HEAVY QUARKS ON THE LATTICE 4:

Controlling and Estimating Errors

Non-Perturbative Renormalization

Error Estimation in Present Calculations

Outlook

Andreas S. Kronfeld

28 August 2007

Non-Perturbative Renormalization

General Considerations

I've said for some time (at least since 2001) that the HQ theory of cutoff effects provides a way to carry out non-perturbative matching calculations.

Here I will show how it is done. None of the formulae are new, but I'm presenting them in a new order to make it explicit.

As elsewhere, an important element will be to reach the continuum limit with brute force (*i.e.*, $m_Q a \ll 1$), which is possible only in a finite volume of extent $L = L_0$.

L_0 is a long-distance scale (by definition). If $L_0 \lesssim 1.5$ fm, it will creep into hadronic masses and matrix elements.

The HQ effective Lagrangian separates long- and short-distance effects; it will hold in finite volume only if $(1/L)/m_Q = (m_Q L)^{-1} \ll 1$.

Thus we need to start with $a_0 \ll L_0$, $m_Q a_0 \ll 1$, $1 \ll m_Q L_0$, or

$$a_0^{-1} \gg \quad m_Q \quad \gg L_0^{-1} \tag{1}$$
$$20 \text{ GeV} \gg 5 \text{ GeV} \gg 625 \text{ MeV} \tag{2}$$

so it seems possible with 32^3 spatial lattices.

Keep in mind that the utility of the heavy-quark expansion in finite volume is an aspect with which we have not much experience.

We will also need light quarks to make heavy-light hadrons.

To have a process in mind, we'll later consider a decay mediated via the vector current $\mathcal{V}^\mu = \bar{u}\gamma^\mu b$.

So let us begin with a fine lattice of spacing a_0 and length L_0.

Lagrangian Matching

We shall start with two lattice gauge theories, which I'll abbreviate "lgt" and "LGT."

The lgt is a lattice gauge theory that is good for light quarks, and it will be used for light quarks throughout. By assumption $m_Q a_0 \ll 1$, so at such a fine spacing we may calculate the heavy-quark with lgt too.

From the Symanzik EFT we have

$$\begin{aligned}\mathcal{L}_{\text{lgt}} \doteq & -\bar{u}(\not{D}+m_u)u + a_0 K^{\text{lgt}}_{\sigma.F}(m_u a_0)\bar{u} i\sigma^{\mu\nu}F_{\mu\nu}u \\ & -\bar{Q}(\not{D}+m_Q)Q + a_0 K^{\text{lgt}}_{\sigma.F}(m_Q a_0)\bar{Q} i\sigma^{\mu\nu}F_{\mu\nu}Q \\ & + \mathrm{O}(m_b^2 a_0^2, m_b\Lambda a_0^2, m_b a_0^2/L_0, \ldots)\end{aligned} \quad (3)$$

where Q is our heavy quark.

We don't know yet when $m_Q = m_b$, so make a good guess for the bare parameters to bracket m_b.

The leading Symanzik coefficient depends on what lgt is:

$$K_{\sigma.F}(ma_0) = \begin{cases} \frac{1}{4}(1 - c_{\text{SW}}) + \text{O}(ma_0) + \text{O}(g^{2(l+1)}) & l\text{-loop PT} \\ \text{O}(a_0/L_0) + \text{O}(a_0\Lambda) + \text{O}(ma_0) & \text{NP} \\ 0 & \text{GW} \end{cases} \tag{4}$$

Since Q is heavy, we also have

$$\begin{aligned} \mathcal{L}_{\text{lgt}} \doteq & -\bar{u}(\not{D} + m_u)u + a_0 K^{\text{lgt}}_{\sigma.F}(m_u a_0)\bar{u}i\sigma^{\mu\nu}F_{\mu\nu}u \\ & -\bar{h}(D_4 + m_1^{\text{lgt}})h + \frac{\bar{h}\boldsymbol{D}^2 h}{2m_2^{\text{lgt}}} + Z^{\text{lgt}}_{\sigma B}\frac{\bar{h}i\boldsymbol{\sigma}\cdot\boldsymbol{B}h}{2m_2^{\text{lgt}}} \\ & + \text{O}(\Lambda^2/m_b^2, \Lambda^2/m_b^2 L, 1/m_b^2 L^2 \ldots) \end{aligned} \tag{5}$$

One can also apply the HQ theory to $\bar{Q}(\not{D} + m_Q)Q$ and $\bar{Q}i\sigma^{\mu\nu}F_{\mu\nu}Q$ on the RHS of Eq. (3). The expression is similar to Eq. (5), but with the continuum short-distance coefficients, plus the Symanzik ones.

Comparing Eqs. (3) and (5), one obtains relations between the short-distance coefficient in the two equations:

$$m_1^{\mathrm{lgt}} = m_Q + \mathrm{O}(a_0^2), \tag{6}$$

$$m_2^{\mathrm{lgt}} = m_Q + \mathrm{O}(a_0^2), \tag{7}$$

$$Z_{\sigma B}^{\mathrm{lgt}} = Z_{\sigma B}^{\mathrm{QCD}} + 4 m_Q a_0 K_{\sigma.F}^{\mathrm{lgt}} C_{\sigma.F}^{\mathrm{QCD}} + \mathrm{O}(a_0^2, 1/m_b), \tag{8}$$

where $C_{\sigma.F}$ appears in

$$\bar{Q} i \sigma^{\mu\nu} F_{\mu\nu} Q \doteq C_{\sigma.F}^{\mathrm{QCD}} \bar{h} i \boldsymbol{\sigma} \cdot \boldsymbol{B} h + \mathrm{O}(1/m_b) \tag{9}$$

If lgt has chiral symmetry, so that $K_{\sigma.F}^{\mathrm{lgt}} = 0$, then we won't have to deal with $C_{\sigma.F}^{\mathrm{QCD}}$.

Exercise: Why does $\doteq$ appear in Eqs. (3), (5) and (9), but $=$ in Eqs. (6)–(8)?

Now bring in LGT, a lattice gauge theory designed for heavy quarks when $m_Q a \not\ll 1$.

By assumption, LGT only has to make sense for heavy quarks, so we cannot assume the validity of a Symanzik description (with $\bar{Q}(\not{D} + m_Q)Q$).

Instead, we have an HQ description of cutoff effects

$$\begin{aligned}\mathcal{L}_{\text{LGT}} \doteq & -\bar{u}(\not{D} + m_u)u + a_0 K^{\text{lgt}}_{\sigma.F}(m_u a_0)\bar{u}i\sigma^{\mu\nu}F_{\mu\nu}u \\ & -\bar{h}(D_4 + m_1^{\text{LGT}})h + \frac{\bar{h}\boldsymbol{D}^2 h}{2m_2^{\text{LGT}}} + Z^{\text{LGT}}_{\sigma B}\frac{\bar{h}i\boldsymbol{\sigma}\cdot\boldsymbol{B}h}{2m_2^{\text{LGT}}} \\ & + \text{O}(\Lambda^2/m_b^2, \Lambda^2/m_b^2 L, 1/m_b^2 L^2 \ldots)\end{aligned} \tag{10}$$

Our goal is to adjust the bare parameters of LGT so that ($\stackrel{!}{=}$ means "adjust to be")

$$m_1^{\text{LGT}}(\kappa_t) \stackrel{!}{=} m_1^{\text{lgt}} = m_Q + \text{O}(a_0^2), \tag{11}$$

$$m_2^{\text{LGT}}(\kappa_s) \stackrel{!}{=} m_2^{\text{lgt}} = m_Q + \text{O}(a_0^2), \tag{12}$$

$$Z^{\text{LGT}}_{\sigma B}(c_{\text{SW}}) \stackrel{!}{=} Z^{\text{lgt}}_{\sigma B} = Z^{\text{QCD}}_{\sigma B} + 4m_Q a_0 K^{\text{lgt}}_{\sigma.F} C^{\text{QCD}}_{\sigma.F} + \text{O}(a_0^2, 1/m_b), \tag{13}$$

where the most important bare parameters in the Fermilab action are illustrated.

This is abstract! It's time to introduce real calculations—of the B, B^*, and Λ_b masses.

From the HQ theory (for both lgt and LGT)

$$M_{J,\mathrm{m}}(L_0) = m_1 + \bar{\Lambda}_\mathrm{m}(L_0) + \frac{\mu^2_{\pi,\mathrm{m}}(L_0)}{2m_2} + \frac{d_J}{3}\frac{Z_{\sigma B}}{2m_2}\mu^2_{G,\mathrm{m}}(L_0) + \mathrm{O}(1/m_b^2, a_0^2), \quad (14)$$

$$M_\mathrm{b}(L_0) = m_1 + \bar{\Lambda}_\mathrm{b}(L_0) + \frac{\mu^2_{\pi,\mathrm{b}}(L_0)}{2m_2} + \mathrm{O}(1/m_b^2, a_0^2). \quad (15)$$

From the meson masses form

$$\bar{M}_\mathrm{m}(L_0) = \tfrac{1}{4}(M_{0,\mathrm{m}} + 3M_{1,\mathrm{m}})(L_0) = m_1 + \bar{\Lambda}_\mathrm{m}(L_0) + \frac{\mu^2_{\pi,\mathrm{m}}(L_0)}{2m_2} + \mathrm{O}(1/m_b^2, a_0^2), \quad (16)$$

$$\Delta M_\mathrm{m}(L_0) = (M_{1,\mathrm{m}} - M_{0,\mathrm{m}})(L_0) = \frac{4}{3}\frac{Z_{\sigma B}}{2m_2}\mu^2_{G,\mathrm{m}}(L_0) + \mathrm{O}(1/m_b^2, a_0^2). \quad (17)$$

Then $(\bar{M}_\mathrm{m} - M_\mathrm{b})(L_0)$ has only m_2; $\Delta M_\mathrm{m}(L_0)$ only m_2 and $Z_{\sigma B}$; $M_{0,\mathrm{m}}(L_0)$ all three.

The universality of the leading HQ effective Lagrangian implies that the HQET matrix elements $\Lambda_{\mathrm{h}}(L_0)$, $\mu^2_{\pi,\mathrm{h}}(L_0)$, and $\mu^2_{G,\mathrm{h}}(L_0)$ $(\mathrm{h} = \mathrm{m}, \mathrm{b})$ are the same for lgt and LGT. The lattice hadron masses differ only at short distances.

Thus, adjusting LGT's bare parameters so that

$$(M_{\mathrm{b}}^{\mathrm{LGT}} - \bar{M}_{\mathrm{m}}^{\mathrm{LGT}})(L_0) \stackrel{!}{=} (M_{\mathrm{b}}^{\mathrm{lgt}} - \bar{M}_{\mathrm{m}}^{\mathrm{lgt}})(L_0), \tag{18}$$

$$\Delta M_{\mathrm{m}}^{\mathrm{LGT}}(L_0) \stackrel{!}{=} \Delta M_{\mathrm{m}}^{\mathrm{lgt}}(L_0), \tag{19}$$

$$M_{0,\mathrm{m}}^{\mathrm{LGT}}(L_0) \stackrel{!}{=} M_{0,\mathrm{m}}^{\mathrm{lgt}}(L_0), \tag{20}$$

yields the bare parameters, as a function of $m_Q a_0$, a_0 fixed. As an exercise, show that Eqs. (18)–(20) reduce to conditions on the short-distance structure.

Dependence on the underlying theories (lgt and LGT) does pollute the matching, because the neglected terms bring in L_0 dependent matrix elements (of Darwin and spin-orbit) that are formally smaller, but (as yet) poorly understood.

Hence, there is a difficult-to-quantify error creeping in.

Now double the lattice spacing, $a_1 = 2a_0$, keeping $L = L_0$.

Matching $\mathrm{LGT}(a_1)$ to $\mathrm{LGT}(a_0)$ determines the bare parameters of $\mathrm{LGT}(a_1)$. Use Eqs. (18)–(20) with $\mathrm{lgt} \to \mathrm{LGT}(a_0)$ on the RHS, and $\mathrm{LGT} \to \mathrm{LGT}(a_1)$ on the LHS.

Now compute the masses for $\mathrm{LGT}(a_1)$ in a larger volume with side $L_1 = 2L_0$.

Match $\mathrm{LGT}(a_2)$, $a_2 = 2a_1$, to $\mathrm{LGT}(a_1)$ with side $L_1 \Rightarrow$ bare parameters of $\mathrm{LGT}(a_2)$.

Compute the masses for $\mathrm{LGT}(a_2)$ with side $L_2 = 2L_1$.

One more iteration, arriving at $\mathrm{LGT}(a_3, L_3)$ with

$$a_3 = (5\ \mathrm{GeV})^{-1} = 0.08\ \mathrm{fm}, \quad L_3 = (5\ \mathrm{GeV})^{-1} = 2.56\ \mathrm{fm}, \tag{21}$$

with L_3 large enough that finite-volume effects are expected to be negligible.

Identify $(M_{\mathrm{b}}^{\mathrm{LGT}} - \bar{M}_{\mathrm{m}}^{\mathrm{LGT}})(L_3) = m_{\Lambda_b} - \bar{m}_B$ to eliminate the bare quark mass in favor of a physical quantity. (Or $M_{0,\mathrm{m}}^{\mathrm{LGT}}(L_3) = m_B$.)

Vector Current

To obtain, say, the form factor for $B \to \pi l \nu$, we need to repeat the same procedure for the vector current $\mathcal{V}^\mu = \bar{u} i \gamma^\mu Q$.

In the Symanzik effective theory

$$V^\mu_{\rm lgt} \doteq Z_V^{-1}(m_u a_0, m_Q a_0)\mathcal{V}^\mu - a K_V(m_u a_0, m_Q a_0)\partial_\nu \bar{u}\sigma^{\mu\nu}Q + \mathrm{O}(m_b^2 a_0^2, m_b \Lambda a_0^2, m_b a_0^2/L_0, \ldots) \tag{22}$$

where $K_V = 0$ and $Z_V = 1$ if lgt has GW chiral symmetry; otherwise K_V can be adjusted and Z_V computed in a way commensurate with the error on the RHS.

Since Q is a heavy-quark, we can apply HQET to $V^\mu_{\rm lgt}$ directly, and also indirectly via $\mathcal{V}^\mu$ and

$$\bar{u}\sigma^{ij}Q \doteq C_{T+}\bar{u}\sigma^{ij}h + \mathrm{O}(1/m_b), \quad \bar{u}\alpha^j Q \doteq C_{T-}\bar{u}\alpha^j h + \mathrm{O}(1/m_b). \tag{23}$$

The HQET description of the currents

$$V^4_{\mathrm{lgt}} \doteq C^{\mathrm{lgt}}_{V_\parallel} i\bar{u}h - B^{\mathrm{lgt}}_{V_1} Q^4_{V_1} - B^{\mathrm{lgt}}_{V_4} Q^4_{V_4} \tag{24}$$

$$V^i_{\mathrm{lgt}} \doteq C^{\mathrm{lgt}}_{V_\perp} \bar{u} i\gamma^i h - \sum_{p=2,3,5,6} B^{\mathrm{lgt}}_{V_p} Q^i_{V_p} \tag{25}$$

where

$$Q^4_{V_1} = -i\bar{u}\boldsymbol{\gamma}\cdot\boldsymbol{D}h \tag{26}$$

$$Q^4_{V_4} = +i\bar{u}\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{D}}h \tag{27}$$

$$Q^i_{V_2} = \bar{u} i\gamma^i \boldsymbol{\gamma}\cdot\boldsymbol{D}h \tag{28}$$

$$Q^i_{V_3} = i\bar{u} iD^i h \tag{29}$$

$$Q^i_{V_5} = \bar{u}\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{D}} i\gamma^i h \tag{30}$$

$$Q^i_{V_6} = \bar{u} i\overleftarrow{D}^i h \tag{31}$$

Of course, the HQ description applies to QCD, lgt, and LGT.

Notes in passing:

- In the Fermilab method, currents are improved not as in the *ALPHA*-Symanzik improvement, but by discretizing the subleading operators $Q^{\mu} \rightarrow Q^{\mu}$.

- This list (taken from the continuum HQET literature) is longer than the list of improvement operators in lattice HQET, which enjoys exact HQS.

Comparing the direct and indirect HQ descriptions of lgt tell us

$$Z^{\text{lgt}}_{V_\parallel} := C^{\text{QCD}}_{V_\parallel}/C^{\text{lgt}}_{V_\parallel} = Z^{\text{lgt}}_{V}, \tag{32}$$

$$1/Z^{\text{lgt}}_{V_\perp} := C^{\text{lgt}}_{V_\perp}/C^{\text{QCD}}_{V_\perp} = 1/Z^{\text{lgt}}_{V} + (m_q + m_b)aK^{\text{lgt}}_{V}C^{\text{QCD}}_{T_-}/C^{\text{QCD}}_{V_\perp}, \tag{33}$$

$$Z^{\text{lgt}}_{V_\parallel}B^{\text{lgt}}_{V1} = B^{\text{QCD}}_{V1} + aZ^{\text{lgt}}_{V}K^{\text{lgt}}_{V}C^{\text{QCD}}_{T_-}, \tag{34}$$

$$Z^{\text{lgt}}_{V_\parallel}B^{\text{lgt}}_{Vi} = B^{\text{QCD}}_{Vi} + aZ^{\text{lgt}}_{V}K^{\text{lgt}}_{V}C^{\text{QCD}}_{T_+}, \quad i = 2, 6, \tag{35}$$

$$Z^{\text{lgt}}_{V_\parallel}B^{\text{lgt}}_{V3} = B^{\text{QCD}}_{V3} - aZ^{\text{lgt}}_{V}K^{\text{lgt}}_{V}C^{\text{QCD}}_{T_+}, \tag{36}$$

$$Z^{\text{lgt}}_{V_\parallel}B^{\text{lgt}}_{V4} = B^{\text{QCD}}_{V4} - aZ^{\text{lgt}}_{V}K^{\text{lgt}}_{V}C^{\text{QCD}}_{T_-}, \tag{37}$$

$$Z^{\text{lgt}}_{V_\parallel}B^{\text{lgt}}_{V5} = B^{\text{QCD}}_{V5} - aZ^{\text{lgt}}_{V}K^{\text{lgt}}_{V}(C^{\text{QCD}}_{T_+} - C^{\text{QCD}}_{T_-}), \tag{38}$$

all $+\mathrm{O}(a_0^2, a_0/m_b, 1/m_b^2)$. Once again the correspondence is simpler if $K^{\text{lgt}}_{V} = 0$.

We (Harada *et al.*) used these relations to extract the one-loop improvement couplings $b^{[1]}_J$ and $c^{[1]}_J$ from our mass-dependent calculations of $Z^{[1]}_J$.

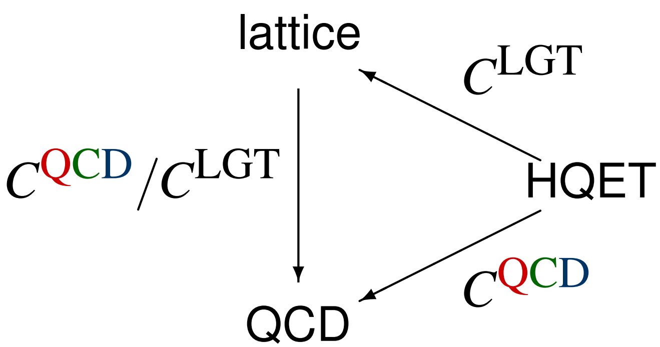
Then the matching proceeds as before, LGT(a_1) [via LGT(a_0)] to lgt(a_0) at L_0; LGT(a_2) to LGT(a_1) at L_1; *etc*.

As before, one needs as many physical quantities as bare parameters in the improved LGT currents.

They may be artificial to the finite volume, such as those provided by a Schrödinger functional.

Since the bare quark mass (and lattice spacing) have already been determined, the large-volume matrix elements of form factors & decay constants are physical outputs of the calculations.

HQET Matching of LGT to QCD



Perturbative Matching

Perturbative Alternative

Non-perturbative matching is a lot of work, and the whole procedure has to be repeated whenever the LGT action is changed.

It also, in practice, pollutes physical results with higher-dimension matrix elements that know about the small volumes.

It also seems impractical if one wants accuracy of order $1/m^2$ in heavy-light hadrons and of order v^6 in quarkonium.

An alternative is to calculate the short-distance behavior in perturbative QCD.

To achieve the desired accuracy, one needs the leading terms (mass and Z factors) to two loops, the subleading terms (c_B and the B coefficients) to one loop, and it probably suffices to have tree-level matching for the sub-subleading terms.

This is also not easy (and not yet achieved).

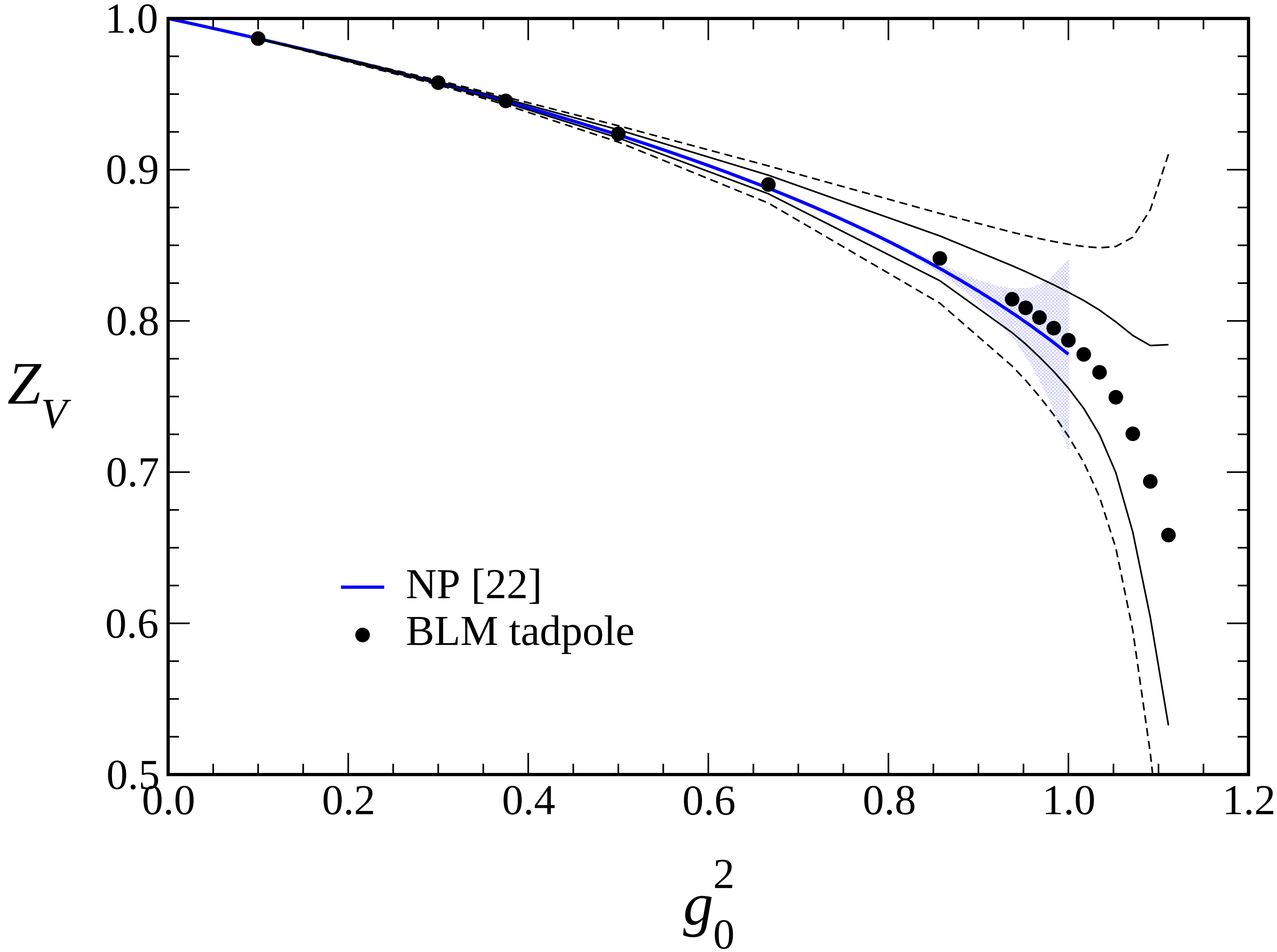
Some two-loop calculations (*e.g.*, mass renormalization for staggered quarks) have been done, using automated perturbation theory.

To assess the prospects, the next few slides compare perturbative and non-perturbative calculations of the standard Wilson O(a) improvement coefficients.

To obtain a one-loop result, one must choose an expansion parameter α_s. Here we use the method of Lepage & Mackenzie, based on the method proposed by Brodsky and them (BLM) for continuum perturbative QCD.

The idea is to avoid potentially large logarithms by choosing a specific scale, q^*, which is the typical gluon momentum in the Feynman diagram. L&M use $\alpha_V(q^*)$ but q^* is scheme-dependent too.

Many comparisons of PT with non-perturbative results cite L&M but fail to use $\alpha_V(q^*)$, which is disingenuous.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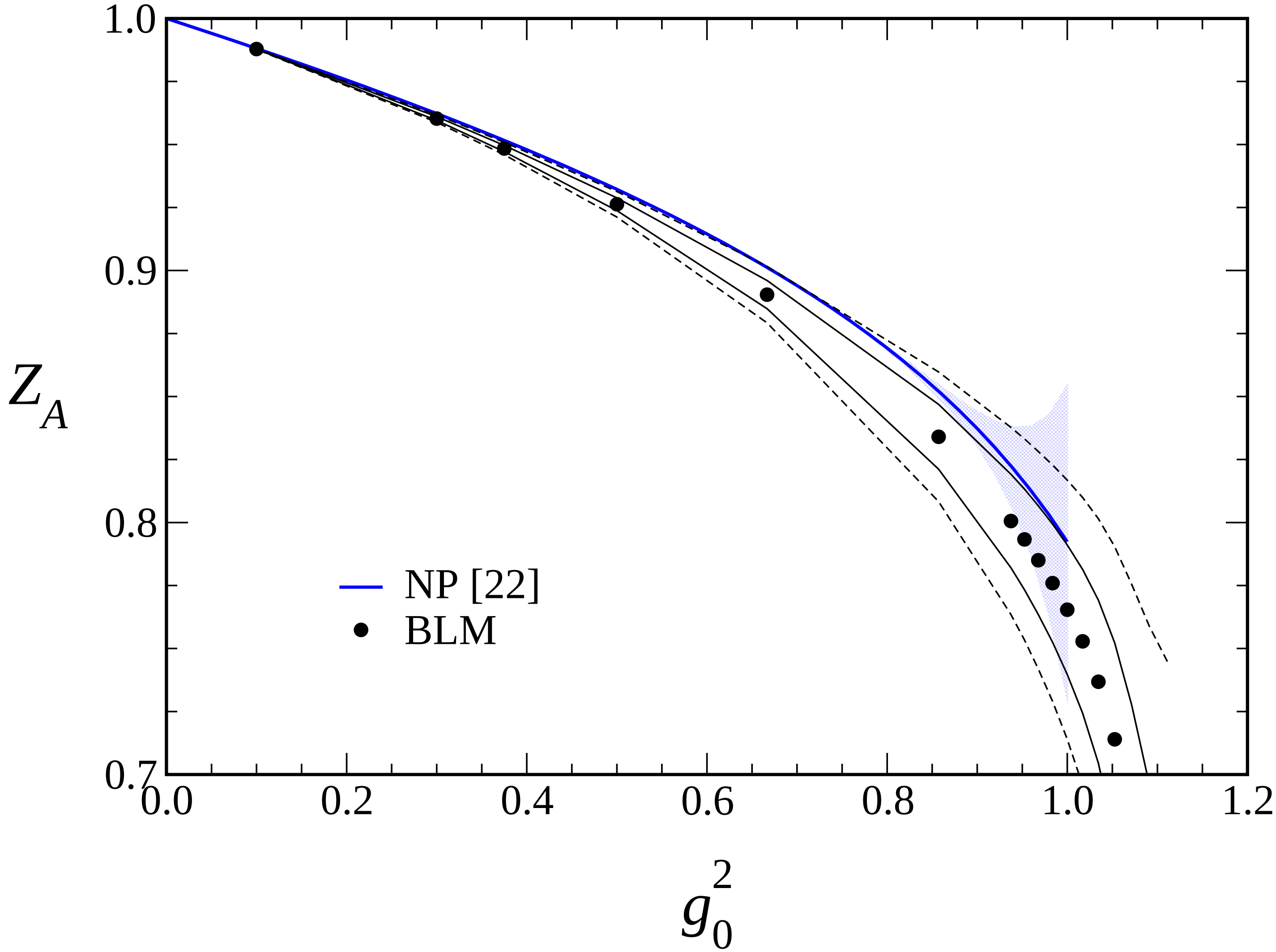
0.9
0.8
0.7
0.6
0.5
Z_V
0.0
0.2
0.4
0.6
0.8
1.0
1.2
g_0^2
NP [22]
BLM tadpole


1.0
0.9
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0.7
Z_A
0.0
0.2
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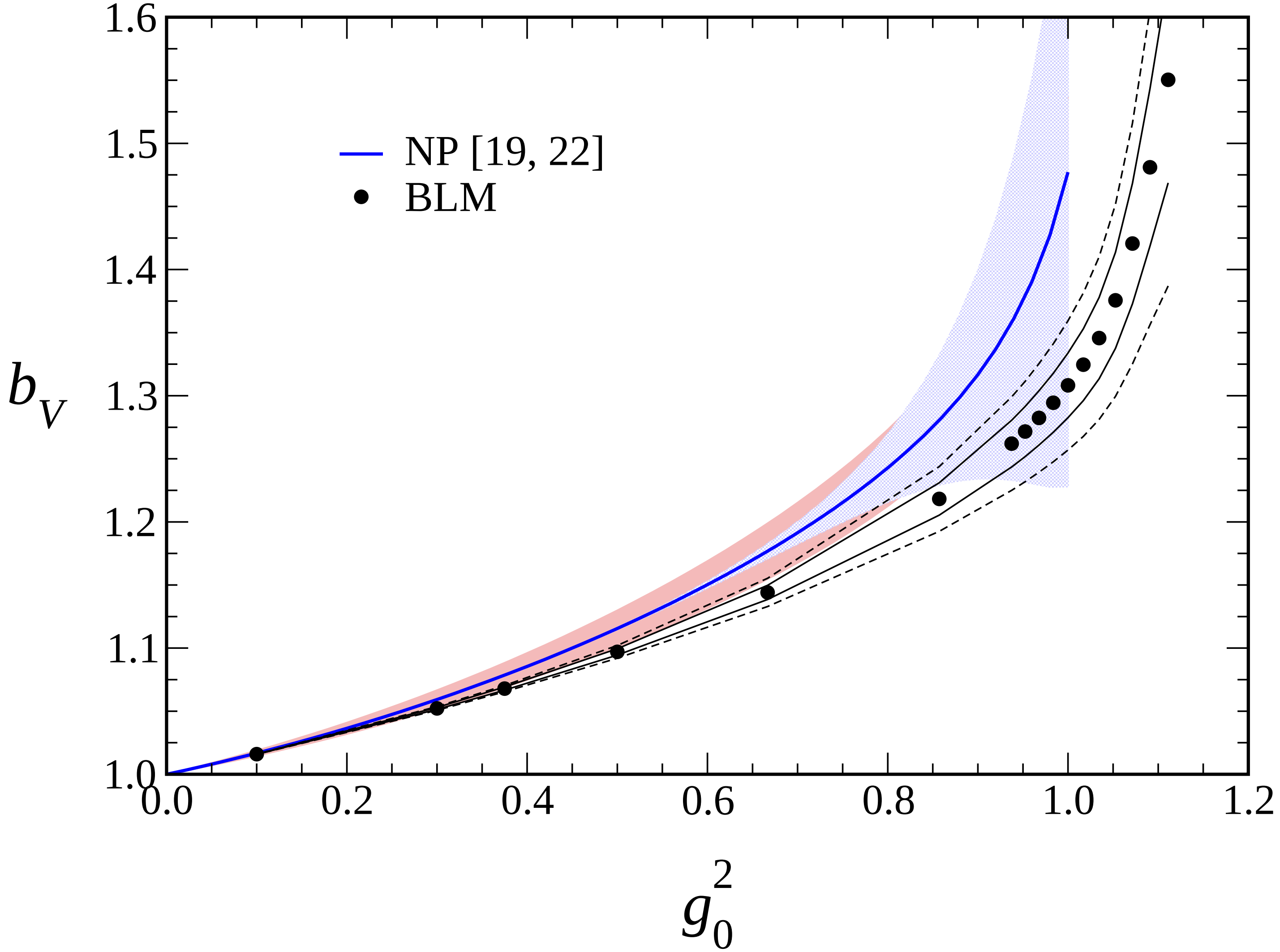
0.6
0.8
1.0
1.2
g_0^2
NP [22]
BLM


NP [19, 22]
BLM
b_V
g_0^2
1.6
1.5
1.4
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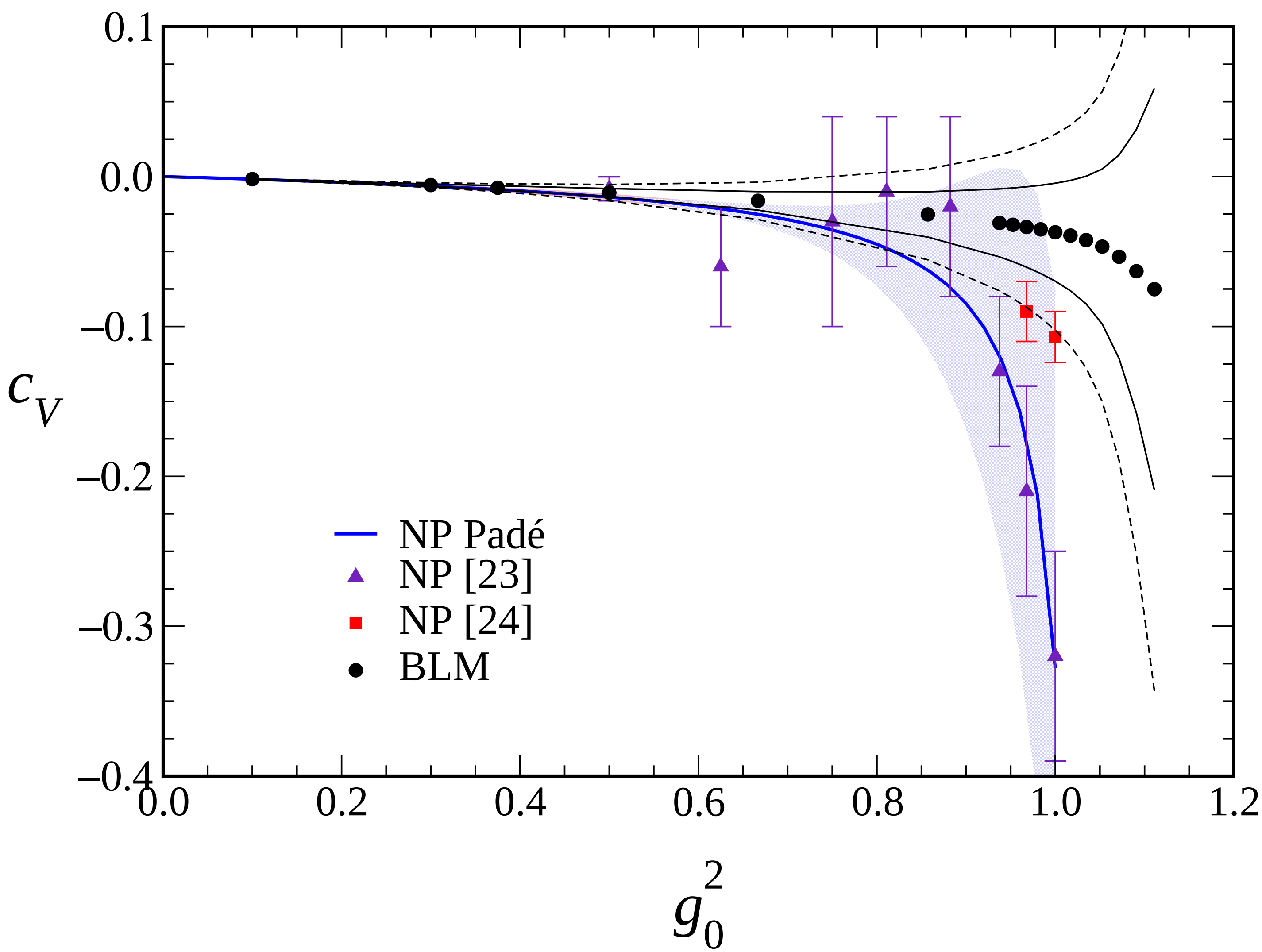
1.2



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–0.1
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0.2
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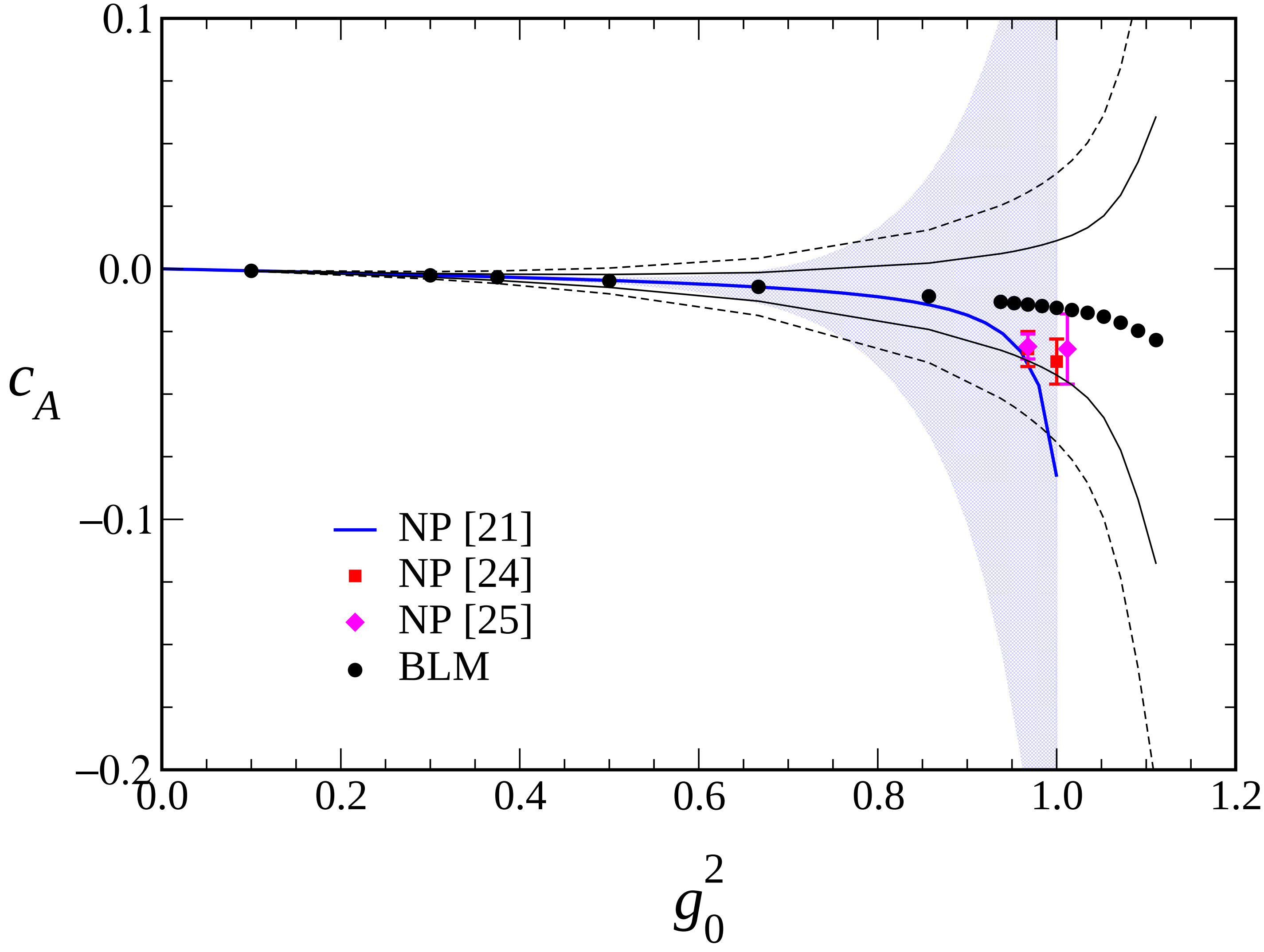
1.2
c_V
g_0^2
NP Padé
NP [23]
NP [24]
BLM



0.1
0.0
−0.1
−0.2
c_A
0.0
0.2
0.4
0.6
0.8
1.0
1.2
g_0^2
NP [21]
NP [24]
NP [25]
BLM

Perspective

For these simple quantities (with no anomalous dimension), BLM perturbation theory works very well.

The differences between PT and NP seem adequately explained by two-loop corrections to PT, and power-law pollution in finite-volume NP.

Radiative and Power-Law Effects

Another angle on perturbative and non-perturbative matching comes from the horridly named "renormalon shadows." Sounds mysterious.

Suppose one can measure experimentally (or compute non-perturbatively) $\mathcal{P}$ and $\mathcal{Q}$. Both are described in an effective field theory including a power correction:

$$\mathcal{P}(Q) = C_{\mathcal{P}}(Q/\mu)\langle O_1 \rangle + B_{\mathcal{P}}(Q/\mu)\langle O_2 \rangle / Q, \tag{39}$$

$$\mathcal{Q}(Q) = C_{\mathcal{Q}}(Q/\mu)\langle O_1 \rangle + B_{\mathcal{Q}}(Q/\mu)\langle O_2 \rangle / Q, \tag{40}$$

where the Cs and Bs are short-distance coefficients. Q is a hard physical scale, and μ is the separation scale.

An example comes from calculations in HQET, where $\mathcal{P}$ is a static calculation, so $B_{\mathcal{P}} = 0$, and $\mathcal{Q}$ is the same but with a kinetic insertion added. $C_{\mathcal{Q}} \neq C_{\mathcal{P}}$ because the insertion generates new UV divergences.

Now one would like to determine

$$\mathcal{R}(Q) = C_{\mathcal{R}}(Q/\mu)\langle O_1 \rangle + B_{\mathcal{R}}(Q/\mu)\langle O_2 \rangle / Q, \tag{41}$$

given $\mathcal{P}$, $\mathcal{Q}$, & approximate short-distance coefficients. What is the uncertainty in $\mathcal{R}$?

In our example, $\mathcal{R}$ is the continuum QCD quantity of interest: $\mathcal{P}$ is static on the lattice, $\mathcal{Q}$ is static+insertion on the lattice, $\mathcal{R}$ is QCD.

Using simple algebra you can eliminate the effective-theory $\langle O_i \rangle$:

$$\mathcal{R} = \frac{C_{\mathcal{R}}}{2}\left[\frac{\mathcal{P}}{C_{\mathcal{P}}} + \frac{\mathcal{Q}}{C_{\mathcal{Q}}}\right] + \frac{\bar{B}_{\mathcal{R}}[\mathcal{P}/C_{\mathcal{P}} - \mathcal{Q}/C_{\mathcal{Q}}]}{B_{\mathcal{P}}/C_{\mathcal{P}} - B_{\mathcal{Q}}/C_{\mathcal{Q}}}, \tag{42}$$

where

$$\bar{B}_{\mathcal{R}} = B_{\mathcal{R}} - \frac{C_{\mathcal{R}}}{2}\left[\frac{B_{\mathcal{P}}}{C_{\mathcal{P}}} + \frac{B_{\mathcal{Q}}}{C_{\mathcal{Q}}}\right]. \tag{43}$$

The second term in Eq. (42) is the power correction: the combination $\mathcal{P}/C_{\mathcal{P}} - \mathcal{Q}/C_{\mathcal{Q}}$ is formally of order Λ_{QCD}/Q.

Suppose that the Cs have been calculated through l loops and the Bs through k loops.

Then the customary assumption is that the uncertainty in the (leading-twist) first bracket is $O(\alpha_s^{l+1})$ and in the power correction $O(\alpha_s^{k+1}\Lambda_{\mathrm{QCD}}/Q)$. Moreover, it seems worthwhile to include the power correction as soon as $\Lambda_{\mathrm{QCD}}/Q \gtrsim \alpha_s^{l+1}$.

But the leading-twist parts of $\mathcal{P}/C_{\mathcal{P}} - \mathcal{Q}/C_{\mathcal{Q}}$ do not cancel perfectly when $C_{\mathcal{P}}$ and $C_{\mathcal{Q}}$ are calculated perturbatively. There is a shadow of order α_s^{l+1}, commensurate with the uncertainty in the leading-twist term.

Unless $\alpha_s^{l+1} \ll \Lambda_{\mathrm{QCD}}/Q$, the shadow obscures the desired power correction, and the second term in Eq. (42) would not improve the accuracy of $\mathcal{R}$.

This certainly applies to lattice HQET, and motivates non-perturbative matching in that method.

Is a shadow cast on the Fermilab method, or NRQCD?

The Fermilab method doesn't start with HQET, but HQET can be used to examine its contents, and compare it to QCD:

$$\Phi^{\mathrm{LGT}} = C^{\mathrm{LGT}} \left[\Phi_\infty + B^{\mathrm{LGT}} \Phi'_\infty / m_Q \right], \tag{44}$$

$$\Phi^{\mathrm{QCD}} = C^{\mathrm{QCD}} \left[\Phi_\infty + B^{\mathrm{QCD}} \Phi'_\infty / m_Q \right]. \tag{45}$$

We want to know the uncertainty, when $Z = C^{\mathrm{QCD}}/C^{\mathrm{LGT}}$ is computed through l loops, and $\delta B = B^{\mathrm{LGT}} - B^{\mathrm{QCD}}$ is matched through k loops.

One finds the relative error

$$1 - \frac{Z^{(l)}\Phi^{\mathrm{LGT}}}{\Phi^{\mathrm{QCD}}} = \frac{\delta Z^{(l+1)}}{Z} - \frac{Z^{(l)}\delta B^{(k+1)}\Phi'_\infty}{Z\Phi_\infty m_Q}, \tag{46}$$

where $Z^{(l)}$ is the l-loop approximation to Z, and $\delta Z^{(l+1)} = Z - Z^{(l)}$.

The first term has truncation error α_s^{l+1}, the second $\alpha_s^{k+1}\Lambda_{\mathrm{QCD}}/m_Q$.

There is no shadow contribution, because the power correction is not obtained by explicit subtraction. Instead, it is present all along, and LGT is adjusted to hit its target, QCD.

One can always worry that the first uncalculated coefficient in $Z^{(l+1)}$ or $\delta B^{(l+1)}$ is large. But those are garden-variety errors from truncating PT. They have nothing to do with renormalons or shadows.

The shadows imply that non-perturbative matching is needed before introducing power corrections in lattice HQET.

On the other hand, since the shadow problem does not apply to the Fermilab method, one has a transparent pattern of PT truncation uncertainties from radiative and power-law corrections.

Estimating Errors

General Framework

The HQ theory of cutoff effects provides a framework for estimating the discretization uncertainties in a heavy-quark calculation.

Discretization effects are isolated in the mismatch of short-distance coefficients:

$$\delta\mathcal{C}_i = \mathcal{C}_i^{\mathrm{LGT}}(\{c_j\}) - \mathcal{C}_i^{\mathrm{QCD}}, \tag{47}$$

and errors from truncation or pollution in

$$Z_J = C_J^{\mathrm{QCD}}/C_J^{\mathrm{LGT}}. \tag{48}$$

In the remainder of this lecture we will use Ansätze for $\delta\mathcal{C}$ of the kinetic and chromomagnetic interactions, and explicit tree-level calculations of $\delta\mathcal{C}$ several further-suppressed contributions.

Extrapolation Method

The extrapolation method identifies the rest mass m_1 with m_Q, which means that the kinetic and chromomagnetic contributions are too small.

The resulting error is

$$\delta \mathcal{C}_2^{(\text{extrapolation})} \langle O_2 \rangle = \left| \frac{p}{2m_2} - \frac{p}{2m_1} \right|_{m_Q = m_1}, \tag{49}$$

where p is a soft scale, taken to be Λ_{QCD} below.

There are further errors: the Z factors have a leading error of order $(m_Q a)^2$, possible suppressed by α_s.

I do not know how to propagate this error through the $1/m$ extrapolation, because the proponents haven't explained how they would like it done. So I'll omit it (with caution).

Lattice NRQCD

In lattice NRQCD the quark mass is adjusted non-perturbatively to the kinetic energy: $m_Q = m_2$.

The first error is in spin-dependent effects. With l-loop matching

$$\delta C_{\mathcal{B}}^{(\mathrm{NRQCD})} \langle O_{\mathcal{B}} \rangle \sim \alpha_s^{l+1} \left(1 + \frac{1}{4m_Q^2 a^2} \right) \frac{p}{2m_Q} \tag{50}$$

has the right asymptotic behavior in $m_Q a$.

The power-law divergence comes from the short-distance part of higher-dimension spin-dependent interactions.

Perturbatively matched lattice HQET would look the same.

Non-perturbatively matched lattice HQET would look the same would have p^2/m_Q^2, where p could be influenced by the small volumes.

Fermilab Method

The Fermilab method chooses $m_Q = m_2$ and, so, eliminates the error (49).

The leading error is NRQCD-like

$$\delta \mathcal{C}_{\mathcal{B}}^{(\text{Fermilab})} \langle O_{\mathcal{B}} \rangle \sim \left. \frac{\alpha_s^{l+1} pa}{2(1+m_0 a)} \right|_{m_Q = m_2}, \tag{51}$$

again with l-loop matching of the chromomagnetic energy.

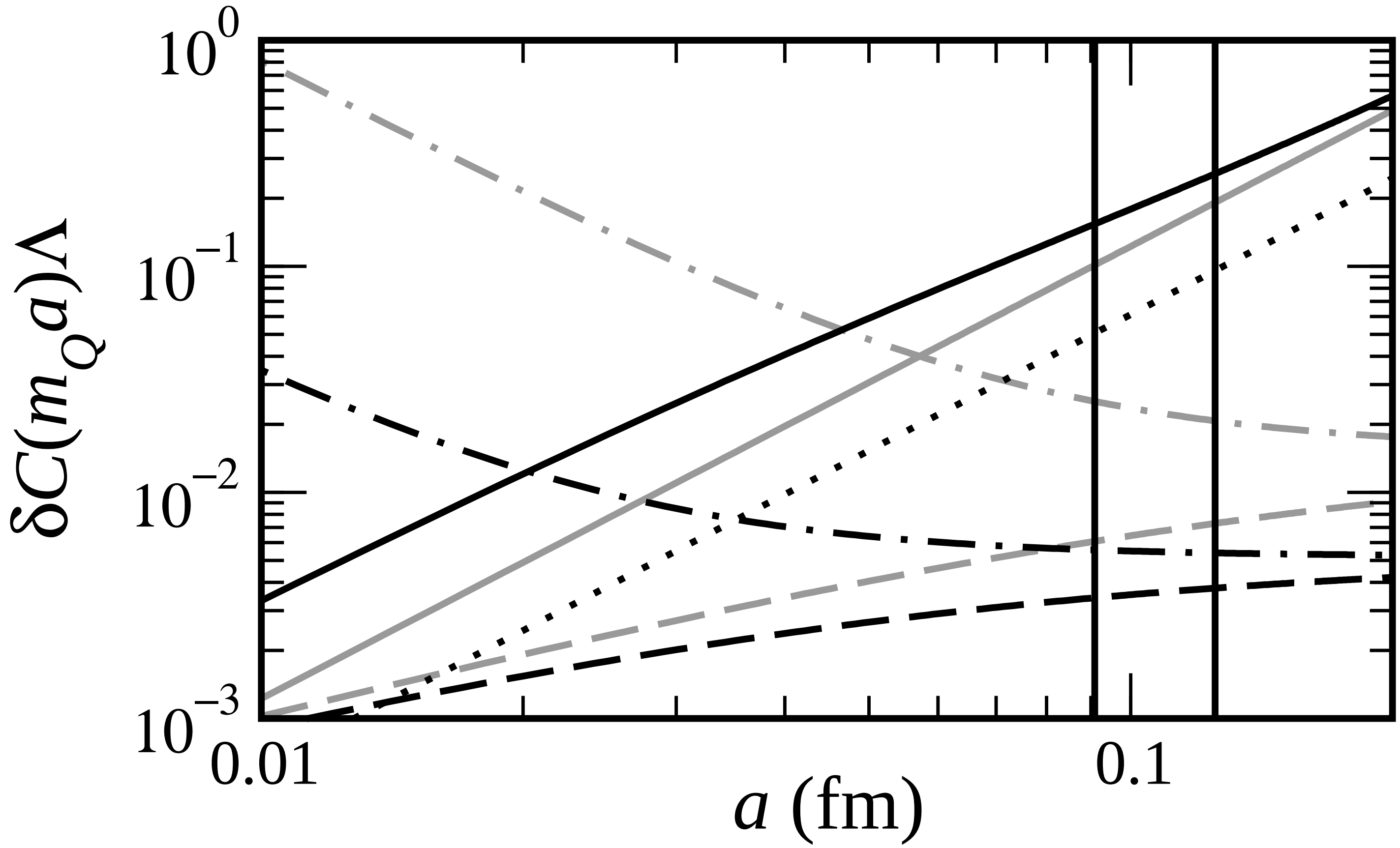
The existence of a continuum limit controls the $a \to 0$ limit.

Parameter Estimates

We now take

$$
\begin{aligned}
p &= 700 \text{ MeV} \gtrsim \Lambda_{\text{QCD}} \\
m_c &= 1400 \text{ MeV} \\
m_b &= 4200 \text{ MeV} \\
\alpha_s &= 0.25 \\
l &= 1
\end{aligned}
$$

The choices are meant to be conservative, but the important thing is that they are the same for all methods.

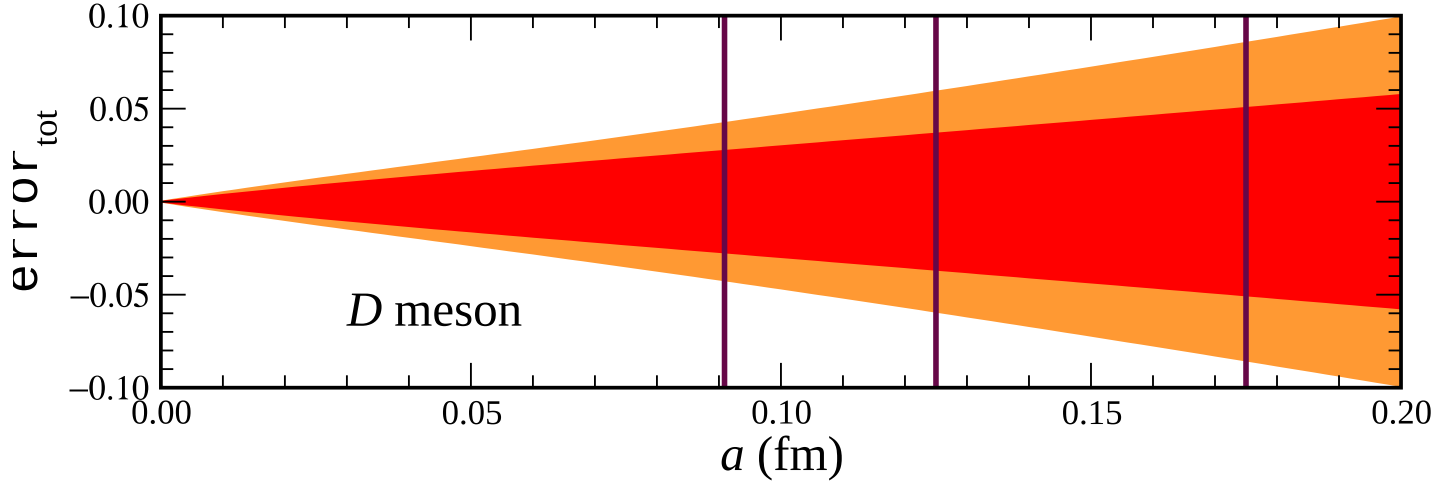


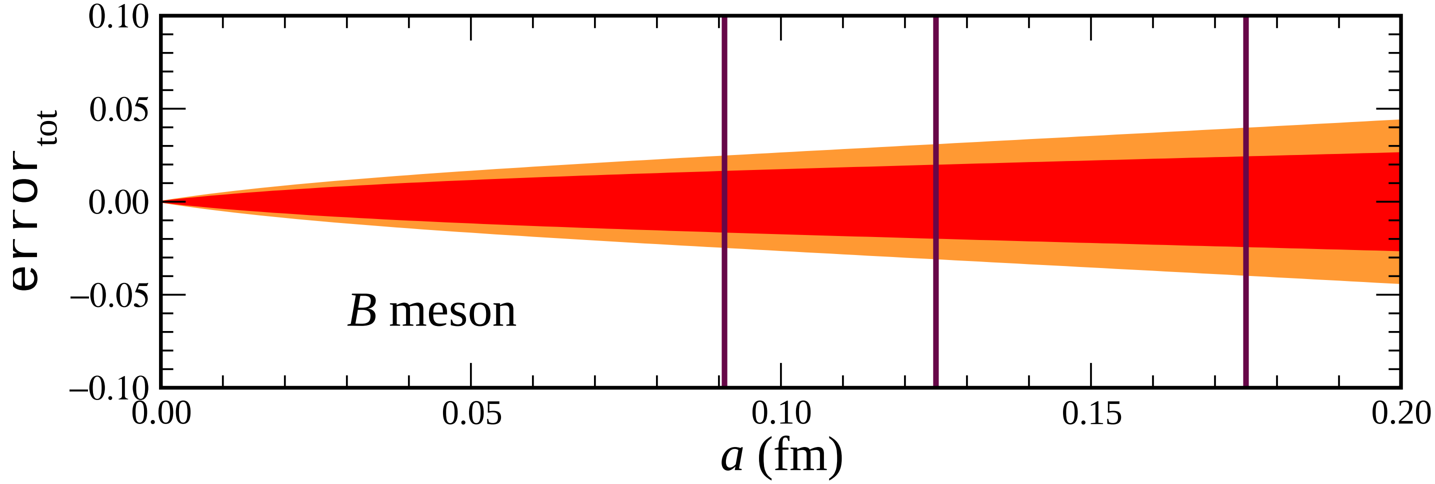
Comparison of errors: Vertical lines are 0.091 fm and 0.125 fm. Black is $Q = b$; gray is $Q = c$. Solid, dashed-dotted, and dashed lines are extrapolation, NRQCD, Fermilab. Dotted lines are light quarks and gluons, $\frac{1}{2}(pa)^2$.

An interesting outcome of this analysis, is that the Fermilab method has smaller discretization errors for b quarks than for c quarks.

The is, essentially, because the the b is heavier, so the power corrections start out small, so errors in their normalization can lead only to small errors.

This persists when looking at the higher-dimension mismatches of Darwin, spin-orbit, and $(\boldsymbol{D}^2)^2$, as well as analogous corrections to the heavy-light currents.





Total heavy-quark discretization error for f_D and f_B from leading and subleading mismatches. $\Lambda = 500$ MeV, $\Lambda = 700$ MeV.

Outlook

Outlook

Lattice QCD is not an easy business, and heavy quarks do not make it easier.

For this reason the particle-physics community will not have full confidence in our calculations unless they are carried out with complementary methods, with different sources of uncertainty.

To make any kind of comparison, every flavor-physics lattice calculation needs a full error budget (or a hint that experimenters should ignore it).

Flavor physics will continue to be relevant in the coming decade with LHC-b and, also, the usefulness of flavor physics in constraining models of (we hope) the new phenomena that will be unraveled at ATLAS and CMS.

Acknowledgments

Karl, Kostas, and Steve for entrusting me with this important topic.

Steve and the INT staff for local organization.

David Kaplan for the use of his bicycle.

You students for listening attentively and asking good questions.

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You can find Lepage&Mackenzie from SPIRES with the search string “f eprint hep-lat and topcite 500+”