

The three-loop β -function of SU(N) lattice gauge theories with Overlap fermions

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ABSTRACT

We calculate the third coefficient of the lattice β -function ($\beta_L(g_0)$) using standard perturbation theory. This calculation is performed using Overlap fermions and Wilson gluons, while the background field technique has been chosen for convenience.

Our results depend explicitly on the number of colors (N) and fermion flavors (N_f). Since the dependence of the β function on the Overlap parameter ρ cannot be extracted analytically, we tabulate our results for different values in the allowed range of ρ ($0 < \rho < 2$), focusing on values which are being used most frequently in simulations. The non-trivial dependence of $\beta_L(g_0)$ on ρ is plotted for 1 and 2 loop contributions.

I. INTRODUCTION

- In later years, use of non-ultralocal actions which preserve chiral symmetry on the lattice has become more viable. The two actions which are being used most frequently are domain-wall fermions [1,2] and Overlap fermions [3–5] based on the Wilson fermion action.
- Overlap fermions are notoriously difficult to study, both numerically and analytically. Many recent promising results from simulations with Overlap fermions have appeared; see, e.g., Refs. [6–12]. Regarding analytical computations, the only ones performed thus far have been either up to 1-loop, such as Refs. [13–19], or vacuum diagrams at higher loops [20,21]. *The present work is the first one involving non-vacuum diagrams beyond the 1-loop level.*
- We compute the 2-loop renormalization of the coupling constant in the presence of Overlap fermions. From this quantity we derive the 3-loop bare beta function on the lattice.

II. THEORY

- The definition and value of renormalized coupling constant g depends on the renormalization scheme (parameterized by a scale μ).

- The dependence $g(\mu)$ is given by the (renormalized) β -function:

$$\beta(g) \equiv \mu \frac{dg}{d\mu}$$

- For the lattice regularization a bare β -function is defined as:

$$\beta_L(g_0) = -a \frac{dg_0}{da} \Big|_{g, \mu}$$

a : lattice spacing
 μ : renormalization scale
 $g(g_0)$: renormalized (bare) coupling constant

- QCD: Asymptotic limit ($g_0 \rightarrow 0$):

$$\beta_L(g_0) = -b_0 g_0^3 - b_1 g_0^5 - b_2^L g_0^7 - \dots$$

* b_0, b_1 : universal constants

$$b_0 = \frac{1}{(4\pi)^2} \left(\frac{11}{3}N - \frac{2}{3}N_f \right)$$

$$b_1 = \frac{1}{(4\pi)^4} \left[\frac{34}{3}N^2 - N_f \left(\frac{13}{3}N - \frac{1}{N} \right) \right]$$

N : number of colors

N_f : number of flavors

* b_2^L : depends on regulator

- Coefficients b_i^L are calculated perturbatively

- **Present work:**

Calculation of b_2^L using the Overlap fermionic action and Wilson gluons

III. METHODOLOGY

$\beta_L(g_0)$ and perturbation theory:

Lattice: $\beta_L(g_0) = -a \frac{dg_0}{da} |_{g,\mu} = -b_0 g_0^3 - b_1 g_0^5 - b_2^L g_0^7 + \dots$

Continuum: $\beta(g) = \mu \frac{dg}{d\mu} |_{a,g_0} = -b_0 g^3 - b_1 g^5 - b_2 g^7 + \dots$

Renormalization of g_0 : $g_0 = Z_g(g_0, a\mu)g$

$$\implies \beta^L(g_0) = \left(1 - g_0^2 \frac{\partial \ln Z_g^2}{\partial g_0^2}\right)^{-1} Z_g \beta(g_0 Z_g^{-1})$$

Perturbative expansion of Z_g^2 :

$$Z_g(g_0, a\mu)^2 = 1 + L_0(a\mu) g_0^2 + \textcircled{L_1(a\mu)} g_0^4 + O(g_0^6)$$

$b_0, b_1, b_2, L_0(a\mu)$: appear in the literature (Refs. [22,19])



Background field technique ([23–25]) for the calculation of $Z_g(g_0, a\mu)$:

$$Z_A(g_0, a\mu) Z_g(g_0, a\mu)^2 = 1$$

$$A_B^\mu(x) = Z_A(g_0, a\mu)^{1/2} A_R^\mu(x)$$

In the formalism of Background field:

$$U_\mu(x) = \underbrace{e^{iag_0 Q_\mu(x)}}_{\text{Quantum field}} \cdot \underbrace{e^{ia A_\mu(x)}}_{\text{Background field}}$$

Calculation of $Z_g(g_0, a\mu)$ \longrightarrow calculation of $Z_A(g_0, a\mu)$

For the above calculation we consider the Background field 2-point function (connected, 1PI), both in the continuum (dimensional regularization) and on the lattice.

Continuum:

$$\Gamma_R^{AA}(p)_{\mu\nu}^{ab} = -\delta^{ab} \left(\delta_{\mu\nu} p^2 - p_\mu p_\nu \right) (1 - \nu_R(p)) / g^2$$

$$\nu_R(p) = g^2 \nu_R^{(1)}(p) + g^4 \nu_R^{(2)}(p) + \dots$$

Lattice:

$$\sum_\mu \Gamma_L^{AA}(p)_{\mu\mu}^{ab} = -\delta^{ab} 3\hat{p}^2 [1 - \nu(p)] / g_0^2$$

$$\nu(p) = g_0^2 \nu^{(1)}(p) + g_0^4 \nu^{(2)}(p) + \dots$$

- $\hat{p}_\mu = (2/a) \sin(ap_\mu/2)$

- $\Gamma^{AA}(p)$: Background field 2-point function

There follows:

$$Z_A = \frac{1 - \nu_R(p, \mu, g)}{1 - \nu(p, a, g_0)}$$

The gauge parameter must also be renormalized (up to one loop), in order to compare lattice and continuum results:

$$\lambda = Z_Q \lambda_0,$$

$$Z_Q = 1 + g_0^2 z_Q^{(1)} + \dots$$

- Z_Q : renormalization constant of Quantum field

Necessary components for the computation of Z_A :

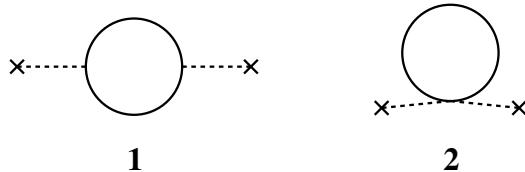
$$\underbrace{\nu_R^{(1)}, \nu_R^{(2)}, z_Q^{(1)}, \nu^{(1)}, \nu^{(2)}}_{\text{already known}}$$

**perturbative computation
(present work)**

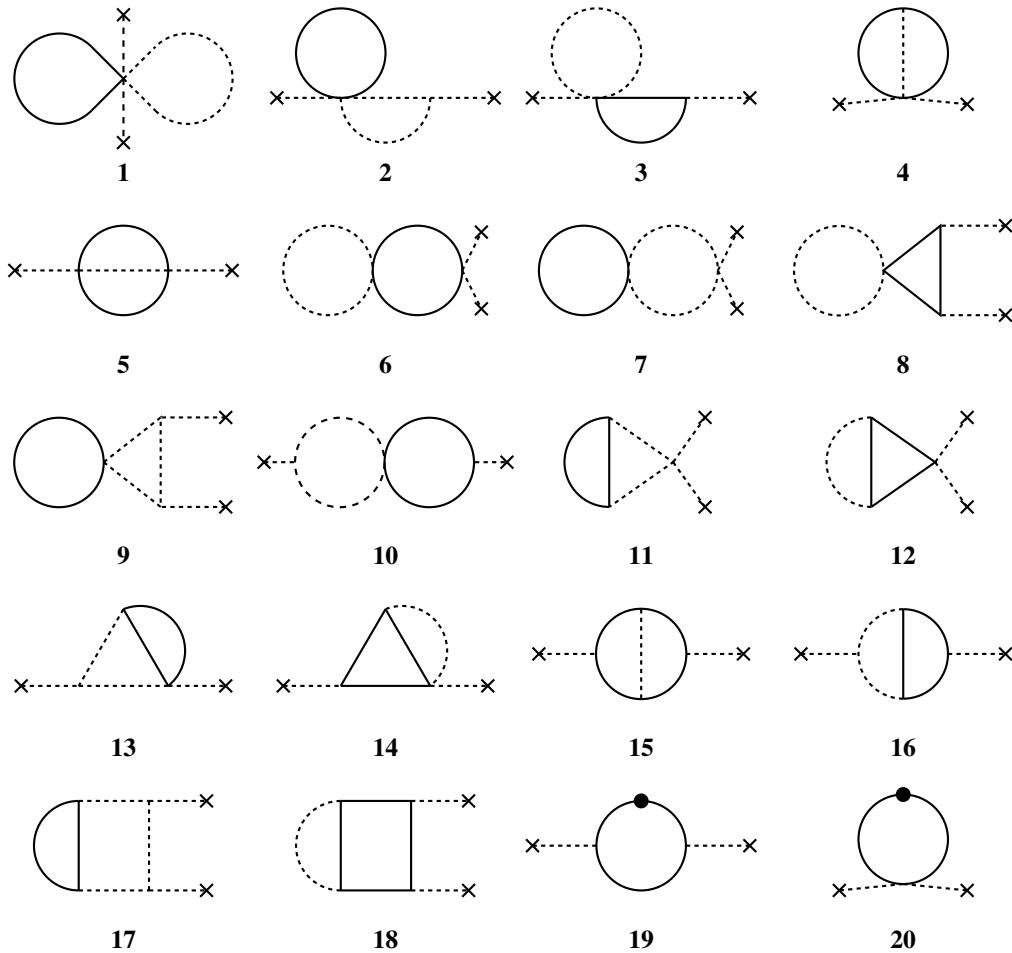
1 and 2-loop diagrams with closed fermionic loop:

The gluonic contributions needed for the present work have been calculated already (Wilson action) while the *fermionic contributions* are associated with the diagrams below:

- $\nu^{(1)}$:



- $\nu^{(2)}$:



- ◊ Diagrams with IR divergences must be grouped together:
 $(6+12, 7+11, 8+18, 9+17)$

- ◊ For overlap fermions, diagrams 19 and 20 both vanish, since the mass counterterm equals zero

- \times -----: Background field
- : Quantum field
- _____ : fermion field
- : mass counterterm

IV. OVERLAP ACTION

$$S_{\text{Overlap}} = a^8 \sum_{n,m} \bar{\Psi}(n) \textcolor{red}{D}_N(n, m) \Psi(m)$$

Overlap-Dirac operator:

$$\textcolor{red}{D}_N(n, m) = \textcolor{red}{\rho} \left[\frac{\delta_{n,m}}{a^4} - \left(\textcolor{blue}{X} \frac{1}{\sqrt{\textcolor{blue}{X}^\dagger \textcolor{blue}{X}}} \right)_{nm} \right]$$

$$\textcolor{blue}{X} = \frac{1}{a^4} (D_W - \textcolor{red}{\rho}), \quad D_W: \text{ Wilson-Dirac operator}$$

$0 < \textcolor{red}{\rho} < 2$ (to guarantee the correct pole structure of D_N)

★ Coupling constant is included inside of X

Momentum space:

$$\begin{aligned} X(q, p) = & \underbrace{\chi_0(p)(2\pi)^4 \delta_P(q - p)}_{\text{tree-level}} + \underbrace{X_1(q, p) + X_2(q, p)}_{\text{1-loop}} \\ & + \underbrace{X_3(q, p) + X_4(q, p)}_{\text{2-loop}} + O(g_0^5) \end{aligned}$$

$\chi_0, \textcolor{blue}{X}_i$: vertices of the Wilson fermion action

$$\chi_0(p) = \frac{i}{a} \sum_\mu \gamma_\mu \sin(ap_\mu) + \frac{r}{a} \sum_\mu (1 - \cos(ap_\mu)) - \frac{\rho}{a}$$

$$X_1(q, p) = g_0 \int d^4 k \delta(q - p - k) \sum_\mu A_\mu(k) V_{1,\mu}\left(\frac{q+p}{2}\right)$$

$$X_2(q, p) = \frac{g_0^2}{2} \int \frac{d^4 k_1 d^4 k_2}{(2\pi)^4} \delta(q - p - k_1 - k_2) \sum_\mu A_\mu(k_1) A_\mu(k_2) V_{2,\mu}\left(\frac{q+p}{2}\right)$$

$$X_3(q, p) = \frac{g_0^3}{3!} \int \frac{d^4 k_1 d^4 k_2 d^4 k_3}{(2\pi)^8} \delta(q - p - \sum_{i=1}^3 k_i) \sum_\mu \prod_{i=1}^3 A_\mu(k_i) \left[-a^2 V_{1,\mu}\left(\frac{q+p}{2}\right) \right]$$

$$X_4(q, p) = \frac{g_0^4}{4!} \int \frac{d^4 k_1 d^4 k_2 d^4 k_3 d^4 k_4}{(2\pi)^{12}} \delta(q - p - \sum_{i=1}^4 k_i) \sum_\mu \prod_{i=1}^4 A_\mu(k_i) \left[-a^2 V_{2,\mu}\left(\frac{q+p}{2}\right) \right]$$

A_μ : general gluonic field

$$V_{1,\mu}(q) = i \gamma_\mu \cos(aq_\mu) + r \sin(aq_\mu)$$

$$V_{2,\mu}(q) = -i \gamma_\mu a \sin(aq_\mu) + a r \cos(aq_\mu)$$

Goal:

To find the perturbative expansion of D_N in powers of g_0 .

This leads to the propagator of zero mass fermions and to gluon-fermion-antifermion vertices (with up to 4 gluons for the needs of the present calculation). The much simpler case of vertices with up to 2 gluons (and no background) can be found in Ref. [26].

After laborious analytical manipulations:

$$D_N(k_1, k_2) = \underbrace{D_0(k_1)}_{\text{inverse propagator}} (2\pi)^4 \delta^4(k_1 - k_2) + \Sigma(k_1, k_2)$$

$$D_0(k_1) = 1 + \frac{\chi_0(k_1)}{\omega(k_1)}, \quad \omega(p) = \sqrt{\left(\sum_\mu \sin^2(p_\mu)\right) + \left(2r \sum_\mu \sin^2(p_\mu/2) - \rho\right)^2}$$

$$\begin{aligned} \frac{\Sigma(k_1, k_2)}{\rho} &= \underbrace{V_1^1(k_1, k_2)}_{\text{1-gluon vertex}} + \underbrace{V_1^2(k_1, k_2) + V_2^2(k_1, k_2)}_{\text{2-gluon vertex}} \\ &\quad + \underbrace{V_1^3(k_1, k_2) + V_2^3(k_1, k_2) + V_3^3(k_1, k_2)}_{\text{3-gluon vertex}} \\ &\quad + \underbrace{V_1^4(k_1, k_2) + V_2^4(k_1, k_2) + V_3^4(k_1, k_2) + V_4^4(k_1, k_2)}_{\text{4-gluon vertex}} + \mathcal{O}(g_0^5) \end{aligned}$$

where we have set $a = 1$, and:

- $V_1^i(k_1, k_2) = \frac{1}{\omega(k_1) + \omega(k_2)} \left[X_i(k_1, k_2) - \frac{1}{\omega(k_1)\omega(k_2)} \chi_0(k_1) X_i^\dagger(k_1, k_2) \chi_0(k_2) \right]$
- $V_2^i(k_1, k_2) = \int \frac{d^4 k_3}{(2\pi)^4} \frac{1}{\omega(k_1) + \omega(k_3)} \frac{1}{\omega(k_1) + \omega(k_2)} \frac{1}{\omega(k_2) + \omega(k_3)} \times$

$$\sum_{\{j>0, k>0\} \atop \{j+k=i\}} \left[-X_j(k_1, k_3) \chi_0^\dagger(k_3) X_k(k_3, k_2) \right.$$

$$-X_j(k_1, k_3) X_k^\dagger(k_3, k_2) \chi_0(k_2)$$

$$-\chi_0(k_1) X_j^\dagger(k_1, k_3) X_k(k_3, k_2)$$

$$\left. + \frac{\omega(k_1) + \omega(k_2) + \omega(k_3)}{\omega(k_1)\omega(k_2)\omega(k_3)} \chi_0(k_1) X_j^\dagger(k_1, k_3) \chi_0(k_3) X_k^\dagger(k_3, k_2) \chi_0(k_2) \right]$$

- $$V_3^i(k_1, k_2) = \int \int \frac{d^4 k_3}{(2\pi)^4} \frac{d^4 k_4}{(2\pi)^4} \frac{1}{4} \left(\prod_{p \in S_4} \frac{1}{\omega(k_{p_1}) + \omega(k_{p_2})} \right) \times$$

$$\sum_{\substack{\{j > 0, k > 0, l > 0 \\ \{j+k+l=i\}}} \left[-\frac{1}{6} \left(\sum_{p \in S_4} \omega(k_{p_1}) \omega(k_{p_2}) \omega(k_{p_3}) \right) X_j(k_1, k_3) X_k^\dagger(k_3, k_4) X_l(k_4, k_2) \right.$$

$$+ (\omega(k_1) + \omega(k_3) + \omega(k_4) + \omega(k_2)) \times$$

$$[\chi_0(k_1) X_j^\dagger(k_1, k_3) X_k(k_3, k_4) X_l^\dagger(k_4, k_2) \chi_0(k_2) + \chi_0(k_1) X_j^\dagger(k_1, k_3) X_k(k_3, k_4) \chi_0^\dagger(k_4) X_l(k_4, k_2)$$

$$+ \chi_0(k_1) X_j^\dagger(k_1, k_3) \chi_0(k_3) X_k^\dagger(k_3, k_4) X_l(k_4, k_2) + X_j(k_1, k_3) \chi_0^\dagger(k_3) X_k(k_3, k_4) X_l^\dagger(k_4, k_2) \chi_0(k_2)$$

$$+ X_j(k_1, k_3) \chi_0^\dagger(k_3) X_k(k_3, k_4) \chi_0^\dagger(k_4) X_l(k_4, k_2) + X_j(k_1, k_3) X_k^\dagger(k_3, k_4) \chi_0(k_4) X_l^\dagger(k_4, k_2) \chi_0(k_2)]$$

$$- \left(\sum_{p \in S_4} \frac{\omega(k_{p_1}) \omega(k_{p_2}) (\omega(k_{p_1})/2 + \omega(k_{p_3})/3)}{\omega(k_1) \omega(k_3) \omega(k_4) \omega(k_2)} \right) \chi_0(k_1) X_j^\dagger(k_1, k_3) \chi_0(k_3) X_k^\dagger(k_3, k_4) \chi_0(k_4) X_l^\dagger(k_4, k_2) \chi_0(k_2)$$
- $$V_4^4(k_1, k_2) = \int \int \int \frac{d^4 k_3}{(2\pi)^4} \frac{d^4 k_4}{(2\pi)^4} \frac{d^4 k_5}{(2\pi)^4} \frac{1}{12} \left(\prod_{p \in S_5} \frac{1}{\omega(k_{p_1}) + \omega(k_{p_2})} \right) \times$$

$$\left[\left(\sum_{p \in S_5} \omega(k_{p_1}) \omega(k_{p_2}) \omega(k_{p_3}) \omega(k_{p_4}) (\omega(k_{p_1})/6 + \omega(k_{p_5})/30) \right) \times \right.$$

$$[(X_1(k_1, k_3) X_1^\dagger(k_3, k_4) X_1(k_4, k_5) X_1^\dagger(k_5, k_2) \chi_0(k_2)$$

$$+ X_1(k_1, k_3) X_1^\dagger(k_3, k_4) X_1(k_4, k_5) \chi_0^\dagger(k_5) X_1(k_5, k_2) + X_1(k_1, k_3) X_1^\dagger(k_3, k_4) \chi_0(k_4) X_1^\dagger(k_4, k_5) X_1(k_5, k_2)$$

$$+ X_1(k_1, k_3) \chi_0^\dagger(k_3) X_1(k_3, k_4) X_1^\dagger(k_4, k_5) X_1(k_5, k_2) + \chi_0(k_1) X_1^\dagger(k_1, k_3) X_1(k_3, k_4) X_1^\dagger(k_4, k_5) X_1(k_5, k_2)]$$

$$- \frac{1}{6} \left(\sum_{p \in S_5} \omega(k_{p_1}) \omega(k_{p_2}) (\omega(k_{p_1}) + \omega(k_{p_3})) \right) \times$$

$$[X_1(k_1, k_3) \chi_0^\dagger(k_3) X_1(k_3, k_4) X_1^\dagger(k_4, k_5) \chi_0(k_5) X_1^\dagger(k_5, k_2) \chi_0(k_2)$$

$$+ X_1(k_1, k_3) \chi_0^\dagger(k_3) X_1(k_3, k_4) \chi_0^\dagger(k_4) X_1(k_4, k_5) \chi_0^\dagger(k_5) X_1(k_5, k_2)$$

$$+ X_1(k_1, k_3) \chi_0^\dagger(k_3) X_1(k_3, k_4) \chi_0^\dagger(k_4) X_1(k_4, k_5) X_1^\dagger(k_5, k_2) \chi_0(k_2)$$

$$+ X_1(k_1, k_3) X_1^\dagger(k_3, k_4) \chi_0(k_4) X_1^\dagger(k_4, k_5) \chi_0(k_5) X_1^\dagger(k_5, k_2) \chi_0(k_2)$$

$$+ \chi_0(k_1) X_1^\dagger(k_1, k_3) \chi_0(k_3) X_1^\dagger(k_3, k_4) \chi_0(k_4) X_1^\dagger(k_4, k_5) X_1(k_5, k_2)$$

$$+ \chi_0(k_1) X_1^\dagger(k_1, k_3) \chi_0(k_3) X_1^\dagger(k_3, k_4) X_1(k_4, k_5) \chi_0^\dagger(k_5) X_1(k_5, k_2)$$

$$+ \chi_0(k_1) X_1^\dagger(k_1, k_3) X_1(k_3, k_4) \chi_0^\dagger(k_4) X_1(k_4, k_5) \chi_0^\dagger(k_5) X_1(k_5, k_2)$$

$$+ \chi_0(k_1) X_1^\dagger(k_1, k_3) X_1(k_3, k_4) \chi_0^\dagger(k_4) X_1(k_4, k_5) X_1^\dagger(k_5, k_2) \chi_0(k_2)$$

$$+ \chi_0(k_1) X_1^\dagger(k_1, k_3) X_1(k_3, k_4) X_1^\dagger(k_4, k_5) \chi_0(k_5) X_1^\dagger(k_5, k_2) \chi_0(k_2)]$$

$$+ \left(\sum_{p \in S_5} \frac{\omega^2(k_{p_1}) \omega(k_{p_2}) \omega(k_{p_3})}{\omega(k_1) \omega(k_2) \omega(k_3) \omega(k_4) \omega(k_5)} (\omega(k_{p_2}) [\omega(k_{p_1})/2 + \omega(k_{p_3})/6] + \omega(k_{p_4}) [\omega(k_{p_1})/3 + \omega(k_{p_2}) + \omega(k_{p_5})/3]) \right) \times$$

$$\chi_0(k_1) X_1^\dagger(k_1, k_3) \chi_0(k_3) X_1^\dagger(k_3, k_4) \chi_0(k_4) X_1^\dagger(k_4, k_5) \chi_0(k_5) X_1^\dagger(k_5, k_2) \chi_0(k_2)]$$

Inserting the Background field into the vertices:

The use of the Background field technique implies that instead of the general gluonic fields (appearing in X_i 's), one has to take all possible combinations of Background (**A**) and Quantum (**Q**) fields.

Hence,

$$\begin{aligned} X_0(p) &= X_0(p) \\ X_1(q, p) &= X_1^Q(q, p) + X_1^A(q, p) \\ X_2(q, p) &= X_2^{QQ}(q, p) + X_2^{QA}(q, p) + X_2^{AA}(q, p) \\ X_3(q, p) &= X_3^{QQQ}(q, p) + X_3^{QQA}(q, p) + X_3^{QAA}(q, p) + X_3^{AAA}(q, p) \\ X_4(q, p) &= X_4^{QQQQ}(q, p) + X_4^{QQQA}(q, p) + X_4^{QQAA}(q, p) + X_4^{QAAA}(q, p) + X_4^{AAAA}(q, p) \end{aligned}$$

Consequences:

- The vertices become extremely lengthy and complicated
- Calculation is computationally expensive and human time demanding

For instance,

- the vertex with $Q\text{-}Q\text{-}A\text{-}\Psi\text{-}\bar{\Psi}$ consists of 9,784 terms
- the vertex with $Q\text{-}Q\text{-}A\text{-}A\text{-}\Psi\text{-}\bar{\Psi}$ has 724,120 terms !!!



Actual procedure for calculating diagrams:

- Contraction of vertices.
- Simplification of color dependence, Dirac matrices and tensors.
- Usage of symmetries of the theory and of the diagrams.
- Extraction of the external momentum q :
 - ★ Isolation of terms that give logarithms.
 - ★ Taylor expansion of convergent terms with respect to q up to $\mathcal{O}(q^2)$.
- Numerical integration over internal momenta for lattices with varying size and for different values of ρ (Overlap parameter).
- Extrapolation of the results to lattice with infinite size, estimation of the systematic errors. IR divergent diagrams must be summed up before proceeding with the extrapolation.

VI. RESULTS

Each Feynman diagram has a dependence on g , N , N_f , ρ and aq according to the following formulas:

* 1-loop contribution:

$$\widehat{a}q^2\nu_i^{(1)}(q) = N_f \left[\textcolor{red}{e}_i^{(0)} + a^2 q^2 \left\{ \textcolor{red}{e}_i^{(1)} + \textcolor{red}{e}_i^{(2)} \frac{\ln a^2 q^2}{(4\pi)^2} \right\} + \mathcal{O}((aq)^4) \right]$$

index i : runs over diagrams

- $\hat{p}^2 = 4 \sum_\mu \sin^2(p_\mu/2)$
- $\textcolor{red}{e}_i^j \equiv e_i^j(\rho)$

* 2-loop contribution:

$$\begin{aligned} \widehat{a}q^2\nu_i^{(2)}(q) = N_f & \left[\textcolor{red}{c}_i^{(0)} + a^2 q^2 \left\{ \textcolor{red}{c}_i^{(1)} + \textcolor{red}{c}_i^{(2)} \frac{\ln a^2 q^2}{(4\pi)^2} + \textcolor{red}{c}_i^{(3)} \left(\frac{\ln a^2 q^2}{(4\pi)^2} \right)^2 \right\} \right. \\ & \left. + \textcolor{red}{c}_i^{(4)} a^2 \sum_\mu \frac{q_\mu^4}{q^2} + \mathcal{O}((aq)^4) \right] \end{aligned}$$

- $\textcolor{red}{c}_i^j \equiv c_i^j(\rho) = [c_i^{(j,-1)}/N + c_i^{(j,1)}N]$

Comparison with continuum limit and usage of Ward Identities, leads to:

- $\sum_i e_i^{(0)} = 0$,
- $\sum_i c_i^{(0)} = 0$,
- $\sum_i c_i^{(2)} = \frac{3}{2}$
- $\sum_i c_i^{(4)} = 0$ (Lorentz invariance)
- $\sum_i c_i^{(2)} = \frac{1}{16\pi^2} \left(3N - \frac{1}{N} \right)$
- $c_{15}^{(3)} = \frac{1}{3N}, \quad c_{16}^{(3)} = \frac{4}{3}N, \quad c_{17}^{(3)} = -\frac{5}{3}N, \quad c_{18}^{(3)} = \frac{N^2 - 1}{3N}$

- 1-loop:

Total $\nu^{(1)}(q)$ for 21 values of the overlap parameter ($0 < \rho < 2$).

Each diagram was integrated for lattice size n^4 , $n \leq 128$.

Once we have the numerical results, we proceed with extrapolation and systematic error calculation.

- 2-loop:

Calculation concerning $\nu^{(2)}(q)$ is done for 21 values of ρ and for $n \leq 28$. Due to the extremely large size of the vertices involved, it is almost impossible to extend the results for larger n .

Typically,

integration of 2000 terms completed in $\simeq 7$ days in 1 CPU.

Present calculation: approximately 3650×2000 terms.

If only a single CPU available:

our work would have required 70 years!!!

* In certain cases with large systematic errors we extended the integration up to $n = 46$ (for particular values of ρ)

A large variety of possible numerical checks has been performed:

- Total contribution to gluon mass adds to zero
- Coefficients of non-Lorentz invariant terms cancel
- Terms with double logarithms correspond to continuum counterparts, diagram by diagram
- Terms with single logarithms add up to their expected value

TABLES

We tabulate the total $\mathcal{O}(q^2)$ contribution of $\widehat{aq}^2 \nu_i^{(1)}(q)$ for different values of the Overlap parameter ρ :

TABLE I. Coefficients $\sum_i e_i^{(1)}$

ρ	$\sum_i e_i^{(1)}$
0.1	-0.020377(7)
0.2	-0.01581702(2)
0.3	-0.0133504717(2)
0.4	-0.01169109515(2)
0.5	-0.0104621922(2)
0.6	-0.0095058191(2)
0.7	-0.00874441051(7)
0.8	-0.00813753230(4)
0.9	-0.00766516396(3)
1.0	-0.00732057894(3)
1.1	-0.00710750173(2)
1.2	-0.00703970232(7)
1.3	-0.0071425543(2)
1.4	-0.0074569183(2)
1.5	-0.00804670459(4)
1.6	-0.0090134204(1)
1.7	-0.010526080(2)
1.8	-0.0128914(2)
1.9	-0.01680(8)

- ★ Results with **green** color correspond to values of ρ that are mostly used in simulations
- $e_i^{(1)}(\rho)$ are **independent** of:
 - N
 - choice of regularization for the pure gluonic part of the action (Symanzik, Iwasaki, etc.)
- ★ The above results are in good agreement with corresponding results of Ref. [19].

Here is presented the 2-loop $\mathcal{O}(q^2)$ contribution for $\rho \in [0.1, 1.8]$:

TABLE II. Coefficients $\sum_i c_i^{(1,-1)}$ and $\sum_i c_i^{(1,1)}$

ρ	$\sum_i c_i^{(1,-1)}$	$\sum_i c_i^{(1,1)}$
0.1	-0.0096(6)	0.124(3)
0.2	-0.0044(1)	0.0118(5)
0.3	-0.00321(6)	0.0045(1)
0.4	-0.00244(4)	0.00302(8)
0.5	-0.00191(1)	0.0022(6)
0.6	-0.001606(6)	0.00176(2)
0.7	-0.001397(3)	0.001454(7)
0.8	-0.001241(1)	0.00124(1)
0.9	-0.001107(1)	0.001051(9)
1.0	-0.000979(1)	0.000872(3)
1.1	-0.000849(2)	0.000710(8)
1.2	-0.000706(3)	0.00052(1)
1.3	-0.000543(4)	0.00033(3)
1.4	-0.000335(7)	0.00007(1)
1.5	-0.00005(1)	-0.0002(1)
1.6	0.00034(1)	-0.0004(1)
1.7	0.00093(6)	-0.0021(5)
1.8	0.0020(1)	-0.02(3)

- ★ For generic values of ρ , we ran our integrations up to lattices $n \leq 28$.
- ★ For $\rho = 1, 1.4, 1.6$ we used large lattices, often up to $n = 36$ or larger, to reduce systematic errors.

GRAPHS

1-loop contribution:

$$\widehat{a}q^2\nu^{(1)}(q) = N_f \left[a^2 q^2 \left\{ \textcolor{red}{e_i^{(1)}} + \frac{3}{2} \frac{\ln a^2 q^2}{(4\pi)^2} \right\} + \mathcal{O}((aq)^4) \right]$$

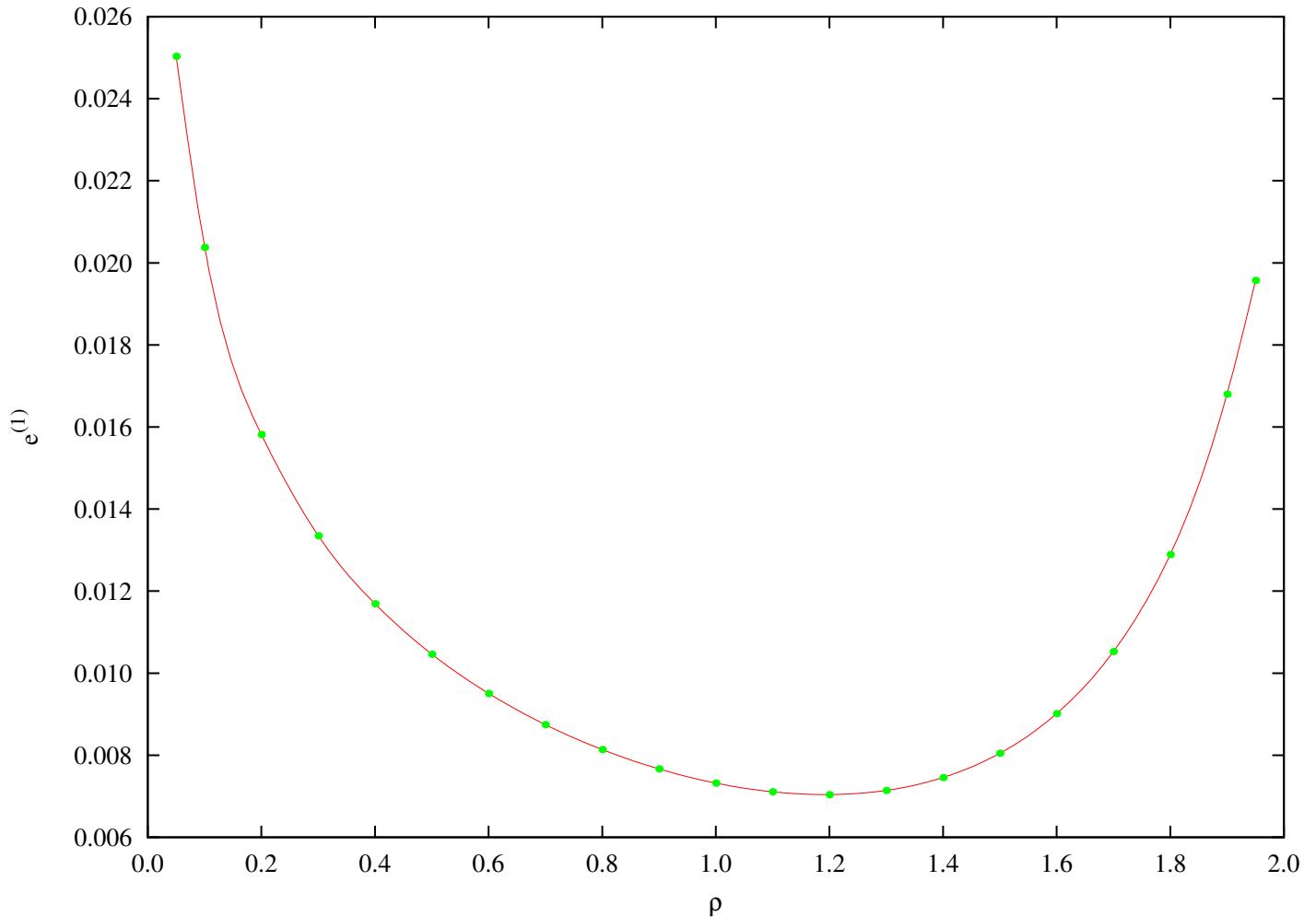


Fig.1: Plot of the total coefficient $e^{(1)} = \sum_i e_i^{(1)}$ for different values of the Overlap parameter ρ

- Errors are too small to be visible at this scale

2-loop contribution:

$$\widehat{a}q^2\nu^{(2)}(q) = N_f \left[a^2 q^2 \left\{ \frac{\textcolor{red}{c}_i^{(1,-1)}}{N} + N c_i^{(1,1)} + \left(3N - \frac{1}{N} \right) \frac{\ln a^2 q^2}{(4\pi)^4} \right\} \right]$$

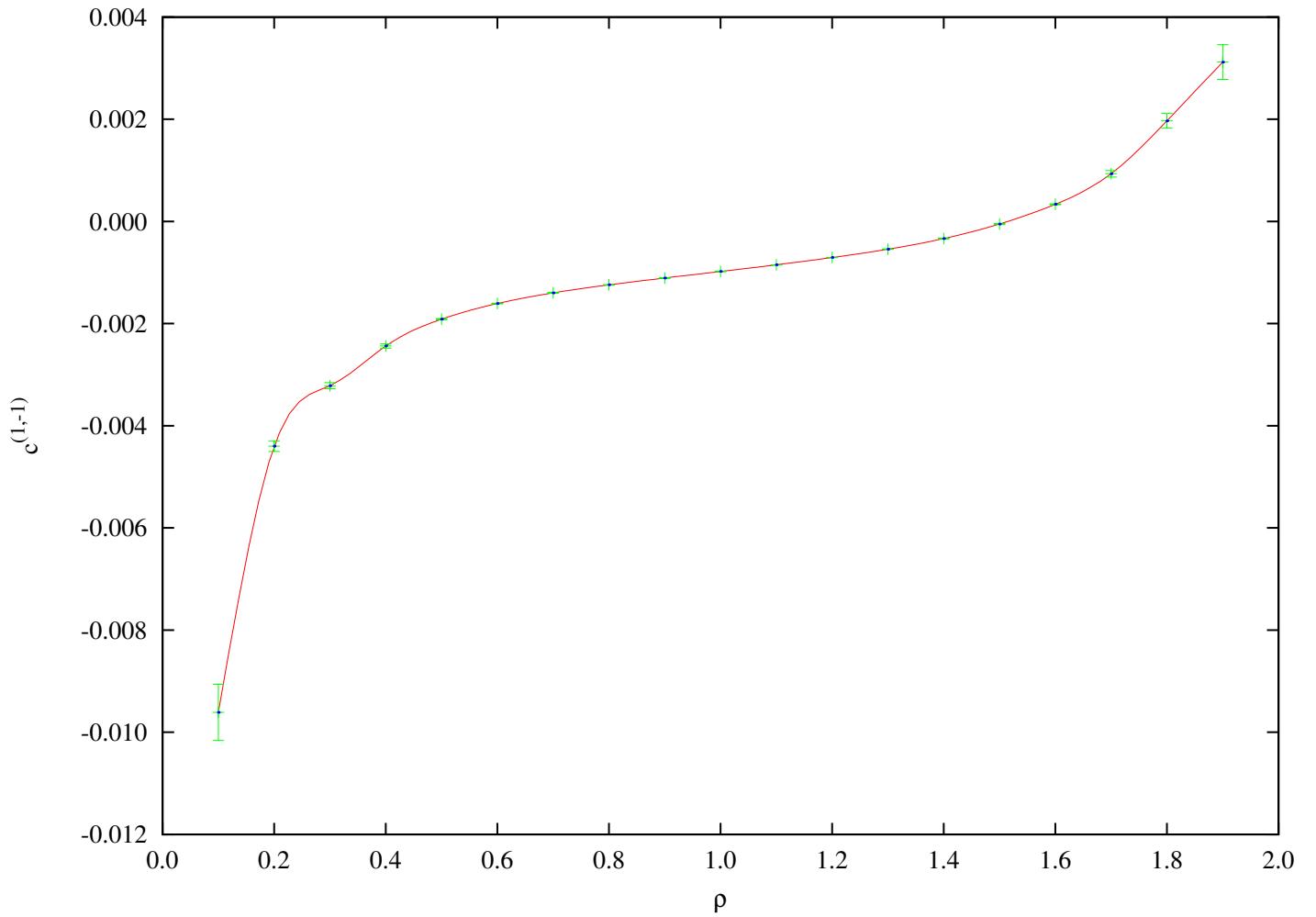


Fig.2: Plot of the total coefficient $c^{(1,-1)} = \sum_i c_i^{(1,-1)}$ for different values of the Overlap parameter ρ

- Errors come from extrapolation to infinite size lattice

$$\widehat{a}q^2\nu^{(2)}(q) = N_f \left[a^2 q^2 \left\{ \frac{c_i^{(1,-1)}}{N} + N \textcolor{red}{c_i^{(1,1)}} + \left(3N - \frac{1}{N} \right) \frac{\ln a^2 q^2}{(4\pi)^4} \right\} \right]$$

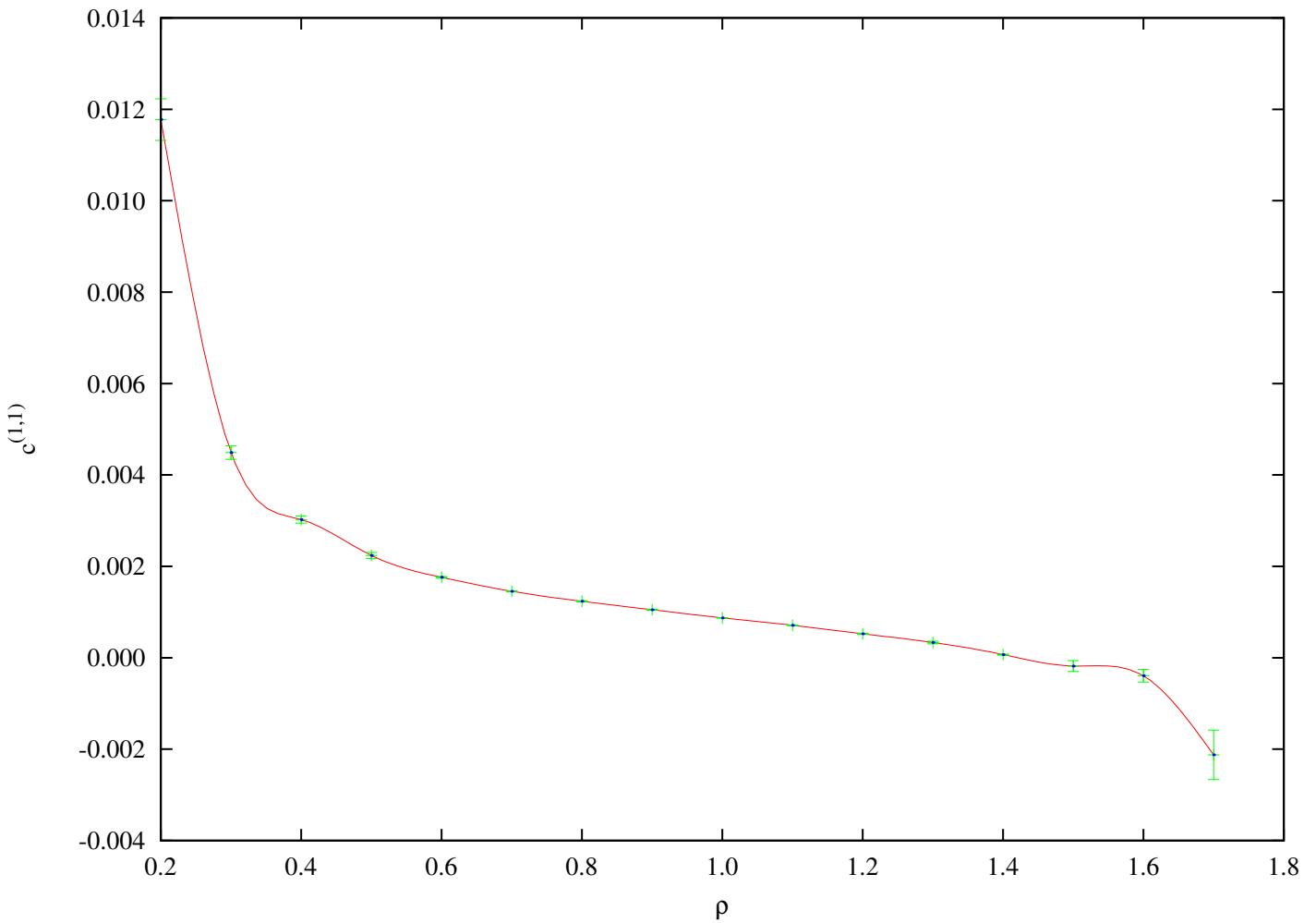


Fig.3: Plot of the total coefficient $c^{(1,1)} = \sum_i c_i^{(1,1)}$ for different values of the Overlap parameter ρ

- Errors come from extrapolation to infinite size lattice

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