# Fundamental constants and electroweak phenomenology from the lattice

# Lecture IV: CKM phenomenology: at tree level

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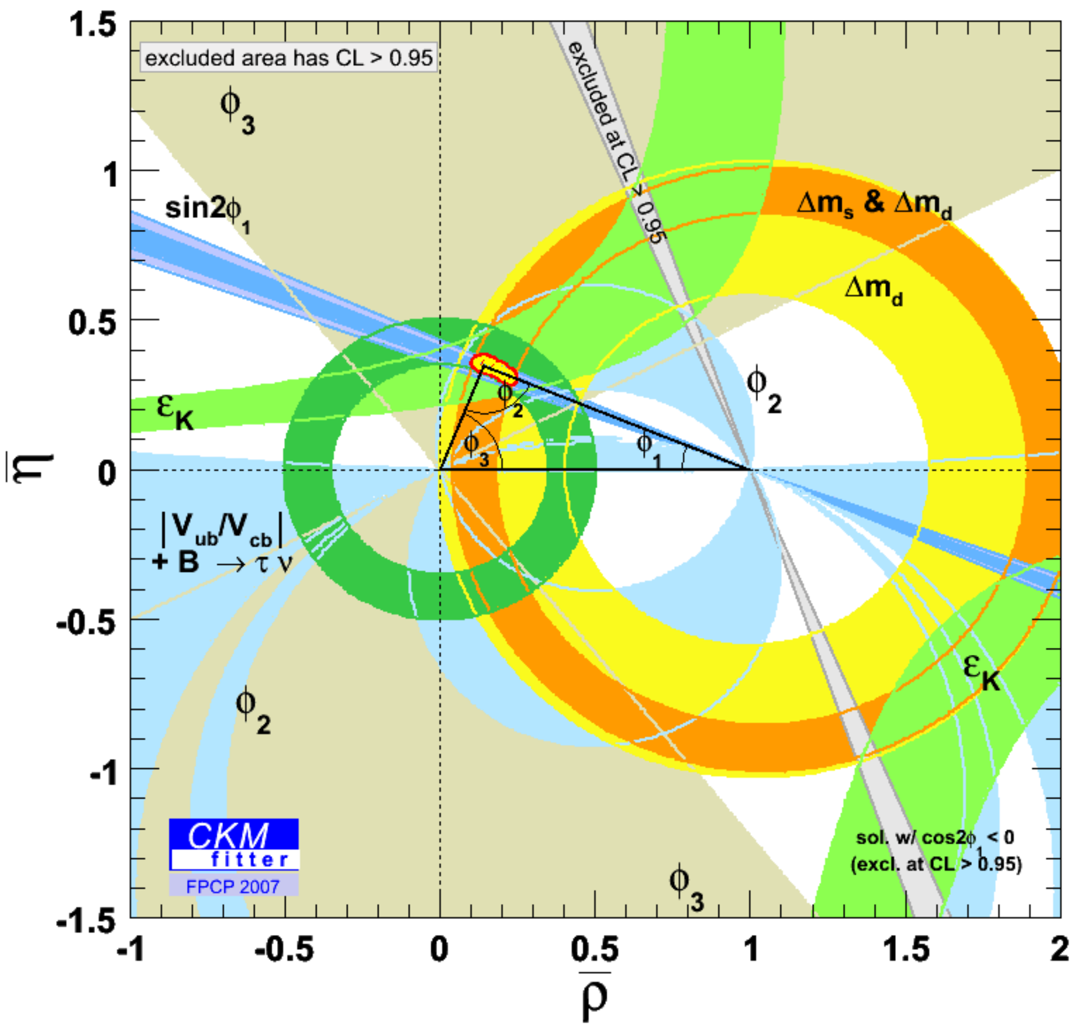
@ INT summer school 2007,
Seattle, August 2007.

# IV. CKM phenomenology: at tree level

1. Quark flavor physics
   - Weak interaction; from W-exchange to four-fermion interactions
   - Quark mixings: the CKM matrix, unitarity triangle
2. $V_{us}$, the Cabibbo angle
   - Flavor SU(3) breaking: one-loop ChPT and higher order corrections; Lattice calculation
3. $V_{cb}$
   - Inclusive and exclusive semi-leptonic decays
   - Heavy quark symmetry; lattice calculation
4. $V_{ub}$
   - Continuum extraction from inclusive decays
   - Lattice calculation for exclusive processes

# IV. CKM Phenomenology: at tree level
# 1. Quark flavor physics

# CKM Physics



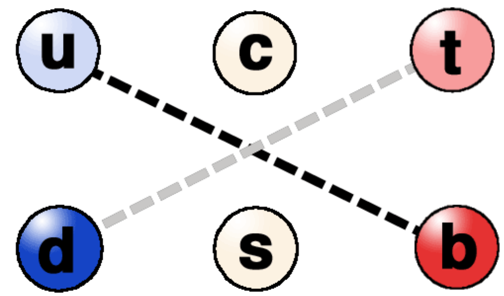

- Our goal:
  - To understand this plot
  - How lattice QCD may contribute to improve it.

# Weak interaction

- Quarks may change their flavor through weak interaction.
  - Active only for left-handed quarks and right-handed anti-quarks.

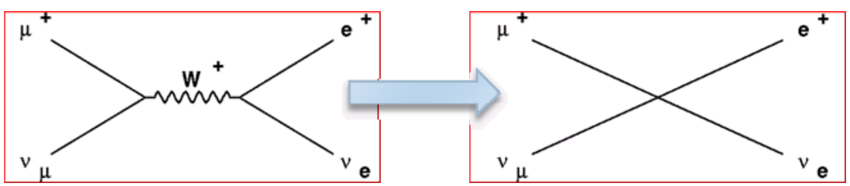
$$\bar{q}^{f'}\gamma_\mu(1-\gamma_5)q^f W^\mu$$

  - Short distance ($\sim 1/M_W$) interaction.




$$\frac{g^2}{q^2-M_W^2} \sim G_F$$

$$G_F = \frac{1}{4\sqrt{2}}\frac{g^2}{M_W^2} = 1.17\times 10^{-5}\ \mathrm{GeV}^{-2}$$




  - Acts on (weak) isospin doublets.

$$\begin{pmatrix} u \\ d' \end{pmatrix}, \begin{pmatrix} c \\ s' \end{pmatrix}, \begin{pmatrix} t \\ b' \end{pmatrix}, \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$$

# Changing flavors

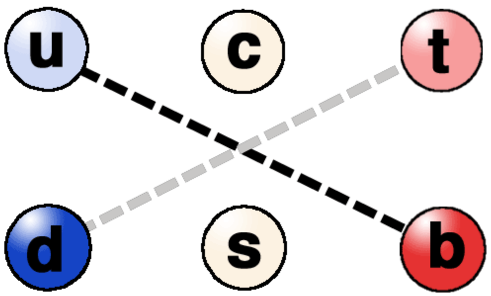
- Quark flavor may change:
  - Weak isospin is not identical with the real isospin.

$$\begin{pmatrix} u \\ d' \end{pmatrix}, \begin{pmatrix} c \\ s' \end{pmatrix}, \begin{pmatrix} t \\ b' \end{pmatrix} \neq \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}$$

  - Related by a 3x3 unitary matrix.

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = V_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

  - Called Cabibbo-Kobayashi-Maskawa matrix.
    - Cabibbo (60s), Kobayashi-Maskawa (1973)



# CKM matrix

- Degrees of freedom
  - NxN complex matrix has $2N^2$ real parameters.
  - Unitarity constraints N (diagonal) + N(N-1) (off-diagonal); thus $N^2$ parameters remain.
  - Quark phases are arbitrary 2N except for 1 (overall phase does not change $V_{CKM}$); thus $(N-1)^2$ remain.
  - N(N-1)/2 are mixing angles.
  - (N-1)(N-2)/2 are CP violating phases.

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

N=3 $\Rightarrow$
3 mixing angles
+ 1 CP phase

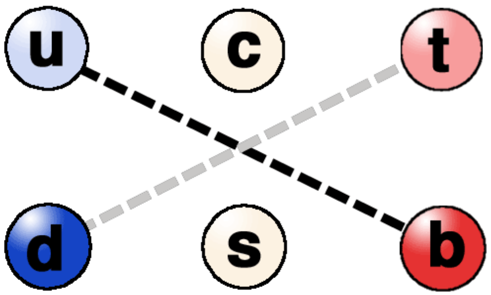
# Mixing angles

- Strength of the weak interaction is different among processes

$$\mu^- \to \nu_\mu + e^- + \bar{\nu}_e \qquad \Rightarrow G_{\mu\nu}^2 = G_F^2$$

$$n \to p + e^+ + \nu_e \qquad \Rightarrow G_{ud}^2 = 0.95 \times G_F^2$$

$$K^- \to \pi^0 + e^- + \bar{\nu}_e \qquad \Rightarrow G_{us}^2 = 0.05 \times G_F^2$$




- $V_{us}=\sin\theta_c$: the Cabibbo angle

$$u = u$$

$$d' = d\cos\theta_c + s\sin\theta_c$$

  - $\sin\theta_c \sim 0.22$

- Other angles
  - 2↔3: Vcb
  - 1 ↔3: Vub
    - Much smaller in magnitude

# CKM unitarity

- Unitary implies…
  - Normalization

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

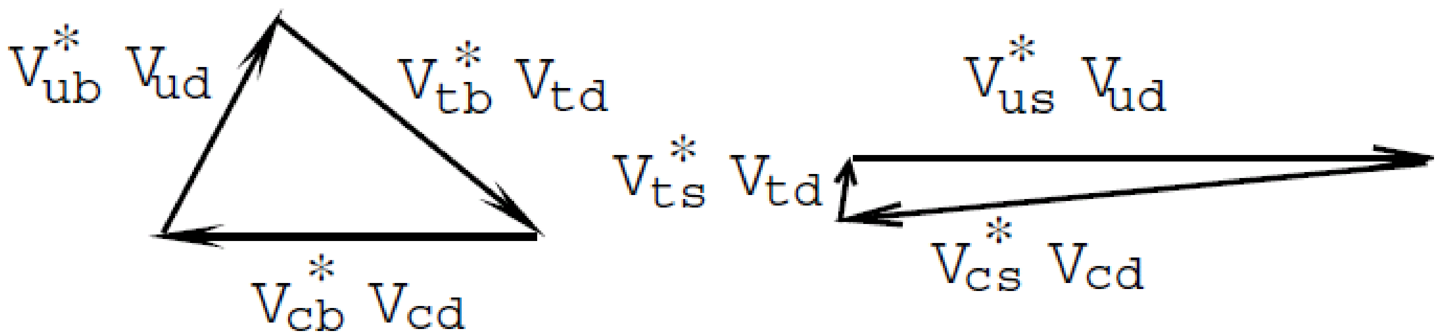
$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

  - Orthogonality

$$V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0 \quad (b \to d)$$

$$V_{us}^* V_{ud} + V_{cs}^* V_{cd} + V_{ts}^* V_{td} = 0 \quad (s \to d)$$

Unitarity triangles



# CKM hierarchy

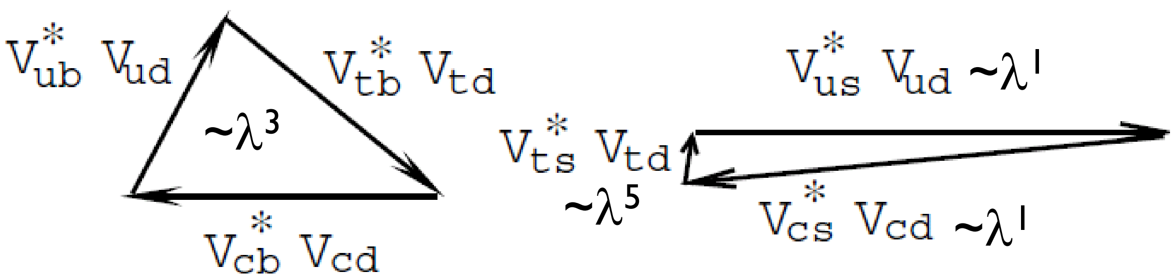
- We don't know why, but the CKM matrix has a hierarchical structure.

$$V_{CKM} = \begin{pmatrix} 0.97 & 0.23 & 0.004 \\ 0.23 & 0.96 & 0.04 \\ 0.007 & 0.04 & >0.8 \end{pmatrix}$$

  - Wolfenstein parametrization: with $\lambda$=0.225,

$$V_{CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

written explicitly by four parameters: $\lambda$, A, $\rho$, $\eta$.

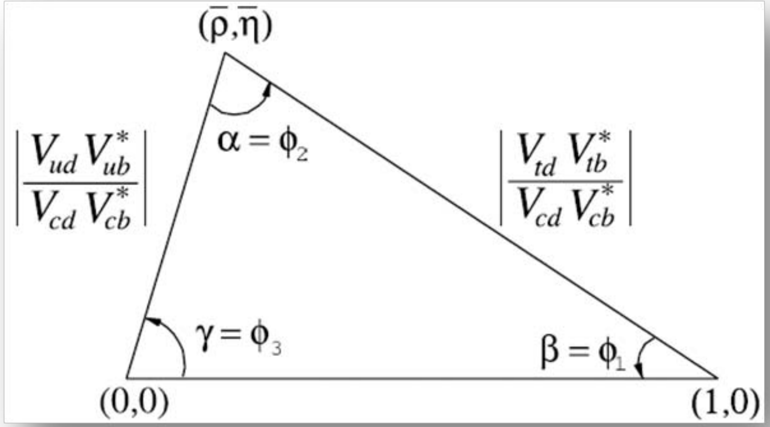


# Unitarity triangle

- Most interesting unitarity condition.

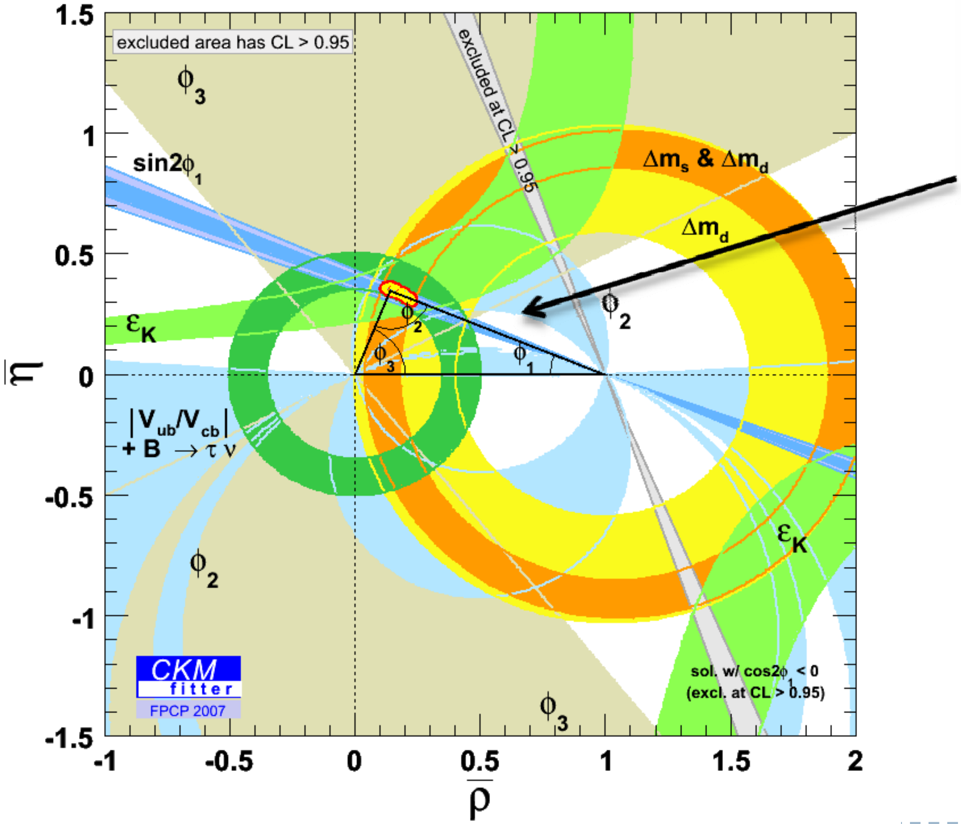
  $$V_{ub}^{*}V_{ud} + V_{cb}^{*}V_{cd} + V_{tb}^{*}V_{td} = 0$$

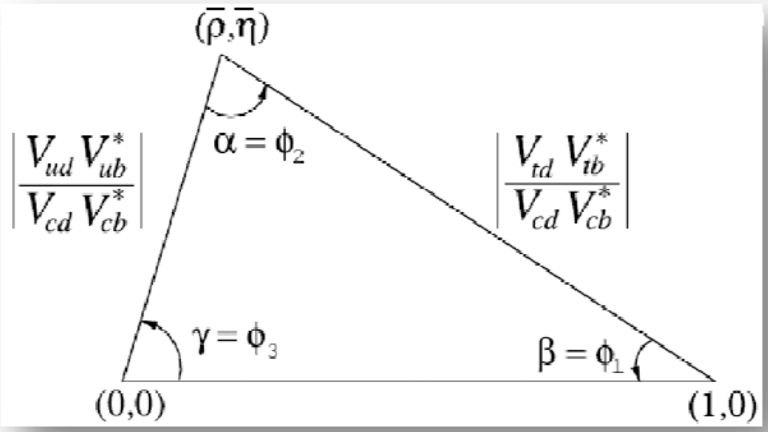
  - Normalized by a better known side $V_{cb}^{*}V_{cd}$.
  - Apex is $(\rho,\eta)$.
  - Defines three angles.




$$\phi_1(=\beta) = \arg\left(-\frac{V_{cd}V_{cb}^{*}}{V_{td}V_{tb}^{*}}\right), \quad \phi_2(=\alpha) = \arg\left(-\frac{V_{td}V_{tb}^{*}}{V_{ud}V_{ub}^{*}}\right), \quad \phi_3(=\gamma) = \arg\left(-\frac{V_{ud}V_{ub}^{*}}{V_{cd}V_{cb}^{*}}\right).$$

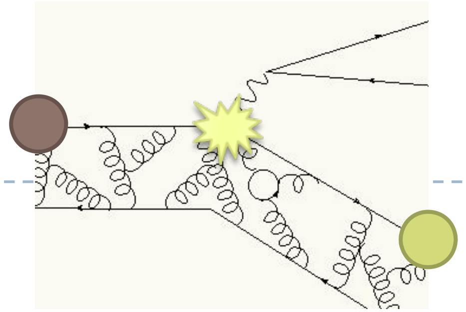
# Unitarity triangle




- Several measurements (sides and angles) can be compared on a single plane of $(\rho,\eta)$.
  - Tree processes (today)
  - Loop processes (tomorrow)

# Tree-level processes



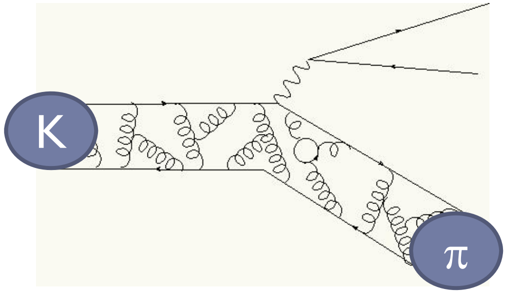
| CKM element | generations | quark level process | exclusive processes |
|---|---|---|---|
| $V_{us}$ | $2 \to 1$ | $s \to ul\nu$ | $K \to \pi l\nu$, $\Lambda \to pl\nu$ |
| $V_{cb}$ | $3 \to 2$ | $b \to cl\nu$ | $B \to D^{(*)} l\nu$, $\Lambda_b \to \Lambda_c l\nu$ |
| $V_{ub}$ | $3 \to 1$ | $b \to ul\nu$ | $B \to \pi l\nu$, $\rho l\nu$, $\omega l\nu$ |

# IV. CKM Phenomenology: at tree level 2. $V_{us}$, the Cabibbo angle

# Cabibbo angle

- The best known mixing angle.
  - Primary information from K→πlν decay, that contains a quark level process s→ulν. (Hyperon decay could also be used.)
  - Decay rate

$$\Gamma_{Kl3} = \frac{G_F^2}{192\pi^3} S_{EW} (1+\delta_K) |V_{us}|^2 f_+^2(0) I_K$$




- $S_{EW}$: short distance EW radiative correction.
- $\delta_K$: long distance EM radiative correction (sub %)
- $f_+(0)$: form factor = QCD soft physics
- $I_K$: phase space integral = contains the info of the form factor shape.

# Form factor

- Matrix element

$$\langle \pi(p_\pi) | \bar{s}\gamma_\mu u | K(p_K) \rangle = f_+(t)(p_K + p_\pi)_\mu + f_-(t)(p_K - p_\pi)_\mu$$

  - Very similar to the pion form factor, but now contains $f_-$ because (initial $\leftrightarrow$ final) exchange symmetry is lost.
  - Instead of $f_-$, one can also use the scalar form factor

$$f_0(t) = f_+(t) + \frac{t}{m_K^2 - m_\pi^2} f_-(t)$$

which is the piece to survive the projection

$$(p_K - p_\pi)^\mu \langle \pi(p_\pi) | \bar{s}\gamma_\mu u | K(p_K) \rangle = (m_K^2 - m_\pi^2) f_0(t)$$

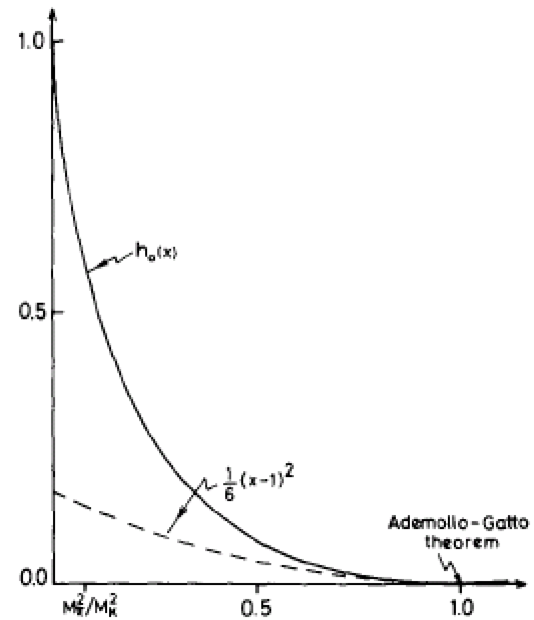
# Analytical constraints

- Before embarking on the hard (and costly) calculations on the lattice, analytically known facts should be used as much as possible.
  - $f_+(t)$ reduces to the pion form factor $G_\pi(t)$ in the limit of degenerate $\pi$ and K.
  - In this limit, $f_+(0)=1$.
  - Away from this limit, there is a correction starting from second order (no $O(m_s-m_u)$ = Ademollo-Gatto theorem, 1964).
    - Simple explanation: $f_+$ is a symmetric piece under the exchange $\pi \leftrightarrow K$. So, $m_s-m_u$ cannot appear.
    - Leutwyler-Roos (1984)

$$f_+(0) = 1+f_2+f_4 = 1 - \underbrace{0.023}_{\text{ChPT}} - \underbrace{0.016(8)}_{\text{quark model}} = 0.961(8)$$

# ChPT

- For small $q^2$=t region, ChPT provides a reliable framework to calculate the form factor.
  - At one-loop, non-analytic dependence on quark mass is predicted (chiral log).
    Gasser-Leutwyler (1985)




$$f_+(0) = 1 + \frac{3}{2} H_{K\pi}(0) + \frac{3}{2} H_{K\eta}(0),$$

$$H_{PQ}(0) = -\frac{1}{128\pi^2 f^2}\left(m_P^2 + m_Q^2\right) h_0\left(\frac{m_P^2}{m_Q^2}\right),$$

$$h_0(x) = 1 + \frac{2x}{1-x^2}\ln x$$

gives $f_2$=-0.023, now two-loop is known (Bijinens et al.)

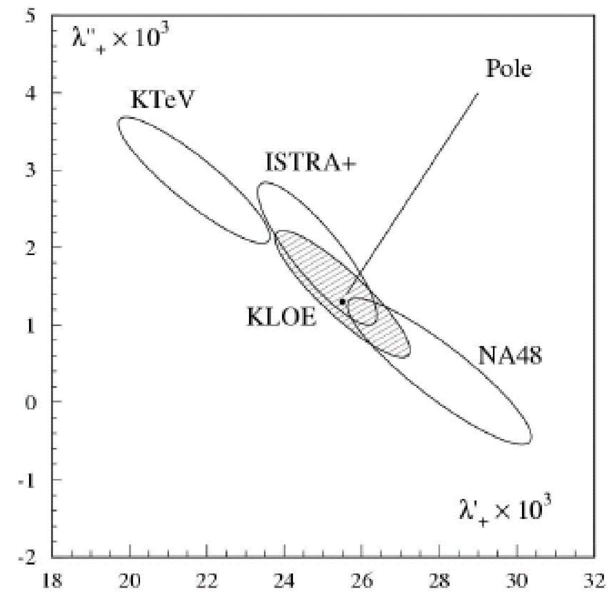
# What lattice can do

- Main target is $f_4$: only a few % contribution, but that is the accuracy one wants to achieve.
  - $f_2$ should also be calculated. Good consistency check with ChPT.
- Many other predictions/cross-checks possible. Mainly the form factor shape,

$$f_{\pm}(q^2) = f_{\pm}(0)[1+\lambda_{\pm}(q^2/m_{\pi}^2)]$$

  - The slope parameter $\lambda_{\pm}$ can be compared with the experimental data.

$f_+$ slope vs curvature (Gatti at Kaon 07)



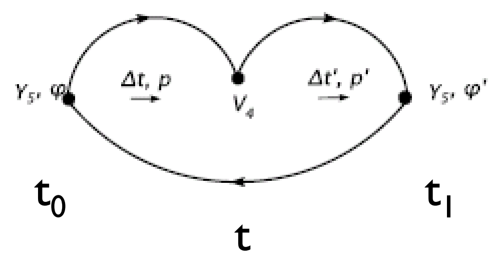
# Lattice calculation

- Very similar to the pion form factor calculation.
  - But need to separate $f_+$ from $f_0$.

$$\langle \pi(p_\pi) | \bar{s}\gamma_\mu u | K(p_K) \rangle = f_+(t)(p_K + p_\pi)_\mu + f_-(t)(p_K - p_\pi)_\mu$$

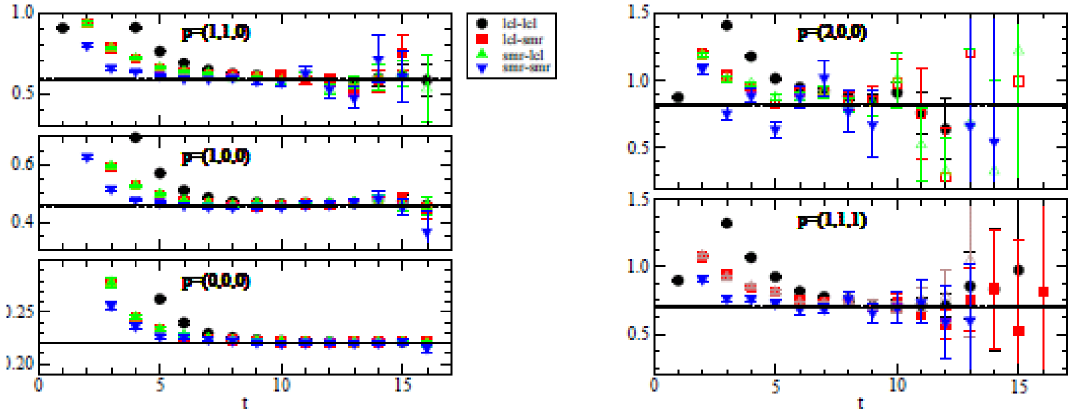
  - Possible by looking at different $\mu$ directions and solving linear equations, but…

  - When both $\pi$ and K are at rest, only $f_0$ can be obtained.
  - Statistical error is larger with finite momentum insertion.



# Statistical noise

- Larger the momentum, larger the noise.




  - Possible to understand as follows (Lepage, 1990)

$$N^2(t) \sim \left\langle \left( \mathrm{Tr}\left[ \Gamma S_q(x,0) \Gamma' S_q(0,x) \right] \right)^2 \right\rangle - \left\langle \mathrm{Tr}\left[ \Gamma S_q(x,0) \Gamma' S_q(0,x) \right] \right\rangle^2$$

$$= \exp\left[ -E_{\pi\pi}(\mathbf{p} \pm \mathbf{p}) t \right] - \exp\left[ -2E_\pi(\mathbf{p}) \right]$$

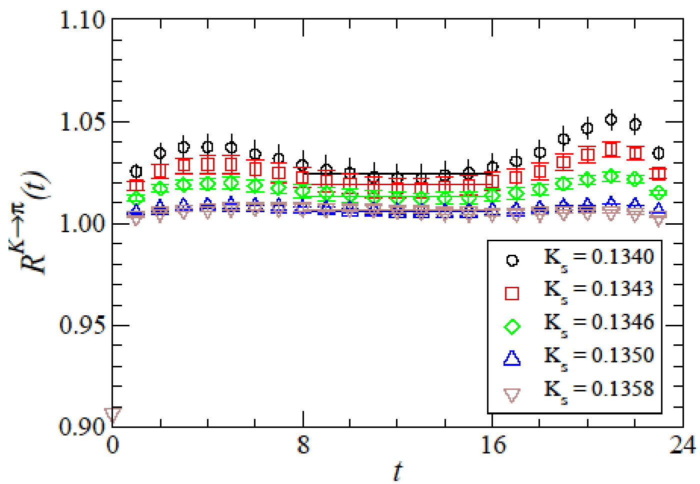
$$N(t)/S(t) \sim \exp\left[ \left( E_\pi(\mathbf{p}) - E_{\pi\pi}(\mathbf{0})/2 \right) t \right]$$

$$\sim \exp\left[ \left( E_\pi(\mathbf{p}) - E_\pi(\mathbf{0}) \right) t \right], \quad E_\pi(\mathbf{p}) = \sqrt{m_\pi^2 + \mathbf{p}^2}$$

# A clever method

- Precision is the key for this quantity. Consider ratios in which the bulk of stat fluctuation cancel.
  - 1st ratio

$$\frac{C^{\pi V_4 K}(t)C^{K V_4 \pi}(t)}{C^{\pi V_4 \pi}(t)C^{K V_4 K}(t)} \to \frac{\langle \pi(0)|V_4|K(0)\rangle\langle K(0)|V_4|\pi(0)\rangle}{\langle \pi(0)|V_4|\pi(0)\rangle\langle K(0)|V_4|K(0)\rangle} = \frac{(m_K+m_\pi)^2}{4m_K m_\pi} f_0(q^2_{max})$$




JLQCD, 2005

- Precisely calculated. Renormalization factors cancel.
- Can be obtained only at $q^2_{max} = (m_K - m_\pi)^2$
- Need to extrapolate back to $q^2=0$.

# Clever ratios

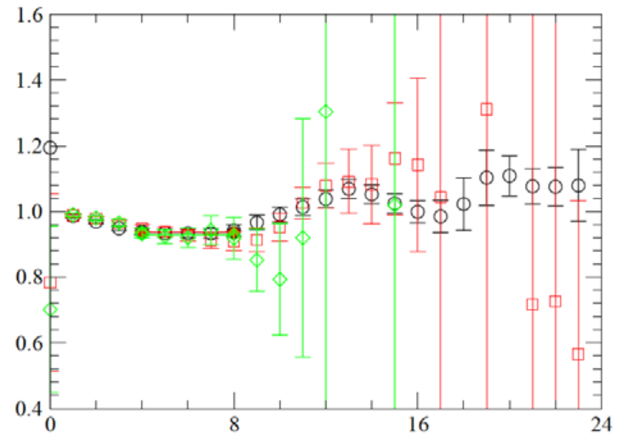
- Extrapolate back to $q^2=0$
  - $2^{nd}$ ratio with finite momentum.

$$\frac{\dfrac{\langle \pi(p)|V_4|K(0)\rangle}{\langle \pi(0)|V_4|K(0)\rangle}}{\dfrac{\langle \pi(p)|P|0\rangle}{\langle \pi(0)|P|0\rangle}} = \frac{m_K+E_\pi}{m_K+m_\pi}\,\frac{f_+(q^2)\left[1+\xi(q^2)\dfrac{m_K-E_\pi}{m_K+E_\pi}\right]}{f_+(q^2_{\max})\left[1+\xi(q^2_{\max})\dfrac{m_K-m_\pi}{m_K+m_\pi}\right]}$$

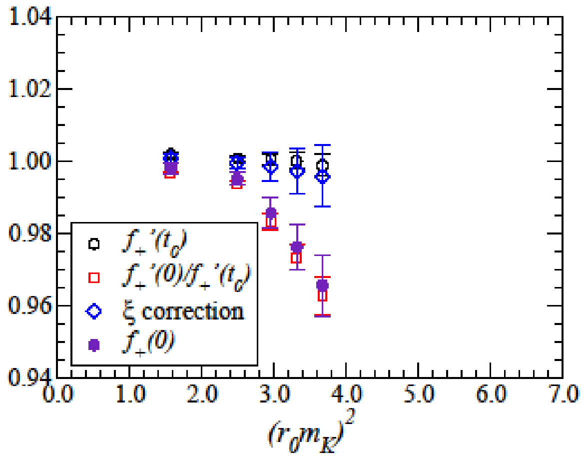
$$\xi(q^2)=\frac{f_-(q^2)}{f_+(q^2)}$$

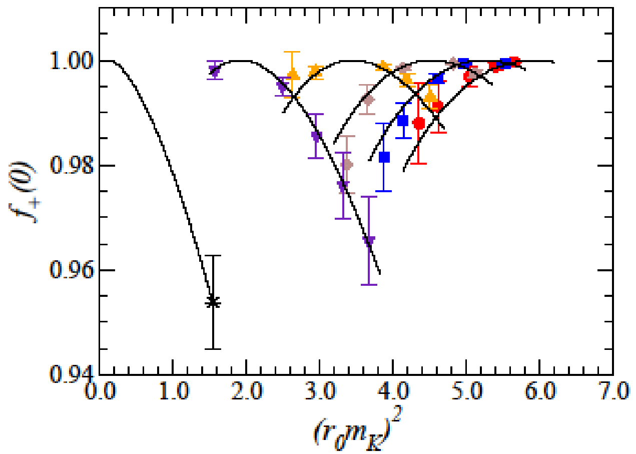
- Subtract $f_-$ to get $f_+$
  - $3^{rd}$ ratio with different $\mu$

$$\frac{\dfrac{\langle \pi(p)|V_k|K(0)\rangle}{\langle \pi(p)|V_4|K(0)\rangle}}{\dfrac{\langle \pi(p)|V_k|\pi(0)\rangle}{\langle \pi(p)|V_4|\pi(0)\rangle}} = \frac{1-\xi(q^2)}{\dfrac{m_K+E_K}{m_\pi+E_\pi}+\xi(q^2)\dfrac{m_K-E_K}{m_\pi+E_\pi}}$$



# $f_+(0)$




- Combine the 3 ratios.
  - $q^2$ conversion is dominant.

- Analysis with one-loop χPT plus an analytic term $(m_K^2-m_\pi^2)^2$.

$$f^+(0) = 1 + \frac{3}{2} H_{K\pi}(0) + \frac{3}{2} H_{K\eta}(0),$$

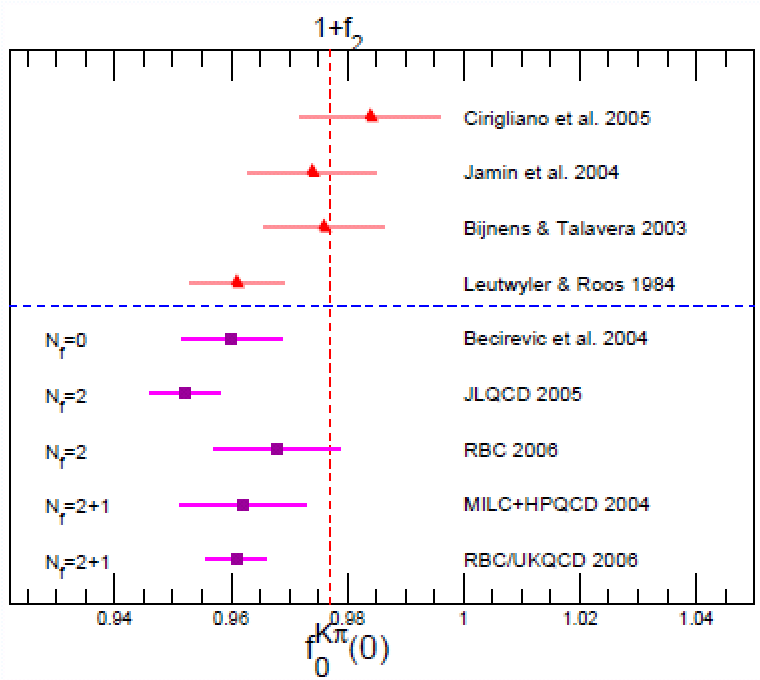
$$H_{PQ}(0) = -\frac{1}{128\pi^2 F^2}(m_P^2 + m_Q^2) h_0\left(\frac{m_P^2}{m_Q^2}\right),$$

$$h_0(x) = 1 + \frac{2x}{1-x^2} \ln x.$$

JLQCD (2005): $f_+(0)=0.954(9)$

- Note: there are several newer calculations…

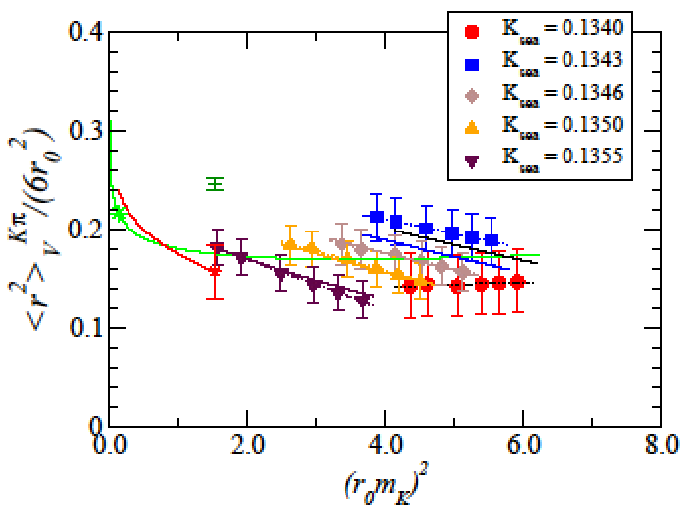
# Other recent results




- Recent results compiled by Juettner at Lattice 2007.
  - Now, several groups are interested in this quantity.
  - Results including 2+1-flavors of dynamical quarks.
  - Light enough sea quarks.
  - Results compatible with the original estimate by Leutwyler-Roos (1984).

# Points to be checked

- Presenting just a final number is not good enough. Form factor shapes (charge radii) contain lots of info.
  - Chiral extrapolation: consistency with $\chi$PT.
  - Analyticity: consistency with the known K* pole.
  - Consistency with the experimental measurements for both $f_+$ and $\xi$.




Ex). Charge radius :

$$f_{K\pi}^{+}(t) = f_{K\pi}^{+}(0)\left[1+\frac{1}{6}\left\langle r^2\right\rangle_V^{K\pi} t + \cdots\right],$$

$$f_{K\pi}^{0}(t) = f_{K\pi}^{0}(0)\left[1+\frac{1}{6}\left\langle r^2\right\rangle_S^{K\pi} t + \cdots\right].$$

Not satisfactory, so far.

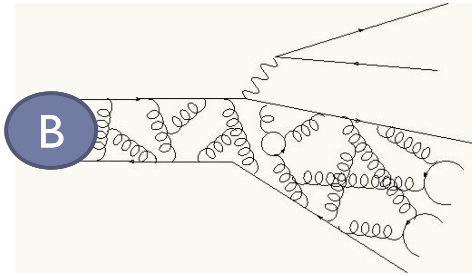
# IV. CKM Phenomenology: at tree level 3. $V_{cb}$

# Heavy-to-heavy

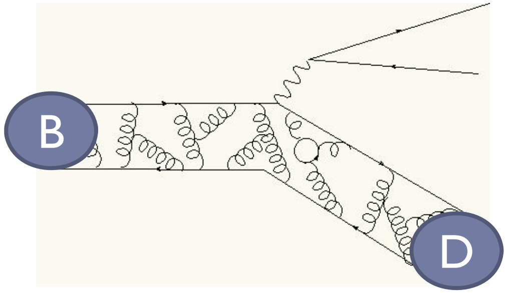
- Second well-known parameter: A

$$V_{CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

  - Inclusive:
    - do not specify the final state (except that it contains a charm).
    - Heavy decays can be well controlled by perturbation theory.

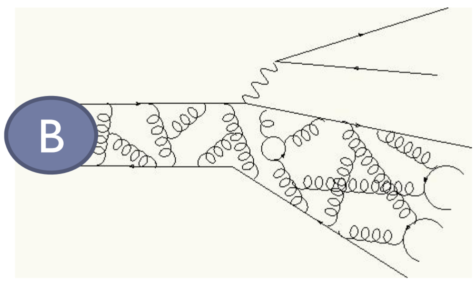



  - Exclusive:
    - treats a definite final state (e.g. D, or D*).
    - Heavy quark symmetry constrains the form factors.



# Inclusive decay

- Perturbation theory
  - Valid when energy scale is large. In this case, provided by the mass difference $m_b$-$m_c$~ 3 GeV.
  - Valid when smeared over final state. Thus, consider inclusive. In this case, the sum is over D, D*, Dπ, D*π, etc.



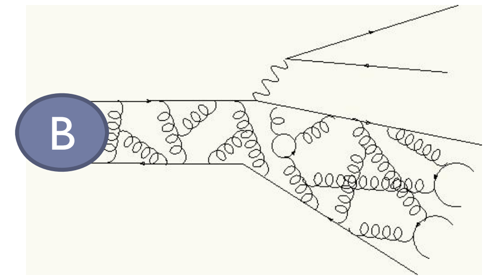

- Decay rate
  - At the quark level,

$$\Gamma_{sl}(b \to c) = \frac{G_F^2 m_b^5(\mu)}{192\pi^3} |V_{cb}|^2 (1 + A_{ew}) A^{pert}(m_c^2/m_b^2, \mu)$$

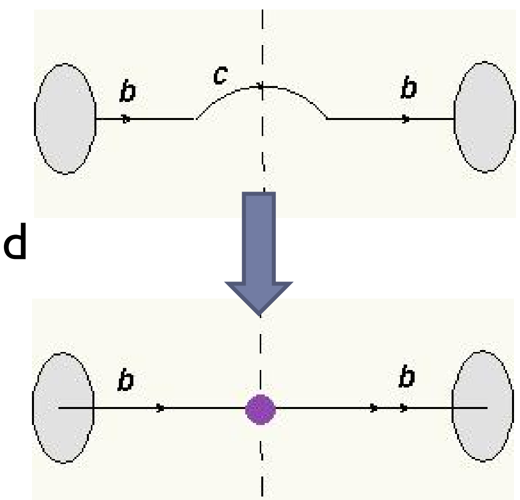
  - Contains $m_b^5$: precise knowledge of $m_b$ is crucial, or to be fitted with exp data.

# Heavy quark expansion

- Initial state is a B meson, not a b quark.
  - Correction can be calculated by the Operator Product Expansion (OPE); in this case, called the Heavy Quark Expansion.




$$\Gamma_{sl}(b \to c) = \frac{G_F^2 m_b^5(\mu)}{192\pi^3} |V_{cb}|^2 (1 + A_{ew}) A^{pert}(r,\mu)$$
$$\times \left[ z_0(r) + z_2(r)\left(\frac{\mu_\pi^2}{m_b^2}, \frac{\mu_G^2}{m_b^2}\right) + z_3(r)\left(\frac{\rho_D^3}{m_b^3}, \frac{\rho_{LS}^3}{m_b^3}\right) + \dots \right]$$




  - B meson matrix elements represent the bound state effects.

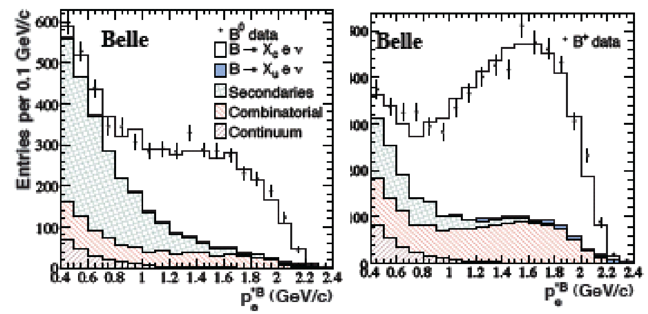
$$\mu_\pi^2 = -\langle B | \bar{b}(iD_\perp)^2 b | B \rangle,$$
$$\mu_G^2 = \langle B | \bar{b}(iD_\perp^\mu)(iD_\perp^\nu)\sigma_{\mu\nu} b | B \rangle,$$

  - Can be fitted with exp data.

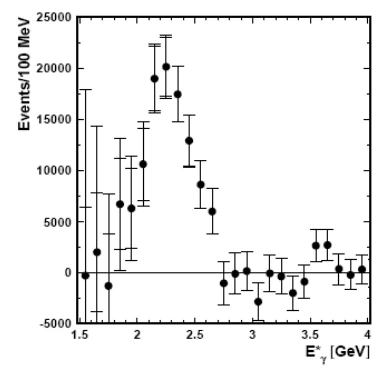
# $V_{cb}$ inclusive

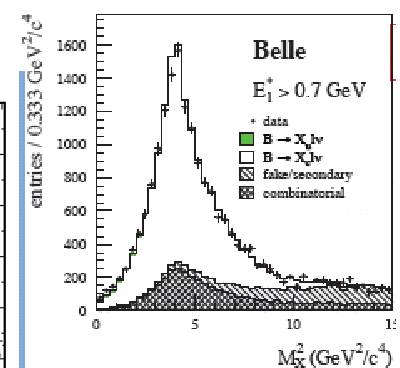
- Several moments
  - Theoretical calculation also possible for differential decay rate. To avoid duality errors, one must use moments, instead.
  - Several moments $\langle M_X{}^n \rangle$ and $\langle E_l{}^n \rangle$ compared with exp data.
  - May also combine with the photon energy spectrum in $B \to X_s \gamma$, which is governed by the same matrix elements.

$E_l$ distribution (Belle, 2006)




$M_X$ distribution (Belle, 2006)



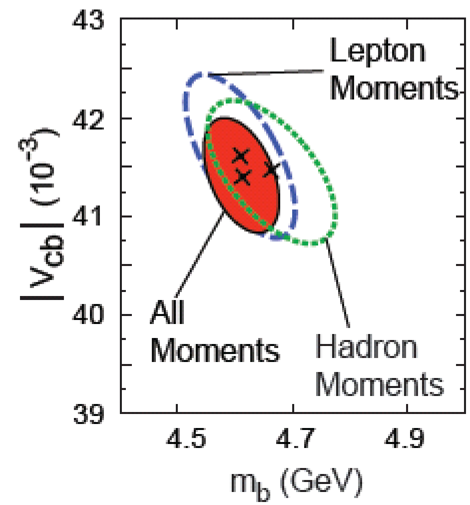



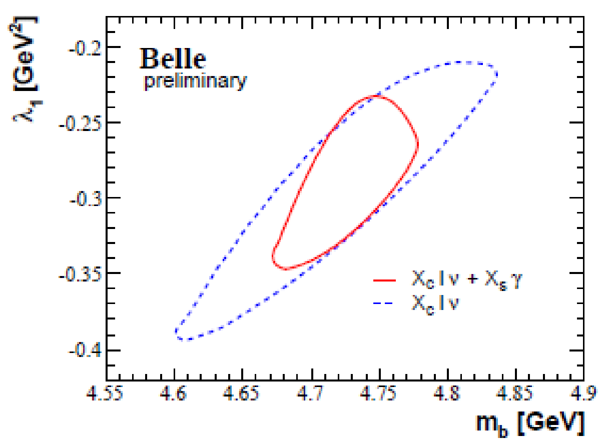

$\gamma$ spectrum (Belle, 2006)

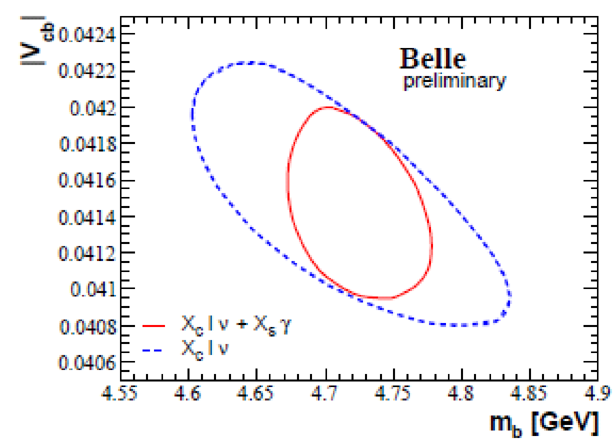
# $V_{cb}$ inclusive

- $|V_{cb}|$ obtained to 1-2%.
  - $|V_{cb}|$=0.0417(7) (PDG 2006)
  - Other parameters, such as $m_b$, $m_c$, $\mu_\pi^2$, etc., can be obtained at the same time. Very strong method!
  - Duality issue?
    - $B\to X_c l\nu$ is dominated by D and D* (80%).

BaBar 2006



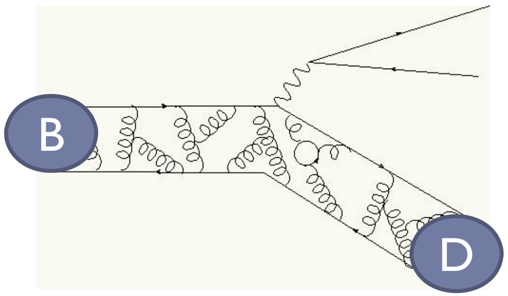



# Exclusive decays

- Use the exclusive decays $B \to D^{(*)} l \nu$ to determine $|V_{cb}|$.
  - Analogous to the $|V_{us}|$ determination through $K \to \pi l \nu$.
  - Need a precise calculation of the form factors = non-perturbative physics.

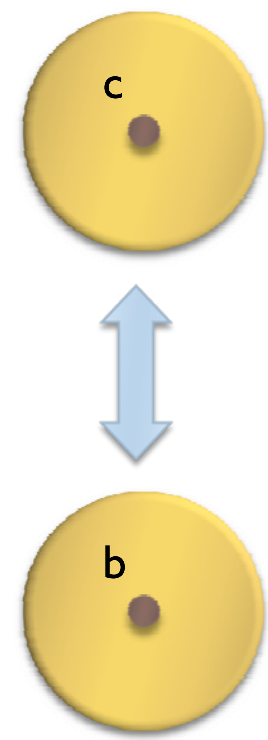
  - Very different systematic effect from the inclusive decays, thus a good cross-check.
  - Heavy quark symmetry plays an important role, like the chiral symmetry (or flavor SU(3)) in $K \to \pi l \nu$.

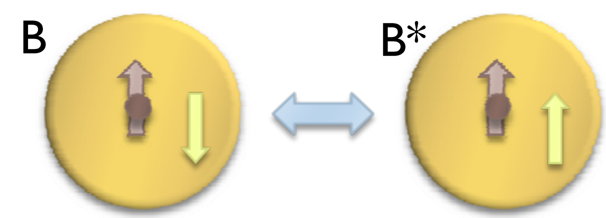


# Heavy quark symmetry

- Heavy quarks look similar…
  - In the heavy-light meson (or heavy-light-light baryon), the heavy quark hardly moves, looks as if a static color source.
  - Therefore, no difference between b and c in the heavy quark limit ($m_Q \to \infty$).
  - Also, there is no difference between spin-up and spin-down heavy quarks, because the spin-(chromo-)magnetic interaction is at $O(1/m_Q)$.

$$H_{sm} = \psi^\dagger \frac{\sigma \cdot \mathbf{B}}{2m_Q} \psi$$

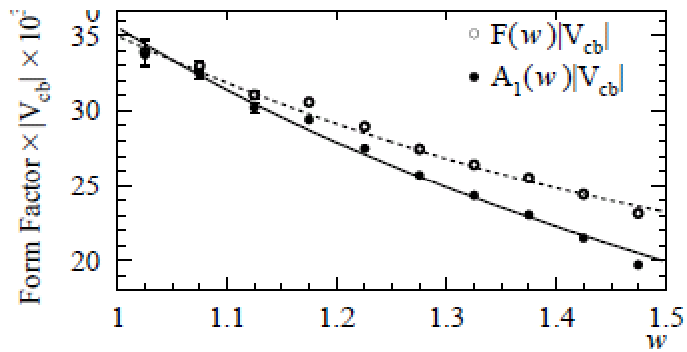


# Heavy quark symmetry

- Symmetry relations among form factors
  - Interchange of b↔c
  - Interchange of ↑↔↓

B →D* , BaBar, 2004




ex) Isgur-Wise function

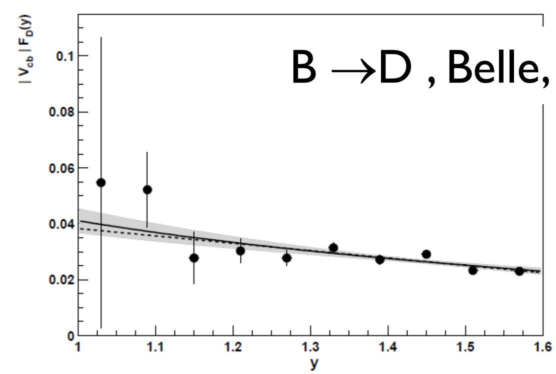
- B→D and B→D* are governed by the same form factor ξ(w), called the Isgur-Wise function.

$$\langle D(v')|\bar{c}_{v'}\gamma_\mu b_v|\bar{B}(v)\rangle = \xi(w)\left(v_\mu + v'_\mu\right),$$

$$\langle D^*(v',\varepsilon)|\bar{c}_{v'}\gamma_\mu\gamma_5 b_v|\bar{B}(v)\rangle = -i\xi(w)\left[(1+w)\varepsilon^*_\mu - (\varepsilon^*\cdot v)v'_\mu\right]$$

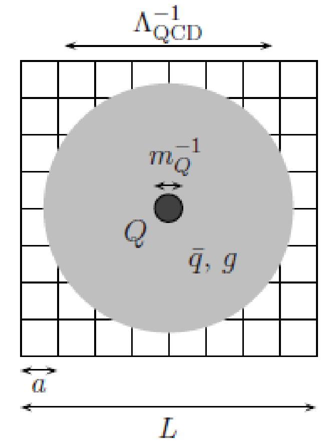
  - A function of w=v.v', see below.

B →D , Belle, 2001



# Scale separation

- Write the momentum of heavy quark as $p = m_Q v + k$
  - $v$ : four-velocity of the heavy quark.
  - $k$: residual momentum




- Heavy quark mass limit:
  - propagator

$$i\frac{\not p + m_Q}{p^2 - m_Q^2 + i\varepsilon} = i\frac{m_Q \not v + m_Q + k}{2m_Q v\cdot k + k^2 + i\varepsilon} \to i\frac{1+\not v}{2}\frac{1}{v\cdot k + i\varepsilon}$$

  - Lagrangian

$$L_Q = \overline{Q}_v (iv\cdot D) Q_v; \quad Q(x) = e^{-im_Q v\cdot x} Q_v(x)$$

Georgi (1990), Eichten-Hill (1990)

  - States are distinguished by the heavy quark velocity.

# Heavy meson states

- Usual normalization

$$\langle H(p')|H(p)\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$$

- HQET normalization

$$\langle H(v',k')|H(v,k)\rangle = 2v^0\delta_{v,v'}(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')$$

$$|H(p)\rangle = \sqrt{m_H}\left[|H(v)\rangle + O(1/m_Q)\right]$$

- Decay constant

$$\langle 0|\bar{q}\gamma_\mu\gamma_5 Q(0)|P(p)\rangle = i\, f_P p_\mu$$

$$\langle 0|\bar{q}\gamma_\mu\gamma_5 Q_v|P(v)\rangle = i\left(f_P\sqrt{m_P}\right)v_\mu$$

- Heavy quark scaling

$$f_P \sim \frac{1}{\sqrt{m_P}}\left[1+O(1/m_P)\right]$$

- Form factors: B→Dlν as an example

$$\langle D(p')|V_\mu|B(p)\rangle = f_+(q^2)(p+p')_\mu + f_-(q^2)(p-p')_\mu = \sqrt{m_B m_D}\left[h_+(w)(v+v')_\mu + h_-(w)(v-v')_\mu\right]$$

$h_+(w) \to \xi(w)$ $\quad$ $h_-(w) \to 0$

given by a function of w=v.v'

# Isgur-Wise function

- Form factors:

$$\langle D(v')|V^\mu|B(v)\rangle = h_+(w)(v+v')^\mu + h_-(w)(v-v')^\mu,$$
$$\langle D^*(v',\varepsilon)|V^\mu|B(v)\rangle = h_V(w)\varepsilon^{\mu\nu\alpha\beta}\varepsilon_\nu^* v'_\alpha v_\beta,$$
$$\langle D^*(v',\varepsilon)|A^\mu|B(v)\rangle = -ih_{A1}(w)(1+w)\varepsilon^{*\mu} + ih_{A2}(w)(\varepsilon^*\cdot v')v^\mu + ih_{A3}(w)(\varepsilon^*\cdot v)v'^\mu$$

- Heavy quark limit: $m_b, m_c \to \infty$

$$h_+(w) = h_V(w) = h_{A1}(w) = h_{A3}(w) = \xi(w)$$ Isgur-Wise function
$$h_-(w) = h_{A2}(w) = 0$$

- Zero recoil limit
  - $\xi(w=1)=1$ because of the vector current conservation (number of heavy quark).
  - A strong constraint, like the $f_+(0)=1$ of pion/kaon form factor.

# $1/m_Q$ corrections

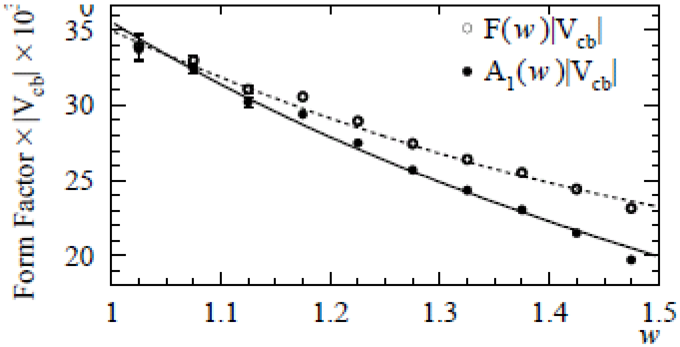
- Luke's theorem
  - The leading correction of $O(1/m_Q)$ vanishes in the zero recoil limit w=1.

$$h_+(1) = \eta_V \left[ 1 - l_P \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 + O(1/m_Q^3) \right],$$

$$h_{A1}(1) = \eta_A \left[ 1 - \frac{l_V}{(2m_c)^2} + \frac{2l_A}{(2m_c)(2m_b)} - \frac{l_P}{(2m_b)^2} + O(1/m_Q^3) \right]$$

  - An analog of the Ademollo-Gatto theorem; comes from the symmetry $\langle D| \leftrightarrow |B\rangle$.
  - Extraction of $|V_{cb}|$ is most precise in the zero recoil limit.

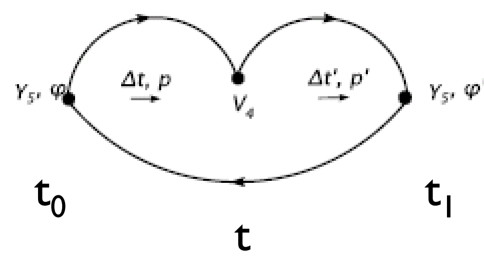
B →D* , BaBar, 2004



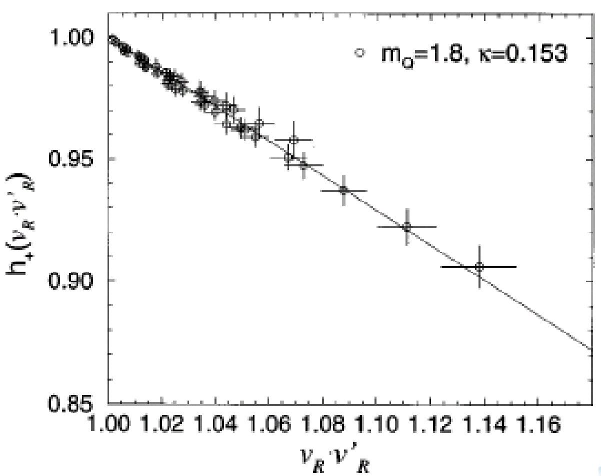
# Lattice calculation: IW function shape

- Lattice calculation is possible with a similar method as used for $K \to \pi$ form factor
  - Except that the heavy quark is heavy: treated by HQET on the lattice (for example).
    - Putting velocity is non-trivial.
  - Because of a S/N issue, NRQCD is better (or the conventional lattice formulation).
    - Sometimes called Moving NRQCD.

For detailed discussion of heavy quark formulations, see Kronfeld's lecture.

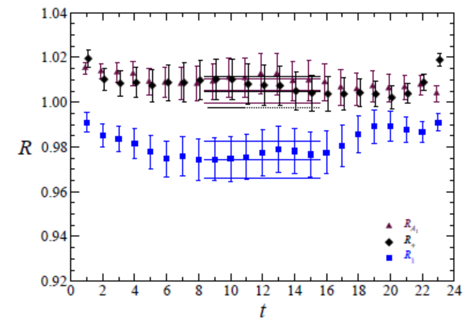



SH and Matsufuru (1996)



# Lattice calculation: zero recoil limit

- In the zero recoil limit, lattice can calculate the $O(1/m_Q^2)$ (or higher) deviation from the heavy quark limit.
  - Clever ratios (again!)




Fermilab group (SH et al.), 2001

$$R_+ = \frac{\langle D|\bar{c}\gamma_4 b|\bar{B}\rangle\langle\bar{B}|\bar{b}\gamma_4 c|D\rangle}{\langle D|\bar{c}\gamma_4 c|D\rangle\langle\bar{B}|\bar{b}\gamma_4 b|\bar{B}\rangle} = |h_+(1)|^2 = \eta_V\left[1 - l_P\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right)^2 + O(1/m_Q^2)\right]$$

$$R_1 = \frac{\langle D^*|\bar{c}\gamma_4 b|\bar{B}^*\rangle\langle\bar{B}^*|\bar{b}\gamma_4 c|D^*\rangle}{\langle D^*|\bar{c}\gamma_4 c|D^*\rangle\langle\bar{B}^*|\bar{b}\gamma_4 b|\bar{B}^*\rangle} = |h_1(1)|^2 = \eta_V\left[1 - l_V\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right)^2 + O(1/m_Q^2)\right]$$

$$R_{A1} = \frac{\langle D^*|\bar{c}\gamma_j\gamma_5 b|\bar{B}\rangle\langle\bar{B}^*|\bar{b}\gamma_j\gamma_5 c|D\rangle}{\langle D^*|\bar{c}\gamma_j\gamma_5 c|D\rangle\langle\bar{B}^*|\bar{b}\gamma_j\gamma_5 b|\bar{B}\rangle} = |\hat{h}_{A1}(1)|^2 = \eta_A\left[1 - l_A\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right)^2 + O(1/m_Q^2)\right]$$

  - These determine the expansion coefficients $l_P$, $l_V$, $l_A$ to reconstruct $h_{A1}(1)$.

$$h_{A1}(1) = \eta_A\left[1 - \frac{l_V}{(2m_c)^2} + \frac{2l_A}{(2m_c)(2m_b)} - \frac{l_P}{(2m_b)^2} + O(1/m_Q^3)\right]$$

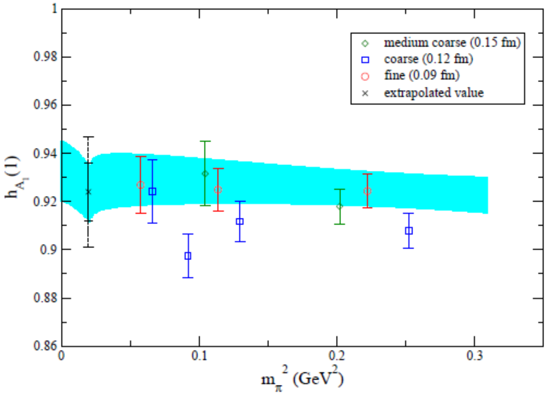
# More recent work

Laiho et al. (at Lattice 2007)

- Including dynamical fermions
  - Asqtad improved staggered (2+1 flavors)
- With a single ratio

$$\frac{\langle D^* | \bar{c}\gamma_j\gamma_5 b | \overline{B}\rangle \langle \overline{B} | \bar{b}\gamma_j\gamma_5 c | D^*\rangle}{\langle D^* | \bar{c}\gamma_4 c | D^*\rangle \langle \overline{B} | \bar{b}\gamma_4 b | \overline{B}\rangle} = |h_{A_1}(1)|^2$$

  - Not equal to one in the heavy quark limit, but the error is still controllable.
- Three lattice spacings



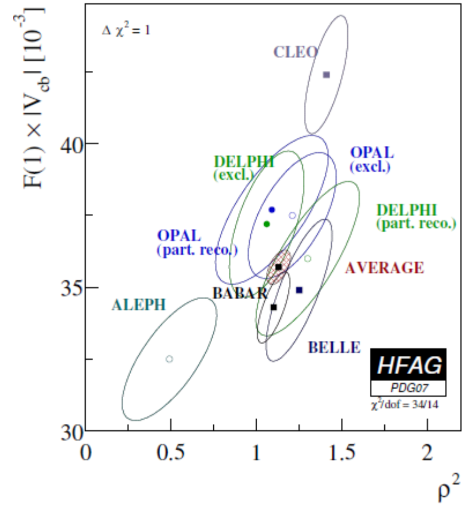

- Result

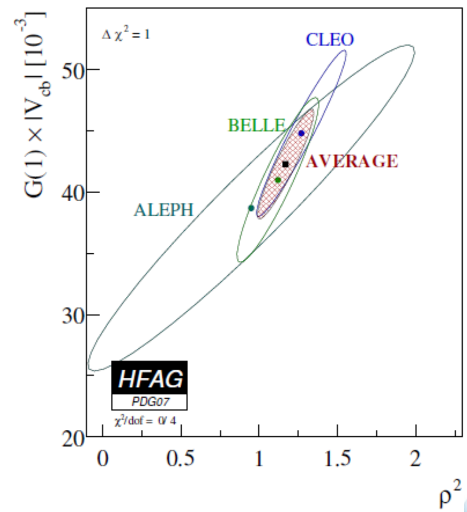
$$h_{A1}(1) = 0.924(11)(19).$$

  - Error is competitive with inclusive.

# “Exclusive” summary

- Experiment
  - $F(1)|V_{cb}|$ is now measured to 2%.
  - Slope of the form factor has not been well measured, but now converging.
- Theory
  - Most recent lattice calculation has got 2%.
  - Theoretical calculation of $F(1)$ can become better than 1%?
- Combined
  - $|V_{cb}|$ = 0.0402(7)(8), compared to 0.0417(7) from inclusive.



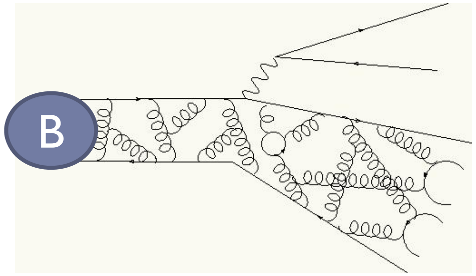
# IV. CKM Phenomenology: at tree level
# 4. $V_{ub}$

# Heavy-to-light

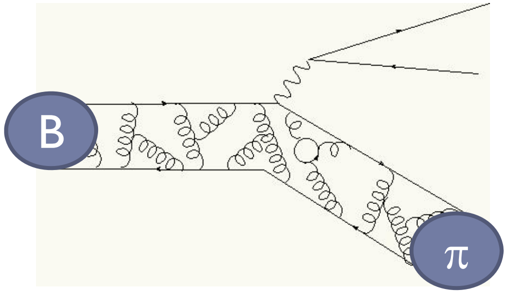
- Less known parameter: $(\rho^2+\eta^2)^{1/2}$

$$V_{CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

  - Inclusive:
    - do not specify the final state (except that it contains a charm).
    - Heavy decays can be well controlled by perturbation theory.

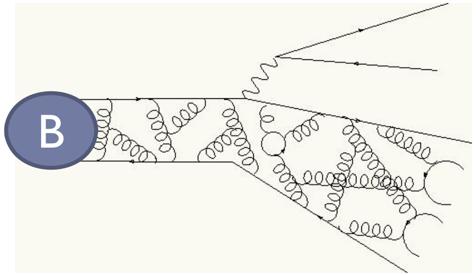



  - Exclusive:
    - treats a definite final state (e.g. π, ρ, ω, …).
    - Heavy quark symmetry does not help a lot…



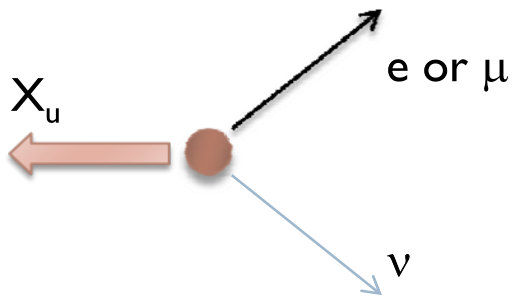
# Inclusive b→u

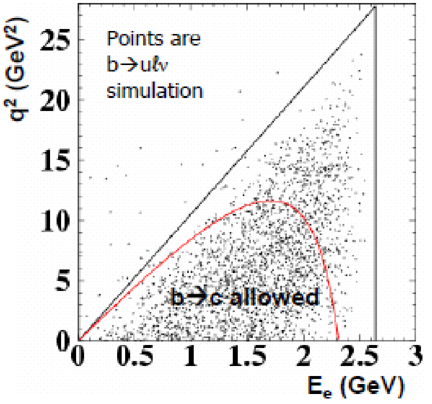
- Perturbation theory
  - Valid when energy scale is large. In this case, provided by the mass difference $m_b$~ 5 GeV; better than b→c
  - Valid when smeared over final state. In this case, the sum is over many final states, thus much safer.
- Experimentally harder
  - Must distinguish b→u from b→c background, which is 100x larger.
  - Needs cut to enhance the signal.

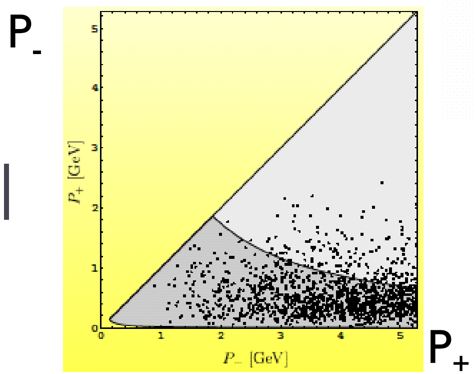


# $B \to X_u l\nu$ kinematics

- 3-body decay characterized by
  - $E_l$: charged lepton energy
  - $q^2$: $l\nu$ invariant mass
  - $m_X$: hadron invariant mass
- Several cuts to enhance $b \to u$
  - $E_l$ cut
  - $m_X$ cut
  - $q^2$ cut
  - Light-cone parameter $P_+ = E_X - |P_X|$



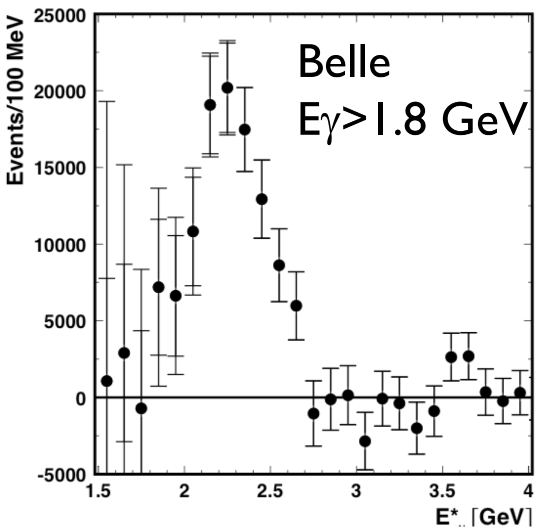



# Shape function

- Non-perturbative physics enters as the shape function
  - An analog of the B meson matrix element in HQE
  - In this case, the distribution in the light-cone variable

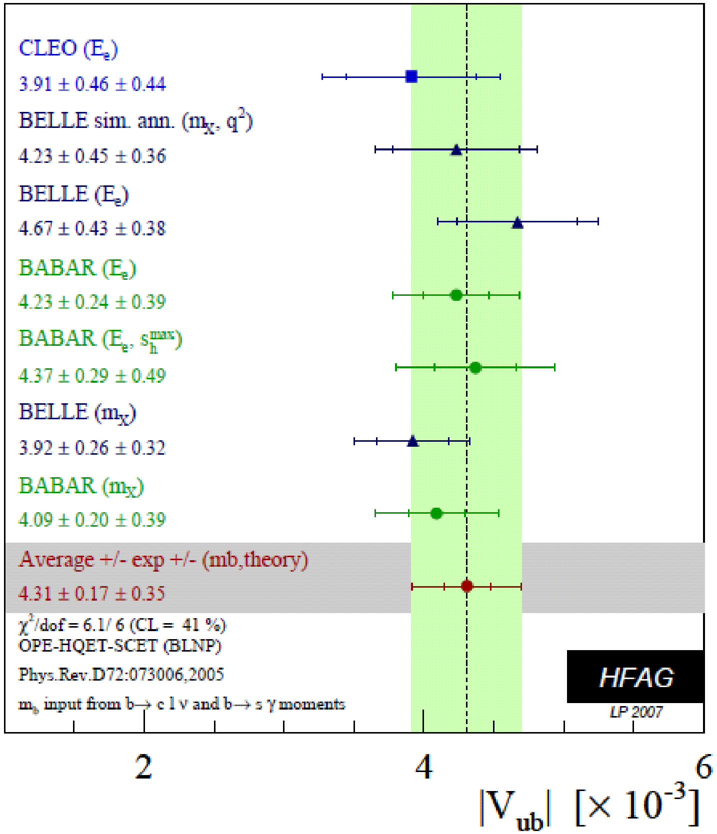
$$f(k_+) = \frac{1}{2m_B} \left\langle \overline{B} \middle| \overline{b}_v \delta(in \cdot D + k_+) b_v \middle| \overline{B} \right\rangle$$

  also observable from $B \to X_s \gamma$.
  - Possible to calculate on the lattice??



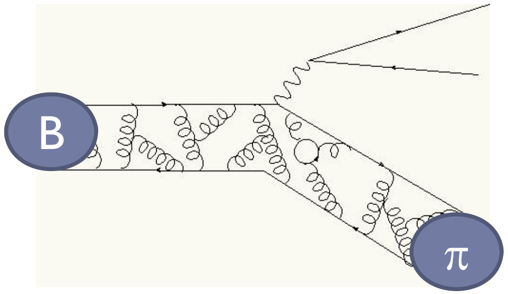
# $|V_{ub}|$ inclusive

- Now, $|V_{ub}|$ is reasonably precise ~ 8%.
  - |Vub| = 0.0440(20)(27) (PDG 2006)
  - Sets the challenge for lattice QCD.



# Exclusive decays

- Use the exclusive decays $B \to \pi l \nu$, $\rho l \nu$, $\omega l \nu$, etc. to determine $|V_{ub}|$.




  - Need a precise calculation of the form factors = non-perturbative physics.
  - Very different systematic effect from the inclusive decays, thus a good cross-check.
  - Heavy quark symmetry does not help a lot. Only the heavy quark scaling is useful.

# Kinematics

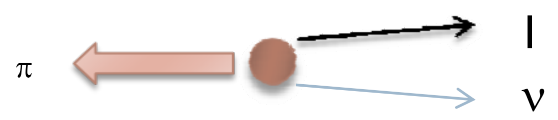
- $B \to \pi l \nu$
  - $q^2$: $l\nu$ invariant mass; $0 \le q^2 \le (m_B - m_\pi)^2$
  - small $q^2$ $\Leftrightarrow$ large recoil of $\pi$
  - large $q^2$ $\Leftrightarrow$ small recoil of $\pi$
- Differential decay rate

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2 |V_{ub}|^2}{24\pi^3} |p_\pi|^3 |f_+(q^2)|^2$$

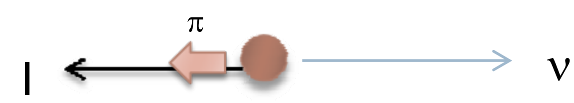
  - Depends only on $f_+$; $f_0$ term is suppressed by small $m_l$.
- Lattice calculation
  - Possible only when both B and $\pi$ have small spatial momenta.

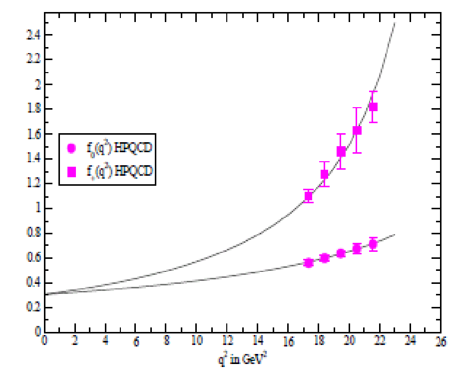
    $\Rightarrow$ large $q^2$ region

small $q^2$



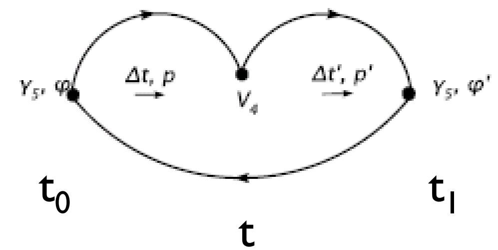

large $q^2$



# Lattice calculation

- Calculation of 3pt function
  - Use the sequential source method with momentum insertion.
  - The clever ratios not so much useful: numerator and denominator are not similar.




- Most difficult among other semi-leptonic decays. Several checks to be done
  - Operator matching
  - Heavy quark scaling
  - Chiral extrapolation
  - Dispersion relation

# Operator matching

- Lattice operators have to be matched to the continuum operator.
  - Usually done using perturbation theory at one-loop. Neglected higher orders could be sizable.
  - For light-light currents the non-perturbative matching is available in many cases.
  - In the double ratios ($K\to\pi$, $B\to D$) the matching factors largely cancel.
  - Cancellation is less precise for heavy-to-light, thus larger systematic error.

$$V_\mu^{(Qq)cont} = Z^{(Qq)} V_\mu^{(Qq)latt}$$

$$\frac{Z^{(qq')} Z^{(q'q)}}{Z^{(qq)} Z^{(q'q')}}$$

# Heavy quark scaling

- HQET normalization

$$\langle \pi(k_\pi)|\bar{q}\gamma^\mu b|B(v)\rangle = 2\left[f_1(v\cdot k_\pi)v^\mu + f_2(v\cdot k_\pi)\frac{k_\pi^\mu}{v\cdot k_\pi}\right]$$

  - Related to the conventional form factors

$$f^+(q^2) = \sqrt{m_B}\left\{\frac{f_2(v\cdot k_\pi)}{v\cdot k_\pi} + \frac{f_1(v\cdot k_\pi)}{m_B}\right\},$$

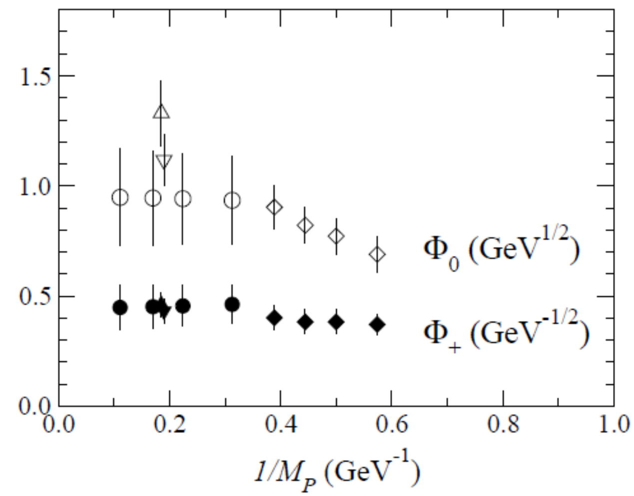
$$f^0(q^2) = \frac{2}{\sqrt{m_B}}\frac{m_B^2}{m_B^2-m_\pi^2}\left\{[f_1(v\cdot k_\pi)+f_2(v\cdot k_\pi)] - \frac{v\cdot k_\pi}{m_B}\left[f_1(v\cdot k_\pi) + \frac{m_\pi^2}{(v\cdot k_\pi)^2}f_2(v\cdot k_\pi)\right]\right\}$$

  - HQET scaling is manifest

$$f^+(q^2) \sim \sqrt{m_B},$$

$$f^0(q^2) \sim \frac{1}{\sqrt{m_B}},$$

JLQCD (2001)




Possible to check the consistency among different heavy quark formulations

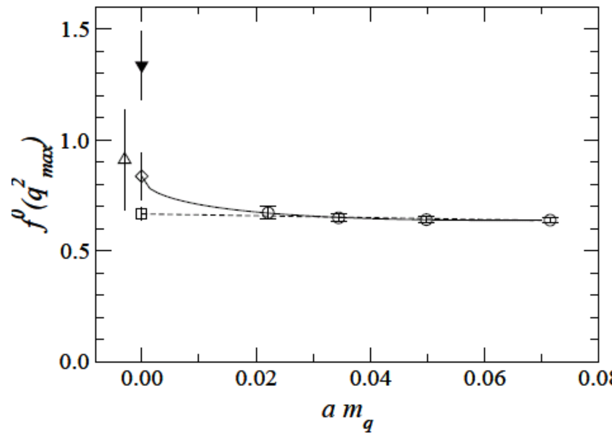
# Chiral extrapolation

- Soft pion theorem

$$f_0(q^2_{\max}) = \frac{f_B}{f_\pi}$$

  - Valid in the chiral limit.
  - Chiral extrapolation is not trivial because $q^2$ changes as $m_q$ changes.

JLQCD (2001)

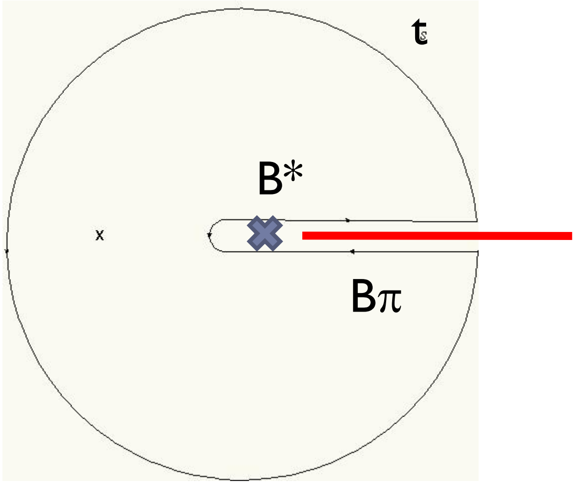



- ChPT predicts the chiral log
  - Calculation exists (Becirevic-Prelovsek-Zupan, 2002), not fully tested so far.

# Dispersion relation

- Near $q^2_{max}$ the $B^*$ pole dominates the dispersion relation.

$$F(q^2) = \frac{1}{2\pi i}\oint dt \frac{F(t)}{t-q^2} = \frac{1}{\pi}\int_{t_0}^{\infty} dt \frac{\mathrm{Im}\, F(t)}{t-q^2}$$




- Using the B*Bπ coupling,

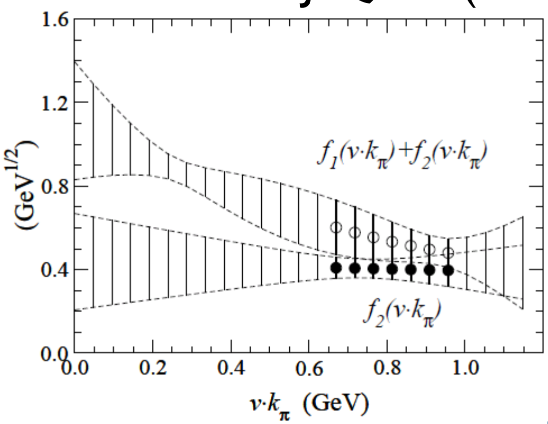
$$\lim_{q^2\to m_B^2} f^+(q^2) = \frac{f_{B^*}}{f_\pi}\frac{g}{1-q^2/m_{B^*}^2}$$

or

$$\lim_{v\cdot k_\pi\to 0} f_2(v\cdot k_\pi) = g\frac{f_{B^*}\sqrt{m_{B^*}}}{2f_\pi}\frac{v\cdot k_\pi}{v\cdot k_\pi+\Delta_B}$$

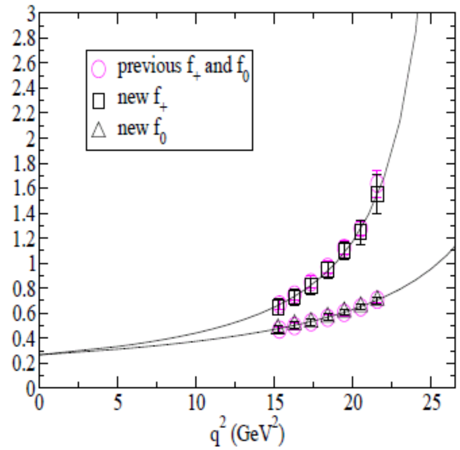
which implies constant $f_2$.

JLQCD (2001)

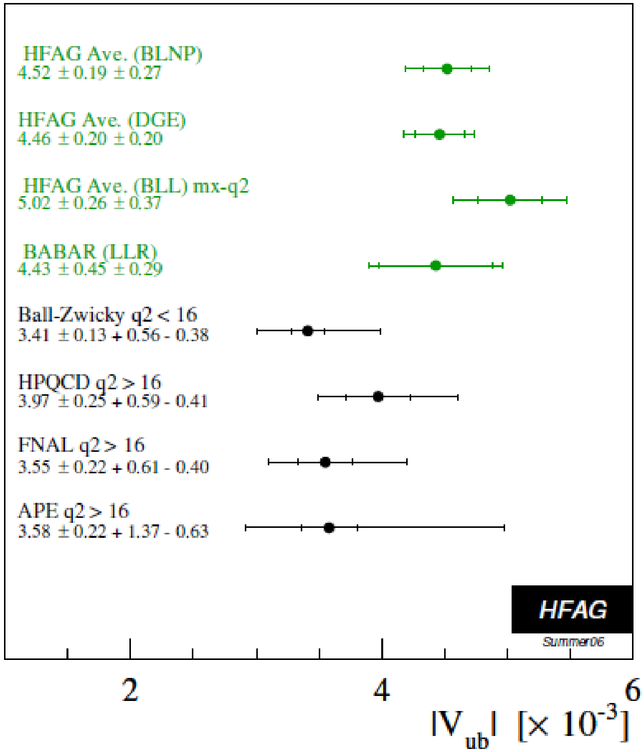


# Most recent results

- Lattice data available only in the large $q^2$ region; take exp data only in that region to extract $|V_{ub}|$.
  - $|V_{ub}|$ = 0.00384(+67-49), to be compared with 0.00440(20)(27) from inclusive.
  - Error is x(2-3) larger, mostly theoretical.




HPQCD 2006



# Semi-leptonic decays…

- Complicated!
  - But good, because there are many different ways to check lattice calculations.
  - All come from symmetries (chiral, heavy quark) and analyticity.

  - Lattice calculation must pass these stringent *theoretical* tests in order to make reliable predictions.
  - Heavy-to-light is the greatest challenge. Need <5% accuracy to be competitive.