HEAVY QUARKS ON THE LATTICE 1:

Effective Field Theories for Heavy Quarks

Heavy Quarks and the Lattice

Physics of Heavy-Quark Systems

Effective HQ Lagrangian

Power Counting

Heavy-Quark Symmetry

Andreas S. Kronfeld

23 August 2007

Heavy Quarks and the Lattice

Quark Masses and the QCD Scale

QCD has a dynamically generated scale Λ_{QCD}:

$$\Lambda_{\mathrm{QCD}} = \begin{cases} \Lambda_{\overline{\mathrm{MS}}} = 250\ \mathrm{MeV} \\ m_\rho = 770\ \mathrm{MeV} \\ m_K^2 m_s^{-1} = 2500\ \mathrm{MeV} \end{cases} \tag{1}$$

Nature has three quarks (*up*, *down*, *strange*) whose masses are less than, indeed much less than, Λ_{QCD}.

Most of lattice gauge theory only entails these three quarks, because their mass scales is firmly in the non-perturbative regime of QCD.

Nature has another three quarks (*charm*, *bottom*, *top*) whose masses are greater than, perhaps much greater than, Λ_{QCD}.

Heavy Quark Masses

Heavy quarks:

$$\begin{aligned} m_{\text{top}} &= 174200\ \text{MeV} \\ m_{\text{bottom}} &= \quad 4200\ \text{MeV} \\ m_{\text{charm}} &= \quad 1300\ \text{MeV} \end{aligned} \tag{2}$$

The top quark is so massive that its lifetime ($\Gamma_{\text{top}}^{-1} \approx 5 \times 10^{-25}$ s) is less than the typical time to hadronize ($\Lambda_{\text{QCD}}^{-1} \sim 2 \times 10^{-24}$ s).

The charmed quark mass lies in the upper range of chromodynamic physics, particularly if one looks at the threshold to produce $c\bar{c}$ pairs.

The tacit assumption behind most of lattice QCD is that short-distance effects of heavy quarks can be absorbed (via renormalization) into the gauge coupling and masses of the $n_f = 3$ QCD Lagrangian.

Typical lattices (*e.g*, MILC) have spacing $0.04\ \text{fm} \lesssim a \lesssim 0.16\ \text{fm}$, so

$$\begin{array}{c} 0.84 \lesssim m_b a \lesssim 3.4 \\ 0.25 \lesssim m_c a \lesssim 1.0 \end{array} \qquad (3)$$

The fact that the heavy quark masses are large in lattice units is the central problem that these lectures have to address.

From a particle-physics perspective, the most urgent application of lattice QCD is flavor physics, especially mixing and decay properties of B and D mesons, so the problem cannot be addressed by waiting or quenching.

The key to resolving the "$m_Q a \not\ll 1$ problem" is that a quark can be considered heavy if and only if its Compton wavelength is a short-distance, like a. Therefore, we effects from this scale are (generalizations of) renormalization problems.

Effective field theories: to analyze heavy-quark methods (including error estimates); to devise heavy-quark methods.

Truthiness and Truth

Truthiness:* it is impossible to simulate a heavy quark directly on the lattice; it literally falls through the holes.

Truth: It is possible to set $m_Q = m_b$ such that the difference between lattice gauge theory and (relativistically invariant) continuum QCD is small, controlled, and estimated by standard tools of quantum field theory.

Of course, if you assume—implicitly or explicitly—that $m_Q a \ll 1$ when, your computer has $m_Q a \not\ll 1$, then you'll introduce several conceptual or quantitative errors.

Since CPU time increases as a large power of a, one should also consider whether it is cost-effective to reduce a until $m_b a \ll 1$.

* truthiness: *reality that is intuitively known, without regard to reason and logic*

Physics of Heavy-Quark Systems

Heavy-Light Hadrons

A “heavy-light” meson (baryon) has one heavy quark and one light anti-quark (two light quarks), plus any amount of gluons and sea quarks, *aka* brown muck.

Consider two hydrogen atoms, one with a proton as its nucleus, and one with a deuteron. How are they different? Why?

Consider two uranium atoms, one with 143 neutrons in its nucleus, and one with 146. 92 electrons! How do their atomic levels differ? How does their chemistry differ?

The key insight is that the lighter objects don’t have enough energy to push the heavy object around, so it just sits there in the bound state. As long as it’s heavy enough, the light dynamics doesn’t depend on how heavy the nucleus is.

Although we cannot solve QCD (or uranium) with pencil and paper, we know that to good approximation the mass of the heavy quark (or the U nucleus) doesn’t matter.

In QCD the heavy-quark is a nearly static source of color.

The light quark(s and gluons) have chromodynamic momenta and energies, Λ_{QCD}. Alternatively, the gauge potential A_μ has typical size Λ_{QCD} (in a smooth gauge); the field strength $F_{\mu\nu}$ has typical size Λ_{QCD}^2.

So one anticipates that a hadron mass has contributions

$$M_{\text{hadron}} = m_{\text{quark}} + \Lambda + \frac{\Lambda^2}{2m} + \cdots \tag{4}$$

from the heavy quark mass, the energy of the brown muck, and the kinetic energy of the heavy quark, *etc*.

Below we shall see how to obtain this result in an effective field theory for heavy-light hadrons (HQET).

Quarkonium

Positronium is a bound state between an electron e^- and a positron e^+.

Quarkonium is a bound state between a heavy quark Q and a heavy antiquark $\bar{Q}$, plus any amount of brown muck.

Consider a binary star system in a dense cloud of dust. Can you "solve" the dynamics? What do you "know" about the dynamics?

Similarly, in quarkonium, the heavy objects orbit each other "slowly" and a hierarchy of scales emerges: $\frac{1}{2}mv^2 \ll mv \ll m$, where v denotes the velocity.

In perturbative QCD, one-gluon exchange generates the potential

$$V(r) = -C_F \frac{\alpha_s(1/r)}{r} + \cdots \qquad (C_F = \tfrac{4}{3} \text{ for QCD.}) \tag{5}$$

with $\alpha_s(1/r)$ and $\cdots$ arising from radiative corrections and multi-gluon exchange.

To the extent that heavy quarkonium is quasi-Coulombic,

$$\langle p^2 \rangle^{1/2} = \mu C_F \alpha_s(p), \quad \mu = \text{reduced mass.} \tag{6}$$

As we shall see below, it is possible to develop a series in v^2, $v = p/m$:

$$\begin{array}{ll} \bar{b}b : v^2 = \frac{1}{10}; & \eta_b, \Upsilon, \chi_{bJ}, \dots \\ \bar{c}c : v^2 = \frac{1}{3}; & \eta_c, J/\psi, \chi_{cJ}, \dots \\ \bar{b}c : v_c^2 = \frac{1}{2},\ v_b^2 = \frac{1}{25}; & B_c, B_c^*, \dots \end{array} \tag{7}$$

Potential models (with the confining part put in by hand) provide a sound phenomenological understanding of quarkonium. In the context of LGT, these models can be used to estimate uncertainties.

The analysis of quarkonium via potentials is made more field-theoretic by developing effective fields theories: first with m^{-1} as a short distance (NRQCD); then with $(mv)^{-1}$ as a short distance (pNRQCD).

Research project: use LGT and pNRQCD to compute non-perturbative corrections to m_t as measured in $e^+e^- \to t\bar{t}$ at threshold.

Baryons with Two Heavy Quarks

Doubly-heavy baryons have two heavy quarks and a light (valence) quark, plus gluons and sea quarks.

The force between two quarks in the $\bar{3}$ (6) channel is attractive (repulsive), so pairs of quarks tend to form diquarks.

Indeed, with SU(3) color the $(qq)_{\bar{3}}$ force is half that of the $(\bar{q}q)_1$ force.

The light valence quark does not have enough energy to push the heavy diquark apart. (NB: Stretch for ccq is longer than for D or ψ.)

Hence, a doubly-heavy baryon can be treated as a quarkonium-like heavy diquark with a light quark whizzing around it: it has quarkonium-like excitations of the heavy-heavy system, and heavy-light excitations of the brown muck.

Derivations of the Effective HQ Lagrangian

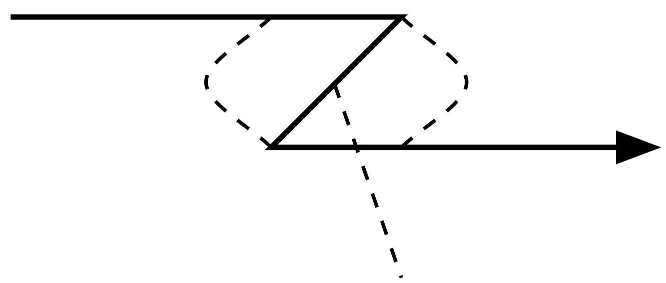
Effective Field Theories

Inside hadrons, gluons (if you can think of gluons) have a characteristic energy Λ_{QCD} (or $m_Q\alpha_s^{1/2}$ in quarkonium) that is insufficient to produce heavy $Q\bar{Q}$ pairs.

Such pairs are far from being on the mass shell.

The Coleman-Norton theorem tells us that singular behavior in Green functions arises only when particles can be on shell.

In particular, any possible zig-zags in a valence quark line produce no singularities.



The whole blob can be shrunk to simple vertex, with power corrections in E/m.

Notation

Euclidean

metric $\delta_{\mu\nu} = \text{diag}(+,+,+,+)$

$\mu,\nu = 1,2,3,4;\ x^4 = x_4 = it$

$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$$

$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \quad \gamma_5^\dagger = \gamma_5$

$d^4x = dx^1 dx^2 dx^3 dx^4$

$e^{-S} = e^{\int d^4x\, \mathcal{L}}$

Minkowski

metric $g_{\mu\nu} = \text{diag}(-,+,+,+)$

$\mu,\nu = 0,1,2,3;\ x^0 = -x_0 = t$

$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\gamma_4$$

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$$

$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \gamma_5^\dagger = \gamma_5$

$d^4x = i dx^0 dx^1 dx^2 dx^3$

$e^{iS} = e^{\int d^4x\, \mathcal{L}}$

Derivation I

The functional integral is

$$\int \mathcal{D}A\mathcal{D}q\mathcal{D}\bar{q} \exp\left(-S_g - \int d^4x\,\bar{q}(\not{D}+m)q\right) \tag{8}$$

Let us insert

$$h^{(\pm)} = \frac{1\pm\gamma_4}{2}q, \qquad \gamma_4 h^{(\pm)} = \pm h^{(\pm)}, \quad q = \begin{pmatrix} h^{(+)} \\ h^{(-)} \end{pmatrix} \tag{9}$$

into the quark Lagrangian, easily finding

$$\begin{aligned} \bar{q}(\not{D}+m)q = \bar{h}^{(+)}(D_4+m)h^{(+)} - \bar{h}^{(-)}(D_4-m)h^{(-)} \\ + \bar{h}^{(+)}\boldsymbol{\sigma}\cdot\boldsymbol{D}h^{(-)} + \bar{h}^{(-)}\boldsymbol{\sigma}\cdot\boldsymbol{D}h^{(+)} \end{aligned} \tag{10}$$

Now integrate out the "lower components" $h^{(-)}$, $\bar{h}^{(-)}$:

$$\mathcal{D}h^{(+)}\mathcal{D}\bar{h}^{(+)} \exp\left[-\int d^4x\,\bar{h}^{(+)}\left(D_4+m+\boldsymbol{\sigma}\cdot\boldsymbol{D}\frac{1}{D_4-m}\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)h^{(+)}\right]. \tag{11}$$

We haven't done anything yet, but now consider that we are interested in *hadrons* with a wavefunction that constrains the momentum and field strength.

We thus (formally) treat A_4 and $\boldsymbol{D}$ as small compared to m.

Near the mass shell $D_4 + m \ll m$, so

$$\bar{h}^{(+)}\boldsymbol{\sigma}\cdot\boldsymbol{D}\frac{1}{D_4 - m}\boldsymbol{\sigma}\cdot\boldsymbol{D}h^{(+)} = -\bar{h}^{(+)}\boldsymbol{\sigma}\cdot\boldsymbol{D}\frac{1}{2m-(D_4+m)}\boldsymbol{\sigma}\cdot\boldsymbol{D}h^{(+)} \tag{12}$$

doesn't commute with

$$\approx -\bar{h}^{(+)}\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{2m}h^{(+)} - \bar{h}^{(+)}\boldsymbol{\sigma}\cdot\boldsymbol{D}\frac{D_4+m}{(2m)^2}\boldsymbol{\sigma}\cdot\boldsymbol{D}h^{(+)} \tag{13}$$

UV regulator!

$$+\cdots$$

Now introduce a change of variables

$$h^{(+)} \mapsto \left[1+\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{8m^2}\right]h^{(+)}, \quad \bar{h}^{(+)} \mapsto \bar{h}^{(+)}\left[1+\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{8m^2}\right]. \tag{14}$$

Changes of variables of integration are always allowed, but watch for Jacobians.

Now collect the terms of order m^{-2}:

$$\frac{1}{8m^2}\bar{h}^{(+)}\left[(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2(D_4+m)+(D_4+m)(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2-2\boldsymbol{\sigma}\cdot\boldsymbol{D}(D_4+m)\boldsymbol{\sigma}\cdot\boldsymbol{D}\right]h^{(+)}=$$

$$-\frac{1}{8m^2}\bar{h}^{(+)}(\boldsymbol{\sigma}\cdot\boldsymbol{D}\boldsymbol{\sigma}\cdot\boldsymbol{E}-\boldsymbol{\sigma}\cdot\boldsymbol{E}\boldsymbol{\sigma}\cdot\boldsymbol{D})h^{(+)}=-\bar{h}^{(+)}\frac{[\boldsymbol{\sigma}\cdot\boldsymbol{D},\boldsymbol{\sigma}\cdot\boldsymbol{E}]}{8m^2}h^{(+)} \qquad (15)$$

where the chromoelectric field $\boldsymbol{E}=[D_4,\boldsymbol{D}]$.

Because the heavy-quark expansion does not commute with the UV regulator (*i.e.*, it does hold at high energies), the derivation misses effects of radiative corrections.

Different spin structures acquire different Wilson coefficients:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2=\boldsymbol{D}^2+i\boldsymbol{\sigma}\cdot\boldsymbol{B}\rightarrow\boldsymbol{D}^2+Z_{\sigma B}(\alpha_s)i\boldsymbol{\sigma}\cdot\boldsymbol{B}, \qquad (16)$$

$$[\boldsymbol{\sigma}\cdot\boldsymbol{D},\boldsymbol{\sigma}\cdot\boldsymbol{E}]\rightarrow Z_{D.E}(\alpha_s)(\boldsymbol{D}\cdot\boldsymbol{E}-\boldsymbol{E}\cdot\boldsymbol{D})+Z_{D\times E}(\alpha_s)i\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D}), \qquad (17)$$

the “kinetic,” “chromomagnetic” (or “hyperfine”), “Darwin,” and “spin-orbit” terms.

Combining Eqs. (11), (13), (15)–(17) yields the heavy-quark effective Lagrangian:

$$\mathcal{L}_{\text{HQ}} = -\bar{h}^{(+)} \left[D_4 + m - \frac{\boldsymbol{D}^2}{2m} - Z_{\sigma B} \frac{i\boldsymbol{\sigma} \cdot \boldsymbol{B}}{2m} - Z_{D \cdot E} \frac{(\boldsymbol{D} \cdot \boldsymbol{E} - \boldsymbol{E} \cdot \boldsymbol{D})}{8m^2} \right. \quad (18)$$
$$\left. - Z_{D \times E} \frac{i\boldsymbol{\sigma} \cdot (\boldsymbol{D} \times \boldsymbol{E} - \boldsymbol{E} \times \boldsymbol{D})}{8m^2} + \cdots \right] h^{(+)}$$

The Hamiltonian corresponding to this Lagrangian is

$$\mathcal{H}_{\text{HQ}} = -\bar{h}^{(+)} \partial_4 h^{(+)} - \mathcal{L}_{\text{HQ}} \quad (19)$$
$$= m\bar{h}^{(+)} h^{(+)} + \bar{h}^{(+)} \left[A_4 - \frac{\boldsymbol{D}^2}{2m} - Z_{\sigma B} \frac{i\boldsymbol{\sigma} \cdot \boldsymbol{B}}{2m} + \cdots \right] h^{(+)} \quad (20)$$

The rest mass term commutes with everything else. Hence, wavefunctions do not depend on the rest mass *except* through the $(1/m)$-suppressed terms.

Manifests the absence of singularities from pair-production.

Exercise: Extend this derivation to the next order in $(D_4 + m)/m$; generalize the change of variable so that the terms suppressed by powers of $1/m$ contain only $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{E}$.

Exercise: Generalize the split between the large and small components as follows. Introduce a "hadron velocity" v. The projectors are

$$\frac{1 \mp i \not{v}}{2}$$

Above we had Minkowski $v = (1, \mathbf{0})$ or Euclidean $v = (\mathbf{0}, i)$; $v^2 = -1$.

As before, treat $v \cdot A$ and $D_\perp^\mu = D^\mu + v^\mu v \cdot D$ as small compared to m.

Exercise: Use two infinitesimally different velocities to show that $D_\perp^2/2m$ receives no radiative correction (or read Manohar & Wise on "reparametrization").

Derivation II

The preceding derivation integrated out the anti-quarks. A second derivation uses only changes of variables and, thus, treats quarks and anti-quarks in the same way.

Introduce the same split $h^{(\pm)}$ as before:

$$-\mathcal{L} = \bar{h}^{(+)}(D_4+m)h^{(+)} - \bar{h}^{(-)}(D_4-m)h^{(-)} + \bar{h}^{(+)}\boldsymbol{\sigma}\cdot\boldsymbol{D}h^{(-)} + \bar{h}^{(-)}\boldsymbol{\sigma}\cdot\boldsymbol{D}h^{(+)} \tag{21}$$

Now change variables (Foldy-Wouthuysen-Tani transformation)

$$h^{(\pm)} \to h^{(\pm)} - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}}{2m}h^{(\mp)}, \quad \bar{h}^{(\pm)} \to \bar{h}^{(\pm)} - \bar{h}^{(\mp)}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}}{2m}. \tag{22}$$

By design, the second two terms go away:

$$
\begin{aligned}
-\mathcal{L} = {} & \bar{h}^{(+)} \left[D_4 + m - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{2m} - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}(D_4+m)\boldsymbol{\sigma}\cdot\boldsymbol{D}}{4m^2} \right] h^{(+)} \\
& + \bar{h}^{(-)} \left[-D_4 + m - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{2m} + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}(D_4-m)\boldsymbol{\sigma}\cdot\boldsymbol{D}}{4m^2} \right] h^{(-)} \\
& + \bar{h}^{(+)} \left[-\frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}}{2m} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^3}{4m^2} \right] h^{(-)} \\
& + \bar{h}^{(-)} \left[\frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}}{2m} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^3}{4m^2} \right] h^{(+)}
\end{aligned}
\tag{23}
$$

And apply (14) again to obtain $\boldsymbol{E}$ on the first two lines:

$$
h^{(+)} \mapsto \left[1 + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{8m^2} \right] h^{(+)}, \quad \bar{h}^{(+)} \mapsto \bar{h}^{(+)} \left[1 + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{8m^2} \right].
$$

$$
\begin{aligned}
-\mathcal{L} = \bar{h}^{(+)} & \left[D_4 + m - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{2m} - \frac{[\boldsymbol{\sigma}\cdot\boldsymbol{D}, \boldsymbol{\sigma}\cdot\boldsymbol{E}]}{8m^2} - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^4}{8m^3} \right] h^{(+)} \\
+ \bar{h}^{(-)} & \left[-D_4 + m - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{2m} + \frac{[\boldsymbol{\sigma}\cdot\boldsymbol{D}, \boldsymbol{\sigma}\cdot\boldsymbol{E}]}{8m^2} - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^4}{8m^3} \right] h^{(-)} \\
+ \bar{h}^{(+)} & \left[-\frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}}{2m} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^3}{4m^2} \right] h^{(-)} \\
+ \bar{h}^{(-)} & \left[\frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}}{2m} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^3}{4m^2} \right] h^{(+)}
\end{aligned}
\tag{24}
$$

Now apply a second FWT transformation:

$$
h^{(\pm)} \to h^{(\pm)} \mp \frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}}{4m^2} h^{(\mp)}, \qquad \bar{h}^{(\pm)} \to \bar{h}^{(\pm)} \pm \bar{h}^{(\mp)} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}}{4m^2},
\tag{25}
$$

$$\mathcal{L}_{\rm HQ} = -\bar{h}^{(+)} \left[D_4 + m - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{2m} - \frac{[\boldsymbol{\sigma}\cdot\boldsymbol{D}, \boldsymbol{\sigma}\cdot\boldsymbol{E}]}{8m^2} - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^4}{8m^3} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{E})^2}{4m^3} \right] h^{(+)} \quad (26)$$
$$- \bar{h}^{(-)} \left[-D_4 + m - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2}{2m} + \frac{[\boldsymbol{\sigma}\cdot\boldsymbol{D}, \boldsymbol{\sigma}\cdot\boldsymbol{E}]}{8m^2} - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^4}{8m^3} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{E})^2}{4m^3} \right] h^{(-)}$$

neglecting terms of order m^{-4} between $\bar{h}^{(\pm)}h^{(\pm)}$ and (commensurately) terms between $\bar{h}^{(\pm)}h^{(\mp)}$ that can be removed by changes of variable of order m^{-3}.

The advantage of this method is that is provides interactions for both quark and anti-quark, which is necessary for quarkonium.

As before, the some steps (here just changes of variable) do not commute with the UV regulator, so this derivation misses non-trivial Wilson coefficients.

Power Counting

Preliminaries

An effective field theory has infinitely many couplings and, hence, is not immediately of much value.

Here the "couplings" correspond to the Wilson coefficients, which encode the short-distance physics. Our situation is not as bad as in, say, χPT: owing to asymptotic freedom, we can expect a reliable calculation in QCD perturbation theory.

Nevertheless, unless we can classify the infinitely-many interactions into a hierarchy of smaller and smaller effects, the effective field theory does not really simplify the underlying theory.

This is achieved by power counting.

The correct power counting can be established (only) by considering the physics of the problem at hand.

Heavy-Light Hadrons

From Eq. (20) we had

$$\mathcal{H}_{\text{HQ}} = \bar{h}^{(+)} \left[m + A_4 - \frac{\boldsymbol{D}^2}{2m} - Z_{\sigma B} \frac{i\boldsymbol{\sigma} \cdot \boldsymbol{B}}{2m} + \cdots \right] h^{(+)}$$

The first issue we must address is the size of the Coulomb potential A_4. In a heavy-light system, it is a property of the light degrees of freedom, so in QCD it is of order $A_4 \sim \Lambda_{\text{QCD}}$.

Next consider $\boldsymbol{D} = \boldsymbol{\partial} + \boldsymbol{A}$. The derivative measures the 3-momentum of the heavy quark; by momentum conservation, this is the same as that of the light degrees of freedom, $p_i \sim \Lambda_{\text{QCD}}$.

The 3-vector potential is also characteristic of the gluons: $A_i \sim \Lambda_{\text{QCD}}$.

In summary, it is easy to assign a power of Λ_{QCD}/m to each term in the effective Lagrangian: $s_O = \dim O - 4$ and split

$$\mathcal{H}_{\text{eff}} = \sum_{s=-1}^{\infty} \mathcal{H}^{(s)}. \tag{27}$$

The full blown "heavy-quark effective theory" then results from treating $\mathcal{H}^{(s)}$, $s > 0$, as perturbations.

In particular,

$$\mathcal{H}^{(1)} = -\bar{h}^{(+)} \left[\frac{\boldsymbol{D}^2}{2m} + Z_{\sigma B} \frac{i\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2m} \right] h^{(+)}, \tag{28}$$

$$\mathcal{H}^{(2)} = -\bar{h}^{(+)} \left[Z_{D\cdot E} \frac{\boldsymbol{D}\cdot\boldsymbol{E} - \boldsymbol{E}\cdot\boldsymbol{D}}{8m^2} + Z_{D\times E} \frac{i\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{D})}{2m} \right] h^{(+)}. \tag{29}$$

Quarkonium

From Eq. (20) and extra terms in Eq. (26), we have

$$\mathcal{H}_{\mathrm{HQ}} = \bar{h}^{(+)} \left[m + A_4 - \frac{\boldsymbol{D}^2}{2m} - \frac{i\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2m} - \frac{[\boldsymbol{\sigma}\cdot\boldsymbol{D}, \boldsymbol{\sigma}\cdot\boldsymbol{E}]}{8m^2} - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{D})^4}{8m^3} + \cdots \right] h^{(+)}, \qquad (30)$$

leaving out Wilson coefficients to keep the equation compact.

Again we must start by addressing the size of the Coulomb potential A_4. Now, however, it is a characteristic of the of the slow dance of two heavy objects:

$$A_4 \sim \frac{\boldsymbol{p}^2}{2m} = \tfrac{1}{2} m v^2. \qquad (31)$$

Consequently the chromoelectric field $\boldsymbol{E} = -\boldsymbol{\partial} A_4 \sim m^2 v^3$.

On the other hand, the 3-vector potential is smaller

$$\partial^2 A_i \sim \underset{v(mv)^4/m}{\alpha_s \bar{h}\partial_i h/m} + \underset{mv(mv^2)^2}{A_4\partial_i A_4} \quad\Rightarrow\quad \boldsymbol{A} \sim mv^3 \Rightarrow \boldsymbol{B} \sim m^2v^4 \tag{32}$$

if, as in the introduction, we set $\alpha_s \sim v$.

Velocity counting imposes a hierarchy on the effective heavy-quark Lagrangian:

$$\mathcal{H}^{(2)} = \bar{h}^{(+)}\left[A_4 - \frac{\boldsymbol{D}^2}{2m}\right]h^{(+)}, \tag{33}$$

$$\mathcal{H}^{(4)} = -\bar{h}^{(+)}\left[\frac{Z_{\sigma B} i\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2m} + \frac{Z_{D\cdot E}[\boldsymbol{D}\cdot,\boldsymbol{E}]}{8m^2} + \frac{Z_{D\times E} i\boldsymbol{\sigma}\cdot\{\boldsymbol{D}\times,\boldsymbol{E}\}}{8m^2} + \frac{Z_{D^4}(\boldsymbol{D}^2)^2}{8m^3}\right]h^{(+)} \tag{34}$$

Treating $\mathcal{H}^{(n)}$, $n > 2$, as perturbations yields the effective theory called non-relativistic QCD (NRQCD).

Strictly speaking, one has to worry about the mixed power counting of $\boldsymbol{D}$, but for LGT purposes, it is more essential to keep gauge invariance simple.

Remarks

NB: I like to use "NRQCD" and "HQET" to distinguish the power counting, which arises from consideration of the physics of the systems.

Thus, HQET is the effective theory for heavy-light hadrons (one for each hadron v, cf., Exercise).

Similarly, NRQCD is the effective theory for heavy quarkonium.

This usage is physical and conforms with that in the continuum heavy-quark literature.

We shall see, however, that the lattice literature has a mind of its own.

Heavy Quark Symmetry

Spin Symmetry

The leading Lagrangians for both NRQCD and HQET possess heavy-quark spin symmetry.

In the formalism, we see that the leading spin-dependent interactions, $\boldsymbol{\sigma} \cdot \boldsymbol{B}$ and $\boldsymbol{\sigma} \cdot (\boldsymbol{D} \times \boldsymbol{E})$ are suppressed in the two power-counting schemes.

Physically, it is hard to change the angular momentum of a heavy object.

Flavor Symmetry

The leading HQET Lagrangian

$$\mathcal{L}^{(0)} = -\bar{h}^{(+)}(D_4 + m_1)h^{(+)} = -\bar{h}^{(+)}(-iv \cdot D + m_1)h^{(+)} \tag{35}$$

possesses a heavy-quark flavor symmetry, which follows from the m-independence of the wave function.

It is not manifest, but consider two flavors and let $\theta = (m_{1c} - m_{1b})v \cdot x$ and consider the generators

$$\tau^1 = \frac{i}{2}\begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}, \quad \tau^2 = \frac{i}{2}\begin{pmatrix} 0 & -ie^{i\theta} \\ ie^{-i\theta} & 0 \end{pmatrix}, \quad \tau^3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{36}$$

satisfying the SU(2) algebra $[\tau^d, \tau^e] = \varepsilon^{dfe}\tau^f$.

The flavor symmetry is

$$h_v^{(+)} \mapsto e^{\tau^a \omega_a} h_v^{(+)}, \quad \bar{h}_v^{(+)} \mapsto \bar{h}_v^{(+)} e^{-\tau^a \omega_a}. \tag{37}$$

The symbol $\mathcal{D}^\mu = D^\mu - im_1 v^\mu$ satisfies $[\mathcal{D}^\mu, \tau^d] = 0$ and is, thus, trivially covariant under the transformation (37).

In the continuum literature, the rest mass is usually dispensed with through a field transformation $e^{-imv\cdot x}$. This is unnecessary, and for Euclidean space problematic.

Derivation III

The emergence of the heavy-quark symmetries leads to a third way to derive the heavy-quark Lagrangians.

Use Coleman-Norton reasoning and symmetry to realize that the physics of heavy quarks is described by two-component spinors, $h^{(\pm)}$.

Write down all interactions of a given order in the power counting, and match the results Lagrangian to QCD.

Symmetry-breaking arise in the matching calculations that yield the Wilson coefficients.

Exercise: Discuss with your friends which of the three derivations is best.