C_L(E, \vec{P}) \equiv \int_L d^4x\, e^{iEt - i\vec{P}\cdot\vec{x}} \langle \Omega | T\left\{ \sigma_{2\pi}(x) \sigma_{2\pi}^{\dagger}(0) \right\} | \Omega \rangle_L$$

  - For fixed $\vec{P}$, poles in C_L occur when $E = E_n$

- Analyze in generic EFT for pions, (kaons, …) working to all orders in (TO)PT
  - For simplicity, assume exact isospin symmetry
  - Restrict kinematic range to $0 < E^* = \sqrt{E^2 - P^2} < 4M_\pi$

$C_L(E, \vec{P}) =$



No need for $i\epsilon$ in finite volume

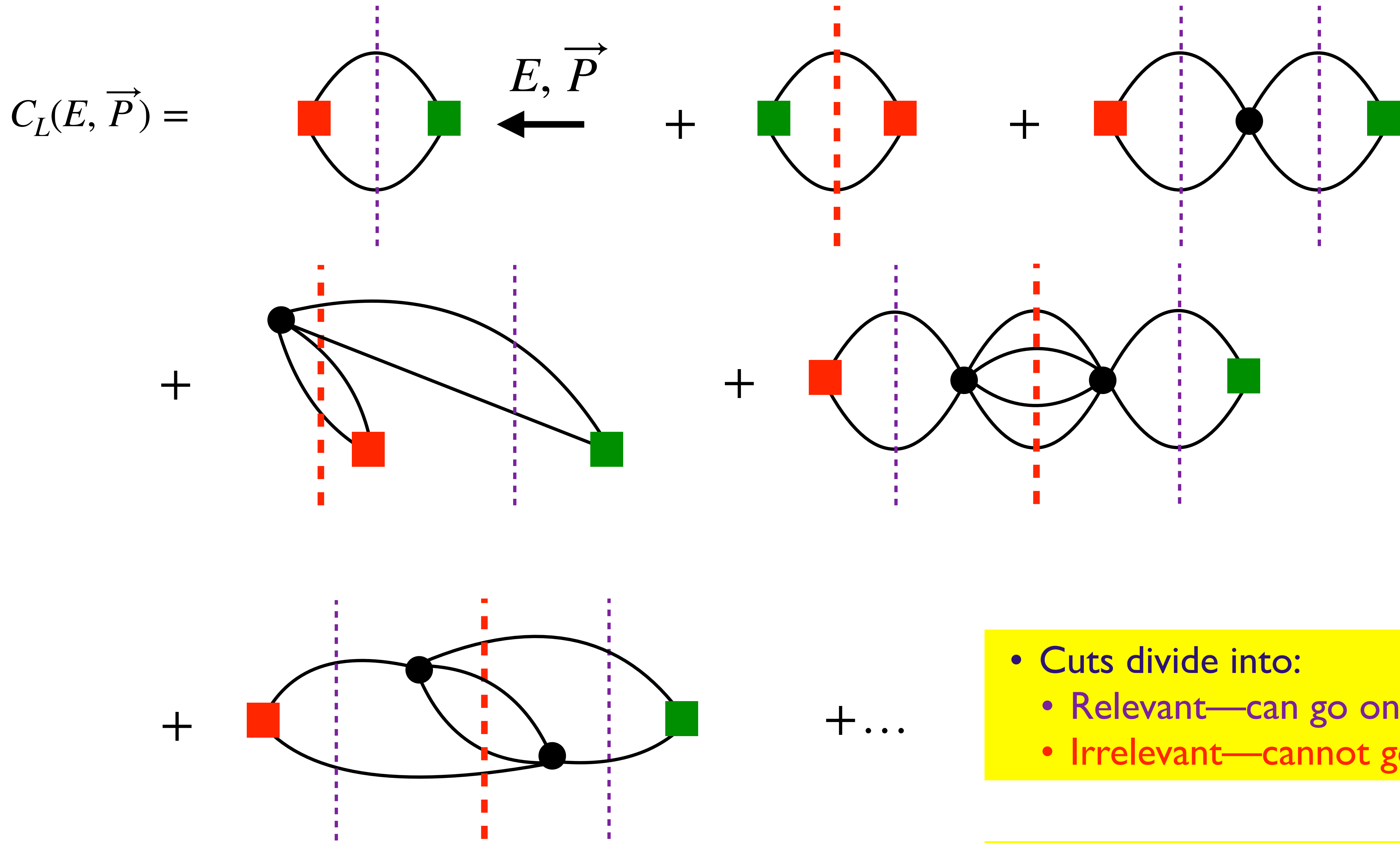
$\frac{1}{E - \omega_1 - \omega_2}$ Can go on shell

$\frac{1}{-E - \omega_1 - \omega_2}$ Cannot go on shell

$\frac{1}{E - \omega_1 - \omega_2}$ Can go on shell

$\frac{1}{E - \omega_1 - \omega_2}$ Can go on shell

Generic relativistic FT (RFT) approach



- Cuts divide into:
  - Relevant—can go on shell
  - Irrelevant—cannot go on shell

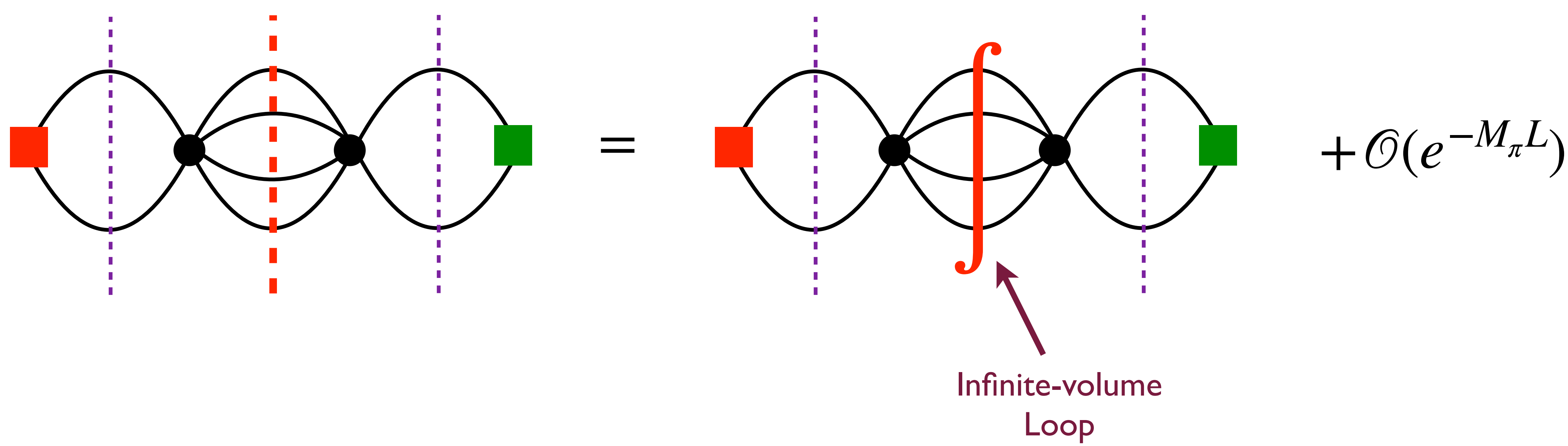
- Three-momenta in loops are summed over finite-volume set

Use Poisson summation formula

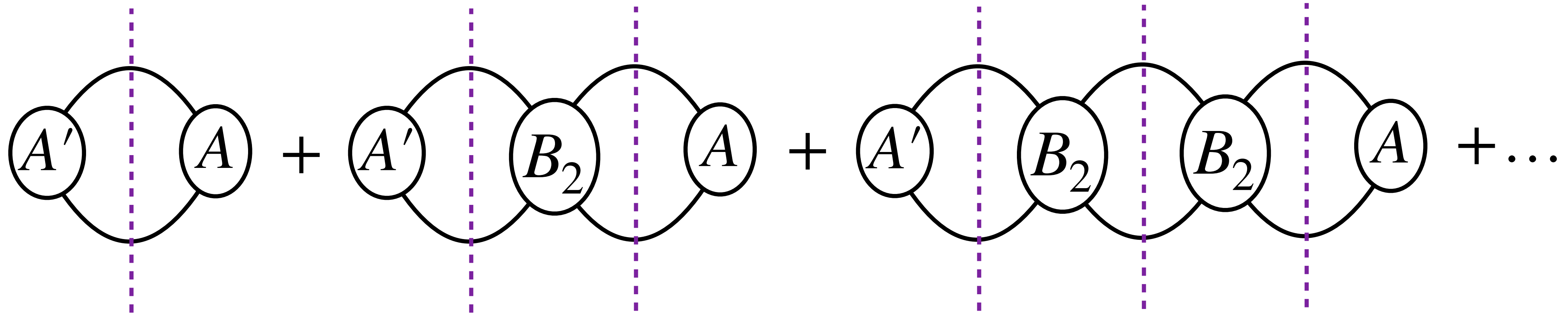
$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \cancel{\sum_{\vec{l}\neq\vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})}$$

Exp. suppressed if $g(\vec{k})$ is smooth
and $g' \sim g/M_\pi$

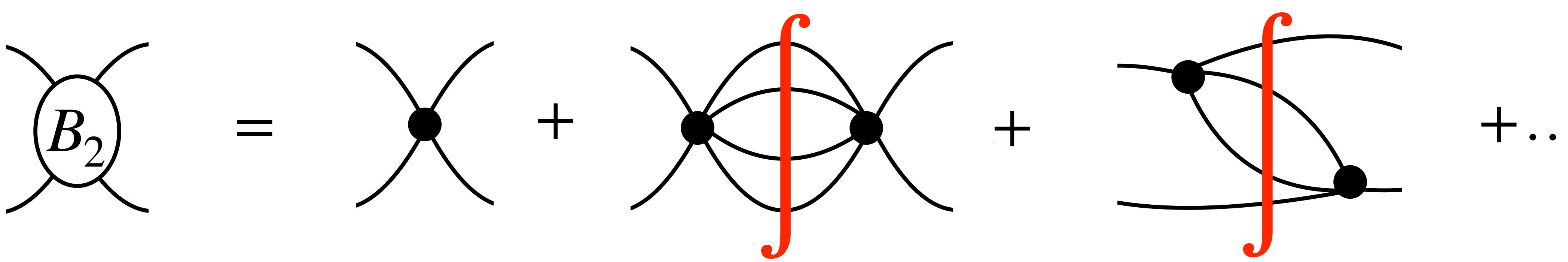
- Replace loop sums with integrals where summand/integrand is nonsingular
  - Drop exponentially suppressed terms ($e^{-M_\pi L}, e^{-(M_\pi L)^2}$, etc.) while keeping power-law dependence



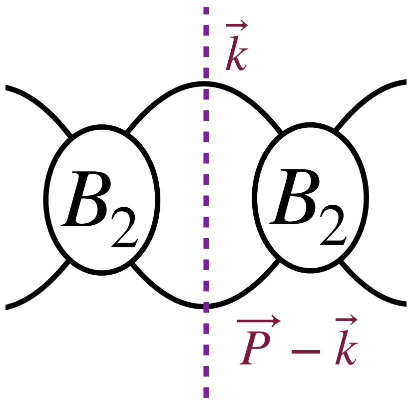
Expansion in relevant cuts

$C_L(E, \vec{P}) = C_L^{(0)}(E, \vec{P}) +$ 

- B_2 is the TOPT version of a Bethe-Salpeter kernel (2PI in s-channel)
    - A' and A are corresponding "endcaps"



Dealing with relevant cuts



$$\frac{1}{L^3}\sum_{\vec{k}} f(E,\vec{P},\vec{k})\frac{1}{2}\frac{1}{4\omega_k\omega_{P-k}}\frac{1}{E-\omega_k-\omega_{P-k}}g(E,\vec{P},\vec{k})$$

$$= \mathrm{PV}\int \frac{d^3k}{(2\pi)^3} f(E,\vec{P},\vec{k})\frac{1}{2}\frac{1}{4\omega_k\omega_{P-k}}\frac{1}{E-\omega_k-\omega_{P-k}}g(E,\vec{P},\vec{k})$$

$$+\sum_{\ell',m';\ell,m} f^{\mathrm{on}}_{\ell'm'}(E^*)\ F_{\ell'm';\ell m}(E,\vec{P},L)\ g^{\mathrm{on}}_{\ell m}(E^*)$$

On-shell projected in pair CM frame, and decomposed into harmonics

- F is a known, calculable kinematic finite-volume function

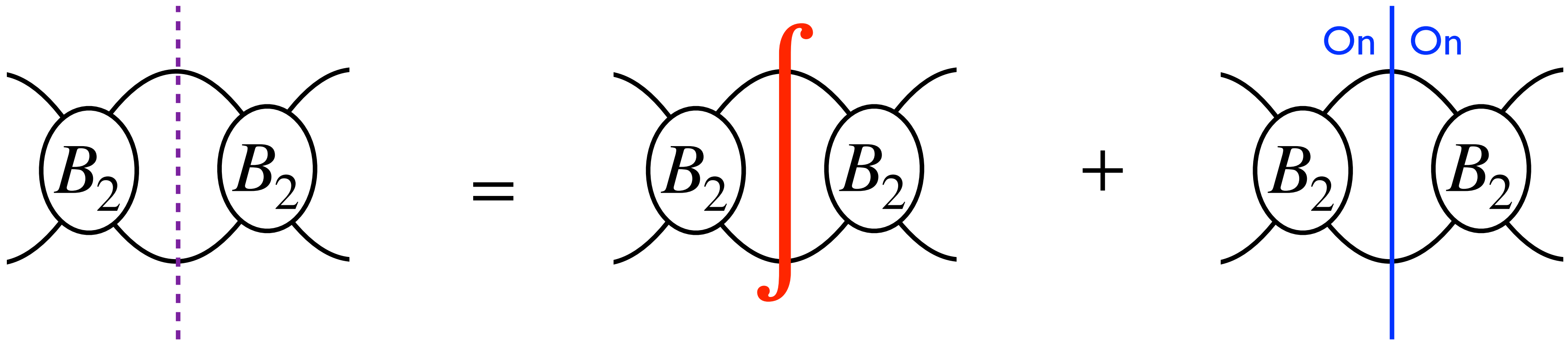
Harmonic polynomial

CM frame relative momentum

$$\boxed{F_{\mathrm{PV};\ell'm';\ell m}(E,\vec{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \mathrm{PV}\int\frac{d^3k}{(2\pi)^3}\right)\frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k})}}$$

UV cutoff

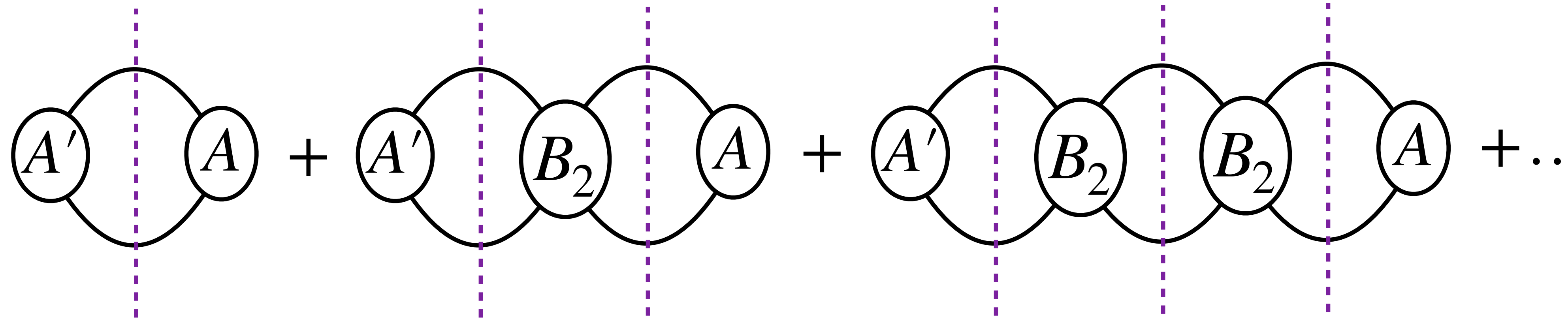
Key move



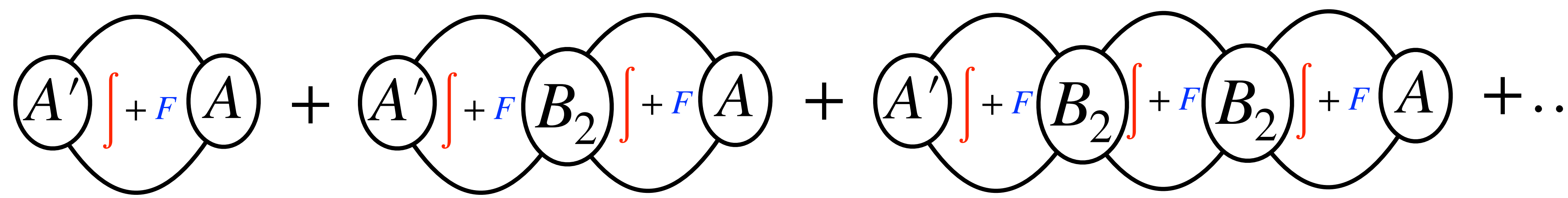
$$\Sigma = \int + F$$

Resummations

$$C_L(E,\vec{P}) = C_L^{(0)}(E,\vec{P}) +$$

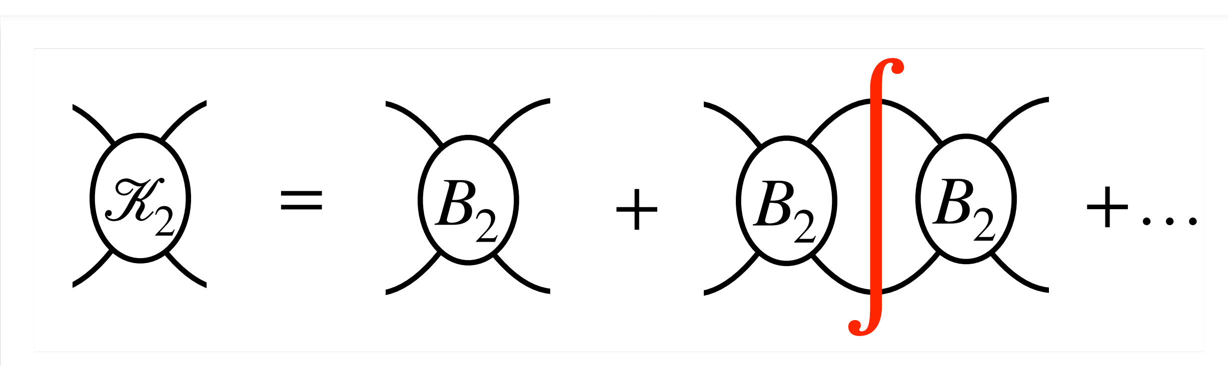


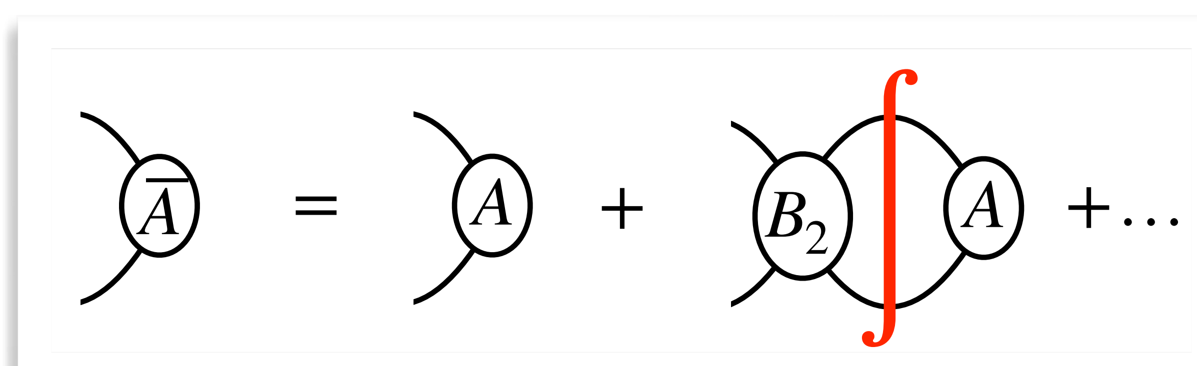
$$= C_L^{(0)}(E,\vec{P}) +$$



$$= C_\infty(E,\vec{P}) \; + \overline{A}' \cdot iF \cdot \overline{A} + \overline{A}' \cdot iF \cdot i\mathscr{K}_2 \cdot iF \cdot \overline{A} + \overline{A}' \cdot iF \cdot i\mathscr{K}_2 \cdot iF \cdot i\mathscr{K}_2 \cdot iF \cdot \overline{A} + \ldots$$

$$= C_\infty(E,\vec{P}) \; + \; \overline{A}' \cdot iF \cdot \frac{1}{1+\mathscr{K}_2 \cdot F} \cdot \overline{A}$$





Quantization condition (QC2)

$$C_L(E, \overrightarrow{P}) = C_\infty(E, \overrightarrow{P}) \; + \; \overline{A}' \cdot iF \cdot \frac{1}{1 + \mathscr{K}_2 \cdot F} \cdot \overline{A}$$

Has no L-dependent poles

Only source of L-dependent poles

- QC2: finite-volume energies occur when

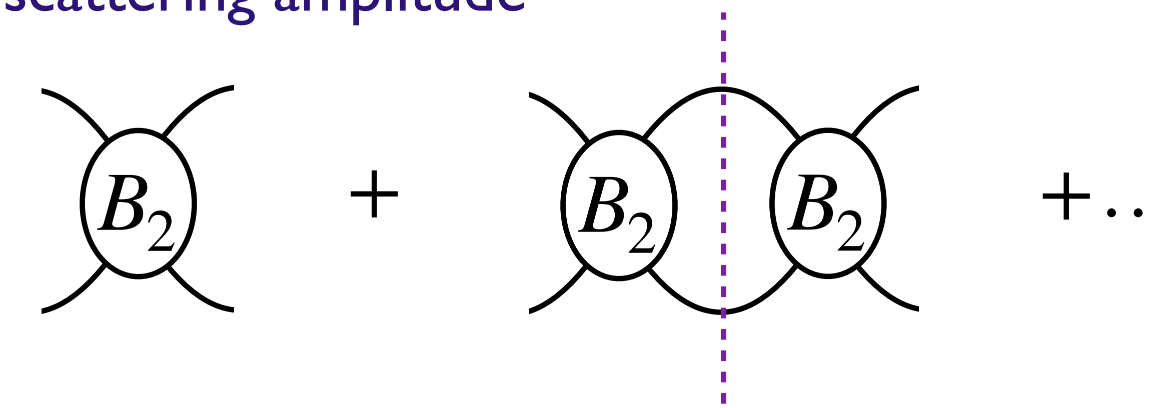
$$\det(F^{-1} + \mathscr{K}_2) = 0$$

  - Matrix indices are CM-frame ℓ, m
  - $\mathscr{K}_2$ is an infinite-volume quantity: diagonal in ℓ, m
  - F depends on finite-volume size & geometry, mixes ℓ, m
  - In practical applications, must truncate in ℓ

Step 2: relating $\mathcal{K}_2$ to $\mathcal{M}_2$

- Consider "finite-volume scattering amplitude"

$$\mathcal{M}^{(\text{off})}_{2,L} =$$



  - Use similar steps as for $C_{2,L}$: project on ℓ, m, project on shell, use "key move"

$$i\mathcal{M}_{2,L} = i\mathcal{K}_2 + i\mathcal{K}_2 \cdot iF \cdot i\mathcal{K}_2 + \ldots = i\mathcal{K}_2 \frac{1}{1 + F\mathcal{K}_2}$$

- Take $L \to \infty$ limit, regularizing integrals with $i\epsilon$ prescription

$$\mathcal{M}_{2,L} \to \mathcal{M}_2, \qquad F_{\ell',m';\ell,m} \to -i\delta_{\ell'\ell}\delta_{m'm}\rho, \qquad \rho = -i\sqrt{q^{*2}}/16\pi E^*$$

  - Leads to standard relation between $\mathcal{M}_2$ & $\mathcal{K}_2$, showing that $\mathcal{K}_2$ is the standard, <u>relativistic</u> two-particle K matrix

$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho\mathcal{K}_2}$$

Matrix structure in QC3

- All quantities are infinite-dimensional matrices with indices $\mathbf{k}\ell m i$ describing 3 on-shell particles

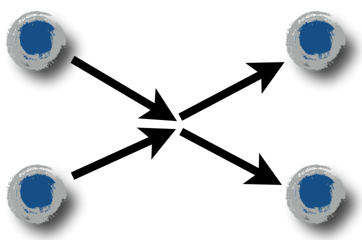
[finite volume "spectator" momentum: $\boldsymbol{k} = (2\pi/L)\boldsymbol{n}$] x [2-particle CM angular momentum: ℓ, m] x [spectator flavor: i]

- For large $\boldsymbol{k}$ (at fixed E, L), the other two particles are below threshold
- Must include such configurations, by analytic continuation, up to a cut-off at $k \approx m$ [Polejaeva & Rusetsky, `12]

Divergence-free K matrix

- $K_{df,3}$ has the same symmetries as M_3: relativistic invariance, particle interchange, T-reversal

M_2, K_2

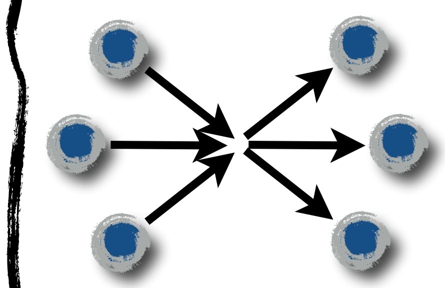


12 momentum components

-10 Poincaré generators

2 degrees of freedom
$s=E^{*2}$ + θ

M_3, $K_{df,3}$



18 momentum components

-10 Poincaré generators

8 degrees of freedom
$s=E^{*2}$ + 7 “angles”

- Need more parameters to describe $\mathscr{K}_{\rm df,3}$ than $\mathscr{K}_2$ (will be discussed in lecture 3)
- Why $\mathscr{K}_2$ and $\mathscr{K}_{\rm df,3}$ appear in QC3, rather than $\mathscr{M}_2$ and $\mathscr{M}_{\rm df,3}$, will be explained shortly

Threshold expansion for $\mathscr{K}_{\mathrm{df},3}$

- $\mathscr{K}_{\mathrm{df},3}$ is a real, smooth function which is Lorentz, P and T invariant
- Expand about threshold in powers of $\Delta = (s - 9M_\pi^2)/9M_\pi^2$, $\tilde{t}_{ij} = (p_i' - p_j)^2/9M_\pi^2, \ldots$

Depend on CM energy

Angular dependence

$$\mathcal{K}_{\mathrm{df},3} = \boxed{\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},\,1}\Delta + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},\,2}\Delta^2} + \boxed{\mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B} + \mathscr{O}(\Delta^3)$$

$$\boxed{\Delta_B = \sum_{i,j=1}^{3} \tilde{t}_{ij}^2 - \Delta^2,}$$

- Can separate terms in fit based on dependence on energy and rotational properties
  - E.g. only $\mathscr{K}_B$ contributes to nontrivial irreps