Modeling pion physics in the ϵ regime of two-flavor QCD using lattice QED

A Poor Man's QCD?

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Motivation

- Current challenge in lattice QCD: Compute low energy hadronic observables with controlled errors.
- Algorithmic difficulties: Difficult to study realistic quark masses.
  Calculations at unphysically large pion masses followed by extrapolations to realistic quark masses with χPT.
- Extrapolations: Are they reliable? Need to know the range over which χPT is applicable.

$\Rightarrow$ Take a model simpler than QCD and study χPT as an effective field theory describing lattice field theory.

Goals:

- To construct lattice field theory to model pions of QCD.
- To understand how χPT emerges in such a theory.
- To understand effects of quark masses on pion scattering.

Model

$N_f = 2$ Strongly coupled lattice QED with staggered fermions.

- Why $N_f = 2$?
  To study a simple model with two light quarks.
- Why staggered fermions?
  To have chiral symmetry and study the chiral limit.
- Why the strong coupling limit?

  Develop efficient Directed Loop Algorithms to study the chiral limit while retaining the qualitative physics (namely chiral symmetry breaking and confinement) of full QCD.
- Why Strong Coupling QED?
  U(1) simpler than SU(3).
  confinement and chiral symmetry breaking also present in U(1) at strong coupling.

Continuum Limit

- Without a way to fine tune lattice artifacts will dominate since at strong coupling $F_\pi \sim a$
- New idea to overcome this:

  Work in $d+1$ dimensions where $d=4$ space time dimensions

  Extra dimension is fictitious temperature which allows fine tuning to a critical point

  Near critical point, where $F_\pi \ll a$.

  As we will see, $F_\pi \sim 100 MeV$ and $a \sim 1 GeV$.

  $\Rightarrow$ Thus, we can still explore physics of continuum limit even in strong coupling limit.

Euclidean Space Action

$$S = -\sum_{x,\mu} \eta_{\mu,x} \left[\mathrm{e}^{i\phi_{\mu,x}} \overline{\psi}_x \psi_{x+\hat{\mu}} - \mathrm{e}^{-i\phi_{\mu,x}} \overline{\psi}_{x+\hat{\mu}} \psi_x \right] - \sum_x \left[m\overline{\psi}_x \psi_x + \frac{\tilde{c}}{2} \left(\overline{\psi}_x \psi_x \right)^2 \right]$$

1. x: lattice site on $d+1$ dimensional hypercubic lattice $L_t \times L^d$
2. μ runs over the temporal and spatial directions $0, 1, 2, ..., d$
3. $\overline{\psi}_x$,ψ_x: 2 component Grassman fields for 2 flavors mass m
4. $\phi_{\mu,x}$: $U(1)$ gauge field through which the fields interact
5. staggered fermion phases: $\eta^2_{0,x} = T$, $\eta^2_{i,x} = 1 \; i = 1, 2, ..., d$
6. T: fictitious temperature.
7. $\tilde{c}$: strength of the anomaly

Symmetries

- Same symmetries as full QCD.
- Sum over lattice sites decomposed into sum over *even* and *odd* sites.
- $\tilde{c}, m = 0$, action has global $SU_L(2) \times SU_R(2) \times U_A(1)$ symmetry.
- Action invariant under $U_A(1)$ and $SU_L(2)$ transformations:

$$\begin{aligned} \overline{\psi}_o &\to \overline{\psi}_o \exp(i\theta) & \psi_o &\to \exp(i\theta)\psi_o \\ \overline{\psi}_e &\to \overline{\psi}_e \exp(-i\theta) & \psi_e &\to \exp(-i\theta)\psi_e \\ \overline{\psi}_o &\to \overline{\psi}_o V_L^\dagger & \psi_o &\to \psi_o \\ \overline{\psi}_e &\to \overline{\psi}_e & \psi_e &\to V_L\psi_e \end{aligned}$$

$SU_R(2)$ obtained by $V_L \Leftrightarrow V_R$ and $o \Leftrightarrow e$.

V_L, V_R $SU(2)$ matrices: $\exp(i\vec{\theta} \cdot \vec{\sigma})$

σ_i Pauli matrix acting on flavor space.

Symmetries

- $\tilde{c} \neq 0$,$m = 0$
  $U_A(1)$ explicitly broken
  $SU_L(2) \times SU_R(2) \times U_A(1) \rightarrow SU_L(2) \times SU_R(2) \times Z_2$.

  $\Rightarrow$ Thus, coupling $\tilde{c}$ induces the effects of the anomaly.

- $c, m \neq 0$
  $U_A(1)$ explicitly broken
  $SU_L(2) \times SU_R(2)$explicitly broken
  $SU_L(2) \times SU_R(2) \times U_A(1) \rightarrow SU_V(2)$.

  $\Rightarrow$ Thus, to mimic real world with u,d quarks: $\tilde{c} \neq 0$ and $m \neq 0$.

- Based on mean field strong coupling calculations, expect the symmetry breaking pattern to be similar to that of full QCD.

MDPI Model

Need algorithm to study the partition function:

1. Integrate over the gauge fields exactly.
2. Use Grassman algebra to simplify the partition function.
3. Interpret the remaining terms as gauge invariant objects: monomers, dimers, and pion loops, and instantons.
4. Express partition function in terms of MDPI configurations.

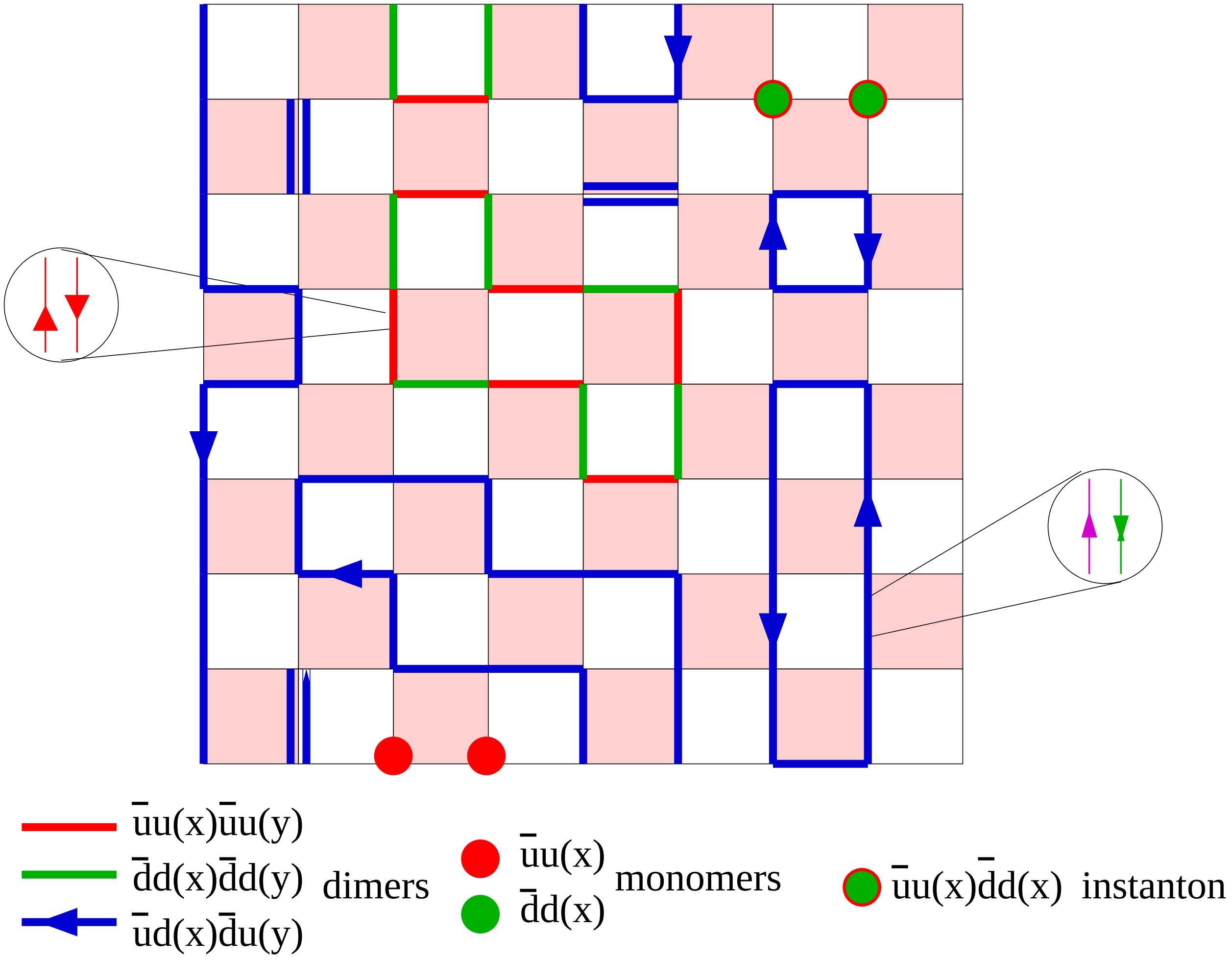
$$Z = \sum_{[I,n^d,n^u,\pi^d_\mu,\pi^u_\mu,\pi^1_\mu]} \prod_{x,\mu} m^{n_d(x)} m^{n_u(x)} c^{I(x)}$$

- $[I, n^d, n^u, \pi^d_\mu, \pi^u_\mu, \pi^1_\mu]$: a MDPI configuration
- $I(x) = 0, 2$: instantons $\quad n_{u,d}(x) = 0, 1$: u, dmonomers
- $\pi^{u,d}_\mu = 0, 1$: u, ddimers $\quad \pi^1_\mu = -1, 0, 1$: $\overline{u}d$or $\overline{d}u$ dimers.

Z is sum over positive definite weights
$\Longrightarrow$ Directed Path Algorithm in MDPI space.

MDPI Configuration Space

An example of an MDPI configuration on the lattice.



Directed Loop Algorithm

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$\Longrightarrow$ Directed Path Algorithm in MDPI space.

- DPA very efficient in studying chiral limit.
- Have tested algorithm in simple case of 2×2 where exact hand calculation of partition function is possible.

Directed Loop Algorithm

Three update routines:

The $u \leftrightarrow d$ flip update:

- Changes $u$quark to $d$quark and vice-versa on pion loop or string.
- $\overline{u}u\overline{u}u$dimer becomes $\overline{d}d\overline{d}d$dimer and vice versa
- $\overline{u}d\overline{d}u$dimer becomes $\overline{d}u\overline{u}d$dimer and vice versa
- $\overline{u}u$monomer becomes $\overline{d}d$monomer.
- Satisfies detailed balance.

Loop swap update

- Swaps neutral pion-loopinto a charged pion-loopand vice versa.
- Satisfies detailed balance.

Directed Loop Algorithm

Directed-path mass update

- Can create and destroy monomers and instantons.
- Can change shape of pion loops.
- Two types of update differ on sites touched.
  charged-pion directed path update can only touch sites containing either charged pion-loops (including double dimers) and instantons
  neutral-pion directed path update can only touch sites containing neutral pion-loops (including double dimers), instantons and monomers.
- Satisfies detailed balance.

All three updates needed for ergodicity .

Directed Loop Algorithm

Directed path fixed monomer update

- Allows the monomers to change positions while keeping the total monomer fixed.
- Satisfies detailed balance.
- Not required for ergodicity but allows us to test additional ϵ regime predictions (as we will see)

Algorithm very efficient and as we will see can be used to study the chiral limit. (Note most current algorithms too inefficient to approach $m = 0$.)

Observables

- Numerous observables can be measured with this algorithm.
- Simplest are three helicity moduli (current susceptibilities). For a conserved current $J^i_\mu(x)$, the helicity modulus (current susceptibility) is defined as:

$$Y^i_w = \frac{1}{dL^d} \left\langle \sum_{\mu=1}^{d} \left(\sum_x J^i_\mu(x) \right)^2 \right\rangle$$

on a $L_t \times L^d$ lattice.
There are three conserved currents in our model: axial, chiral, and vector:

$$\begin{aligned} J^A_\mu(x) &= (-1)^x \left[\pi^u_\mu(x) + \pi^d_\mu(x) + |\pi^1_\mu(x)|\right] \\ J^C_\mu(x) &= (-1)^x \left[\pi^u_\mu(x) - \pi^d_\mu(x)\right] \\ J^V_\mu(x) &= \pi^1_\mu(x) \end{aligned}$$

Observables

- Can measure correlation functions defined as:

$$
\begin{aligned}
G_\pi^a(x,y) &= \frac{1}{2}\langle \overline{\psi}_x i\sigma^a(-1)^x\psi_x\ \overline{\psi}_y i\sigma^a(-1)^y\psi_y\rangle \\
G_\sigma(x,y) &= \frac{1}{2}\langle \overline{\psi}_x\psi_x\ \overline{\psi}_y\psi_y\rangle \\
G_\eta(x,y) &= \frac{1}{2}\langle \overline{\psi}_x i(-1)^x\psi_x\ \overline{\psi}_y i(-1)^y\psi_y\rangle \\
G_\delta^a(x,y) &= \frac{1}{2}\langle \overline{\psi}_x \sigma^a\psi_x\ \overline{\psi}_y \sigma^a\psi_y\rangle
\end{aligned}
$$

- The corresponding susceptibilities, χ_π and χ_η are:

$$\chi = \frac{1}{L_t L^d}\sum_{x,y} G(x,y)$$

- The directed path algorithm allows a straightforward measurement of $G(x,y)$ and χ.

Chiral Lagrangian

In the phase with broken chiral symmetry and large anomaly, low energy physics of our model described by:

$$\mathcal{L} = \frac{F^2}{4}\mathrm{tr}\Big(\partial_\mu U^\dagger \partial_\mu U\Big) - m\Sigma tr\Big(U + U^\dagger\Big)$$

F: pion decay constant in the chiral limit

Σ: chiral condenstate

$U \in SU(2)$: pion field.

ϵ regime: limit where L (size of 4d hypercube) is large such that $FL \ll 1$ but $m\Sigma L^4$ is held fixed

$\Longrightarrow$ To apply χPT to our model in the ϵ regime choose $c = 0.3$ and $m = 0$ to be in the broken phase. Tuned T to near critical point at $T = 1.733$.

χPT ϵ regime

Dependence of Y_c and Y_v(Hansen):

$$Y_c = \frac{F^2}{2}\Big(\{1 + \frac{\beta_1}{(FL)^2} + \frac{a'}{(FL)^4} + ...\} + \frac{u^2}{24}\{1 + \frac{3\beta_1}{(FL)^2} + \frac{b_c}{(FL)^4} + ...\} + \mathcal{O}(u^4)\Big)$$

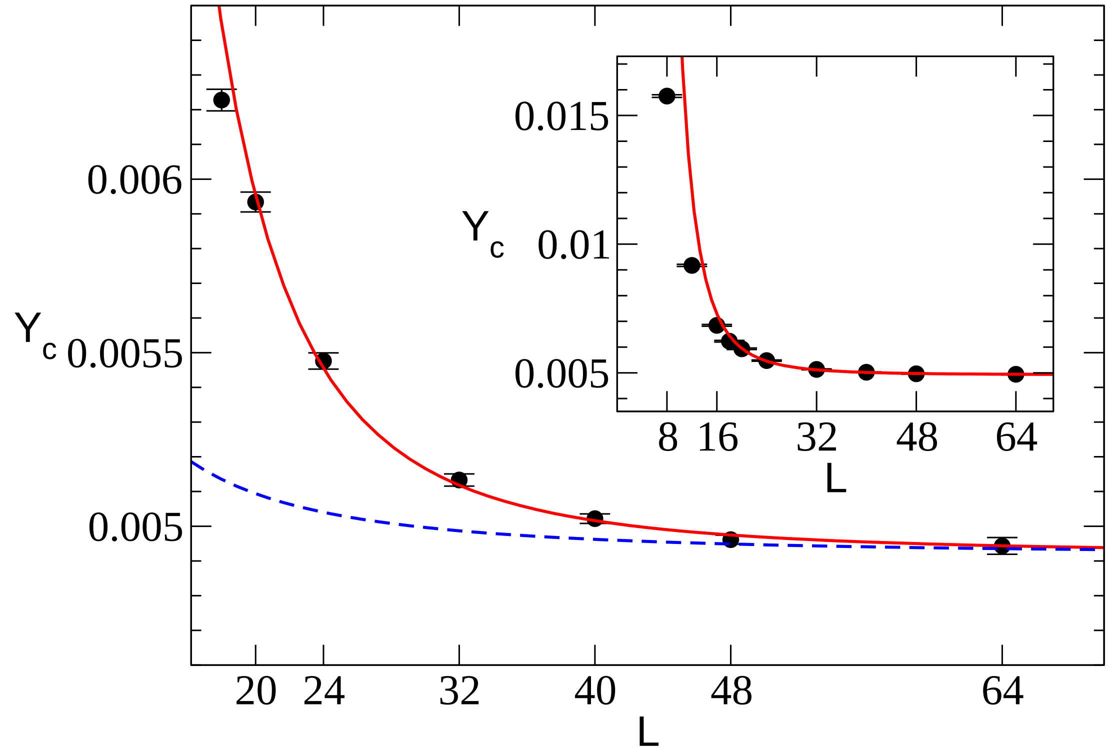
$$Y_v = \frac{F^2}{2}\Big(\{1 + \frac{\beta_1}{(FL)^2} + \frac{a'}{(FL)^4} + ...\} - \frac{u^2}{24}\{1 + \frac{3\beta_1}{(FL)^2} + \frac{b_v}{(FL)^4} + ...\} + \mathcal{O}(u^4)\Big)$$

for small $u = \Sigma m L^4[1 + 3\beta_1/(2(FL)^2)]$.

$\beta_1 = 0.14046$(4d shape coefficient)

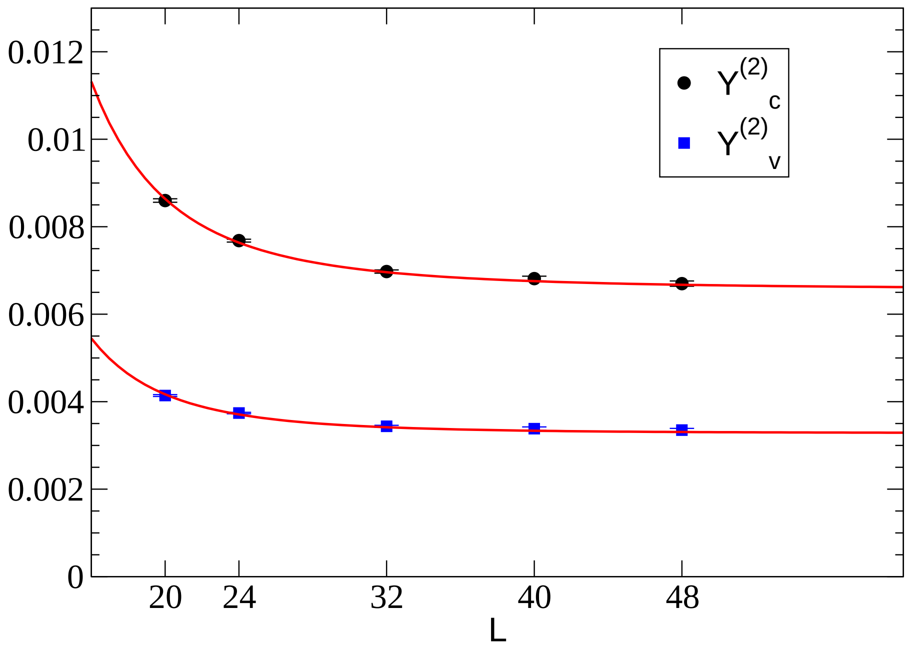
a' b_c,$b_v$depend on higher order low energy constants.

Chiral current susceptibility Y_c



Y_c as function of L at $c = 0.3$, $T = 1.733$, $m = 0$. Solid line shows the fit with $F = 0.0992(1)$, $a' = 2.7(1)$, $\chi^2/DOF = 0.8$. Dotted line shows same curve but with $a' = 0$.

Fixed monomer number



Plot of $Y_c^{(2)}$ and $Y_v^{(2)}$, evaluated in two monomer sector as function of L at $T = 1.733$, $c = 0.3$,$m = 0$. Solid lines are fits to same formulas as before but with $m \neq 0$.

χPT ϵ regime

- Behavior of χ_π as a function of L at $m = 0$ for $O(N)$ model (Hasenfratz et al) $N = 4$ result:

$$\chi_\pi = \Sigma^2 \frac{L^4}{4}\left(1 + \frac{3\beta_1}{(FL)^2} + \frac{a}{(FL)^4} + \ldots\right)$$

- $\log L$ corrections (Gockeler et al)

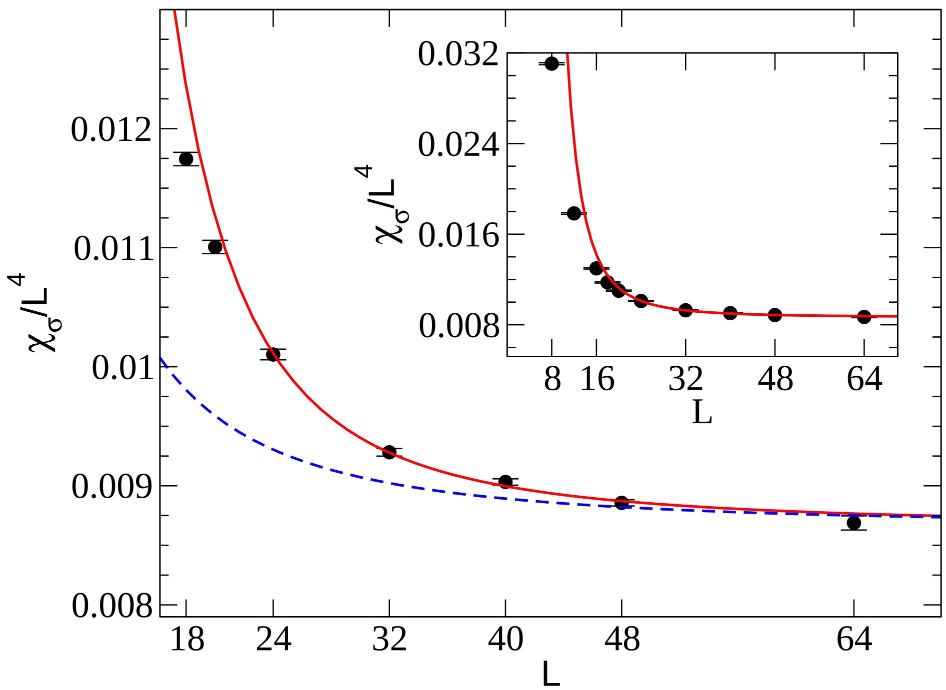
$$\chi_\pi = \Sigma^2 \frac{L^4}{4}\left[1 + \frac{3\beta_1}{F^2L^2} + \frac{1}{F^4L^4}\left\{\alpha + \frac{3}{16\pi^2}(\log F^2L^2)\right\} + \ldots\right]$$

where $\alpha = -3(\beta_1^2 - 3\beta_1 - 4\beta_2)/4 + 3\log\left(\Lambda_M \Lambda_\Sigma / F^2\right)$

$\beta_2 = -0.020305$another shape coefficient

mass scales $\Lambda_M, \Lambda_\Sigma$ encode non-universal information
$\Rightarrow$ logarithmic dependence of M_π, Σ on quark mass m

Chiral condensate susceptibility χ_σ



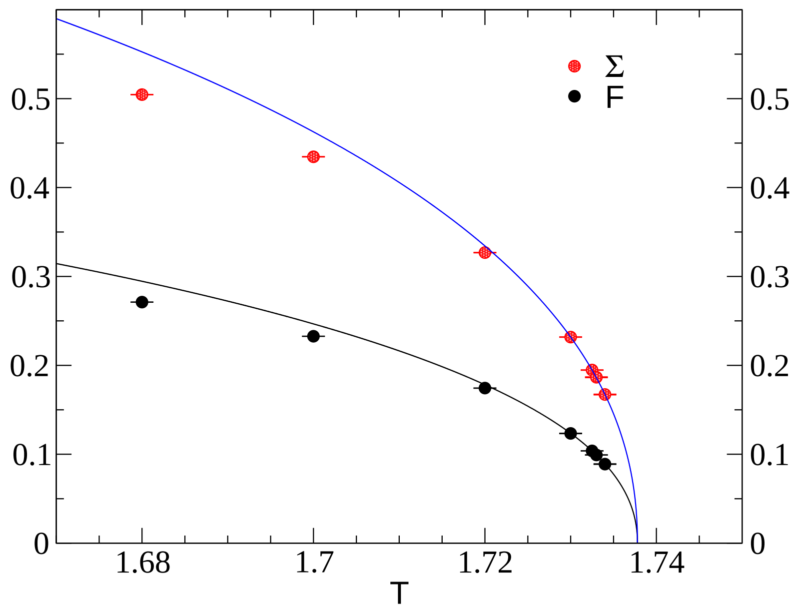
Chiral condensate susceptibility χ_σ,χ_π as function of lattice size L at $T = 1.733$, $c = 0.3$,$m = 0$ [$\chi_\sigma = \chi_\pi$]. Solid line shows fit with $\Sigma = 0.1866(2)$, $F = 0.0992$, $a = 3.0(2)$, $\chi^2/DOF = 1.3$. Dotted line shows the same curve but with $a = 0$.

$\Rightarrow$ Not sensitive to $\log L$ corrections

Critical point

At $c = 0.3$,$T = 1.733$ $F \sim 0.1$. MFT $\Rightarrow$ 2nd order transition.

$$F \sim A_F(T_c - T)^{\frac{1}{2}}|ln(T_c - T)|^{\frac{1}{4}} \qquad \Sigma \sim A_\Sigma(T_c - T)^{\frac{1}{2}}|ln(T_c - T)|^{\frac{1}{4}}$$



$T \geq 1.73$,
$A_F = 0.943(4)$,$A_\Sigma = 1.769(4)$,$T_C = 1.73779(4)$,$\chi^2/DOF = 0.7$.

Conclusions

1. Developed a strong coupling lattice QED model of pions in $N_f = 2$ QCD.
2. Have shown that mapping to a dimer model leads to very efficient algorithms that can be used to study the chiral limit and close to it.
3. Able to confirm the low energy predictions of χPT in the ϵ regime.
4. Have found $F \ll 1$ by tuning fictitious temperature to a 2nd order phase transition and approached continuum limit.

Future Work

With an efficient algorithm now available to study the $SU_L(2) \times SU_R(2)$ lattice QED model at strong coupling and having established consistency with the ϵ regime of χPT we plan:

1. To complete a study of chiral perturbation theory in the p regime.
2. To compute the effects of quark mass on pion scattering by measuring two and four point correlation functions and extracting scattering phase shifts and lengths via Lüscher's method.

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