HEAVY QUARKS ON THE LATTICE 3:

Heavy-Quark Techniques

Recapitulation of Lecture 2

Aspirations for Flavor Physics

Lattice HQET

Lattice NRQCD

Methods with Wilson Fermions

Andreas S. Kronfeld

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Recapitulation of Lecture 2

EFT for LGT

My second lecture (and Sint's lectures) discussed how gauge theories can be described by an effective Lagrangian:

$$\mathcal{L}_{\rm LGT} \doteq \mathcal{L}_{\rm Sym}, \qquad \text{read } \doteq \text{ as "has the same physics as".} \tag{1}$$

The left-hand side is in your computer; it is built out of lattice gauge fields $U_\mu(x)$ and lattice fermions fields $\psi(x)$. The right-hand side is an analysis tool (*i.e.*, in your mind); it is built out of continuum gauge fields $A_\mu(x)$ and continuum fermions fields $q(x)$.

Assuming LGT has a good heavy-quark limit, I argued that

$$\mathcal{L}_{\rm Sym} = \cdots - \bar{q}\left(\gamma_4 D_4 + \sqrt{\frac{m_1}{m_2}}\boldsymbol{\gamma}\cdot\boldsymbol{D} + m_1\right) q + \mathcal{L}'_I, \tag{2}$$

which is not the hoped-for "QCD + small corrections," owing the deviation from unity of the factor $\sqrt{m_1/m_2}$. By design, $\mathcal{L}'_I$ is small as long as $\Lambda_{\rm QCD} a \lesssim \boldsymbol{p} a \ll 1$.

We are free, however, to abandon $\mathcal{L}_{\rm Sym}$ in favor of $\mathcal{L}_{\rm HQ}$, the effective Lagrangian for heavy-quarks:

$$\mathcal{L}_{\rm LGT} \doteq \mathcal{L}_{\rm Sym}^{\rm muck} + i\bar{h}_v^{(+)} v\cdot D h_v^{(+)} - m_1 \bar{h}_v^{(+)} h_v^{(+)} + \frac{\bar{h}_v^{(+)} D_\perp^2 h_v^{(+)}}{2m_2} + \cdots, \qquad (3)$$

where the $\cdots$ are of the form $C_{s,i}^{\rm LGT} O_{s,i}$.

The logic and structure parallels precisely the heavy-quark description of QCD:

$$\mathcal{L}_{\rm QCD} \doteq \mathcal{L}_{\rm cont}^{\rm muck} + i\bar{h}_v^{(+)} v\cdot D h_v^{(+)} - m \bar{h}_v^{(+)} h_v^{(+)} + \frac{\bar{h}_v^{(+)} D_\perp^2 h_v^{(+)}}{2m} + \cdots. \qquad (4)$$

In the $\cdots$, the operators are the same, but the coefficients are different.

Framework can be established order-by-order in perturbative QCD, and is thought to hold non-perturbatively as well.

Comparing the two, we see that in quarkonium (heavy-light) it is essential (efficient) to identify the kinetic mass m_2 with the physical quark mass m.

External electroweak operators (currents, 4-quark operators for mixing) can be built up the same way (for example):

$$\bar{\psi}_q i\gamma^\mu \psi_b \doteq C_{V_\parallel}^{\mathrm{LGT}} v^\mu \bar{q} h_v^{(+)} + C_{V_\perp}^{\mathrm{LGT}} \bar{q} i\gamma_\perp^\mu h_v^{(+)} - \sum_{s,i} B_{s,i}^{\mathrm{LGT}}(\mu) V_{s,i}^\mu(\mu) \tag{5}$$

$$\bar{q} i\gamma^\mu b \doteq C_{V_\parallel}^{\mathrm{QCD}} v^\mu \bar{q} h_v^{(+)} + C_{V_\perp}^{\mathrm{QCD}} \bar{q} i\gamma_\perp^\mu h_v^{(+)} - \sum_{s,i} B_{s,i}^{\mathrm{QCD}}(\mu) V_{s,i}^\mu(\mu). \tag{6}$$

with lattice fields ψ, $\bar{\psi}$ and continuum fields b, $h_v^{(+)}$, $\bar{q}$.

In the end, discretization effects for heavy quarks are captured by a non–dynamical mass shift $m_2 - m_1$, multiplicative factors $Z_J = C_J^{\mathrm{QCD}}/C_J^{\mathrm{LGT}}$, and mismatches

$$\delta C_{s,i} = C_{s,i}^{\mathrm{LGT}} - C_{s,i}^{\mathrm{QCD}}, \tag{7}$$

$$\delta B_{s,i} = Z_V B_{s,i}^{\mathrm{LGT}} - B_{s,i}^{\mathrm{QCD}}. \tag{8}$$

Our objective tomorrow will be to assess how large these (bounded) errors are.

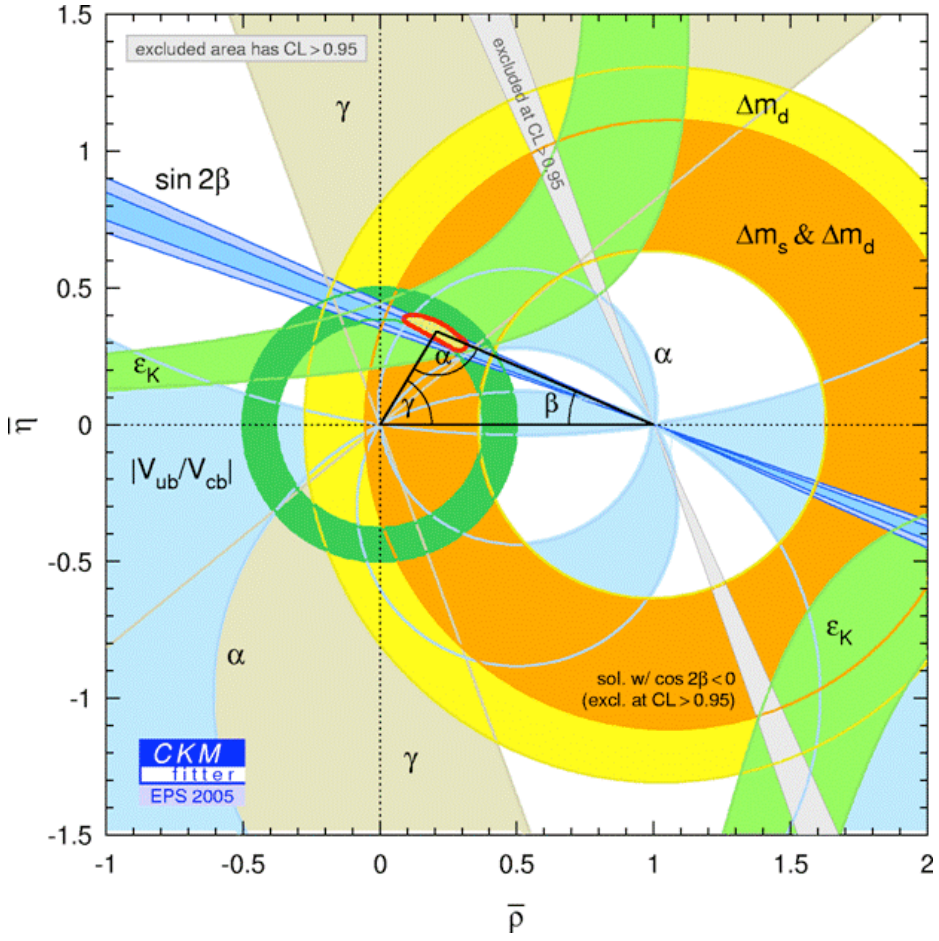
Aspirations for Phenomenology

CKM Unitarity Triangle

Shoji Hashimoto told you about the CKM unitarity triangle. Taxpayers have spent $\sim \$10^9$ on it.

Here we show the plot when there was only an lower bound on the frequency Δm_s of $B_s^0 \leftrightarrow \bar{B}_s^0$ oscillations.

Lattice calculations enter in several places: the green, yellow, and orange annuli, and the lime hyperbola.



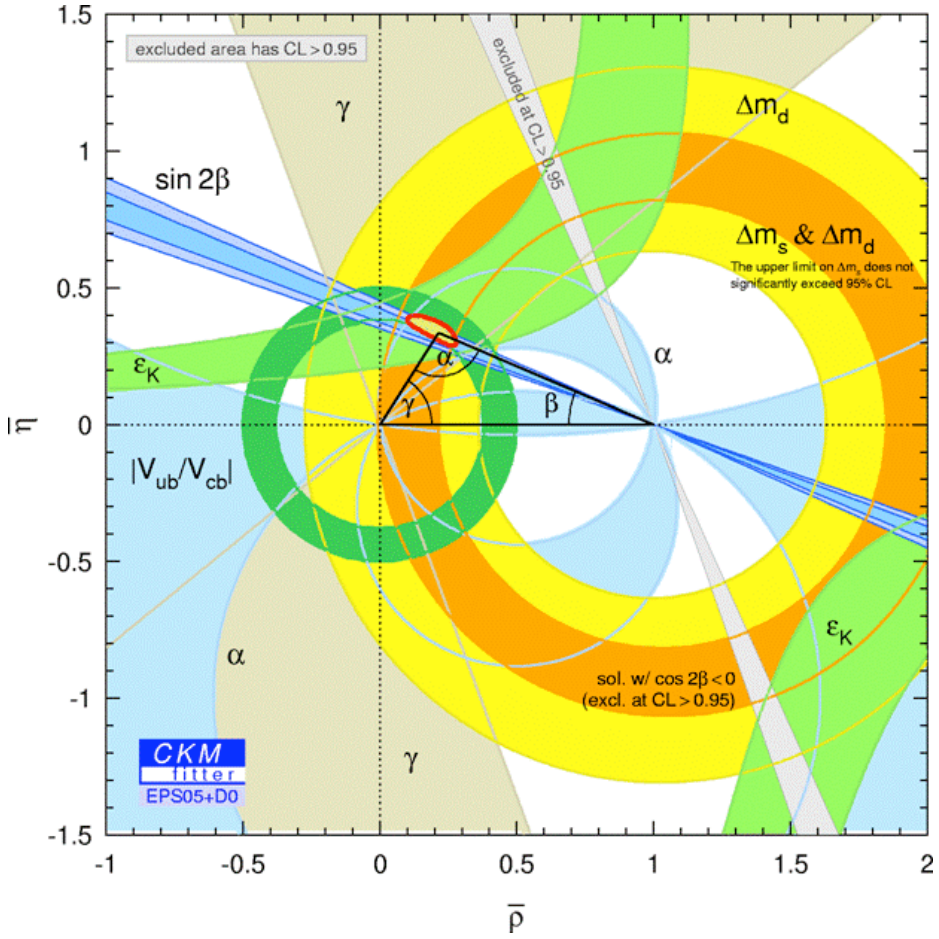
CKM Unitarity Triangle

Shoji Hashimoto told you about the CKM unitarity triangle. Taxpayers have spent $\sim \$10^9$ on it.

Here we show the plot after there was a two-sided bound on the frequency Δm_s of $B_s^0 \leftrightarrow \bar{B}_s^0$ oscillations.

Lattice calculations enter in several places: the green, yellow, and orange annuli, and the lime hyperbola.

DØ's two-sided bound yields a 10% experimental constraint.



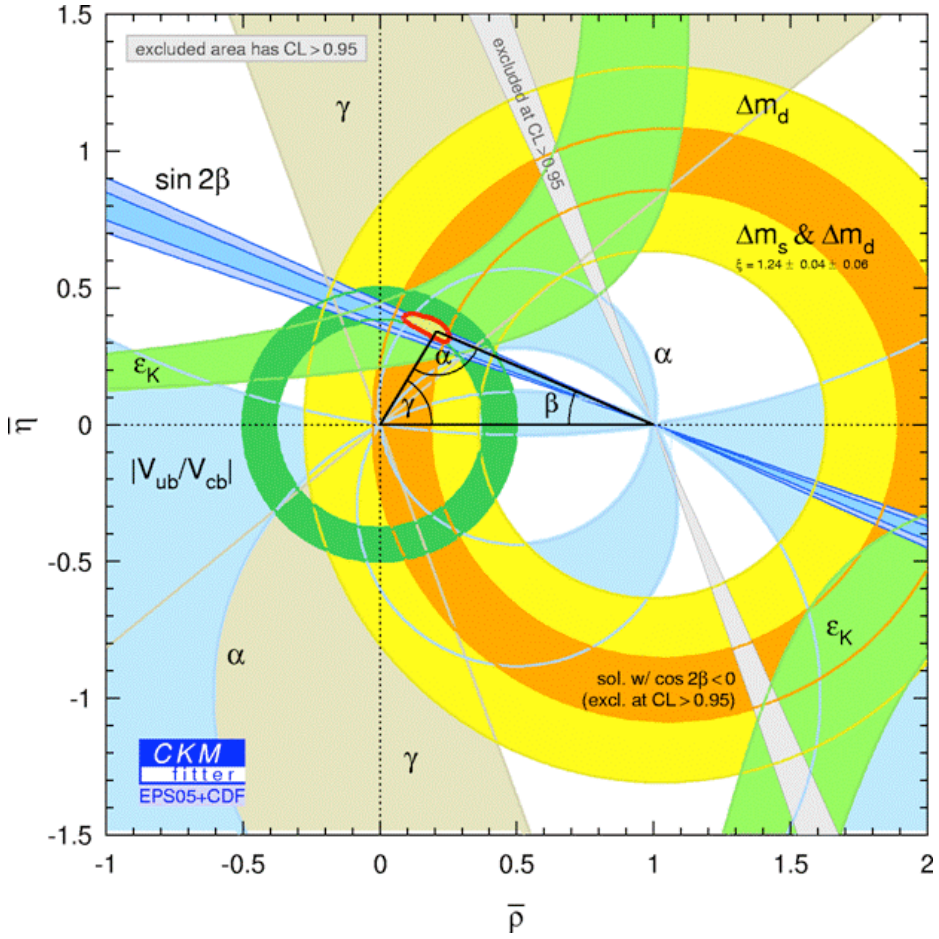
CKM Unitarity Triangle

Shoji Hashimoto told you about the CKM unitarity triangle. Taxpayers have spent $\sim \$10^9$ on it.

Here we show the plot after there was a real measurement frequency Δm_s of $B_s^0 \leftrightarrow \bar{B}_s^0$ oscillations.

Lattice calculations enter in several places: the green, yellow, and orange annuli, and the lime hyperbola.

CDF's 1% measurement of Δm_s yields hardly any change!



Goals for Flavor Physics

The real goal of flavor physics is to see if new phenomena contribute to B, D, and K mixing and decay.

One way to see it is to determine the CKM matrix from processes that are lowest-order in the weak interactions (trees), and then compare to processes that are higher order in the weak interaction. The tree triangle vs. the loop triangle.

Interesting (and achievable) goals are 1% for V_{cb}, 1–2% for $\Delta m_s/\Delta m_d$, and 2–4% for V_{ub} and Δm_q, and 4–5% for B_K.

Every lattice paper on this subject must have a full error budget: compute the easy ones (*e.g.*, statistical) and estimate the others.

Yes: estimate the error from $n_f = 2$; estimate the error from the lattice spacing you use, not the one you plan to use next year.

Lattice HQET

Static Approximation

But let us turn to methods for heavy-quarks, and the seminal conference in Seillac, Lattice 1987, when Karl, Steve, and I all had more hair (length, area, volume).

Eichten introduced the static approximation for heavy quarks

$$S_{\text{static}} = a^3 \sum_x \bar{\psi}(x) \frac{1+\gamma_4}{2} \left[(1+m_0 a)\psi(x) - U_4^\dagger(x - a\hat{e}_4)\psi(x - a\hat{e}_4) \right]. \qquad (9)$$

Eichten omitted the bare mass term $m_0 a$ (but we shall see below why to keep it).

The original derivation [*Nucl. Phys.* B *Proc. Suppl.* **4**, 170 (1988)] focussed on the physics of heavy-light systems, leading to a static continuum theory, discretized to yield Eq. (9).

Historical note: this paper inspired much more famous work by Isgur & Wise on HQS, and by Georgi on the velocity-dependent HQET. HQET was thus conceived at Fermilab and born in Seillac.

The quark propagator is obtained from a initial-value prescription $G(x - a\hat{e}_4, x) = 0$, to impose forward propagation in time.

Let $n = (x_4 - y_4)/a$. Then

$$G_{\text{static}}(x,y) = \theta(x_4 - y_4)\delta_{\boldsymbol{x},\boldsymbol{y}}(1 + m_0 a)^{-(n+1)} \prod_{\ell=0}^{n-1} U_4^\dagger(y + \ell a\hat{e}_4)\frac{1+\gamma_4}{2} \qquad (10)$$

where $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$.

This convention may seem odd, but it is consistent with that normally used with light (Wilson or staggered) fermions. So, one ties this to a normal quark propagator to obtain a "static-light" meson.

The "Wilson line" develops a UV divergence proportional its length (in lattice units). The bare mass is included here to absorb this divergence: $m_0 \sim g_0^2/a$. By HQS it does not need to include the physical quark mass; that is up to you, and it is usually not done.

Meson Propagators

The pseudoscalar meson propagator (b static, l light)

$$\sum_{\boldsymbol{x}} \langle \bar{\psi}_l \gamma_5 \psi_b(x) \bar{\psi}_b \gamma_5 \psi_l(0) \rangle = \sum_{\boldsymbol{x}} \langle \mathrm{tr}\,[G_{\mathrm{static}}(x,0)\gamma_5 G_l(0,x)\gamma_5] \rangle_U$$
$$\sim (1+m_0 a)^{-1} e^{-x_4[\ln(1+m_0 a)/a + \delta m + \bar{\Lambda}]}, \qquad (11)$$

where δm is the UV divergent part and $\bar{\Lambda}$ is the physical binding energy (in the static approximation, cf. Lecture 2). The $\delta_{\boldsymbol{x},\boldsymbol{0}}$ in G_{static} eliminates $\sum_{\boldsymbol{x}}$.

The (square of the) noise

$$\left\langle \left[\sum_{\boldsymbol{x}} \bar{\psi}_l \gamma_5 \psi_b(x) \bar{\psi}_b \gamma_5 \psi_l(0) \right]^2 \right\rangle = \left\langle \left\{ \sum_{\boldsymbol{x}} \mathrm{tr}\,[G_{\mathrm{static}}(x,0)\gamma_5 G_l(0,x)\gamma_5] \right\}^2 \right\rangle_U$$
$$\sim (1+m_0 a)^{-2} e^{-2x_4 m_\pi}, \qquad (12)$$

so signal/noise falls steeply.

Recent work (*ALPHA*) substitutes a smeared link $U_4 \to W_4$ in action & propagator.

Exercise: Derive Eq. (10) by solving

$$\sum_z \left[(1+m_0 a)\delta_{x,z} - U_4^\dagger(x-a\hat{e}_4)\delta_{x-a\hat{e}_4,z} \right] G_{\text{static}}(z,y) = \delta_{x,y} \tag{13}$$

Show that the initial value $G(x-a\hat{e}_4,x)=0$ is needed to eliminate a solution that grows exponentially for $n<0$.

Exercise: Find the propagator for the alternative action

$$S_{\text{static}} = a^3 \sum_x \bar{\psi}(x) \frac{1-\gamma_4}{2} \left[(1+m_0 a)\psi(x) - U_4(x)\psi(x+a\hat{e}_4) \right]. \tag{14}$$

Exercise: Compute the UV divergence $\delta m^{[1]}$ at one loop. (Solution in Eichten & Hill).

$1/m$ Corrections

In B physics the heavy-quark expansion parameter is $\Lambda_{\rm QCD}/m_b \sim 10\%$. Thus, it is necessary to include contribution of order $1/m$, and to reach percent-level uncertainties one may need those of order $1/m^2$.

The quark mass is in this sense easier than, say, f_B. It starts at order $(1/m)^{-1}$, so the leading binding energy $\bar{\Lambda}$ is essential and the kinetic correction may be needed.

Exercise: How far in the $1/m$ expansion must you go to reach 1% for charmed quarks?

In lattice HQET, the kinetic and chromomagnetic interactions are treated as insertions:

$$\langle\Phi\rangle = \frac{1}{Z}\int \mathcal{D}U\,\mathcal{D}\psi\mathcal{D}\bar{\psi}\,\Phi\left[1 + C_2\sum_x O_2 + C_B\sum_x O_B\right] e^{-S_{\rm light}-S_{\rm static}} \qquad (15)$$

and the kinetic operator O_2 induces additional UV divergences.

EFTs and Cutoffs

The derivations of EFTs for heavy quarks hold only for energies well below that of the heavy-quark mass.

In the context of an explicit UV regulator, that means that we should only hope for validity for $a^{-1} \sim m$.

The kinetic energy is the first example (there would be more at order m^{-2}) of the more UV-singular behavior.

With lattice HQET, however, each UV-divergent operator is inserted only once (or perhaps twice), so the severity of the divergence is held in check. Moreover, the terms needed to absorb the divergences—*e.g.*, $m_0 a$ for δm—are always already there.

This means that lattice HQET is compatible with a continuum limit, provided the UV divergences can be handled non-perturbatively.

The key tool for devising a non-perturbative renormalization program is the Ansatz

$$\mathcal{L}^{\mathrm{LGT}}_{\mathrm{static}} \doteq -\bar{h}^{(+)}(D_4 + m_1)h^{(+)}, \tag{16}$$

$$O_2^{\mathrm{LGT}} \doteq \bar{h}^{(+)}(\boldsymbol{D}^2 + a^{-1}C_{-1}D_4 + a^{-2}C_{-2})h^{(+)}, \tag{17}$$

where the right-hand sides of $\doteq$ are a continuum effective theory, and $m_1 a = \ln(1 + m_0 a) + a\delta m$.

In this case the right-hand sides are simultaneously an example of the general HQ theory of cutoff effects, and the "obvious" Symanzik EFT for static quarks.

We shall see tomorrow how non-perturbative renormalization works in more detail.

Until then, it is perhaps worth pointing out two disadvantages: (1) the $1/m^2$ corrections will be difficult to implement; (2) lattice HQET is unacceptable for quarkonium, so it gives up all those experimental results as confidence-building cross-checks.

Lattice NRQCD

Basic Discretization

At Seillac, Lepage introduced lattice NRQCD, generalizing and discretizing his work with Caswell on NRQED. The first lattice NRQCD action is

$$S_{\mathrm{NRQCD}} = a^3 \sum_x \bar{\psi}(x) \frac{1+\gamma_4}{2} \left[(1+m_0 a)\psi(x) - U_4^\dagger(x-a\hat{e}_4)\psi(x-a\hat{e}_4) \right] + \frac{a^2}{2m_0'} \sum_{x,i} \bar{\psi} \left[2\psi(x) - U_i(x)\psi(x+a\hat{e}_i) - U_{-i}(x)\psi(x-a\hat{e}_i) \right]. \qquad (18)$$

Lepage omitted the bare rest mass $m_0 a$, but we shall keep it for the same reasons as in lattice HQET. The bare kinetic mass m_0' is the one that sets the non-static dynamics.

In principle (and in practice), the lattice NRQCD action includes all interactions through order v^4, supplemented with $\mathrm{O}(v^6)$ spin-dependent interactions.

This action, like the static action, has a propagator derived from an initial-value problem; with the $\delta_{x,y}$ being replaced by something propagating.

Improved Discretization

For various technical reasons, the basic discretization is no longer used, although the basic ingredients are the same.

It is simplest to write down the propagator. Define

$$H_0 = -\frac{\triangle}{2m_0'}, \quad \triangle\psi(x) = U_i(x)\psi(x+a\hat{e}_i) + U_{-i}(x)\psi(x-a\hat{e}_i) - 2\psi(x). \tag{19}$$

Then the lattice NRQCD propagator evolves from $G(\boldsymbol{x},0) = 1$ to

$$G(\boldsymbol{x},t+a) = \left(1-\frac{a\tilde{H}_0}{2n}\right)^n \left(1-\frac{aH_1}{2}\right) U_4^\dagger(\boldsymbol{x},t) \left(1-\frac{aH_1}{2}\right) \left(1-\frac{a\tilde{H}_0}{2n}\right)^n G(\boldsymbol{x},t) \tag{20}$$

where n is a so-called "stability parameter" H_1 is a (Symanzik-improved) discretization of the chromomagnetic and other subleading interactions, and $\tilde{H}_0 = H_0 - aH_0^2/4n$ Symanzik-improves the time-evolution.

Time-reversal invariant, unlike Eq. (18).

The stability parameter n is introduced to prevent instabilities for large momenta, namely to keep

$$\frac{aH_0}{2n} = -\left.\frac{a\triangle}{4m_0' n}\right|_{\rm UV} \sim \frac{1}{4m_0' an} \text{ “} \ll \text{” } 1 \tag{21}$$

(and similarly for $\tilde{H}_0$).

Thus, n must be increased when reducing the quark mass down to that of the charmed quark, or when moving to finer lattice spacing.

Currents

Currents in lattice NRQCD are straightforward discretizations of the continuum effective theory, developed to the needed order in the heavy-quark expansion.

Compare and Contrast with Lattice HQET

The central difference between lattice HQET and lattice NRQCD is that the latter includes the kinetic term in the propagator.

This means that there are UV divergences at every order in perturbation theory, $g^{2\ell}/(ma)^{\ell}$ stemming from the momentum in the vertex of the spatial gluon. They are less severe with NRQCD propagators instead of static ones, because of the higher power of momentum in the denominator.

Exercise: In the continuum, the spatial gluon vertex has UV behavior k_i/m, so the one-loop quark self-energy is

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{\boldsymbol{k}^2}{m^2}\frac{1}{i(p+k)_4+m_0+(\boldsymbol{p}+\boldsymbol{k})^2/2m_2}+\text{tadpole} \tag{22}$$

Carry out the integration over k_4 (contour over upper half plane), and then examine the UV behavior for divergences. Compare the NRQCD case (kinetic energy kept in denominator) to HQET (kinetic energy dropped from denominator).

As always, the key to renormalizing lattice NRQCD is the effective theory describing cutoff effects.

Even if the matching is done non-perturbatively, any mismatch in practice will grow as $a \to 0$. Thus (as stated in the first papers), lattice NRQCD is a viable method only when $ma \sim 1$.

Discretization effects are removed by Symanzik improvement. Remaining discretization errors are estimated and included in the error budget.

Even when this method is applied to heavy-light hadrons, and HQET power counting is imposed on H_1, everyone still calls it "lattice NRQCD." A salient advantage is that calculations of quarkonium can be used to help gain confidence that errors are what one estimates, before turning to B physics.

Moving NRQCD and HQET

As discretizations of the continuum effective field theories, NRQCD and HQET can be formulated in a moving frame with general v.

Goes back to Mandula & Ogilvie, Hashimoto & Matsufuru, and Sloan.

Killer application is $B \to \pi l \nu$ for V_{ub}.

See Stefan Meinel for details.

Methods with Wilson Quarks

Extrapolation Method

Once again at Seillac, Bernard and also Maiani *et al.* proposed a method for heavy quarks, based on the Wilson action

$$S = \sum_n \left\{ \bar{\psi}_n \psi_n - \kappa \sum_\mu \left[\bar{\psi}_n (1-\gamma_\mu) U_{n,\mu} \psi_{n+\mu} + \bar{\psi}_n (1+\gamma_\mu) U^\dagger_{n-\mu,\mu} \psi_{n-\mu} \right] \right\}, \quad (23)$$

They didn't know what would go wrong if $ma \not\ll 1$, but they expected that something would, so they proposed working with $m_0 a < 1$ and extrapolating using heavy-quark scaling up m_b.

(A footnote in Eichten's writeup notices that the static limit is reached as $\kappa \to 0$, but the leading corrections in κ are incorrect.)

Because the momentum in quarkonium is $\sim m\alpha_s$, this method is geared more to heavy-light hadrons, where the corrections are $\sim \Lambda/m$.

I call this method the "extrapolation method," because the extra extrapolation in $1/m$ is what distinguishes it from all other methods.

The central problem is that it wants the quark mass to be both large and small.

At fixed lattice spacing, details of (mass dependent) normalization and, indeed, improvement can send the extrapolation into widely different places. For example,

$$V^{\mu}_{\rm lat} = \begin{cases} \bar{\psi}(x)\gamma^{\mu}\psi(x) & \text{Wilson} \\ (1+\frac{1}{2}m_0 a)^2\bar{\psi}(x)\gamma^{\mu}\psi(x) & \text{"improved"} \\ (1+m_0 a)\bar{\psi}(x)\gamma^{\mu}\psi(x) & \text{Fermilab} \end{cases} \tag{24}$$

extrapolate to 0, ∞, and the static limit, respectively.

This problem can be avoided by taking the continuum limit first, and only then carrying out the $1/m$ extrapolation. Then, quark masses are limited to smallish m, where the $1/m$ expansion may break down, yet in a way that will not be signaled by the numerical data: the hadronic wavefunction smears out the threshold singularity.

Fermilab Method

A systematic study (1993-'96) of the heavy-quark limit of Wilson fermions was carried out by El-Khadra, Mackenzie and me (KKM). We uncovered several aspects and proposed a general strategy. Various combinations of these findings and idea are now called the "Fermilab method."

Wilson quarks have a smooth heavy-quark limit, consistent with HQS (already noticed by Eichten, Lepage, several others; easily inferred from Lüscher's transfer matrix).

The leading defect $\sqrt{m_1/m_2} \neq 1$ is easily remedied with two hopping parameters:

$$\begin{aligned} S_{\text{Fermilab}} = &\sum_n \bar{\psi}_n \psi_n - \kappa_t \sum_n \left[\bar{\psi}_n (1-\gamma_4) U_{n,4} \psi_{n+\hat{4}} + \bar{\psi}_n (1+\gamma_4) U^\dagger_{n-\hat{4},4} \psi_{n-\hat{4}} \right] \\ &- \kappa_s \sum_{n,i} \left[\bar{\psi}_n (r_s - \gamma_i) U_{n,\mu} \psi_{n+i} + \bar{\psi}_n (r_s + \gamma_i) U^\dagger_{n-i,i} \psi_{n-i} \right], \end{aligned} \tag{25}$$

where $\kappa_s \neq \kappa_t$ and, for convenience later, $r_s \neq 1$ even though it is redundant.

In a non-perturbative setting, the two κs can be adjusted so that a hadron's kinetic mass equals its rest mass. This is needed if you want charmed sea quarks and $m_c a \not\ll 1$ but < 1: sea quarks must have the threshold in the right place.

The remaining discretization effects are small, and can be corrected with Symanzik-like improvements, provided higher temporal derivatives are not used.

Higher temporal derivatives are not necessary.

Couplings of all improvement terms (similarly, short-distance coefficients in the effective continuum theories) are functions of $m_0 a$. For example, $c_{\rm SW} \to c_B(m_0 a), c_E(m_0 a)$.

Heavy valence quarks are non-relativistic, so the tuning of m_1 (and hence, κ_t) is not essential here. In this "non-relativistic interpretation," one may set $\kappa_t = \kappa_s$.

Being based on Wilson fermions, one is guaranteed to flow smoothly to the Wilson continuum limit as $a \to 0$. In particular the linear UV divergence in the mass can be handled non-perturbatively via

$$m_0 a = \frac{1}{2\kappa} - 4 \to \frac{1}{2\kappa} - \frac{1}{2\kappa_c} \tag{26}$$

Therefore, this is a method that works comparably well for bottom and charm.

Several aspects of this work clarified by the heavy-quark theory of cutoff effects.

Exercise: Derive the free propagator for the Fermilab action. Integrate over p_4 to obtain the energy, as a function of 3-momentum, defined by the pole. Derive the condition on the κs such that $m_2 = m_1$.

Tsukuba & Columbia

The Fermilab action has been "rediscovered" twice.

A group at Tsukuba U. proposed a "relativistic heavy-quark action." that is identical to the Fermilab action except $r_t \neq 1$ (a bad idea). It asserts that r_s is not redundant (wrong). It presents an argument complementary to that of KKM that powers $(-\gamma_4 D_4 a)^n$ can be traded for $(m_0 a)^n$, and used in Lecture 2. It advocates a correct tuning of r_s already in KKM.

A group at Columbia U. re-examined the contents of the Fermilab and Tsukuba papers, giving yet another derivation showing that powers of $m_0 a$ are under control. They emphasized that HQET counting implies that c_E is a sub-subleading effect for heavy-light hadrons, so one may as well set $c_E = c_B$. They cast the lattice theory in a way anticipating non-perturbative matching calculations (but did not appeal to the HQ theory of cutoff effects).

Methods with Staggered Quarks

Basic Considerations

Staggered quarks are usually not considered as candidates for heavy quarks, because the theoretical cost of the extra tastes does not seem to buy anything.

In particular, I don't know of any systematic study, analogous to KKM, of the heavy quark limit of staggered fermions (or of overlap or domain-wall fermions).

For the charmed quark, however, one may choose to dispense with heavy quark ideas, and simply improve a light-quark action.

The HPQCD Collaboration has recently advocated this strategy for "highly improved staggered quarks" (HISQ).

An interesting feature is that the coupling of the leading Symanzik improvement term (first introduced by Naik) becomes mass-dependent, such that mass-dependent discretization errors obtain a suppression $\boldsymbol{p}/m$.

What Should You Use?

Faced with all these methods for heavy quarks, which one is best? Which one do I recommend?

Of course, I have a personal stake, derived from thinking about this problem a long time.

In the end, your choice depends on what you see as your physics goals.

In particular, if you want to publish a number that is consider relevant to particle physics phenomenology, you must publish a full error budget.

The full error budget is more important than the method; if everyone did that, we could decide together what works where.