NLC QCD On The Lattice

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Near light-cone coordinates

Prokhvatilov et. al, Sov. J. of Nucl. Phys.49 (688); Lenz et. al, Annals of Physics 208 (1-89)

- Transition to NLC coordinates is a two step process
  - Lorentz boost to a fast moving frame with relative velocity

$$x'^0 = \gamma(x^0 - \beta x^3)$$
$$x'^3 = \gamma(x^3 - \beta x^0)$$

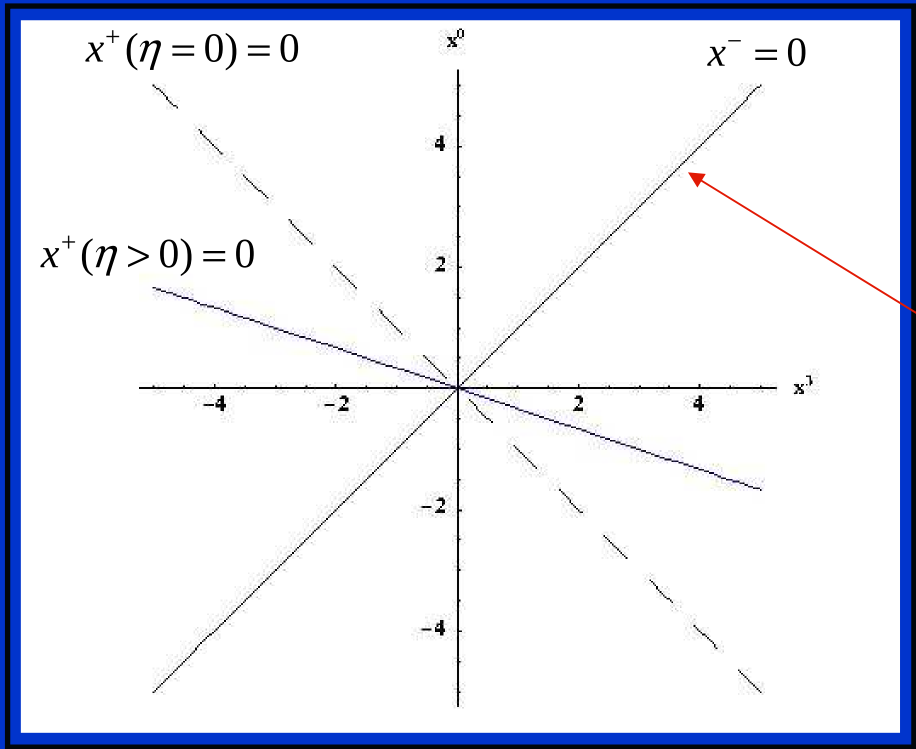
$$\beta = \frac{1 - \eta^2/2}{1 + \eta^2/2}$$

  - Rotation in the x'^0-x'^3-plane

$$x^+ = \frac{1}{\sqrt{2}} \left[\left(1 + \frac{\eta^2}{2}\right) x'^0 + \left(1 - \frac{\eta^2}{2}\right) x'^3 \right]$$
$$x^- = \frac{1}{\sqrt{2}} \left[x'^0 - x'^3 \right]$$

- Allows interpolation between equal-time $\eta^2 = 2$ and light-cone quantization $\eta^2 = 0$
- Introduced to investigate light-cone quantization as a limiting procedure of equal time theories

Near light-cone coordinates



Light Cone time axis along which the system evolves

- scalar product

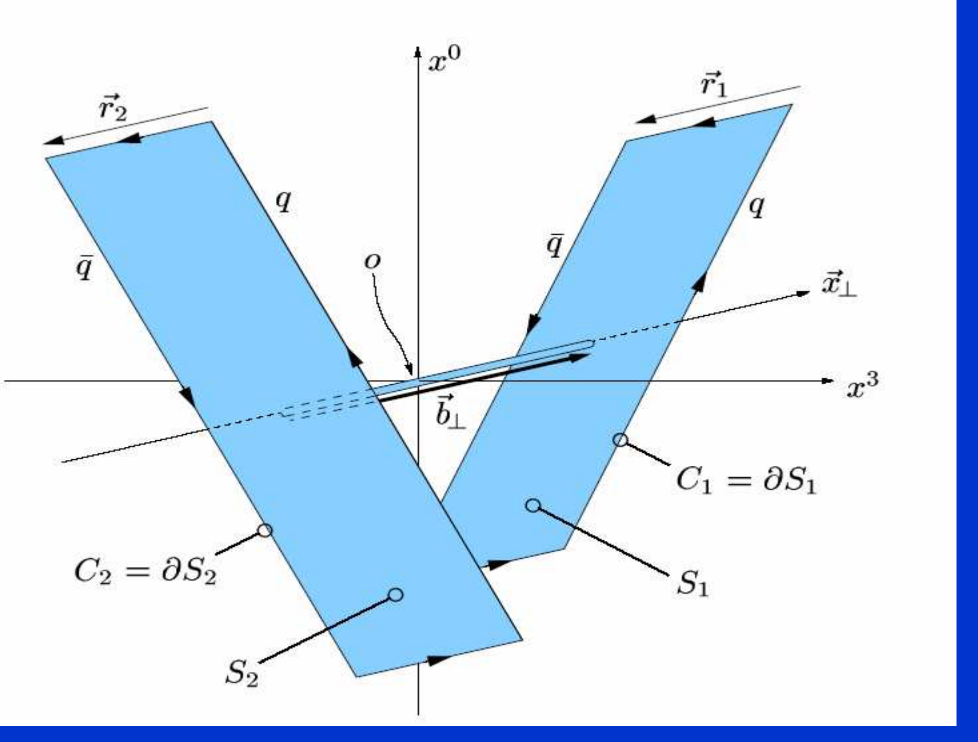
$$\begin{aligned} x_\mu y^\mu &= x^- y^+ + x^+ y^- - \eta^2 x^- y^- - \vec{x}_\perp \vec{y}_\perp \\ &= x_- y_+ + x_+ y_- + \eta^2 x_+ y_+ - \vec{x}_\perp \vec{y}_\perp \end{aligned}$$

- spatial distance ($\Delta x^+ = 0$)

$$\Rightarrow R^2 = -\eta^2 (\Delta x^-)^2 - (\Delta \vec{x}_\perp)^2$$

Why Near Light Cone QCD ?

- Near light cone Wilson loop correlation functions determine the dipole-dipole cross section in QCD. By taking into account the hadronic wave functions one obtains hadronic cross sections



A. I. Shoshi, F. D. Steffen and H. J. Pirner, Nucl. Phys. A **709** (2002) 131.
O. Nachtmann, Annals Phys. **209** (1991) 436.

Motivation

- Near light-cone coordinates are a promising tool to investigate high energy scattering on the lattice
- NLC make high lab frame momenta accessible on the lattice with small a_-

$$P_- = -\eta P_3$$

$$P_j = \frac{2\pi}{N_j a_j} n_j \qquad n_j = 0, \ldots, N_j - 1$$

- Near light-cone QCD has a nontrivial vacuum which cannot be neglected even in the light-cone limit

Problems of LGT near the LC

- Euclidean gluonic Lagrange density

$$x^+ = -\mathrm{i}\, x_E^+ \qquad S = \mathrm{i}\int d^4x_E \mathcal{L}_E \equiv \mathrm{i} S_E \qquad Z = \int DA e^{-S_E}$$

$$\mathcal{L}_E \equiv \frac{1}{2}F^a_{+-}F^a_{+-} + \sum_k \left(\frac{\eta^2}{2}F^a_{+k}F^a_{+k} - \mathrm{i}F^a_{+k}F^a_{-k}\right) + \frac{1}{2}F^a_{12}F^a_{12}$$

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g\, f^{abc} A_\mu{}^b A_\nu{}^c$$

- a complex action remains (similar to finite baryonic density) -> sign problem
- Possible way out: Hamiltonian formulation

$\Longrightarrow$ Sampling of the ground state wavefunctional with guided diffusion quantum Monte-Carlo

$$\begin{aligned} |\Psi_0\rangle &= \lim_{t\to\infty} \exp\left[-t\left(\widehat{H}_0 - E\right)\right]|\Phi\rangle \\ &= \lim_{\substack{\Delta t\to 0\\ N\Delta t\to\infty}} \prod_{n=1}^{N}\left\{\exp\left[-\Delta t\left(\widehat{H}_0 - E\right)\right]\right\}|\Phi\rangle \end{aligned}$$

Continuum Hamiltonian and momentum

- Perform Legendre transformation of the Lagrange density for $A_+ = 0$:

$$\mathcal{L} = \sum_a \left[\frac{1}{2} F^a_{+-} F^a_{+-} + \sum_{k=1}^{2} \left(F^a_{+k} F^a_{-k} + \frac{\eta^2}{2} F^a_{+k} F^a_{+k} \right) - \frac{1}{2} F^a_{12} F^a_{12} \right]$$

$$\Pi^a_k = \frac{\delta \mathcal{L}}{\delta \partial_+ A^a_k} = \frac{\delta \mathcal{L}}{\delta F^a_{+k}} = F^a_{-k} + \eta^2 F^a_{+k}$$

$$\Pi^a_- = \frac{\delta \mathcal{L}}{\delta \partial_+ A^a_-} = \frac{\delta \mathcal{L}}{\delta F^a_{+-}} = F^a_{+-}$$

$$\left[\Pi^a_m(\vec{x}), A^b_n(\vec{y})\right] = -\mathrm{i}\delta^{ab}\delta_{mn}\delta^{(3)}(\vec{x}-\vec{y})$$

- Then, the Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}\sum_a \left[\Pi^a_- \Pi^a_- + F^a_{12} F^a_{12} + \sum_{k=1}^{2} \frac{1}{\eta^2} \left(\Pi^a_k - F^a_{-k} \right)^2 \right]$$

- The Hamiltonian has to be supplemented by Gauss law

$$\left(D^{ab}_- \Pi^b_-(\vec{x}) + \sum_{k=1}^{2} D^{ab}_k \Pi^b_k(\vec{x}) \right) |\Psi\rangle = 0 \quad \forall\, \vec{x}, a$$

$$E'_{\parallel}, B'_{\parallel} = E_{\parallel}, B_{\parallel}$$

$$E'_{\perp}, B'_{\perp} \propto \gamma E_{\perp}, \gamma B_{\perp}$$

$$\gamma = \frac{1}{\eta}$$

- Problem: Linear momentum operator term $\Pi^a_k F^a_{-k} + F^a_{-k} \Pi^a_k$ disturbs QDMC

- Solution: Momentum operator (obtained via the energy-momentum tensor)

$$\mathcal{P}_- = \frac{1}{2}\left(\Pi_k^a F_{-k}^a + F_{-k}^a \Pi_k^a\right)$$

- Commutation relations

$$[H, P_-] = 0$$
$$[H, G] = 0$$

  - Gauss law and P_- are constants of motion

- Choose trial state translation invariant -> $P_- |\Phi\rangle = 0$

$$|\Psi_0\rangle = \lim_{\tau\to\infty} \exp\left[-\left(\widehat{H} - E_0\right)\tau\right]|\Phi\rangle$$

-> Translation invariant ground state

- It is sufficient to consider H_{eff} for translation invariant trial states

$$\mathcal{H}_{eff} = \mathcal{H} + \frac{1}{\eta^2}\mathcal{P}_-$$
$$= \frac{1}{2}\sum_a \left[\Pi_-^a \Pi_-^a + F_{12}^a F_{12}^a + \sum_{k=1}^{2} \frac{1}{\eta^2}\left(\Pi_k^a \Pi_k^a + F_{-k}^a F_{-k}^a\right)\right]$$

$$|\Psi_0\rangle = \lim_{\tau\to\infty} \exp\left[-\left(H_{eff} - E_{eff}\right)\tau\right]|\Phi\rangle$$

- First explorative approach:
  Investigate H_{eff} variationally on the lattice

The lattice Hamiltonian

- $$U_i(x) \equiv \mathcal{P} \exp\left(i\, g \int_x^{x+\hat{e}_i} dy_\mu\, A^a_\mu(y)\, \frac{\sigma_a}{2} \right)$$ $$\xi \equiv \frac{a_-}{a_\perp}$$ $$\lambda \equiv \frac{4}{g^4}$$
- Derivation of the lattice Hamiltonian from the path integral formulation with the transfer matrix-method (Creutz Phys. Rev. D 15, 1128):

$$\mathcal{H}_{\text{lat}} = \frac{1}{N_- N_\perp^2} \frac{1}{a_\perp^4} \frac{2}{\sqrt{\lambda}} \sum_{\vec{x}} \Bigg\{ \sum_a \frac{1}{2}\, \Pi^a_-(\vec{x})^2 + \frac{1}{2}\, \lambda\, \mathrm{Tr}\left[\mathbb{1} - \mathrm{Re}\Big(U_{12}(\vec{x}) \Big) \right] + \sum_{k,a} \frac{1}{2} \frac{1}{\xi^2 \eta^2} \left[\Pi^a_k(\vec{x}) - \sqrt{\lambda}\, \mathrm{Tr}\left[\frac{\sigma_a}{2}\, \mathrm{Im}\Big(U_{-k}(\vec{x}) \Big) \right] \right]^2 \Bigg\} . \quad (5$$

$$\left[\Pi^a_j(\vec{x}), U_{j'}(\vec{x}')\right] = \frac{\sigma_a}{2}\, U_j(\vec{x})\, \delta_{j,j'}\, \delta_{\vec{x},\vec{x}'} ,$$
$$\left[\Pi^a_j(\vec{x}), U^\dagger_{j'}(\vec{x}')\right] = -U^\dagger_j(\vec{x})\, \frac{\sigma_a}{2}\, \delta_{j,j'}\, \delta_{\vec{x},\vec{x}'}$$

- Hamiltonian does only depend on the product $\tilde{\eta} \equiv \xi \cdot \eta$
  - Only one effective parameter is needed
  - Light cone limit might be interpreted in two ways:
    - Light cone limit with equal lattice constants $\xi = 1$
    - Effective equal time theory with vanishing anisotropy $\eta = 1$

• Effective lattice Hamiltonian

$$\mathcal{H}_{\text{eff,lat}} = \frac{1}{N_- N_\perp^2} \frac{1}{a_\perp^4} \frac{2}{\sqrt{\lambda}} \sum_{\vec{x}} \left\{ \frac{1}{2} \sum_a \Pi_-^a(\vec{x})^2 + \lambda \, \mathrm{Tr}\left[\mathbb{1} - \mathrm{Re}\Big(U_{12}(\vec{x}) \Big) \right] + \sum_{k,a} \frac{1}{2} \frac{1}{\tilde{\eta}^2} \left[\Pi_k^a(\vec{x})^2 + \lambda \, \mathrm{Tr}\left[\frac{\sigma_a}{2} \mathrm{Im}\Big(U_{-k}(\vec{x}) \Big) \right]^2 \right] \right\}$$

• Additional global Z_2 invariance in comparison to the full lattice Hamiltonian

$$U_k(\vec{x}_\perp, x^-) \rightarrow z \, U_k(\vec{x}_\perp, x^-) \quad \forall \, \vec{x}_\perp \text{ and } x^- \text{ fixed} \,, \quad z \in Z_2$$
$$U_{-k}(\vec{x}_\perp, x^-) \rightarrow z \, U_{-k}(\vec{x}_\perp, x^-)$$

• Restrict to the spontaneously broken phase

$$\Big\langle \mathrm{Tr}\Big[\mathrm{Re}\Big(U_{-k} \Big) \Big] \Big\rangle \begin{cases} = 0 \; Z_2 \text{ symmetric phase} \\ \neq 0 \; Z_2 \text{ broken phase} \end{cases}$$

← order parameter

Analytic asymptotic solutions

- Strong coupling wavefunctional (perturbation theory)

$$|\Psi_0\rangle = \prod_{\vec{x}} \exp\left\{\frac{1}{3}\,\lambda\,\tilde{\eta}^2 \mathrm{Tr}\left[\mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right]\right.$$
$$\left.\frac{1}{16}\,\frac{\lambda}{1+\tilde{\eta}^2}\sum_k \Big(\mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x})\Big)\right]\Big)^2\right\}\left|\Psi_0^{(0)}\right\rangle + \mathcal{O}(\lambda^2)$$

$$\lambda = \frac{4}{g^4}$$

  - Product state of single plaquette wavefunctionals

- Weak coupling wavefunctional

$$\Psi_0 = \exp\left\{-\sqrt{\lambda}\sum_{\vec{x},\vec{x}'}\sum_a \frac{1}{2}\vec{B}^a(\vec{x})\mathbf{\Gamma}_{\tilde{\eta}}(\vec{x}-\vec{x}')\frac{1}{2}\vec{B}^a(\vec{x}')\right\}$$

$$\mathbf{\Gamma}_{\tilde{\eta}}(\vec{x}-\vec{x}') \equiv \begin{pmatrix} \gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') & 0 & 0 \\ 0 & \gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') & 0 \\ 0 & 0 & \tilde{\eta}^2\gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') \end{pmatrix}$$

$$U_{ij}(\vec{x}) = \exp\Big(\mathrm{i}F^a_{ij}(\vec{x})\lambda^a\Big)$$
$$F^a_{ij}(\vec{x}) = \epsilon_{ijk}B^a_k(\vec{x}) + g\, f^{abc}A^b_i(\vec{x})A^c_j(\vec{x})$$
$$B^a_k(\vec{x}) = \epsilon_{klm}\left[A^a_m(\vec{x}) - A^a_m(\vec{x}-\vec{e}_l)\right]$$

  - Multivariate Gaussian wavefunctional

  - LC limit: transversal dynamics decouple
    -> Effective reduction to a 2-dim (spatial) theory

Variational optimization

- Trial wavefunctional

$$\Psi_0(\rho,\delta) = \prod_{\vec{x}} \exp\left\{\sum_{k=1}^{2} \rho\,\mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x})\Big)\right] + \delta\,\mathrm{Tr}\left[\mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right]\right\}$$

  - Restrict to product of single plaquette wavefunctionals

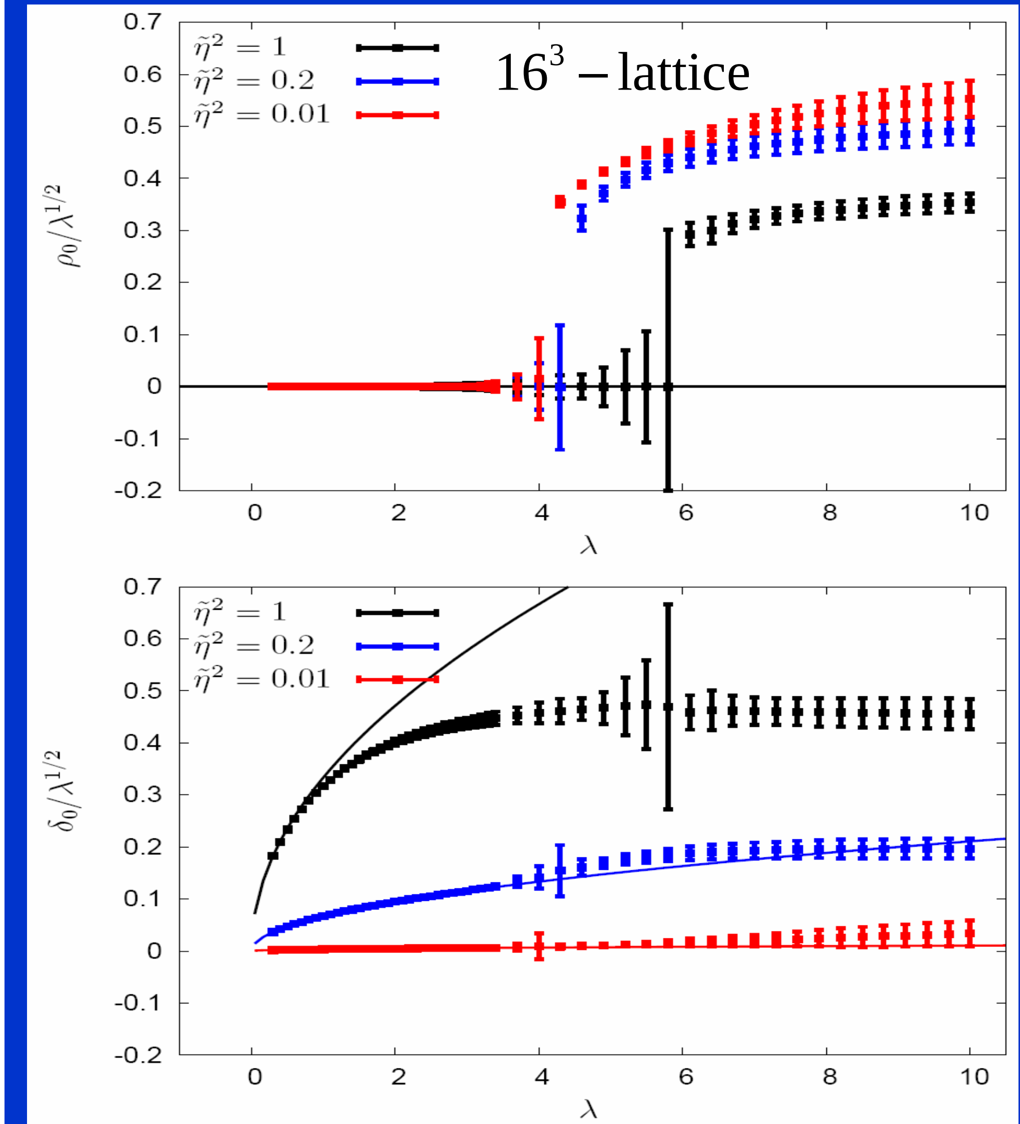
- Optimize the energy density with respect to ρ and δ

$$\begin{aligned}\epsilon_0(\rho,\delta) = & \frac{1}{N_- N_\perp^2}\frac{1}{a_\perp^4}\frac{2}{\sqrt{\lambda}}\sum_{\vec{x}}\left[\left(\frac{3}{4}\frac{\delta}{\tilde{\eta}^2}-\frac{\lambda}{2}\right)\left\langle \mathrm{Tr}\left[\mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right]\right\rangle_{\Psi_0(\rho,\delta)} + \lambda\right] \\ & + \frac{1}{N_- N_\perp^2}\frac{1}{a_\perp^4}\frac{2}{\sqrt{\lambda}}\sum_{\vec{x},k}\left[\frac{3}{8}\rho\left(1+\frac{1}{\tilde{\eta}^2}\right)\left\langle \mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x})\Big)\right]\right\rangle_{\Psi_0(\rho,\delta)}\right. \\ & \left. + \frac{\lambda}{2}\frac{1}{\tilde{\eta}^2}\left(1-\frac{1}{4}\left\langle\left(\mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x})\Big)\right]\right)^2\right\rangle_{\Psi_0(\rho,\delta)}\right)\right] \end{aligned} \quad (5.3)$$

- The expectation values are computed via the prob. measure

$$dP(U) = |\Psi_0(a,b)|^2 \prod_{\vec{x},j} \mathcal{D}U_j(\vec{x})$$

  - produced by a standard local heatbath algorithm



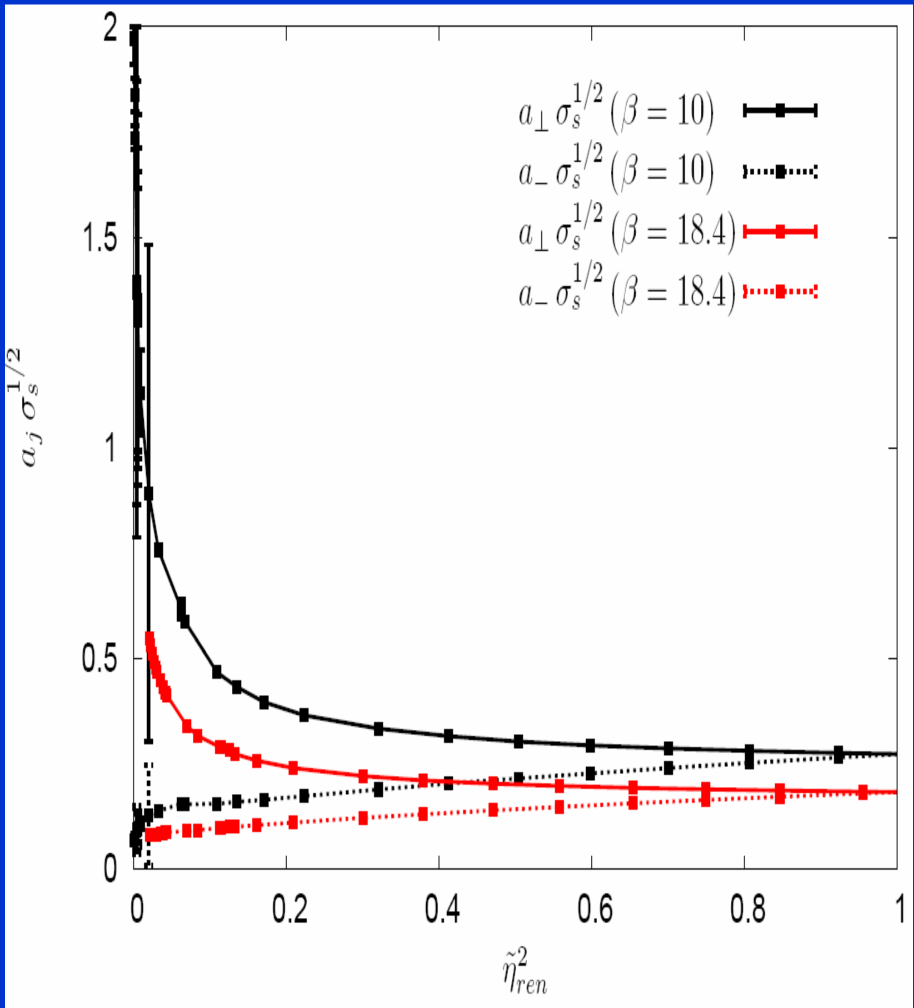
Optimal ρ_0 and δ_0 (strong coupling)

- Strong coupling limit is reproduced (solid line)
  One sees the effect of the phases associated with the center symmetry

$$\left\langle \mathrm{Tr}\left[\mathrm{Re}\left(U_{-k}\right)\right]\right\rangle \begin{cases} = 0 \; Z_2 \text{ symmetric phase} \\ \neq 0 \; Z_2 \text{ broken phase} \end{cases}$$

$$U_k(\vec{x}_\perp, x^-) \rightarrow z\, U_k(\vec{x}_\perp, x^-)$$

Lattice spacings



- Extract heavy quark potential in lattice units from loops extended in the (1,2)- and (-,k)-plane

$$\Rightarrow \left.\begin{matrix} V_{\bar{q}q}(\Delta x^-) \\ V_{\bar{q}q}(\Delta \vec{x}_\perp) \end{matrix}\right\} \Rightarrow \begin{matrix} \sigma_s\, a_-\, a_\perp \\ \sigma_s\, a_\perp^2 \end{matrix}$$

- $a_\perp = a_\perp(\beta, \eta) \quad \Rightarrow$
  the transversal lattice constant $a_\perp$ is varying with the boost parameter η

- Should introduce two different couplings for the longitudinal and transversal part of the Hamiltonian

- $\Longrightarrow$ three couplings $\lambda_-, \lambda_\perp, \eta$ which can be tuned in such a way that $a_\perp$ is independent of η_{ren}

a_- is $a_- = \eta_{ren}\, a_\perp$

Conclusions:

- Near light cone coordinates are a promising tool to calculate high energy scattering on the lattice
- Euclidean path integral as well as Diffusion Quantum Monte Carlo treatments of the theory are not possible due to complex phases during the update process
- An effective lattice Hamiltonian, however, avoiding this problem can be derived
- Simple guidance wavefunctionals have been constructed for strong and weak coupling and optimized variationally
- Work is in progress to correct for unwanted dependences of

  the lattice constants on the near light cone parameter , then a

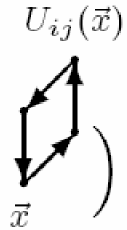
  calculation of a dipole- plaquette cross section is feasible

Thank you for your attention…

$$\Psi_0 \;\propto\; \exp\left\{ \sqrt{\lambda} \sum_{\vec{x},\vec{x}'} \sum_{k} \Gamma^{kk}_{\tilde{\eta}}(\vec{x}-\vec{x}')\, R_k(\vec{x},\vec{x}') \right\}$$

$$R_k(\vec{x},\vec{x}') \;=\; \frac{1}{2}|\epsilon_{kij}| \cdot$$

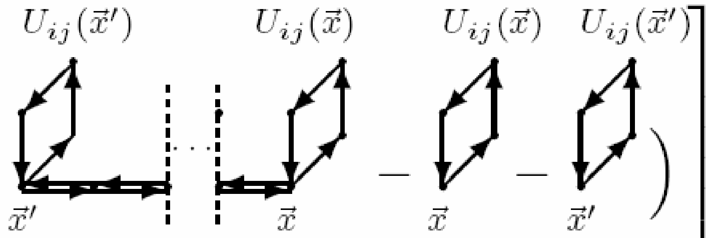
$\mathrm{Tr}\left[\mathrm{Re}\left(\right.\right.$



$\left.\left.\right)\right]$

for $\vec{x} = \vec{x}'$ and

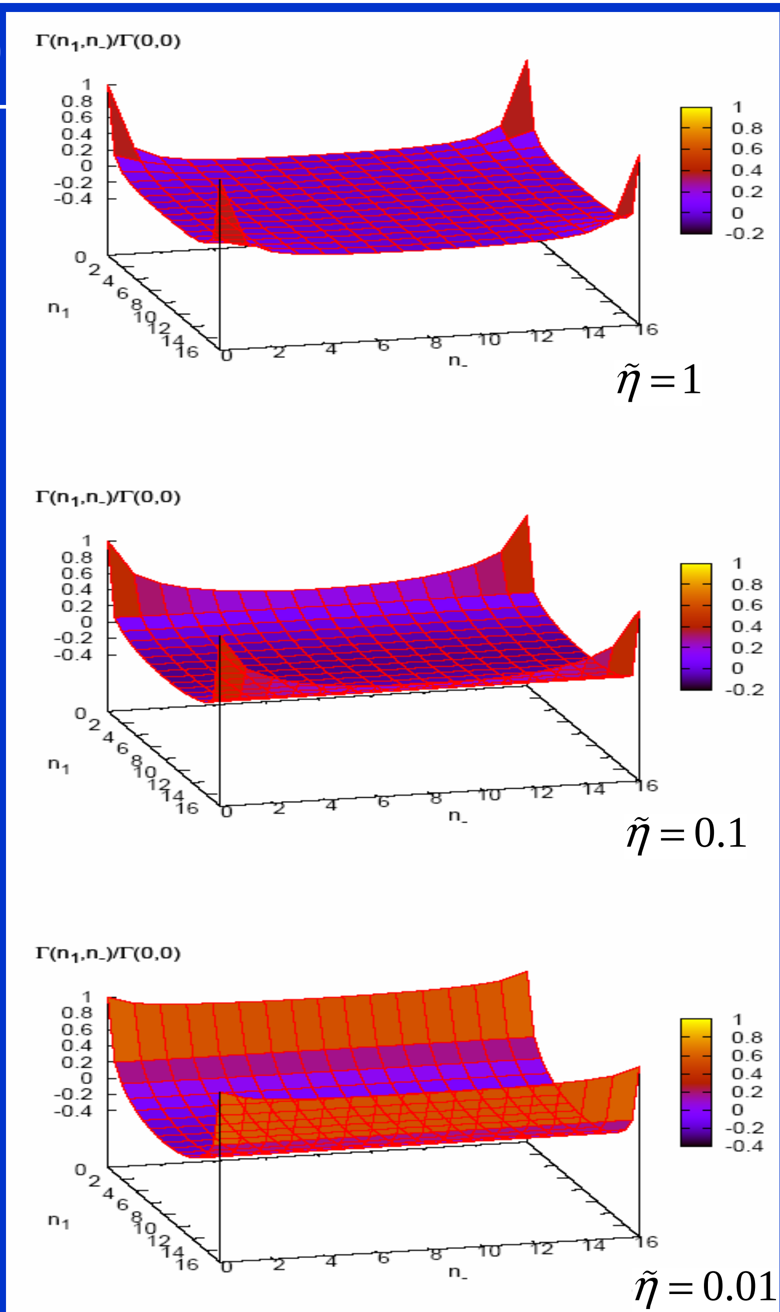
$\frac{1}{\#p} \sum_{\forall\, p} \frac{1}{2} \mathrm{Tr} \left[\mathrm{Re} \left(\right.\right.$



$\left.\left.\right)\right]$

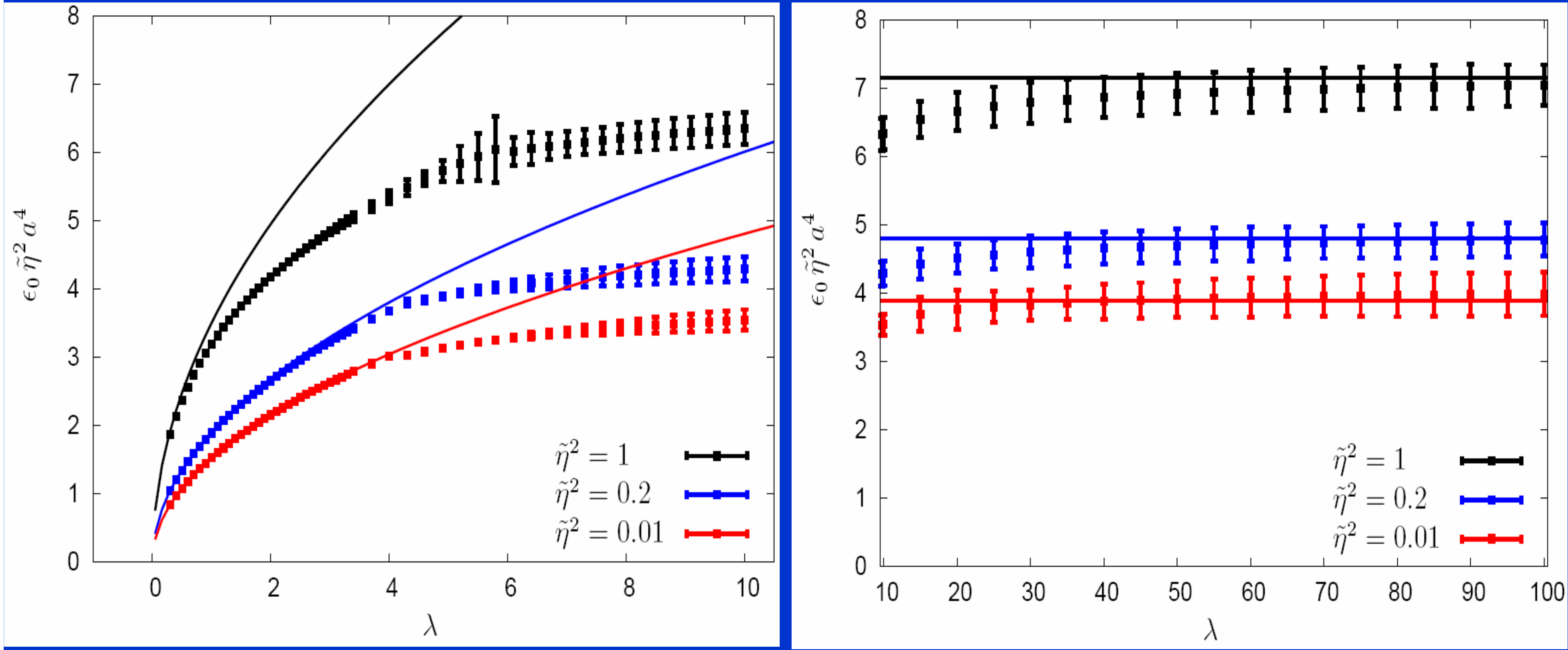
for $\vec{x} \neq \vec{x}'$

$$\frac{\gamma_{\tilde{\eta}}(\Delta \vec{x})}{\gamma_{\tilde{\eta}}(\vec{0})}$$



- Equal time theories:
  Covariance matrix weakly off-diagonal
  - Product of single site wavefunctionals suitable
- Decreasing $\tilde{\eta}$
  - Correlations among longitudinally separated plaquettes become increasingly important
- LC Limit
  - Each plaquette is equally correlated with every other longitudinally separated plaquette

Lattice Energy density 16x16x16



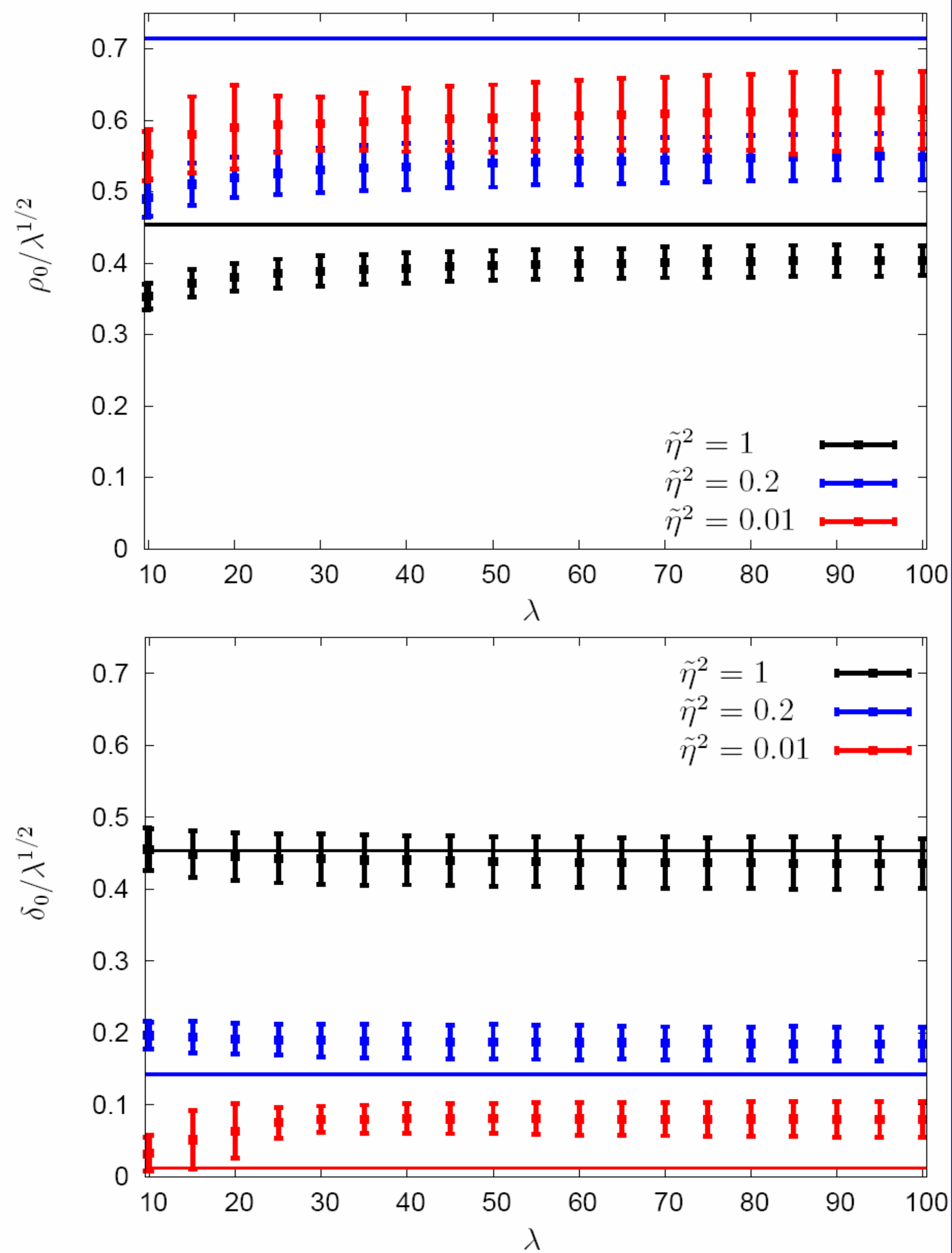
- Nice agreement between prediction/measurement for all values of $\tilde{\eta}$

Strong coupling prediction (solid line):

$$\epsilon_0 = \frac{2}{a_\perp^4 \tilde{\eta}^2}\left(\frac{3}{4} + \tilde{\eta}^2\right)\sqrt{\lambda} + \mathcal{O}(\lambda^{3/2})$$

Weak coupling prediction (solid line):

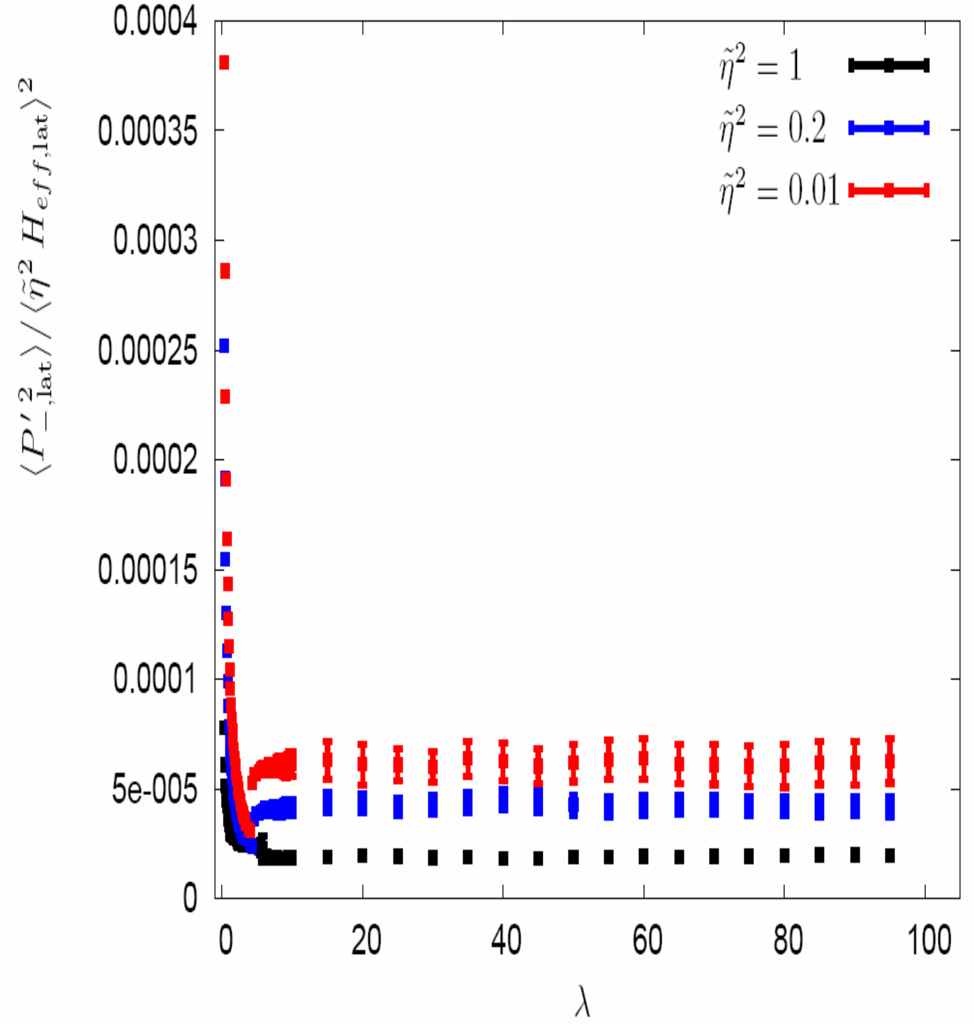
$$\epsilon_0 = \frac{1}{a_\perp^4}\frac{6}{\tilde{\eta}^2}\sum_{\vec{k}}\frac{1}{N_- N_\perp^2}\left(\tilde{\eta}^2 s_1^2 + \tilde{\eta}^2 s_2^2 + s_3^2\right)^{1/2}$$



Optimal ρ_0 and δ_0 (weak coupling)

- Assymptotic weak coupling behavior seen for $\tilde{\eta} = 1$
- Increasing disagreement in the weak coupling regime for decreasing $\tilde{\eta}$
    - Only effective description possible

P_- as translation operator :



- How close is P_- to the exact generator of lattice translations ? (important for the applicability of QDMC)
- For every purely real valued wave-functional ψ_0 we have

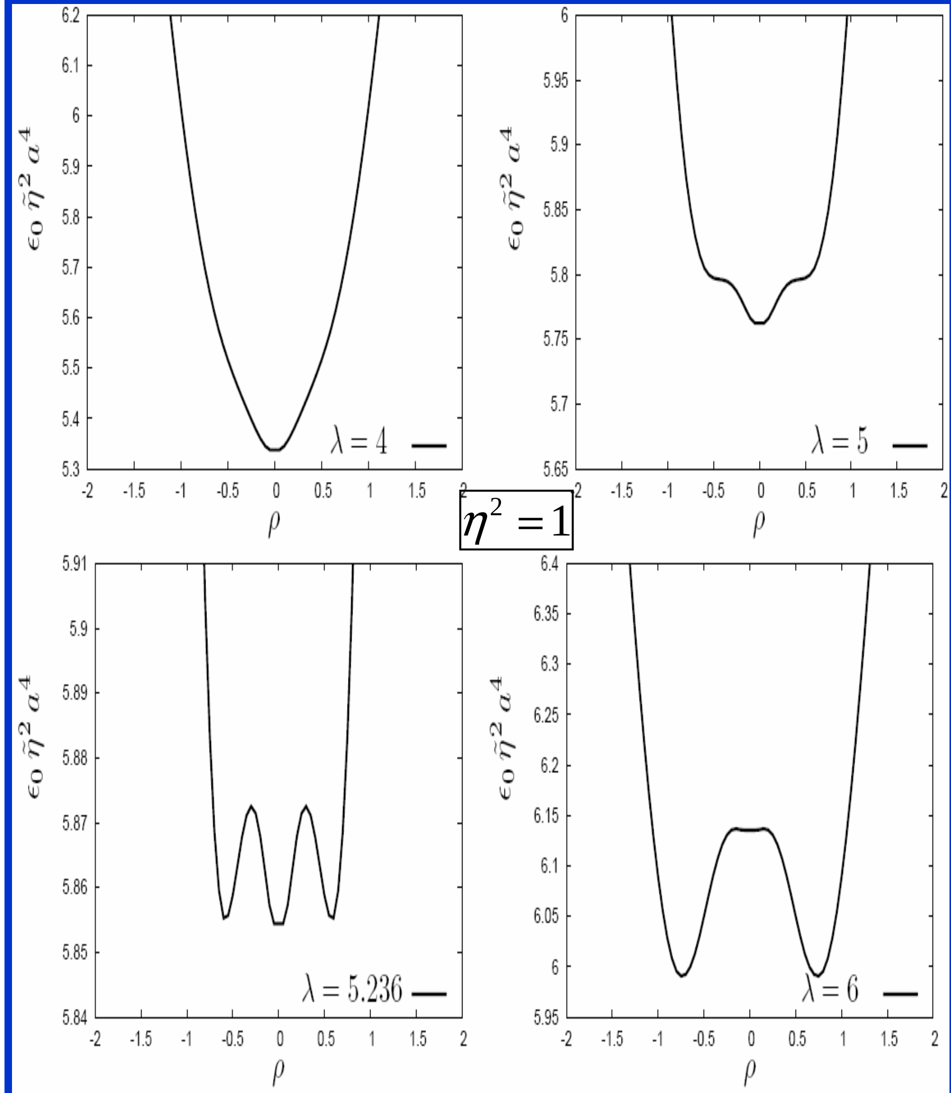
$$\langle \Psi_0 | \, P'_{-,\text{lat}} \, | \Psi_0 \rangle \;=\; 0$$

which follows from partial integration and is consistent with an exact eigen-state

$$\langle \Psi_0 | \, \Pi^a_j(\vec{y}) \; g(U) \, | \Psi_0 \rangle \;=\; - \langle \Psi_0 | \, g(U) \; \Pi^a_j(\vec{y}) \, | \Psi_0 \rangle$$

- Look at the second moment $\langle \Psi_0 | \, P'^{\,2}_{-,\text{lat}} \, | \Psi_0 \rangle$
- Fluctuations of P_- are always less than 1% of the total energy around its mean value equal to zero and may be neglected in realistic computations

Phase transition



- Energy density for fixed optimal δ_0 as a function of ρ for different values of λ
- Z_2 trafo corresponds to $\rho \to -\rho$
- single minimum turns into two degenerate minima which differ by a Z_2 trafo
- 1st order phase transition in accordance with the Ehrenfest classification
- Analytic estimate (strong coupling)

$$\left\langle \left(\frac{1}{2} \mathrm{Tr} \left[\mathrm{Re} \left(U_{-k} \right) \right] \right) \right\rangle_{\Psi_0(\rho,\delta)} \approx \rho \left(1 - \frac{2}{3}\rho^2 + \frac{2}{3}\rho^4 \right)$$

$$\left\langle \left(\frac{1}{2} \mathrm{Tr} \left[\mathrm{Re} \left(U_{-k} \right) \right] \right)^2 \right\rangle_{\Psi_0(\rho,\delta)} \approx \frac{1}{4} + \frac{1}{2}\rho^2 - \frac{1}{2}\rho^4 + \frac{8}{15}\rho^6$$

$$\Longrightarrow \quad \lambda_c(\tilde{\eta}^2) \approx 3\left(1 + \tilde{\eta}^2\right)$$

- By choosing $\lambda > \lambda_c$ fixed we are able to decrease η without crossing the critical line

Renormalization

- Euclidean LGT: String tension by expectation values of extended timelike Wilson loops
- Lorentz invariance $\Longrightarrow$ time is not a special coordinate
  heavy quark potential may be extracted from spacelike Wilson loops, too
- Hamiltonian dynamics $\Longleftrightarrow$ Lagrangean dynamics

$$\Longrightarrow \quad \begin{aligned} \langle W_{ij}(n,m)\rangle &= e^{-n\,a_i\,V(m\,a_j)} \qquad , n >> m \\ V(r) &= \sigma_s\,r + V_0 + \frac{c}{r} \end{aligned}$$

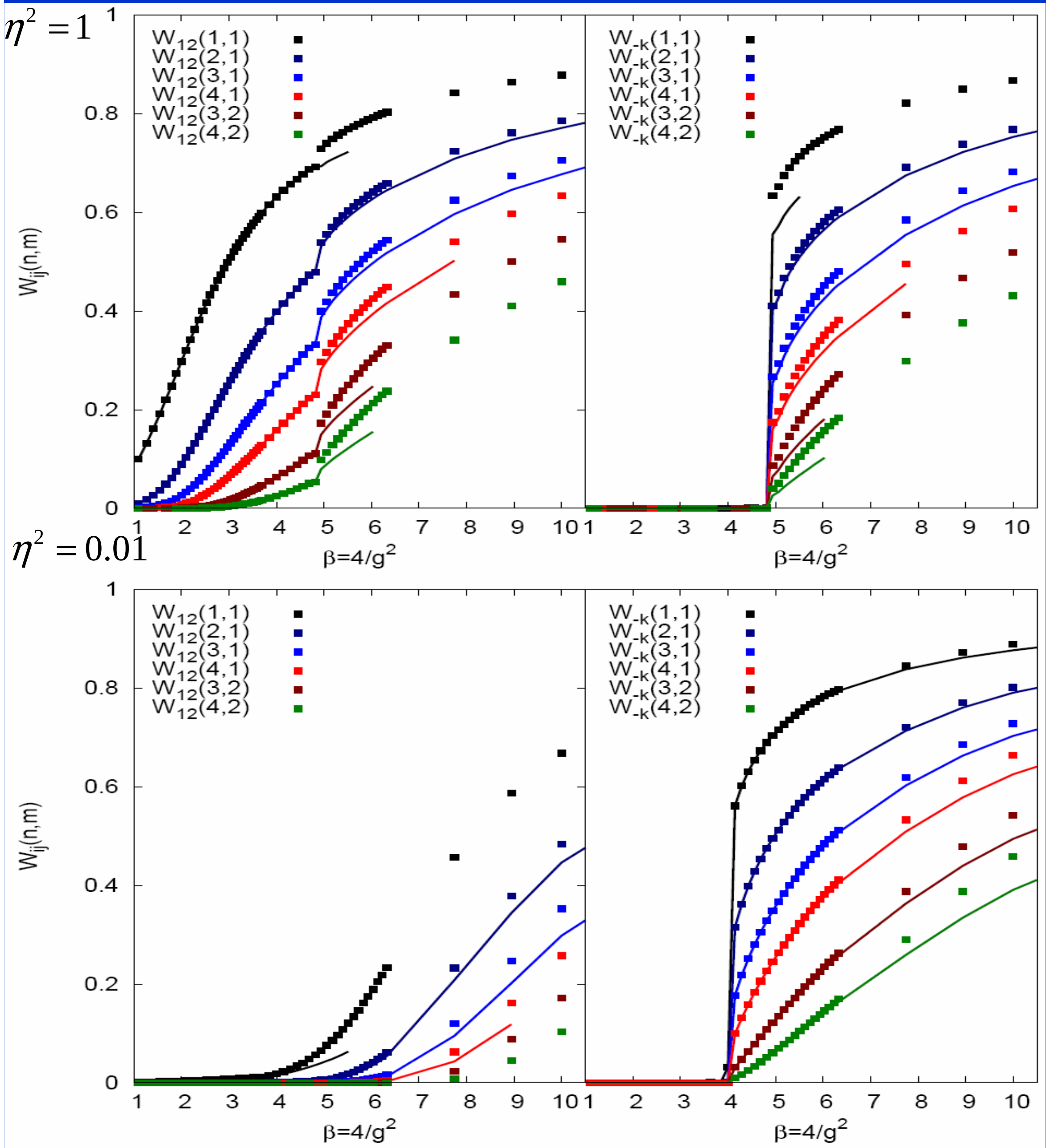
The string tension is obtained by fitting the exponential fall off of Wilson loops elongated along the long side n and the short side m to $V(r)$

- There are two equivalent ways to extract the string tension for quarks which are seperated along the 2-axes

$$\text{either } W_{12} \Longrightarrow K_\perp = \sigma_s a_\perp^2$$

$$\text{or } W_{-2} \Longrightarrow K_- = \sigma_s a_- a_\perp$$

- From these one can extract $\tilde{\eta}_{ren} \equiv \frac{a_-}{a_\perp} = \frac{K_-}{K_\perp}$ a_- and $a_\perp$



Wilson loop expectation values

$$\left\langle \frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\left(U_{-k}\right)\right]\right\rangle_{\Psi_0(\rho_0,\delta_0)} = \frac{I_2(4\rho_0)}{I_1(4\rho_0)}$$

$$\left\langle W_{ij}(n,m)\right\rangle_{\Psi_0(\rho_0,\delta_0)} = \left\langle \frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\left(U_{ij}\right)\right]\right\rangle^{n\cdot m}_{\Psi_0(\rho_0,\delta_0)}$$

- Phase transition obvious
- Nice strong coupling behavior
- Better agreement to strong coupling for smaller values of $\eta \Rightarrow$ effective reduction to a 2-d decoupled theory