Light-cone Hadron Structure from Lattice QCD

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INT Summer School on Lattice QCD and its Applications

Outline

Physics on the light-cone

Many experimental situations dominated by physics at short light-cone distances:

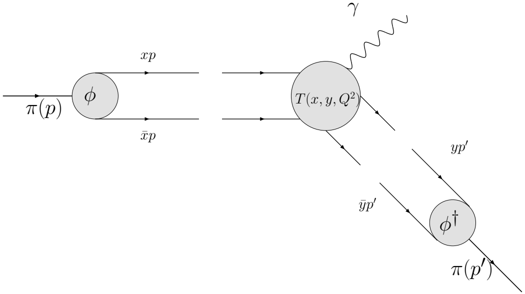
- Hard exclusive processes
- Deep inelastic scattering
- Deeply virtual Compton scattering
- Drell-Yan (pp) scattering
- Semi-inclusive deep inelastic scattering
- Heavy vector boson production
- ...

All involve non-perturbative QCD

Hard Exclusive Processes

Exclusive processes at large $Q^2 \to \infty$ can be factorised into:

- perturbative hard scattering amplitude (process dependent)
- non-perturbative wave functions describing the hadron's overlap with (lowest) Fock state



Example: $F_\pi(Q^2)$

$x + \bar{x} = 1$

$$F(Q^2) = \int_0^1 dx \int_0^1 dy \phi^\dagger(y, Q^2) T(x, y, Q^2) \phi(x, Q^2) \left[1 + \mathcal{O}(m^2/Q^2)\right]$$

Distribution Amplitudes

Distribution amplitudes $\phi_\pi, \phi_K, \ldots$ are universal (process independent):

- exclusive non-leptonic decays $(B \to \pi\pi, KK)$
- semi-leptonic decays $(B \to \pi l \nu)$
- electromagnetic form factors
- vector meson production, etc.

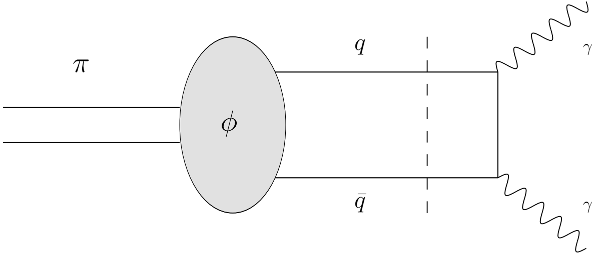
Distribution Amplitude:

- Related to the meson's Bethe–Salpeter wave function by an integral over transverse momenta

$$\phi_\Pi(x, \mu^2) = Z_2(\mu^2) \int^{|k_\perp|<\mu} d^2 k_\perp \, \phi_{\Pi,BS}(x, k_\perp).$$

- Describes the momentum distribution of the valence quarks in the meson Π

Distribution Amplitudes



Amplitude for converting a pion into $q\bar{q}$ pair separated by lightcone distance z

$$\langle 0|\bar{q}(0)\gamma_\mu\gamma_5[0,z]u(z)|\Pi^+(p)\rangle = if_\Pi p_\mu \int_{-1}^{1} d\xi\, e^{-i\xi p\cdot z}\phi_\Pi(\xi,\mu^2)\,,$$

where $z^2 = 0$ and $\xi = x - \bar{x}$

Normalisation:

$$\int_{-1}^{1} d\xi\, \phi_\Pi(\xi,\mu^2) = 1\,.$$

Distribution Amplitudes

Separate transverse and longitudinal variables

- transverse – scale dependence
- longitudinal – Gegenbauer polynomials $C_n^{3/2}(\xi)$

$$\phi_\Pi(\xi,\mu^2) = \frac{3}{4}(1-\xi^2)\left(1+\sum_{n=1}^{\infty} a_n^\Pi(\mu^2)\, C_n^{3/2}(\xi)\right).$$

- a_n contain non-perturbative information $\Longrightarrow$ lattice QCD

- At LO a_n renormalise multiplicatively: $a_n(\mu^2) = L^{\gamma_n^{(0)}/(2\beta_0)}\, a_n(\mu_0^2)$

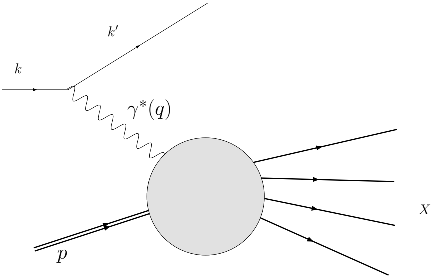
  [$L \equiv \alpha_s(\mu^2)/\alpha_s(\mu_0^2)$, $\beta_0 = 11 - 2N_f/3$]
- Anomalous dimensions $\gamma_n^{(0)}$ rise with spin, n, $\Rightarrow$ higher-order contributions are suppressed at large scales

$$\phi(\xi,\mu^2\to\infty) = \phi_{as}(\xi) = \frac{3}{4}(1-\xi^2).$$

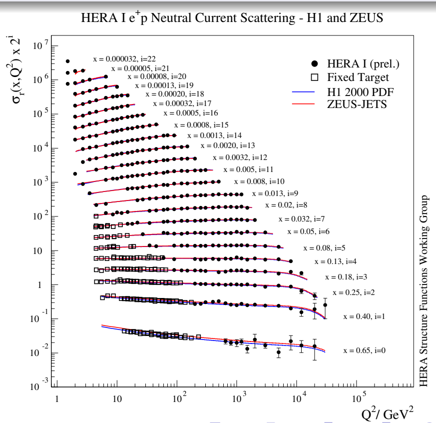
Deep-inelastic scattering (DIS)

Foundation of QCD: $e\, p \to X$

- 1960s – DIS experiments at SLAC "see" partons: Bjorken scaling
- 1970s – QCD postdicts scaling and predicts scaling violations
- . . . 2007 – beautifully confirmed in precision experiments



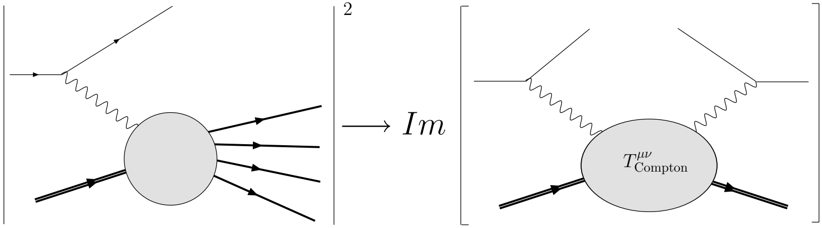
$$Q^2 = -q^2 \qquad x = \frac{Q^2}{2p\cdot q}$$



Deep-inelastic scattering (DIS)

Optical Theorem

- Hadronic part of DIS determined by forward Compton tensor



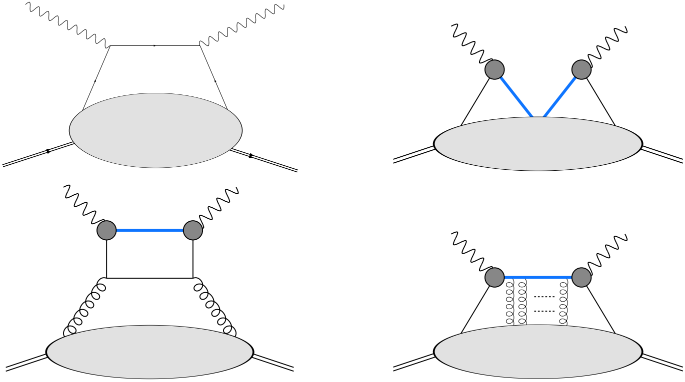
Hadronic tensor: $\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{Q^4}\frac{E'}{E} L_{\mu\nu} W^{\mu\nu} \Rightarrow$ structure functions

$$W^{\mu\nu}(p,q) \sim g^{\mu\nu} F_1(x,Q^2) + \frac{p^\mu p^\nu}{p^2} F_2(x,Q^2)$$
$$+\frac{\epsilon^{\mu\nu\rho\sigma} q^\rho}{p\cdot q}\left[S_\sigma g_1(x,Q^2) + \left(S_\sigma - p_\sigma \frac{S\cdot q}{p\cdot q}\right) g_2(x,Q^2)\right]$$

Deep-inelastic scattering (DIS)

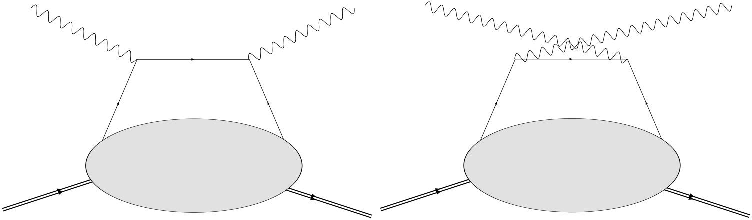
Compton tensor at large Q^2

- Factorisation: proven in perturbative QCD
  - Perturbative kernel
  - Non-perturbative parton distributions



Compton tensor at large Q^2

- Factorisation: proven in perturbative QCD
  - Perturbative kernel
  - Non-perturbative parton distributions
- Handbag dominates as $Q^2 \to \infty$
- Leading-twist (leading power in $\frac{1}{Q^2}$): take leading singularity in quark propagator



Parton distributions

Non-perturbative structure encoded in parton distributions

- Three quark PDFs at leading twist

$$\mathcal{F}(x,\mu^2) \sim \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle P,S|\overline{q}(0)\ [0,\lambda n]\,\Gamma_{\mathcal{F}}\, q(\lambda n)|P,S\rangle$$

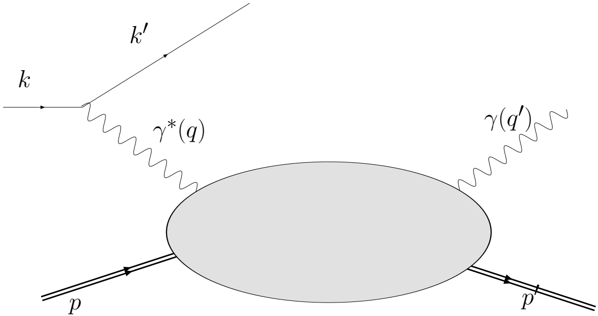
  where n^μ is light-like ($n^2=0$) and $[0,\lambda n] = e^{-ig\int_\lambda^0 n\cdot A(\lambda' n) d\lambda'}$

- DIS: unpolarised $q(x)$ for $\Gamma_{\mathcal{F}} = n\cdot\gamma$, helicity $\Delta q(x)$ for $\Gamma_{\mathcal{F}} = n\cdot\gamma\gamma_5$
  - Unpolarised and helicity determine F_1 and g_1 structure functions at leading twist
- Transversity $\delta q(x)$ for $\Gamma_{\mathcal{F}} = n^\mu t^\nu \sigma_{\mu\nu}$ suppressed by $\frac{m_u}{|Q|}$ in DIS (*c.f.* Drell-Yan)
- Also three leading twist gluon PDFs
- Scale dependence determined by DGLAP evolution equations

Deeply-virtual Compton scattering (DVCS)

$e(k)\, T(p) \longrightarrow T(p')\, e(k')\, \gamma(q')$

- A detailed probe of hadron structure
- Deep kinematics: $Q^2 = -(k'-k)^2 \gg M_T$ and $\Delta^2 = (p'-p)^2 \ll Q^2$
- Related processes: e.g. deeply-virtual meson production, $e\,\gamma \to e\,\pi\,\pi$



$t = \Delta^2 = (q'-q)^2$

$x = -\frac{q^2}{2p\cdot q}$

$\xi = -2\frac{n\cdot\Delta}{n\cdot p}$

[D. Müller *et al.*, '94; X. Ji '97; A. Radyushkin '97]

Generalised parton distributions (GPDs)

A detailed probe

- More information than ever about internal structure of hadrons
- Non-perturbative structure parameterised by GPDs
  - Functions of x, skewness (ξ) and momentum transfer (t)
  - Nucleon has 8 GPDs (2 unpol, 2 helicity, 4 transversity), pion has two

E.g. Unpolarised quark GPDs (Dirac structure $n \cdot \gamma$)

Defined by matrix elements of non-local lightcone operators

$$H_q(x,\xi,t)\overline{u}(p')n\cdot\gamma u(p) + E_q(x,\xi,t)\overline{u}(p')\frac{i\sigma^{\alpha\beta}n_\alpha\Delta_\beta}{2M}u(p) \sim \int \frac{d\lambda}{2\pi}e^{i\lambda x}\left\langle p'\left|\overline{q}\,(0)\,n\cdot\gamma\,[0,\lambda n]\,q\,(\lambda n)\right|p\right\rangle$$

Large experimental programs to explore these observables

- JLab, DESY (HERMES) and CERN (COMPASS)

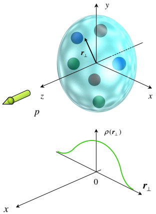
A synthesis of hadron structure

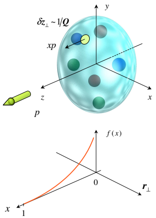
- Forward limit gives PDFs

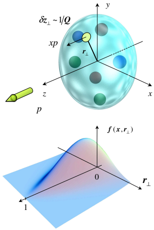
$$H_q(x,0,0) = q(x) \qquad \tilde{H}_q(x,0,0) = \Delta q(x)$$

- Integration over x gives Pauli and Dirac form-factors

$$\int_0^1 dx\, H_q(x,\xi,t) = F_1(t) \qquad \int_0^1 dx\, E_q(x,\xi,t) = F_2(t)$$







Generalised parton distributions

A synthesis of hadron structure

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$$\int_0^1 dx\, H_q(x,\xi,t) = F_1(t) \qquad \int_0^1 dx\, E_q(x,\xi,t) = F_2(t)$$

- Lorentz structure implies *e.g.*

$$\int_0^1 dx\, x^n\, H_q(x,\xi,t) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \xi^{2i} A^q_{n+1,2i}(t) + (1-(-1)^n)\xi^{n+1} C^q_{n+1}(t)$$

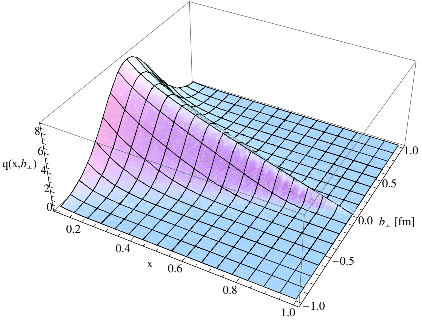
with $A^q_{n,i}(t)$ and $C^q_n(t)$ being generalised form factors $(E_q \to B_q(t))$

Transverse structure and hadron spin

Impact parameter dependent PDFs at $\xi \sim n \cdot \Delta = 0$

- Fourier transform of GPDs [Burkardt '02]

$$q(x, b_\perp) = \int \frac{d\Delta_\perp^2}{(2\pi)^2} e^{-ib_\perp \cdot \Delta_\perp} H_q(x, 0, -\Delta_\perp^2)$$



Ji sum rule

- GPDs allow for decomposition of proton spin

$$\frac{1}{2} = J_g + \sum_q J_q$$

$$J_q = \Sigma_q + L_q = \frac{1}{2}[A_q(0) + B_q(0)]$$

- Allows measurement of L_q

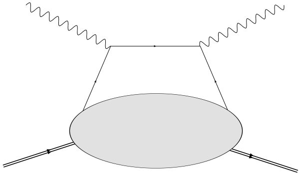
Operator product expansion on the light-cone

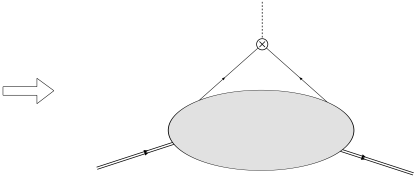
Light-cone on the lattice?

- Euclidean space: light cone rotated to complex direction
- Direct calculation of *e.g.* parton distributions non-trivial

Wilson's operator product expansion comes to the rescue

$$\mathcal{O}(x,0) \xrightarrow{x\to 0} \sum_{i,n} C_{i,n}(x^2) x^n \mathcal{O}_{i,n}$$





Towers of operators: "operator Taylor expansion"

$$\overline{q}(0)\,[0,z]\,z\cdot\Gamma\, q(z) \longrightarrow \sum_i z_{\mu_1}\ldots z_{\mu_n}\overline{q}\Gamma^{\{\mu_1}D^{\mu_2}\ldots D^{\mu_n\}}q - \text{traces}$$

Moments of Distribution Amplitudes

n^{th} moment of a meson distribution amplitude

$$\langle \xi^n \rangle \equiv \int \mathrm{d}\xi \, \xi^n \, \phi(\xi, Q^2), \quad \xi = x_q - x_{\bar{q}}$$

- extracted from matrix elements of twist-2 (symmetric, traceless) operators

$$\langle 0 | (-i)^n \overline{\psi} \gamma_{\{\mu_0} \gamma_5 \overleftrightarrow{D}_{\mu_1} \ldots \overleftrightarrow{D}_{\mu_n\}} \psi | \pi(p) \rangle = f_\pi p_{\{\mu_0} \ldots p_{\mu_n\}} \langle \xi^n \rangle$$

- $\langle \xi^0 \rangle = 1$, $\langle \xi^1 \rangle_\pi = 0$, $\langle \xi^1 \rangle_K \neq 0$

Expansion in terms of Gegenbauer polynomials

$$\phi(x, \mu^2) = 6x(1-x) \sum_{n=0}^{\infty} a_n(\mu^2) C_n^{\frac{3}{2}}(2x-1)$$

e.g.: $a_1 = \frac{5}{3}\langle \xi \rangle$ and $a_2 = \frac{7}{12}\left(5\langle \xi^2 \rangle - 1\right)$

Moments of PDFs and GPDs

Three towers of twist-two quark operators

$$\mathcal{O}^{\{\mu_1\ldots\mu_n\}} = \overline{q}\gamma^{\{\mu_1} \overset{\leftrightarrow}{D}^{\mu_2} \ldots \overset{\leftrightarrow}{D}^{\mu_n\}} q - \text{traces}$$

$$\widetilde{\mathcal{O}}^{\{\mu_1\ldots\mu_n\}} = \overline{q}\gamma^{\{\mu_1}\gamma_5 \overset{\leftrightarrow}{D}^{\mu_2} \ldots \overset{\leftrightarrow}{D}^{\mu_n\}} q - \text{traces}$$

$$\mathcal{O}_T^{\alpha\{\mu_1\ldots\mu_n\}} = \overline{q}\sigma^{\alpha\{\mu_1} \overset{\leftrightarrow}{D}^{\mu_2} \ldots \overset{\leftrightarrow}{D}^{\mu_n\}} q - \text{traces}$$

Forward matrix elements: moments of PDFs

$$\langle P| n_{\mu_1} \ldots n_{\mu_n} \widetilde{\mathcal{O}}^{\{\mu_1\ldots\mu_n\}} |P\rangle = \langle x^n\rangle_{\Delta q} (n\cdot P)^{n-1} \overline{u}(P) \not{n} \gamma_5 u(P)$$

where $\langle x^n\rangle_{\Delta q} = \int_{-1}^{1} dx x^n \Delta q(x)$

Off-forward matrix elements: generalised form factors

$$\langle P+\Delta| n_{\mu_1}\ldots n_{\mu_n} \mathcal{O}^{\{\mu_1\cdots\mu_n\}} |P\rangle = \overline{u}(P')\not{n} u(P) \sum_{i=0,2}^{n-1} (n\cdot P)^{n-i-1} (n\cdot\Delta)^i A^q_{n,i}(t)$$

$$+ i\frac{\overline{u}(P')\sigma^{\alpha\beta} n_\alpha \Delta_\beta u(P)}{2M} \sum_{i=0,2}^{n-1} (n\cdot P)^{n-i-1} (n\cdot\Delta)^i B^q_{n,i}(t)$$

Anatomy of a lattice calculation: Technical aspects

1. Lattice operators
2. Extraction of matrix elements
3. Renormalisation: perturbative or non-perturbative
4. Extrapolations: mass, volume and continuum

Twist-two operators on the lattice

Classification of operators

- Usual classification of operators by twist = dimension - spin (transformation under Lorentz group) is modified
- Lattice theory formulated in Euclidean space: SO(3,1) $\rightarrow$ O(4)
- Finite lattice spacing (and volume) breaks O(4) $\rightarrow$ W_4

W_4 *a.k.a.* H(4): the hypercubic group

- Finite dimensional group of $\pi/2$ rotations and reflections

$$W_4 = \{(a, \pi) | a \in \mathbb{Z}_2^4,\ \pi \in S_4\}$$

- 20 irreps: $4 \cdot \mathbf{1} \oplus 2 \cdot \mathbf{2} \oplus 4 \cdot \mathbf{3} \oplus 4 \cdot \mathbf{4} \oplus 4 \cdot \mathbf{6} \oplus 2 \cdot \mathbf{8}$
- See Baake *et al.* '82, Mandula *et al* '83, Göckeler *et al.* '96

A simple example

Momentum fraction $\langle x \rangle$

- Continuum operator $\mathcal{O}_{\mu\nu} = \overline{q}\gamma_{\{\mu} \overleftrightarrow{D}_{\nu\}} q$ (symmetric, traceless) belongs to [see Weinberg v1ch5.6 for notation]

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right) \otimes \left(\tfrac{1}{2},\tfrac{1}{2}\right) = (0,0) \oplus [(1,0) \oplus (0,1)] \oplus (1,1)$$

- Hypercubic decomposition: $\mathbf{4}_1 \otimes \mathbf{4}_1 = \mathbf{1}_1 \oplus \mathbf{3}_1 \oplus \mathbf{6}_1 \oplus \mathbf{6}_3$,

$$\mathcal{O}_{14} + \mathcal{O}_{41}, \qquad \mathcal{O}_{44} - \tfrac{1}{3}\left(\mathcal{O}_{11} + \mathcal{O}_{22} + \mathcal{O}_{33}\right)$$

- Both operators have same continuum limit; $\mathbf{6}_3$ op. requires $\mathbf{p} \neq 0$
- No mixing with other operators of same or lower dimension

Operator improvement to $\mathcal{O}(a)$

- Need to consider additional irrelevant operators of same symmetry

$$\mathcal{O}_{\{\mu\nu\}} \quad \rightarrow \quad (1 + a\, m_q\, c_0)\mathcal{O}_{\{\mu\nu\}} + i\, a\, c_1 \overline{q}\sigma_{\mu\rho} \overleftrightarrow{D}_{[\nu} \overleftrightarrow{D}_{\rho]}\, q$$
$$+ a\, c_2 \overline{q} \overleftrightarrow{D}_{\{\mu} \overleftrightarrow{D}_{\nu\}}\, q + i\, a\, c_3 \partial_\rho \left(\overline{q}\sigma_{\mu\rho} \overleftrightarrow{D}_\nu\, q\right)$$

Complications: $\langle x^2 \rangle$ and beyond

Higher spin $\left(\frac{n}{2}, \frac{n}{2}\right)$ operators [see Göckeler '95 for details]

- Automatically mix with others of same dimension; *e.g.* $n = 3$ operator $\overline{q}\gamma_{\{\mu} \overleftrightarrow{D}_\nu \overleftrightarrow{D}_{\rho\}} q$

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right) \otimes \left(\tfrac{1}{2},\tfrac{1}{2}\right) \otimes \left(\tfrac{1}{2},\tfrac{1}{2}\right) = 4 \cdot \left(\tfrac{1}{2},\tfrac{1}{2}\right) \oplus 2 \cdot \left(\tfrac{3}{2},\tfrac{1}{2}\right) \oplus 2 \cdot \left(\tfrac{1}{2},\tfrac{3}{2}\right) \oplus \left(\tfrac{3}{2},\tfrac{3}{2}\right)$$

$$\Downarrow$$

$$\mathbf{4}_1 \otimes \mathbf{4}_1 \otimes \mathbf{4}_1 = 4 \cdot \mathbf{4}_1 \oplus \mathbf{4}_2 \oplus \mathbf{4}_4 \oplus 3 \cdot \mathbf{8}_1 \oplus 2 \cdot \mathbf{8}_2$$

  - $\mathbf{4}_2$: $\mathcal{O}_{\{123\}}$ requires $\mathbf{p}_1 \neq \mathbf{p}_2 \neq \mathbf{p}_3 \neq 0$!
  - $\mathbf{4}_1$: $\mathcal{O}_{111}$ mixes with $\overline{q}\gamma_1 q$, coefficient $\sim a^{-2}$!!
  - $\mathbf{8}_1$: $\mathcal{O}_{\{441\}} - \frac{1}{2}(\mathcal{O}_{\{221\}} + \mathcal{O}_{\{331\}})$ since three occurrences, mixes under renormalisation but forward matrix elements of mixing ops. must vanish

- For $n > 4$, operators necessarily mix with lower dimensional operators

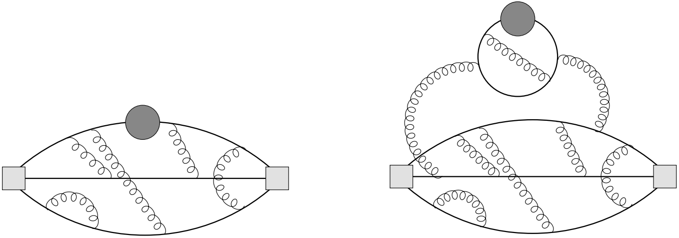
Off-forward case: total derivative operators now relevant!!

$$\mathcal{O}^{\partial\partial}_{\mu\nu\rho} = \partial_{\{\mu}\partial_\nu \overline{q}\gamma_{\rho\}} q$$

Extracting matrix elements: PDFs and GPDs

Three-point functions

$$C_{\mathcal{O}}(t,\tau;\mathbf{p},\mathcal{T}) = \sum_{\mathbf{x},\mathbf{y}} \mathcal{T}_{\beta\alpha} e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}} \langle 0|\chi_\beta(\mathbf{x},t)\mathcal{O}(y,\tau)\overline{\chi}_\alpha(0,0)|0\rangle$$



Three-point correlator has two Wick contractions

- Connected contraction: sequential propagator
- Disconnected contraction requires $S(x,x)\ \forall x$
  - Statistically demanding and usually omitted

Propagators

Sequential propagators

- First compute propagator from source (*e.g.* delta function)

$$M(x,y)M^{-1}(y,0) = \delta(x)$$

  where M is the quark matrix

- Now multiply by the operator and use the resulting object as a source for the sequential inversion

$$M(x,y)M^{-1}(y,z)\mathcal{O}(z)M^{-1}(z,0) = \mathcal{O}(x)M^{-1}(x,0)$$

- Two choices: through operator or through sink
- Can be repeated (*c.f.* polarisability calculations of Engelhardt '07)

Disconnected contractions

- Becoming practical with all-to-all propagators (M. Peardon's lectures)

Extracting matrix elements: PDFs and GPDs

Ratios of correlators

- Straightforward to show for each operator

$$\frac{C_{\mathcal{O}}(t,\tau;\mathbf{p},\mathcal{T})}{C_2(t,\mathbf{p},\mathcal{T})} \xrightarrow{t\gg\tau\gg 0} \langle H(\mathbf{p})|\mathcal{O}|H(\mathbf{p})\rangle$$

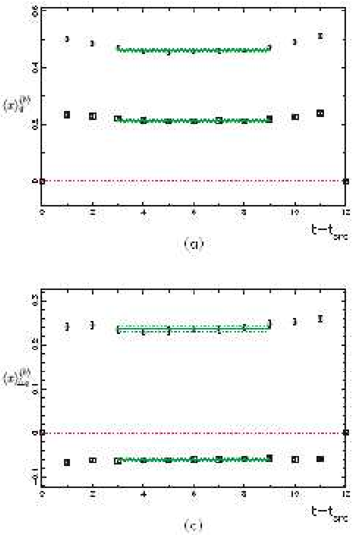
where $C_2(t,\mathbf{p},\mathcal{T}) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}\mathcal{T}_{\beta\alpha}\langle 0|\chi_\beta(\mathbf{x},t)\overline{\chi}_\alpha(0,0)|0\rangle$

For momentum transfer at the operator: GPDs

- Complicated ratios required

$$\frac{C_{\mathcal{O}}(\tau,P',P)}{C_2(\tau_{\text{snk}},P')}\left[\frac{C_2(\tau_{\text{snk}}-\tau+\tau_{\text{src}},P)\;C_2(\tau,P')\;C_2(\tau_{\text{snk}},P')}{C_2(\tau_{\text{snk}}-\tau+\tau_{\text{src}},P')\;C_2(\tau,P)\;C_2(\tau_{\text{snk}},P)}\right]^{1/2}$$

Plateaux for PDF moments: hep-lat/0201021 [LHP/TXL]



Extracting Matrix Elements: moments of DAs

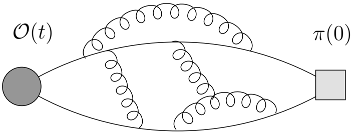
Two point function: flavour changing twist-two operators

$$C^{\mathcal{O}}(t,\vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle 0|\mathcal{O}_{\{\mu_0\ldots\mu_n\}}(\vec{x},t)\left[\overline{q}(0)\,(\gamma_4)\,\gamma_5 u(0)\right]^\dagger|0\rangle$$

$$\rightarrow \quad \frac{A}{2E}\langle 0|\mathcal{O}_{\{\mu_0\ldots\mu_n\}}(0)|\Pi(p)\rangle \left[e^{-Et} + \tau_{\mathcal{O}}\tau_{(4)5}\, e^{-E(L_t-t)}\right], \quad 0 \ll t \ll L_t$$

where

$$A = \langle \Pi(p)|\left[\overline{q}(0)\,(\gamma_4)\,\gamma_5 u(0)\right]^\dagger|0\rangle$$



Example: second moment

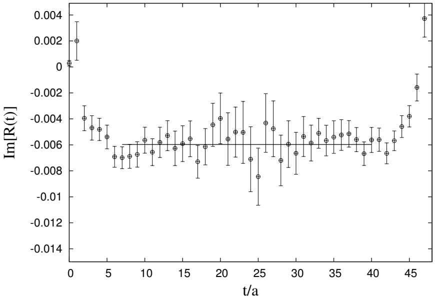
$$R^{2a} = \frac{C^{\mathcal{O}^a_{4ij}}(t)}{C^{\mathcal{O}_4}(t)} = -p_i p_j \,\langle \xi^2\rangle_a$$

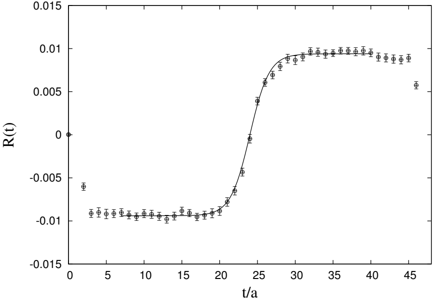
$$R^{2b} = \frac{C^{\mathcal{O}^b_{4ii}}(t)}{C^{\mathcal{O}_4}(t)} = p_i^2 \,\langle \xi^2\rangle_b$$

Ratios for $\langle \xi \rangle$ [QCDSF/UKQCD hep-lat/0606012]

$$R^{1a} = -i\, p_i \, \langle \xi \rangle_a$$

$$R^{1b} = -\frac{E_{\vec{p}}^2 + \frac{1}{3}\vec{p}^2}{E_{\vec{p}}} \, \langle \xi \rangle_b \tanh\left[E_{\vec{p}}(t - L_t/2)\right]$$





Operator Renormalisation [See S. Sint lectures]

- Ideally mult. renormalise operators in scheme, $\mathcal{S}$ and at scale, μ

$$\mathcal{O}^{\mathcal{S}}(\mu) = Z_{\mathcal{O}}^{\mathcal{S}}(\mu)\mathcal{O}_{bare}$$

- Often there are other operators with same quantum numbers and same or lower dimension:

$$\mathcal{O}_i^{\mathcal{S}}(\mu) = \sum_j Z_{\mathcal{O}_i\mathcal{O}_j}^{\mathcal{S}}(\mu, a)\mathcal{O}_j(a)$$

- Renormalisation Group Invariant quantities are defined as

$$\mathcal{O}^{\rm RGI} = Z_{\mathcal{O}}^{\rm RGI}\mathcal{O}_{bare} = \Delta Z_{\mathcal{O}}^{\mathcal{S}}(M)\mathcal{O}^{\mathcal{S}}(M) \qquad \mathcal{S} = \overline{\rm MS}, \text{ MOM, LAT}, \ldots$$

(LHS is independent of scale) with

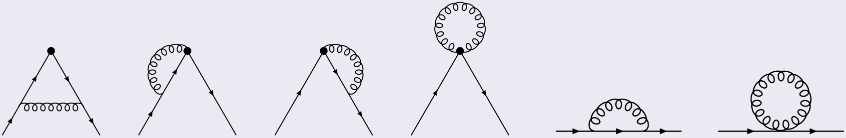
$$[\Delta Z_{\mathcal{O}}^{\mathcal{S}}(\mu)]^{-1} = \left[2b_0 g^{\mathcal{S}}(\mu)^2\right]^{-\frac{d_0}{2b_0}} \exp\left\{\int_0^{g^{\mathcal{S}}(\mu)} d\xi \left[\frac{\gamma^{\mathcal{S}}(\xi)}{\beta^{\mathcal{S}}(\xi)} + \frac{d_0}{b_0\xi}\right]\right\}$$

- Schrödinger functional for operators with derivatives??

Operator renormalisation

Perturbative renormalisation [See Capitani review '05]

- Perturbative matching from lattice to *e.g.* $\overline{\mathrm{MS}}$ scheme
- Compute amputated Green functions in both schemes



- More sophisticated schemes involve tadpole-improved-, RG-improved- and boosted perturbation theory
- NB: lattice side is ugly; gluon propagator given by . . .

Gluon propagator [Weisz, Wohlert '83]

$$G_{\mu\nu}(k) = \frac{1}{(\widehat{k}^2)^2}\left(\alpha \widehat{k}_\mu \widehat{k}_\nu + \sum_\sigma (\widehat{k}_\sigma \delta_{\mu\nu} - \widehat{k}_\nu \delta_{\mu\sigma})\widehat{k}_\sigma A_{\sigma\nu}(k)\right),$$

with

$$\begin{aligned} A_{\mu\nu}(k) &= A_{\nu\mu}(k) = (1-\delta_{\mu\nu})\,\Delta(k)^{-1}\Bigg[(\widehat{k}^2)^2 - c_1\widehat{k}^2\Big(2\sum_\rho \widehat{k}_\rho^4 + \widehat{k}^2 \sum_{\rho\neq\mu,\nu} \widehat{k}_\rho^2\Big) \\ &\quad + c_1^2\Big(\Big(\sum_\rho \widehat{k}_\rho^4\Big)^2 + \widehat{k}^2 \sum_\rho \widehat{k}_\rho^4 \sum_{\tau\neq\mu,\nu} \widehat{k}_\tau^2 + (\widehat{k}^2)^2 \prod_{\rho\neq\mu,\nu} \widehat{k}_\rho^2\Big)\Bigg], \end{aligned}$$

where

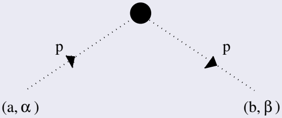
$$\begin{aligned} \Delta(k) &= \Big(\widehat{k}^2 - c_1 \sum_\rho \widehat{k}_\rho^4\Big)\Bigg[\widehat{k}^2 - c_1\Big((\widehat{k}^2)^2 + \sum_\tau \widehat{k}_\tau^4\Big) + \frac{1}{2}c_1^2\Big((\widehat{k}^2)^3 + 2\sum_\tau \widehat{k}_\tau^6 - \widehat{k}^2 \sum_\tau \widehat{k}_\tau^4\Big)\Bigg] \\ &\quad - 4c_1^3 \sum_\rho \widehat{k}_\rho^4 \sum_{\tau\neq\rho} \widehat{k}_\tau^2 . \end{aligned}$$

. . . and vertices are worse . . .

Non-perturbative renormalisation:

- "Rome-Southhampton Method" [Martinelli et al., hep-lat/9411010]
  - mimics (continuum) perturbation theory in a RI'-MOM scheme

Amputated Green's function in a particular gauge:

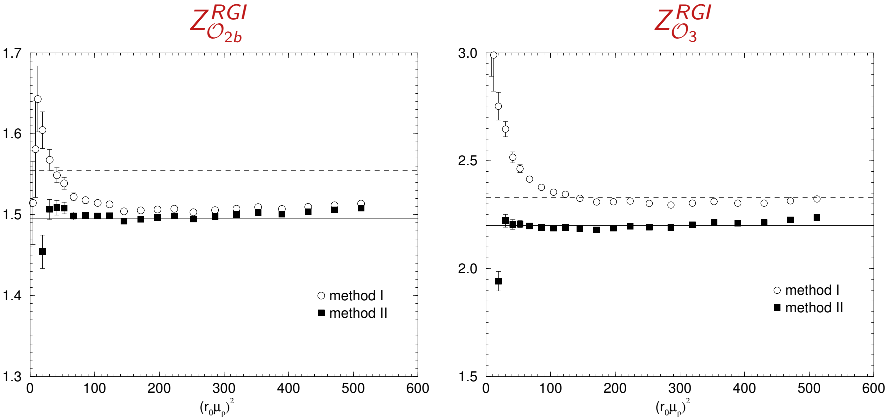


$$\Gamma_{\mathcal{O}}(p) = S^{-1}(p) C_{\mathcal{O}}(p) S^{-1}(p)$$

$$Z_{\mathcal{O}}^{RI'-MOM}(ap, g_0) = \frac{Z_q^{RI'-MOM}(ap', g_0)}{\frac{1}{12}\mathrm{tr}\left[\Gamma_{\mathcal{O}}(ap')\Gamma_{\mathcal{O},Born}^{-1}(ap')\right]|_{p'^2=p^2}}$$

  - *Born* → Fourier transform of free operator
  - scheme valid both pert. and non-pert
- Convert to RGI form perturbatively $\Delta Z_{\mathcal{O}}^{RI'-MOM}(p)$
- Switch to $\overline{MS}$ scheme with a perturbative calculation of $[\Delta Z_{\mathcal{O}}^{\overline{MS}}(\mu)]^{-1}$

Operator renormalisation [QCDSF hep-lat/0410187]



Example: n=2 moment of meson DAs

Renormalise bare lattice operators in scheme, $\mathcal{S}$ and at scale, M

$$\mathcal{O}^{\mathcal{S}}(M) = Z^{\mathcal{S}}_{\mathcal{O}}(M)\mathcal{O}_{bare}$$

$$\langle \xi^n \rangle^{\mathcal{S}}(M) = \frac{Z^{\mathcal{S}}_{\mathcal{O}}(M)}{Z^{\mathcal{S}}_{\mathcal{O}_4}(M)} \langle \xi^n \rangle_{bare}$$

Non-forward matrix elements: hep-lat/0410009

Mix with operators containing external ordinary derivatives

$$\mathcal{O}^{a,\partial\partial}_{412} = \partial_{\{4}\partial_1 \left(\bar{q}\gamma_{2\}}\gamma_5 q\right)$$

$$\mathcal{O}^{\mathcal{S}}_{412} = Z^{\mathcal{S}}_{412}\mathcal{O}^{a}_{412} + Z^{\mathcal{S}}_{\text{mix}}\mathcal{O}^{a,\partial\partial}_{412}$$

$$\langle \xi^2 \rangle = \frac{Z^{\mathcal{S}}_{412}}{Z_{\mathcal{O}_4}} \langle \xi^2 \rangle^{\text{bare}} + \frac{Z^{\mathcal{S}}_{\text{mix}}}{Z_{\mathcal{O}_4}}$$

Extrapolations

Chiral perturbation theory (see C. Bernard's lectures)

- Low energy effective theory of QCD
- Describe small m_q, large L and small p dependence of QCD correlation functions
- Incorporate *some* lattice discretisation effects

A few cautionary remarks:

- Quark masses can be too large: particularly relevant for baryons as odd powers of m_π/Λ_χ occur in chiral expansions (*c.f.* meson sector)
  - Spin-$\frac{3}{2}$ states **required**, otherwise $\Lambda_\chi \to M_\Delta - M_N \sim 300$ MeV
- Volumes can be too small: for an effective hadronic description to be valid, $\Lambda_{QCD} L \sim f_\pi L \gg 1 \Rightarrow L > 1.4$ fm
- Cannot do controlled mass/volume extrapolations for large momentum transfer, *e.g.* in GPDs

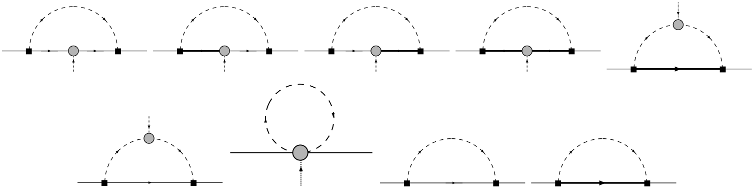
Example: momentum fraction $\langle x \rangle_{u-d}$

Chiral perturbation theory prediction

One-loop calculation (here in DR, but other regularisations also valid):

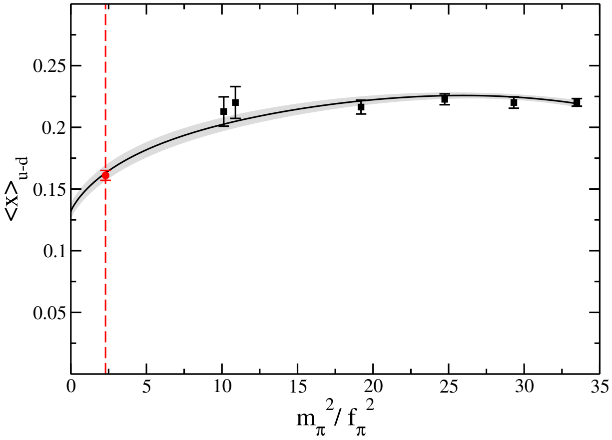
$$\langle x^n \rangle_{u-d} = \langle x^n \rangle_{u-d}^{bare} \left(1 - \frac{3g_A^2+1}{(4\pi f)^2} m^2 log \frac{m^2}{\mu^2}\right) + \langle x^n \rangle_{u-d}^{\Delta, bare} \frac{g_{\Delta N}^2}{(4\pi f)^2} \mathcal{F}(m,\Delta) + c_2(\mu) m^2 + \ldots$$

where $\mathcal{F}(m,\Delta)$ is an ugly function of the mass splitting $\Delta = M_\Delta - M_N$



- Use lattice data to determine parameters ($\equiv$ low energy constants)
- Form also known in PQχPT (involves a superset of LECs)
- Discretisation effects simple to include for GW fermions [Walker-Loud *et al.*]

Example: momentum fraction $\langle x \rangle_{u-d}$ [LHP 0610007]



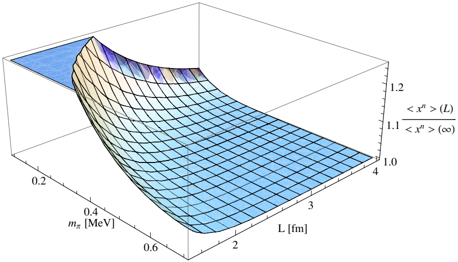
Finite volume effects

χPT describes large volume behaviour of (single-particle) correlations

- LECs independent of volume [Gasser+Leutwyler '80s]
- Momentum integrals become mode sums [R×T^3, $\vec{k} = \frac{2\pi}{L}\vec{n}$, $n_i \in \mathbb{Z}$] , *e.g.*:

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2+m^2)^n} \longrightarrow \int \frac{d\,k_4}{2\pi} \frac{1}{L^3} \sum_{\vec{k}} \frac{1}{(k_4^2+|\vec{k}|^2+m^2)^n}$$

- Expect FV corrections $\sim \exp(-m_\pi L)$



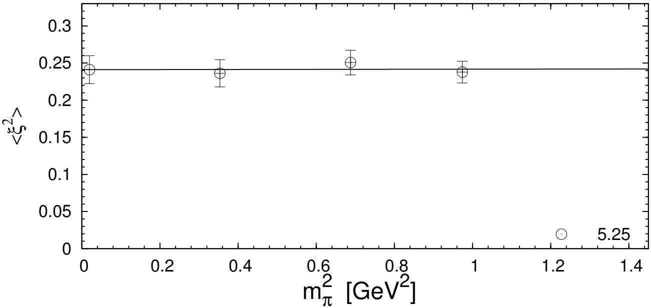
Small volumes

- For $m_\pi L$ of $\mathcal{O}(1)$ or less: pion zero modes enhanced (ϵ, δ and ϵ' regimes)
- Non-perturbative treatment [talk by J. Wasem]

Example: moments of pseudoscalar meson DAs

Chiral perturbation theory prediction

- No non-analytic dependence to two-loops! [Chen *et al.*]
- Extrapolate linearly in m_π^2



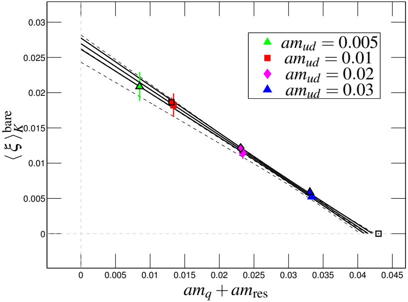
[QCDSF/UKQCD hep-lat/0606012]

Some Recent Results: Lattice 2007

[See also Plenary Talk by Philipp Hägler]

Pseudoscalar meson DA: Sachrajda [UKQCD/RBC]

Results for $\langle \xi \rangle_K^{\text{bare}}$ (16^3 and 24^3 results - χ extrapolation)



$$\langle \xi \rangle_K^{\text{bare}} = 0.0270(8)$$

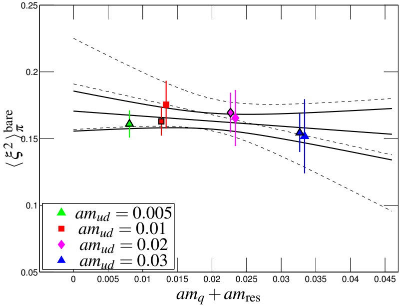
- Quadratic Fit - $\langle \xi \rangle_K^{\text{bare}} = 0.0271(26)$.
- Renormalization Factor computed perturbatively $\Rightarrow$

$$\langle \xi \rangle_K^{\overline{\text{MS}}}(2\,\text{GeV}) = 0.033(2)\,.$$

- Error dominated by the perturbative renormalization constant.

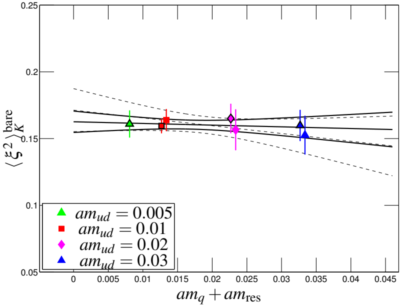
Pseudoscalar meson DA: Sachrajda [UKQCD/RBC]

Results for $\langle \xi^2 \rangle_\pi^{\text{bare}}$ (16^3 and 24^3 results - χ extrapolation)



$$\langle \xi^2 \rangle_\pi^{\text{bare}} = 0.171(15)$$

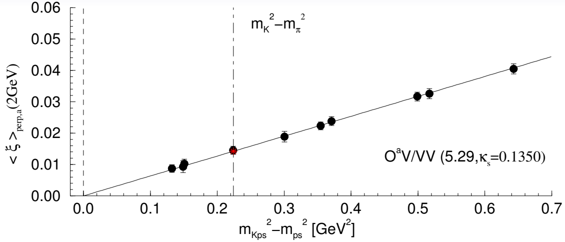
Results for $\langle \xi^2 \rangle_K^{\text{bare}}$ (16^3 and 24^3 results - χ extrapolation)

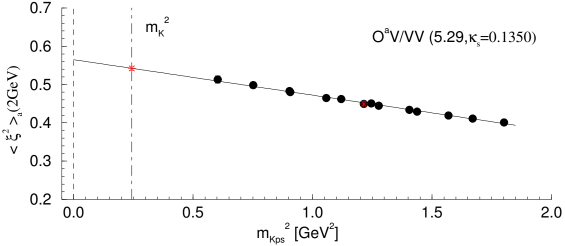


$\langle \xi^2 \rangle_K^{\text{bare}} = 0.163(8)$

Vector meson DA: Horsley [UKQCD/QCDSF]

A (typical) result: $\langle \xi \rangle_\perp$, $\langle \xi^2 \rangle$

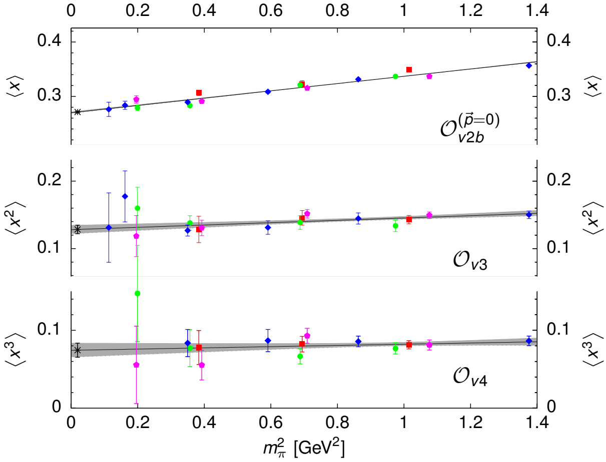




Moments of Pion PDFs: Brömmel [QCDSF]

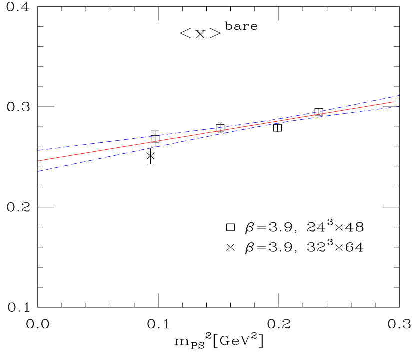
Moments of Forward Distributions

Linear Extrapolation in m_π



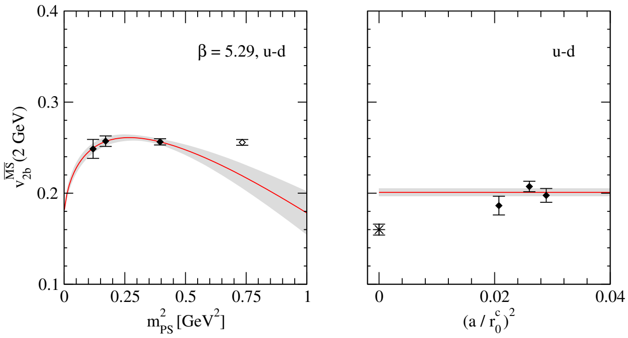
Moments of Pion PDFs: Liu [ETM]

preliminary results



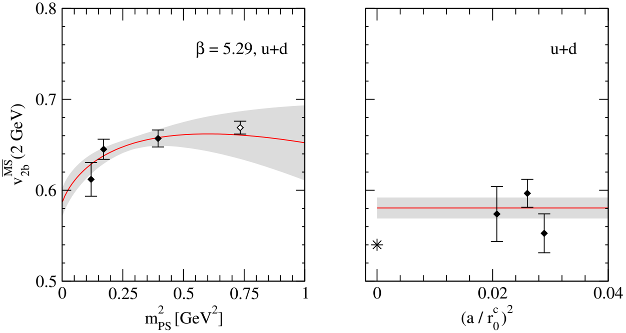
- Finite size effects are not big.
- Linear extrapolation in m_π^2 gives $\langle x \rangle^{\text{bare}}$=0.246(10).

Comparison with lattice results: isovector case



- Small number of parameters $\rightarrow$ fit results for same β
- Discretisation errors seem to be small

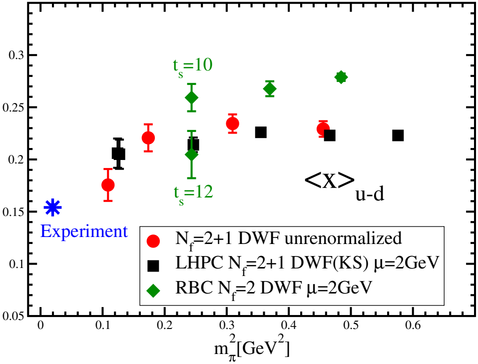
Comparison with lattice results: isoscalar case



- Discretisation errors again small
- Indications for “bending-down”
- But: neglected disconnected contributions

Moments of Nucleon PDFs: $\langle x \rangle_{u-d}$: Ohta [RBC]

RBC/UKQCD (2+1)-flavor, Iwasaki+DWF dynamical, compared with with LHPC/MILC 2+1f and RBC 2f



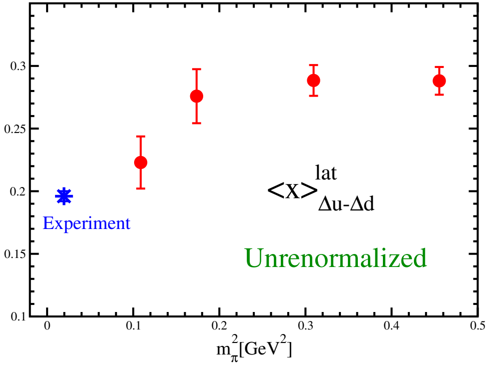
- Renormalization close to 1?
- Light sea quarks are important.
- Light valence quarks are important: otherwise cannot share much momentum with sea quarks and gluons.
- Finite-volume effect beginning at $m_\pi L \sim 4.5$? Not necessarily.

Moments of Nucleon PDFs: $\langle x \rangle_{\Delta u - \Delta d}$: Ohta [RBC]

RBC/UKQCD (2+1)-flavor, Iwasaki+DWF dynamical, $a^{-1} = 1.73(2)$ GeV, $m_{\mathrm{res}} = 0.00315(2)$, $m_{\mathrm{strange}} = 0.04$,

- $m_\pi = 0.67, 0.56, 0.42$ and 0.33 GeV; $m_N = 1.56, 1.39, 1.25$ and 1.15 GeV,

Helicity fraction, $\langle x \rangle_{\Delta u - \Delta d}$, yet to be renormalized



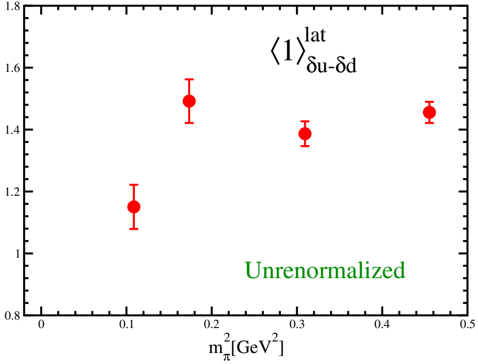
Absolute values seem to have improved, trending to the experimental values, but yet to be renormalized (typically 10% effect at these cuts off).

Moments of Nucleon PDFs: $\langle 1 \rangle_{\delta u - \delta d}$: Ohta [RBC]

RBC/UKQCD (2+1)-flavor, Iwasaki+DWF dynamical, $a^{-1} = 1.73(2)$ GeV, $m_{\rm res} = 0.00315(2)$, $m_{\rm strange} = 0.04$,

- $m_\pi = 0.67$, 0.56, 0.42 and 0.33 GeV; $m_N = 1.56$, 1.39, 1.25 and 1.15 GeV,

Helicity fraction, $\langle 1 \rangle_{\delta u - \delta d}$, yet to be renormalized



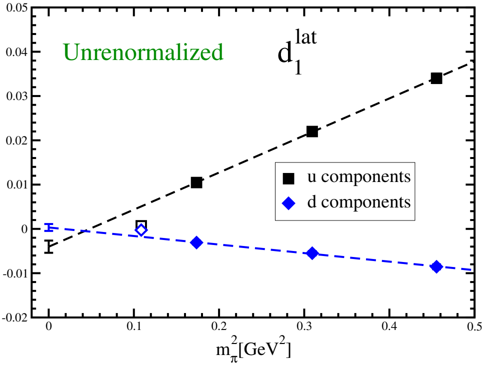
Yet to be renormalized (typically 10% effect at these cuts off).

Twist-3 d_1:Ohta [RBC]

RBC/UKQCD (2+1)-flavor, Iwasaki+DWF dynamical, $a^{-1} = 1.73(2)$ GeV, $m_{\rm res} = 0.00315(2)$, $m_{\rm strange} = 0.04$,

- $m_\pi = 0.67$, 0.56, 0.42 and 0.33 GeV; $m_N = 1.56$, 1.39, 1.25 and 1.15 GeV,

Twist-3, d_1, yet to be renormalized



Chirally well-behaved, small, and in consistency with Wandzura-Wilczek relation.

Moments of Pion GPDs: Hägler [QCDSF/UKQCD]

Spin structure of the pion

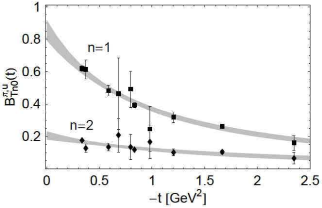
QCDSF/UKQCD, to be published (D. Brömmel Wed 11:20)

longitudinal spin structure of the pion is trivial

what about the transverse spin structure?

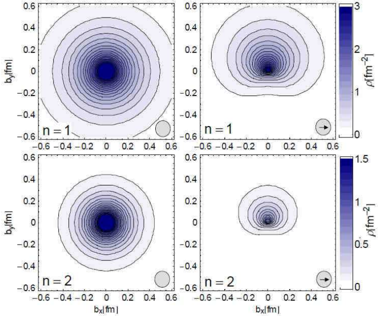
$$\rho_T^{\pi}(b_\perp; s_\perp) = \frac{1}{2}\left\{A_{n0}^{\pi}(b_\perp^2) - \epsilon_{ij} s_\perp^i b_\perp^i \frac{1}{m_\pi} B_{Tn0}^{\pi\prime}\right\}$$

but is $B_{Tn0}^{\pi\prime}$ non-zero?



discretization effects are smaller than statistical errors

finite size effects are up to 20%



the pion has a non-trivial (transverse) spin structure

impact on DY

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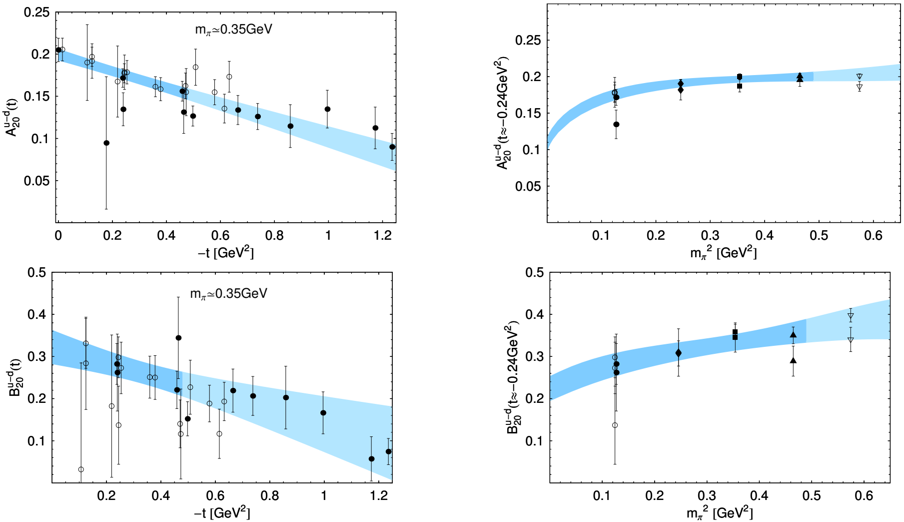
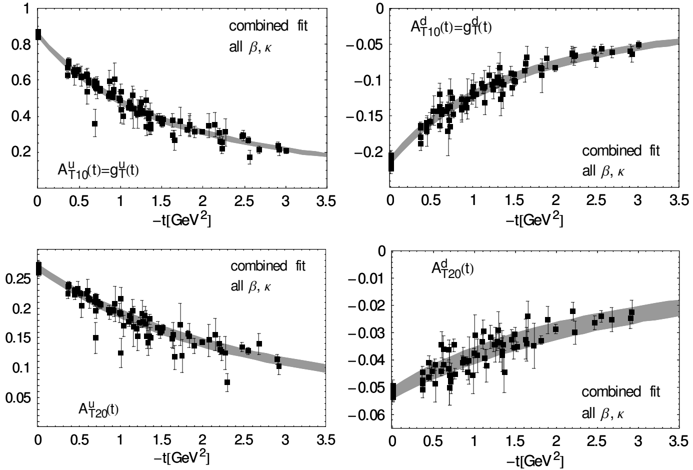
Moments of Nucleon Vector GPDs [LHP]

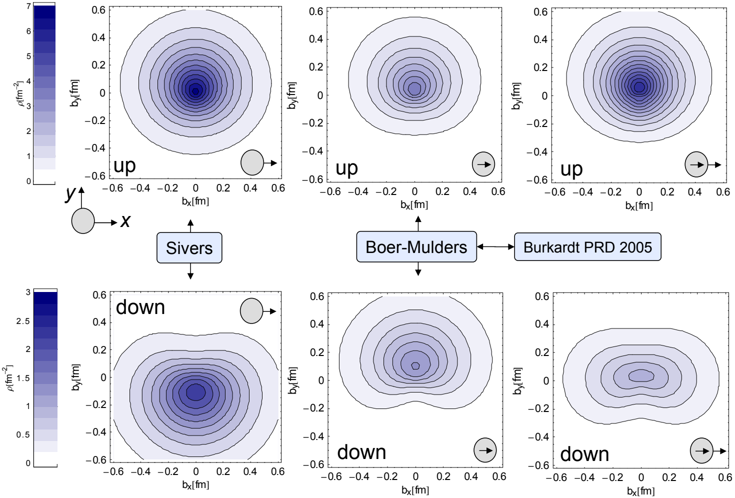


Moments of Nucleon Tensor GPDs [QCDSF/UKQCD]



The generalized formfactors A_{T10} and A_{T20} together with dipole fits.

Preliminary results for the **lowest moment** of the transverse spin density



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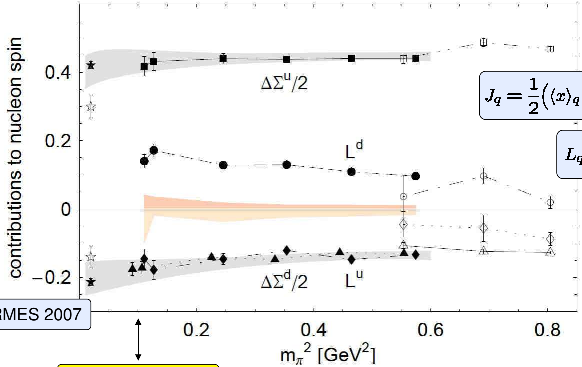
Quark spin and OAM contributions to the nucleon spin

LHPC, arXiv:0705.4295 (D. Renner Thu 14:00); hybrid Asqtad sea + DW valence



$$J_q = \frac{1}{2}\left(\langle x\rangle_q + B_{20}^q(t=0)\right)$$

$$L_q = J_q - \frac{1}{2}\Delta\Sigma$$

⋆ HERMES 2007

$m_\pi \simeq 350$ MeV

$L^d \approx -L^u \approx 30\%$ of $\frac{1}{2}$

$J^u = 0.26(1)$

$J^d = -0.02(1)$

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Summary and Future directions

Hadron structure

- LQCD will have a big impact on our understanding of hadron structure
- Extensive program underway

. . . but much more to do. . .

- Flavour singlet quark PDFs and GPDs
  - major weakness of lattice hadron structure
  - requires statistically limited disconnected diagrams
  - some progress recently: $\langle x \rangle_s$ [Deka/Liu]
- Gluon distributions: recent calculation of $\langle x_g \rangle_\pi$ [Meyer/Negele]
- Higher twist contributions
- Transverse momentum dependent PDFs (talk today by B. Musch)
- x-dependence: inverting moments, other approaches (D. Grunewald)
- Nuclear PDFs/GPDs: the EMC effect