Effective Field Theories for lattice QCD:
Lecture 4

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Outline of Lectures

1. Overview & Introduction to continuum chiral perturbation theory (ChPT)

2. Illustrative results from ChPT; SU(2) ChPT with heavy strange quark; finite volume effects from ChPT and connection to random matrix theory

3. Including discretization effects in ChPT

4. Partially quenched ChPT and applications, including a discussion of whether $m_u=0$ is meaningful
Outline of lecture 4

- Partial quenching and PQChPT
  - What is partial quenching and why might it be useful?
  - Developing PQChPT
  - Results and status
- $m_u=0$ and the validity of PQ theories (and the rooting prescription)
Additional References for PQChPT

- S. Sharpe & N. Shoresh [PQChPT general properties], Phys. Rev. D64 (2001) 114510 [hep-lat/0108003]
What is partial quenching?

- Explain with example of pion correlator:

\[ C_\pi(\tau) = -\left\langle \sum_{\vec{x}} \bar{u}\gamma_5 d(\vec{x}, \tau) \bar{d}\gamma_5 u(0) \right\rangle \]

\[ \equiv -\frac{1}{Z} \int DU \prod_q Dq D\bar{q} e^{-S_{\text{gauge}}} \int x \sum_q \bar{q}(\not{p} + m_q) q \sum_{\vec{x}} \bar{u}\gamma_5 d(\vec{x}, \tau) \bar{d}\gamma_5 u(0) \]

\[ = \frac{1}{Z} \int DU \prod_q \det(\not{p} + m_q) e^{-S_{\text{gauge}}} \sum_{\vec{x}} \text{tr} \left[ \gamma_5 \left( \frac{1}{\not{p} + m_d} \right)_{x0} \gamma_5 \left( \frac{1}{\not{p} + m_u} \right)_{0x} \right] \]

\[ = \left\langle \sum \gamma_5 \begin{array}{c} x \\ u_v \\ d_v \\ 0 \end{array} \begin{array}{c} \gamma_5 \\ \gamma_5 \end{array} \right\rangle \propto f_\pi^2 e^{-m_\pi \tau} + \text{exp. suppressed} \]

- “sea” quarks in determinant; “valence” in propagators

- **Partial Quenching:** \( m_{\text{val}} \neq m_{\text{sea}} \) — many different \( m_{\text{val}} \) for each \( m_{\text{sea}} \)

- Numerically cheap—can we make use of this extra information?

\[ \Rightarrow \text{Many (but not all) numerical calculations use PQing} \]
PQQCD is unphysical

- Intuitively clear that unitarity is violated, since intermediate states differ from external states, e.g. $\pi V \pi V \rightarrow \pi S \pi$ $\pi V \rightarrow \pi V \pi V$

- Extent and impact of unphysical nature will become clearer when give a formal definition of PQ theory
Why partially quench?

- Use PQQCD to learn about physical, unquenched QCD
- This is possible only within an EFT framework
  - Use partially quenched ChPT (PQChPT)
  - Requires that one works in “chiral regime”
  - PQChPT needs very few extra LECs compared to ChPT
  - Extends range over which can match to ChPT
- Comparison with PQChPT is “anchored” by fact that theory with $m_v = m_s$ is physical
- PQQCD is needed to predict properties of small eigenvalues of Dirac operator & connect with Random Matrix Theory
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Nomenclature

- Why called partially quenched? Why not partially unquenched?
- Bad old days: quenched approximation $m_{\text{sea}} \to \infty$
  - $\det(\bar{\psi} + m_q) \to \text{constant}$
  - No quark loops
  - $Z_{\text{QCD}} \to Z_{\text{QQCD}} = \int DU e^{-S_{\text{gauge}}} = Z_{\text{gauge}}$
- Unphysical nature of quenched QCD shows up various ways, e.g. $\langle \bar{\psi}\psi \rangle \to \infty$ as $m_{\text{val}} \to 0$
- Partial quenching is in one sense a less extreme version of quenching, and thus the name
- If $m_{\text{sea}} \gg \Lambda_{\text{QCD}}$ then PQQCD, like quenched QCD, only qualitatively related to QCD
- Consider here only the case when $m_{\text{sea}} \ll \Lambda_{\text{QCD}}$ so one can use $\chi$PT and relate PQCD to QCD quantitatively
Morels’ formulation of (P)QQCD

- **IDEA:** commuting spin-$\frac{1}{2}$ fields (ghosts) $\tilde{q}$ give determinant which cancels that from valence quarks
  \[
  \int D\bar{q} Dq \ e^{-\bar{q}(\not{D}+m_q)q} = \det(\not{D} + m_q)
  \]
  \[
  \int D\bar{q}^\dagger D\bar{q} \ e^{-\bar{q}^\dagger(\not{D}+m_q)\bar{q}} = \frac{1}{\det(\not{D} + m_q)}
  \]

- To formulate PQQCD need three types of “quark”
  - valence quarks $q_{V1}, q_{V2}, \ldots q_{VN_V}$ ($N_V = 2, 3, \ldots$)
  - sea quarks $q_{S1}, q_{S2}, \ldots q_{SN}$ ($N = 2, 3$)
  - ghosts $\tilde{q}_{V1}, \tilde{q}_{V2}, \ldots \tilde{q}_{VN_V}$ ($N_V = 2, 3, \ldots$)

- Ghosts are degenerate with corresponding valence quarks

- Convergence of ghost integral requires $m_q > 0$ (since $\not{D}$ antihermitian)
  - Some subtleties in extending to non-hermitian lattice Wilson-Dirac operator
Morels’ formulation of (P)QQCD

- Partition function reproduces that which is actually simulated

\[
Z_{PQ} = \int DU e^{-S_{\text{gauge}}} \prod_{i=1}^{N_V} \left( D\bar{q}_V Dq_V D\bar{q}_V^+ Dq_V^+ \right) \prod_{j=1}^{N} \left( D\bar{q}_S Dq_S \right) \times \\
\times \exp \left[ -\sum_{i=1}^{N_V} \bar{q}_V (\bar{\Psi} + mV_i) q_V - \sum_{j=1}^{N} \bar{q}_S (\bar{\Psi} + mS_j) q_S - \sum_{k=1}^{N_V} \bar{q}_V^+ (\bar{\Psi} + mV_k) q_V^+ \right] \\
= \int DU e^{-S_{\text{gauge}}} \prod_{i=1}^{N_V} \left( \frac{\det(\bar{\Psi} + mV_i)}{\det(\bar{\Psi} + mV_i)} \right) \prod_{j=1}^{N} \det(\bar{\Psi} + mS_j) \\
= \int DU e^{-S_{\text{gauge}}} \prod_{j=1}^{N} \det(\bar{\Psi} + mS_j) \\
= Z_{\text{QCD-like}}
\]

- Adding valence fields leads to desired valence propagators
Condensed notation

- Collect all fields into \((N + 2N_V)\)-dim vectors:

\[
Q = \begin{pmatrix}
q_1, q_2, \ldots, q_{N_V}, & q_1, q_2, \ldots, q_{N_V}, & \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_{N_V}
\end{pmatrix}
\]

valence  
sea  
ghost

\[
\overline{Q}^{tr} = \begin{pmatrix}
\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_{N_V}, & \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_{N_V}, & \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_{N_V}
\end{pmatrix}
\]

valence  
sea  
ghost

\[
\mathcal{M} = \begin{pmatrix}
m_1, m_2, \ldots, m_{N_V}, & m_1, m_2, \ldots, m_{N_V}, & m_1, m_2, \ldots, m_{N_V}
\end{pmatrix}
\]

valence  
sea  
ghost = valence

- Then can write action and partition function as:

\[
S_{PQ} = S_{\text{gauge}} + \overline{Q}(\overline{\Psi} + \mathcal{M})Q
\]

\[
Z_{PQ} = \int \mathcal{D}U \mathcal{D}Q \mathcal{D}Q \ e^{-S_{PQ}}
\]
Formal representation of PQ correlator

\[ Q = \left( q_{V1}, q_{V2}, \ldots, q_{VN}, q_{S1}, q_{S2}, \ldots, q_{SN}, \bar{q}_{V1}, \bar{q}_{V2}, \ldots, \bar{q}_{VN} \right) \]

\[ C_{\pi}^{PQ}(\tau) = \left\langle \sum \gamma_5 \right\rangle \]

\[ = Z_{PQ}^{-1} \int D U \prod_{j=1}^{N} \det(\overline{\partial} + m_{S_j})e^{-S_{\text{gauge}}} \]

\[ \times \sum_{\bar{x}} \text{tr} \left[ \gamma_5 \left( \frac{1}{\overline{\partial} + m_{V_d}} \right)_{x_0} \gamma_5 \left( \frac{1}{\overline{\partial} + m_{V_u}} \right)_{0x} \right] \]

\[ = Z_{PQ}^{-1} \int DUDQDQ e^{-S_{PQ}} \sum_{\bar{x}} \bar{u}_V \gamma_5 d_V(\bar{x}, \tau) \bar{d}_V \gamma_5 u_V(0) \]
Anchoring to QCD

\[ Q = \left( qV_1, qV_2, \ldots, qV_{N_V}, qS_1, qS_2, \ldots, qS_N, \tilde{q}V_1, \tilde{q}V_2, \ldots, \tilde{q}V_{N_V} \right) \]

- valence
- sea
- ghost
Anchoring to QCD

Set valence and sea masses equal

\[ Q = \left( \begin{array}{c} qV_1, qV_2, \ldots, qV_{N_V}, \quad qS_1, qS_2, \ldots, qS_N, \quad \tilde{q}V_1, \tilde{q}V_2, \ldots, \tilde{q}V_{N_V} \end{array} \right) \]

valence, sea, ghost
Anchoring to QCD

\[ Q = \left( \begin{array}{l}
qV_1, qV_2, \ldots, qV_{N_V} \\
qS_1, qS_2, \ldots, qS_N \\
\tilde{q}V_1, \tilde{q}V_2, \ldots, \tilde{q}V_{N_V}
\end{array} \right) \]

- Set valence and sea masses equal

- If \( m_{Vu} = m_{Sj} \) and \( m_{Vd} = m_{Sk} \) then valence correlator is physical:

\[
C_{\pi}^{PQ}(\tau) = Z_{PQ}^{-1} \int DU D\bar{Q} DQ e^{-S_{PQ}} \sum_{x} \bar{u}_V \gamma_5 d_V (x, \tau) \bar{d}_V \gamma_5 u_V (0)
\]

\[
= Z_{PQ}^{-1} \int DU D\bar{Q} DQ e^{-S_{PQ}} \sum_{x} \bar{q}_{Sj} \gamma_5 q_{Sk} (x, \tau) \bar{q}_{Sk} \gamma_5 q_{Sj} (0)
\]

\[
= Z_{QCD}^{-1} \text{-like} \int DU \prod_{i=1}^{N} Dq_{Si} D\bar{q}_{Si} e^{-S_{QCD} \text{-like}} \times \sum_{x} \bar{q}_{Sj} \gamma_5 q_{Sk} (x, \tau) \bar{q}_{Sk} \gamma_5 q_{Sj} (0)
\]

\[
= C_{\pi}^{QCD \text{-like}}(\tau)
\]

- Example of enhanced (V ↔ S) symmetry in PQ theory
Summary so far

- **PQQCD** is a well-defined, local Euclidean statistical theory
  - Describes $m_v \neq m_s$ and allows formal definition of individual Wick contractions
- Morel's formulation restores "unitarity", but at the cost of introducing ghosts
  - Violate spin-statistics theorem, so Minkowski-space theory violates causality & positivity, and may have a Hamiltonian with spectrum unbounded below
  - For $m_v \neq m_s$, can show (under mild assumptions) that flavor-singlet "pion" correlators develop manifestly unphysical double-poles [Sharpe & Shoresh]
- Can generalize to include discretization errors & to mixed actions (different discretizations of valence & sea quarks, e.g. "overlap on twisted mass")
- To make practical use of PQQCD, need to develop PQChPT
  - Is this possible given the unphysical features?
  - Do we need to have a healthy Minkowski theory to justify EFTs?
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Methods for developing PQChPT

- “Supersymmetric” method based on Morel’s formulation [Bernard & Golterman]
- “Replica” method adjusting loop contributions by adjusting $N_{\text{sea}}$ [Damgaard & Splittorf]
  - Formalizes “Quark-line” method accounting by hand for quarks in loops [Sharpe]
- Give same results to date—likely equivalent
- Use supersymmetric method here
Symmetries of PQQCD

\[ Q = \left( q_{V1}, q_{V2}, \ldots, q_{VN_V}, s_{S1}, s_{S2}, \ldots, s_{SN}, \tilde{q}_{V1}, \tilde{q}_{V2}, \ldots, \tilde{q}_{VN_V} \right) \]

- Action of PQQCD looks like QCD
  \[ S_{PQQCD} = S_{\text{gauge}} + \overline{Q}(\overline{\Psi} + M)Q \]

- Naively, when \( M \to 0 \) have graded version of QCD chiral symmetry:
  \[ Q_{L,R} \to U_{L,R}Q_{L,R}, \quad \overline{Q}_{L,R} \to \overline{Q}_{L,R}U_{L,R}^\dagger, \quad U_{L,R} \in SU(N_V + N|N_V) \]

- Apparent symmetry is \( SU(N_V + N|N_V)_L \times SU(N_V + N|N_V)_R \times U(1)_V \)

- In fact, there are subtleties in the ghost sector, but can ignore in perturbative calculations [Sharpe & Shoresh]

Subtleties have been understood in calculations leading to connection with random matrix theory [Damgaard et al]
Brief primer on graded groups

- $U$ is graded: contains both commuting and anticommuting elements:

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{N_V+N|N_V}, \quad A, D \text{ commuting}, \quad B, C \text{ anticommuting}$$

- If $U \in U(N_V + N|N_V)$ (fundamental representation) then

$$UU^\dagger = U^\dagger U = 1, \quad [\text{with } (\eta_1 \eta_2)^* \equiv \eta_2^* \eta_1^*]$$

- Supertrace maintains cyclicity:

$$\text{str}U \equiv \text{tr}A - \text{tr}D \quad \Rightarrow \quad \text{str}(U_1 U_2) = \text{str}(U_2 U_1)$$

- For $U \in SU(N_V + N|N_V)$, superdeterminant is unity:

$$\text{sdet}U \equiv \exp[\text{str}(\ln U)] = \frac{\det(A - BD^{-1}C)}{\det(D)} \quad \Rightarrow \quad \text{sdet}(U_1 U_2) = \text{sdet}U_1 \text{sdet}U_2$$
Examples of $SU(N_V+N|N)$ matrices

\[
U = \begin{pmatrix}
SU(N_V + N) & 0 \\
0 & SU(N_V)
\end{pmatrix} \quad \Rightarrow \quad \text{sdet}U = 1
\]

\[
U = \begin{pmatrix}
e^{i\theta N_V} & 0 \\
0 & e^{i\theta (N+N_V)}
\end{pmatrix} \quad \Rightarrow \quad \text{sdet}U = \frac{(e^{i\theta N_V})^{N+N_V}}{(e^{i\theta (N+N_V)})^{N_V}} = 1
\]

- An overall phase rotation is not in $SU(N_V + N|N)$

\[
U = \begin{pmatrix}
e^{i\theta} & 0 \\
0 & e^{i\theta}
\end{pmatrix} \quad \Rightarrow \quad \text{sdet}U = \frac{e^{i\theta (N+N_V)}}{e^{i\theta N_V}} = e^{i\theta N}
\]

- Thus $U(N_V + N|N_V) = [SU(N_V + N|N_V) \otimes U(1)]/Z_N$

- Group structure different if $N = 0$ (quenched theory)
Constructing the EFT

- Follow the same steps as for standard ChPT as closely as possible
  
  - Expand about theory with $M=0$ where symmetry is maximal
    
    - Strictly speaking, need to keep (arbitrarily) small mass to avoid PQ divergences & to use Vafa-Witten
    
    - A posteriori find divergences if $m_v \rightarrow 0$ at fixed $m_s$, so must take chiral limit with $m_v/m_s$ fixed
    
    - Symmetry is $G = SU(N_V + N|N_V)_L \times SU(N_V + N|N_V)_R$
  
  - For $M$ real, diagonal, positive [Vafa-Witten] theorem implies that graded vector symmetry is not spontaneously broken [Sharpe & Shoresh; Bernard & Golterman]
    
    - Quark and ghost condensates equal if $m_v = m_s \rightarrow 0$
  
  - We know chiral symm. breaks spontaneously in QCD with non-zero condensate
    
    - Since QCD is inside PQQCD $\Rightarrow$ we know form of PQ condensate & symmetry breaking
    
    - Order parameter $\Omega_{ij} = \langle Q_{L,i,\alpha,c} \overline{Q}_{R,j,\alpha,c} \rangle_{PQ} \rightarrow g U_L \Omega U_R^\dagger$
    
    - With standard masses $\Omega = \omega \times 1$ so vacuum manifold is now $SU(N_V + N|N_V)$
    
    - Symmetry breaking is $G \rightarrow H = SU(N_V + N|N_V)_V$
Constructing the EFT

- Still following the same steps as for standard ChPT as closely as possible ...

- Can derive Ward identities in PQQCD, & Goldstone’s thm. for 2-pt functions
  - $(N+2N_V)^2-1$ Goldstone “particles” created by $\bar{Q}\gamma_\mu\gamma_5T^aQ$ with $T^a$ a generator of graded group

- **New**: can construct transfer matrix for PQQCD including ghosts & show that, despite not being hermitian, it can be diagonalized and has a bounded spectrum
  - [Bernard & Golterman]

  - Energies can be real or come in complex-conjugate pairs (PT symmetry)
  - Have a complete set of states, although left- and right- eigenvectors are different
  - In free theory, correlators fall exponentially (up to powers from double-poles) but can be of either sign

- This result, if it holds up to scrutiny, makes the foundation of PQChPT essentially as strong as that of ChPT, since can follow a line of argument due to [Leutwyler]
  - which uses cluster decomposition and does not explicitly rely on unitarity

  - In particular, the existence of a transfer matrix etc. means that the spectrum deduced from 2-pt functions holds also for all other correlators (assuming no other light particles)
Constructing the EFT

- Sketch of [Leutwyler]'s argument
  - Existence of bounded transfer matrix + assumption of unique vacuum implies that PQ theory satisfies cluster decomposition
  - Integrating out heavy states (which might have complex energies?) still leads to local vertices which can be connected by Goldstone propagators
  - This leads to the same results as a general effective local Lagrangian in terms of Goldstone fields
  - Implementing local symmetry of generating functional with sources (up to anomalies) leads to result that effective Lagrangian can be chosen to be invariant under local symmetry group

- Bottom line: write down the most general local Lagrangian with sources consistent with local \( SU(N_V+N|N_V) \times SU(N_V+N|N_V) \) symmetry
Generalization of $\Sigma$ in PQChPT

- Follow method used for QCD:
  \[
  \Omega/\omega \rightarrow \Sigma(x) \in SU(N_V + N|N), \quad \Sigma \xrightarrow{g} U_L \Sigma U_R^t
  \]

- For standard masses, $\langle \Sigma \rangle = 1$, so define Goldstones by
  \[
  \Sigma = \exp \left[ \frac{2i}{f} \Phi(x) \right], \quad \Phi(x) = \begin{pmatrix}
  \phi(x) & \eta_1(x) \\
  \eta_2(x) & \tilde{\phi}(x)
  \end{pmatrix}
  \]

  - $s\text{det}\Sigma = 1 \Rightarrow s\text{tr}\Phi = \text{tr}\phi - \text{tr}\tilde{\phi} = 0$

- QCD GBs contained in $\Phi$
  \[
  \Phi(x) = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & \pi(x) & 0 \\
  N_V & 0 & 0
  \end{pmatrix}
  \quad \Rightarrow \quad \Sigma = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \Sigma_{\text{QCD}} & 0 \\
  0 & 0 & 1
  \end{pmatrix}
  \]

- Building blocks for PQXPT as for $\chi$PT, e.g.
  \[
  L_\mu = \Sigma D_\mu \Sigma^t \rightarrow U_L L_\mu U_L^t, \quad s\text{tr}(L_\mu) = 0
  \]

- Power counting as in $\chi$PT
PQ Chiral Lagrangian at NLO

[ Bernard & Golterman; Sharpe & Van de Water ]

- General form consistent with graded symmetries

\[
\mathcal{L}^{(2)} = \frac{f^2}{4} \text{str} \left( D_\mu \Sigma D_\mu \Sigma^\dagger \right) - \frac{f^2}{4} \text{str} (\chi \Sigma^\dagger + \Sigma \chi^\dagger) \\
\mathcal{L}^{(4)} = -L_1 \text{str}(D_\mu \Sigma D_\mu \Sigma^\dagger)^2 - L_2 \text{str}(D_\mu \Sigma D_\nu \Sigma^\dagger) \text{tr}(D_\mu \Sigma D_\nu \Sigma^\dagger) \\
+ L_3 \text{str}(D_\mu \Sigma D_\mu \Sigma^\dagger D_\nu \Sigma D_\nu \Sigma^\dagger) \\
+ L_4 \text{str}(D_\mu \Sigma^\dagger D_\mu \Sigma) \text{str}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) + L_5 \text{str}(D_\mu \Sigma^\dagger D_\mu \Sigma)[\chi^\dagger \Sigma + \Sigma^\dagger \chi] \\
- L_6 \left[ \text{str}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) \right]^2 - L_7 \left[ \text{str}(\chi^\dagger \Sigma - \Sigma^\dagger \chi) \right]^2 - L_8 \text{str}(\chi^\dagger \Sigma \chi^\dagger \Sigma + \text{p.c.}) \\
+ L_9 \text{istr}(L_{\mu \nu} D_\mu \Sigma D_\nu \Sigma^\dagger + \text{p.c.}) + L_{10} \text{str}(L_{\mu \nu} \Sigma R_{\mu \nu} \Sigma^\dagger) \\
+ H_1 \text{str}(L_{\mu \nu} L_{\mu \nu} + \text{p.c.}) + H_2 \text{str}(\chi^\dagger \chi) + \text{WZW}_{PQ} \\
+ L_{PQ} \mathcal{O}_{PQ}
\]

- \( \chi = 2B_0 M \)

- Same form as for QCD with \( \text{tr} \rightarrow \text{str} \) plus one extra term (\( \mathcal{O}_{PQ} \))

- How do the LECs relate to those of QCD?
Anchoring PQChPT to ChPT

- If choose $\Sigma$ to lie in QCD subspace

$$\Sigma = \begin{pmatrix}
1 & 0 & 0 \\
0 & \Sigma_{QCD} & 0 \\
0 & 0 & 1 \\
\end{pmatrix}$$

and sources do not connect subspaces, then

$$\mathcal{L}_{PQ\chi PT}^{(2,4,\ldots)}(\Sigma) \rightarrow \mathcal{L}_{\chi PT}^{(2,4,\ldots)}(\Sigma_{QCD})$$

- If external fields in correlation function are from sea sector, then can show that all valence and ghost contributions cancel in intermediate states

  $\Rightarrow$ $\Sigma$ takes the form given above

  $\Rightarrow$ PQ$\chi$PT calculation collapses to one in $\chi$PT

- Thus LECs in PQ$\chi$PT are equal to those in $\chi$PT

  $\Rightarrow$ Results in the chiral regime from PQQCD give information about physical LECs
Additional PQ operator: $O_{PQ}$

- Starting at NLO, at each order there are an increasing number of PQ operators that vanish on QCD subspace.

- At NLO, only one such operator [Sharpe & Van de Water]

\[
O_{PQ} = \text{str}(D_\mu \Sigma D_\nu \Sigma^\dagger D_\mu \Sigma D_\nu \Sigma^\dagger) \\
- \frac{1}{2} \text{str}(D_\mu \Sigma D_\mu \Sigma^\dagger)^2 - \text{str}(D_\mu \Sigma D_\nu \Sigma^\dagger)\text{str}(D_\mu \Sigma D_\nu \Sigma^\dagger) \\
+ 2\text{str}(D_\mu \Sigma D_\nu \Sigma^\dagger D_\mu \Sigma D_\nu \Sigma^\dagger)
\]

- Vanishes if $\Sigma \rightarrow \Sigma_{QCD}$ due to Cayley-Hamilton relations for $3 \times 3$ matrices.

- Does not vanish for general $\Sigma_{PQ}$.

- Appears in $L_{PQ\chi}^{(4)}$ with additional LEC.

- Same is true for standard $\chi$PT if $N \geq 4$.

- $O_{PQ}$ contributes to $\pi\pi$ scattering at NLO, but to $m_\pi$ and $f_\pi$ only at NNLO.
Why is $O_{PQ}$ present?

- Because PQQCD allows isolation of individual Wick contractions, unlike QCD
- For example, $\pi^+ K^0$ scattering in QCD has two contractions

- Can separate these contractions in PQQCD, e.g.

- $O_{PQ}$ contributes to the PQQCD process, but not that in QCD
- Shows how PQQCD differs from QCD even if $m_V = m_S$
Calculating in PQChPT

- PQ Lagrangian at LO:

\[ \mathcal{L}^{(2)} = \frac{f^2}{4} \text{str} \left( D_\mu \Sigma D_\mu \Sigma^\dagger \right) - \frac{f^2}{4} \text{str} \left( \chi \Sigma^\dagger + \Sigma \chi^\dagger \right) \]

- Insert expansion in Goldstone fields:

\[ \Sigma = \exp \left[ \frac{2i}{f} \Phi(x) \right], \quad \Phi(x) = \begin{pmatrix} \phi(x) & \eta_1(x) \\ \eta_2(x) & \bar{\phi}(x) \end{pmatrix}, \quad \text{str} \Phi = 0 \]

\[ \mathcal{L}^{(2)} = \text{str}(\partial_\mu \Phi \partial_\mu \Phi) + \text{str}(\chi \Phi^2) + \ldots \]

\[ = \text{tr}(\partial_\mu \phi \partial_\mu \phi + \partial_\mu \eta_1 \partial_\mu \eta_2 - \partial_\mu \eta_2 \partial_\mu \eta_1 - \partial_\mu \bar{\phi} \partial_\mu \bar{\phi}) \]

\[ + \text{tr} \left[ (\phi^2 + \eta_1 \eta_2) \begin{pmatrix} m_V & 0 \\ 0 & m_S \end{pmatrix} \right] - \text{tr}(\bar{\phi}^2 m_V) - \text{tr}(\eta_2 \eta_1 m_V) \]

- \( \phi \) part is like in QCD, except includes both valence and sea quarks

- Propagator for “charged” meson \( \bar{q}_1 q_2 \) (either valence of sea) is

\[ \frac{1}{(p^2 + m_{12}^2)}, \quad m_{12}^2 = (\chi_1 + \chi_2)/2 \]
Calculating in PQChPT

\[ \mathcal{L}^{(2)} = \operatorname{tr}(\partial_\mu \phi \partial_\mu \phi + \partial_\mu \eta_1 \partial_\mu \eta_2 - \partial_\mu \eta_2 \partial_\mu \eta_1 - \partial_\mu \tilde{\phi} \partial_\mu \tilde{\phi}) \]

\[ + \operatorname{tr}\left[ (\phi^2 + \eta_1 \eta_2) \begin{pmatrix} m_V & 0 \\ 0 & m_S \end{pmatrix} \right] - \operatorname{tr}(\tilde{\phi}^2 m_V) - \operatorname{tr}(\eta_2 \eta_1 m_V) \]

- \tilde{\phi} terms have wrong signs
  - Naively, propagator for “charged” ghost mesons \( \tilde{q}_1 \tilde{q}_2 \) is \(-1/(p^2 + m_{12}^2)\),
    \[ m_{12}^2 = (\chi_1 + \chi_2)/2 \]
  - But potential not minimized and functional integral not convergent!
  - More careful treatment of symmetries of PQQCD, maintaining convergence of ghost functional integral, concludes that naive result is OK in perturbation theory (but not non-perturbatively, e.g. in \( \epsilon \)-regime, where should change \( \tilde{\phi} \rightarrow i\tilde{\phi}, \Sigma^\dagger \rightarrow \Sigma^{-1} \)) [Sharpe & Shoresh]

- Goldstone fermion propagators can have either sign (no convergence problems); actual signs important for cancellations
Implementing stracelessness

- How implement \( \text{str}(\Phi) = \text{tr}(\phi) - \text{tr}(\bar{\phi}) = 0? \)
  1. Use a basis of generators which is straceless:
     \( \Phi = \sum_a \Phi_a T^a \) with \( \text{str}(T^a) = 0 \)
     - Analogous to not including the \( \eta' \) in QCD \( \chi \)PT
  2. Include identity component but then “integrate out”
     \( \Phi \rightarrow \Phi + \Phi_0/\sqrt{N} \) so that \( \text{str} \Phi = \sqrt{N} \Phi_0 \)
     \( \mathcal{L}_{\chi PT} \rightarrow \mathcal{L}_{\chi PT} + m_0^2 \text{str}(\Phi)^2 / N \)
     - Calculate propagators, then send \( m_0^2 \rightarrow \infty \) within them
     - To make formally correct, must regularize with a cut-off (e.g. lattice)
       so that \( (\partial_\mu \Phi_0)^2 < m_0^2 \Phi_0^2 \) (trivial decoupling)
     - Really just a trick to implement stracelessness

- Introducing \( \Phi_0 \) has advantage of allowing use of “quark line” basis:
  \( \Phi_{ij} \sim Q_i \bar{Q}_j \) for all \( i, j \)
“Charged” particle propagators are simple:

\[ \langle \Phi_{ij} \Phi_{ji} \rangle = \pm \frac{1}{p^2 + (x_i + x_j)/2} = \]

Neutral propagators have double poles:

\[ \mathcal{L}^{(2)} = \sum_{j=1}^{N+2N_V} \epsilon_j (\partial_\mu \Phi_{jj} \partial_\mu \Phi_{jj} + m_j \Phi_{jj}^2) + (m_0^2/N)(\sum_j \epsilon_j \Phi_{jj})^2 \]

\[ \epsilon_j = \begin{cases} +1 & \text{valence or sea quarks} \\ -1 & \text{ghosts} \end{cases} \]

Can simply invert with linear algebra tricks. Schematically, for external valence quarks have “hairpin” sum:

\[ \text{V} + \text{V} + \text{V} + \text{S} + \text{V} + \ldots \]
Quark lines & double poles

\[ \frac{\nu}{\nu} + \frac{\nu}{\nu} \pm \frac{\nu}{\nu} + \frac{\nu}{\nu} = \frac{\nu}{\nu} + \frac{\nu}{\nu} + \frac{\nu}{\nu} + \frac{\nu}{\nu} + \ldots \]

- Result after \( m_0^2 \to \infty \) for \( N = 3 \) [Bernard & Golterman; Sharpe & Shoresh]

\[
\langle \Phi_{ij} \Phi_{ji} \rangle = \frac{\epsilon_i \delta_{ij}}{p^2 + \chi_i} \frac{1}{N} \frac{1}{(p^2 + \chi_i)(p^2 + \chi_j)} \frac{(p^2 + \chi_{s1})(p^2 + \chi_{s2})(p^2 + \chi_{s3})}{(p^2 + M_{\pi_0}^2)(p^2 + M_{\eta}^2)}
\]

- Simplifies for degenerate sea quarks:

\[
\langle \Phi_{ij} \Phi_{ji} \rangle = \frac{\epsilon_i \delta_{ij}}{p^2 + \chi_i} \frac{1}{N} \frac{1}{(p^2 + \chi_i)(p^2 + \chi_j)} \frac{(p^2 + \chi_s)}{p^2 + \chi_i}(p^2 + \chi_j)
\]

- Manifestly unphysical double pole for \( \chi_i = \chi_j \)
- Residue is then \( (\chi_i - \chi_s)/N \), so vanishes for physical subspace
- Can show from symmetries of PQQCD that if charged propagators have single poles, then neutral have double (and no higher) poles [Sharpe & Shoresh]
Outline of lecture 4

- Partial quenching and PQChPT
  - What is partial quenching and why might it be useful?
  - Developing PQChPT

- Results and status

- $m_u=0$ and the validity of PQ theories (and the rooting prescription)
Sample calculation: $m_{\pi}$

- Calculations are straightforward extension of standard $\chi$PT
- Mass-squared of “pion” composed of valence quarks $V_1, V_2$
- Quark-line diagrams for 1-loop contributions

- LO four-pion vertices have single strace, so are ”connected”
- Manifest cancellation between contributions from commuting and anticommuting particles
Sample calculation: $m_{\pi^2}$

To simplify expression for loop contributions, assume $N$ degenerate sea quarks and $m_{V1} = m_{V2} \neq m_S$

$$m_{VV}^2 = \chi_V \left( 1 + \frac{1}{N} \frac{2\chi_V - \chi_S}{\Lambda^2} \ln(\chi_V/\mu^2) + \frac{\chi_V - \chi_S}{N\Lambda^2} \right)$$

$$+ \frac{8}{f^2} \left[ (2L_8 - L_5)\chi_V + (2L_6 - L_4)N\chi_S \right]$$

- Reduces to QCD-like result when $\chi_V \to \chi_S$
- $\chi_V$ and $\chi_S$ provide separate dials for determining $2L_8 - L_5$ and $2L_6 - L_4$
- Result in PQ mass-plane depends on physical LECs
- Unphysical nature of result clear from divergence in $\chi_S \ln \chi_V$ as $\chi_V \to 0$
- In practice, expansion breaks down only for very small $\chi_V$
Status of PQChPT calculations

- It is now standard to extend any χPT calculation to PQχPT
  - Many quantities considered at NLO: pions, baryons, vector mesons, scalar mesons, heavy-light hadrons, weak matrix elements (\(B_K\), \(K \rightarrow \pi \pi\)), NEDM, pion scattering, ...
  - First calculations at NNLO for pion properties
  - PQ effects also included in tmχPT, staggered χPT and mixed action χPT
  - Most non-trivial example is baryons, where need to use a set-up in which all three quark lines are explicit
  - Most striking result is for scalar meson correlators, where hairpin propagators lead to unphysical negative contributions at long distances

- In general, can use PQχPT to determine form of expected results for individual contractions (e.g. connected and disconnected contributions to \(\pi_0\) propagators in tmLQCD) [Hansen & Sharpe]

- Most extensive practical use is in MILC improved staggered simulations

- PQChPT can be used to estimate size of disconnected contribs, e.g. g-2 [Juettner]

- Generalization to \(ε\)-regime allows predictions for small eigenvalues & connection with RMT including discretization errors
  - Recent discovery of constraints on signs of some LECs in WChPT [Damgaard, Splittorff, Verbaarschot; Kieburg et al.; Hansen & Sharpe]
Outline of lecture 4

- Partial quenching and PQChPT
  - What is partial quenching and why might it be useful?
  - Developing PQChPT
  - Results and status

- $m_u=0$ and the validity of PQ theories (and the rooting prescription)
Some additional references for $m_u=0$

Including some on the rooting controversy

- T. Banks, Y. Nir & N. Seiberg [additive mass renorm & strong CP problem], hep-ph/9403203
Ambiguity in $m_u=0$?

- Consider QCD with $m_d$ and $m_s$ fixed (e.g. at their physical values), but send $m_u \to 0$.
  - No increase in symmetry
  - $m_{\pi}^2 \propto (m_u + m_d) + \text{NLO}$ does not vanish

- Contrast this with sending both $m_u, m_d \to 0$:
  - $SU(2)_L \times SU(2)_R$ becomes exact, and $m_{\pi}^2 \to 0$

- But doesn’t $m_u \to 0$ have unambiguous meaning at the level of the lattice action?
  - Naively would seem so if use fermions with exact chiral symmetry (e.g. overlap)
  - But there are (infinitely) many choices for overlap kernel, which assign different topological charges to “rough” configurations

- If we set $m_u = 0$ using two different kernels, will we obtain, in the continuum limit, the same value for mass ratios, e.g. $m_{\pi_0}/m_{\text{proton}}$?
  - The standard answer is YES
  - [Creutz, PRL 92, 162003 (2004)] argues NO!

- This is the potential ambiguity.

Also with Wilson fermions using PCAC masses [Sommer’s lectures]
Restatement in $N_f=1$ QCD

- Can formulate the issue also in $N_f = 1$ QCD, a simpler setting
- No PGBs: spectrum consists of "$\eta$", "$\Delta$", etc.
- With two overlap operators having different kernels, if one sets $m = 0$, and takes the continuum limit (not an easy task in practice!) will one get the same value for $m_\eta/m_\Delta$?
  - The standard answer is YES
  - [Creutz, PRL 92, 162003 (2004)] argues NO
  - Note that for $a \neq 0$ will certainly have "kernel-dependent" discretization errors—the issue is what happens when $a \to 0$.

- Use this formulation in subsequent discussion:
  - Note that $\langle \bar{\psi} \psi \rangle \neq 0$, although this breaks no symmetry
The standard argument

- In perturbation theory, if we have chiral symmetry (as with overlap), quark mass is renormalized multiplicatively, to all orders

\[ m(a) = M g(a)^{\gamma_0/\beta_0} [1 + O(g^2)] \]

\[ a\Lambda = e^{-1/(2\beta_0 g^2)} g^{-\beta_1/\beta_0^2} [1 + O(g^2)] \]

\[ \beta_0 = (11 - 2N_f/3)/(16\pi^2) \]

- This is uncontroversial. If it were the whole story, it would imply that, once \( g(a) \) is small enough (so the universal parts of the \( \beta \)-function and anomalous dimension dominate) setting \( M = 0 \) \( \Rightarrow m(a) = 0 \) leads to universal long-distance physics, irrespective of the overlap kernel.
  - Just as different gauge actions give a Symanzik effective action that differs by \( a^2 \times \) irrelevant dim-6 operators, so two different \( m = 0 \) theories will differ by irrelevant dim > 4 operators

- What about non-perturbative contributions to the running?
  - The 't Hooft vertex!
The standard argument

- In one flavor QCD, the 't Hooft vertex is bilinear, and leads to additive shift of quark mass
- Instanton calculations are not reliable when instantons are large, since $g(\rho)$ is not small
- However, what is needed for the RG evolution between scale $1/a$ and $1/(a + da)$ are instantons of size $\rho \sim a$
- If $a$ is small enough, the semi-classical result should be reliable:

$$\frac{dm}{d \ln a} \approx m\gamma_0 g^2 + \text{const} \times (1/a)e^{-8\pi^2/g^2} g^n$$
$$\approx m\gamma_0 g^2 + \text{const} \times \Lambda(a\Lambda)^{28/3}$$

For $N_f=3$

$$\frac{m_d m_s}{\Lambda} (a\Lambda)^{10}$$

[Georgi & Macarthy 1981] [Choi, Kim, Sze, PRL 61, 794 (1988)]
[ Banks, Nir & Seiberg, hep-ph/9403203]

- Additive contribution present, which can only calculate approximately
  - However, it vanishes as $a \sim 9$
Example of running

\[ \frac{dm}{d\ln a} \approx m_0 g^2 + \text{const} \times \Lambda(a\Lambda)^{28/3} \]

Running with 't Hooft vertex

Effect invisible except in IR

\[ N_f=1, \ m(30 \text{ GeV})=2 \text{ MeV} \]
\[ \Lambda=300 \text{ MeV} \]
The standard argument

\[ \frac{dm}{d \ln a} \approx m \gamma_0 g^2 + \text{const} \times \Lambda (a \Lambda)^{28/3} \]

- There is an uncertainty in the running of \( m \)
  - At a given \( a \), for
    \[ |m(a)| \gtrsim m_{cr} \approx \frac{(a \Lambda)^{28/3} \Lambda}{g(a)^2 \gamma_0} \]
    
    the RG evolution to smaller \( a \) will be essentially unaffected by the additive term, and thus unambiguous
  - For \( |m(a)| \lesssim m_{cr} \) evolution to smaller \( a \) is not controlled
  - In this sense there is an ambiguity in \( m(a) \) of size \( m_{cr} \)

- As \( a \to 0 \), however, this ambiguity shrinks rapidly to zero, much faster than the standard logarithmic decrease of \( m(a) \) and faster than other disc. errors

- Thus, in the standard view, we do know, in a regularization invariant way, what \( m = 0 \) means in the continuum limit
  - In particular, we can simply take \( a \to 0 \) holding \( m(a) = 0 \)
Running if set $m(a^*) = 0$

$N_f = 1$, $\Lambda = 300 \text{ MeV}$, $c = 1$

Log scale, so here $1/a = 10^{21} \text{ GeV} !!!$

-Log($a\Lambda$)

$m(a) (\text{MeV})$

depth drops $\sim (a^*)^{28/3}$

$1/a^* = 1.5 \text{GeV}$

$1/a^* = 1 \text{GeV}$

-asymptotes to zero logarithmically
(this is NOT the vanishing we are claiming)

-precise depth depends on details of NP term so uncertain

S. Sharpe, “EFT for LQCD: Lecture 4” 3/27/12 @ “New horizons in lattice field theory”, Natal, Brazil
\[ m(1\text{TeV}) \text{ given that } m(a^*) = 0 \]

\[ \frac{1}{a^*} \text{ (GeV)} \]

\[ \log_{10}[-m(1\text{TeV})/\text{MeV}] \]

falls as \( \sim (a^*)^{28/3} \)

\[ m(1\text{TeV}) \sim 10^{-22} \text{ MeV} \]
Mike Creutz’s view (my summary)

- [Creutz, PRL 92, 162003 (2004)] finds this argument unconvincing
- The argument certainly relies on the assumption that we know the form of the non-perturbative terms at short distances
  - Note that the value of $m(a)$ for the massless theory at $a \approx \Lambda_{\text{QCD}}^{-1}$
    - (the “constituent quark mass”) is unknown, since the additive term certainly dominates by this scale
  - But this is irrelevant for $m(a)$ as $a \to 0$
- Creutz makes some qualitative arguments, but does not directly address the standard argument given above
  - Please read and draw your own conclusions
- It would be very interesting to test Creutz’s proposed breakdown in universality numerically
Relation to PQQCD

- PQ extensions of QCD-like theories provide a way of using symmetries to unambiguously define "$m_u = 0$" [Farchioni et al., 0706.1131, 0710.4454]

- Consider the PQ $N_f = 1$ theory, with $N_V$ valence quarks (and corresponding ghosts) degenerate with the sea quark
  - Enlarged theory now has an approximate chiral symmetry $SU(N_V + 1|N_V)_L \times SU(N_V + 1|N_V)_R$
  - This symmetry becomes exact when $m \to 0$
  - The fact that $\langle \bar{\psi}\psi \rangle \neq 0$ in $N_f = 1$ QCD implies that the chiral symmetry of the PQ extension is spontaneously broken
  - One can thus write down the corresponding PQ\chiPT, and $m = 0$ at quark level unambiguously maps to $m = 0$ at the chiral level in order to match the symmetries
  - There are thus PG bosons and fermions with $m_\pi^2 \propto m$
  - Thus $m = 0$ is unambiguously selected by vanishing PQ pion mass, just as $m_u = m_d = 0$ is picked out by vanishing physical pion mass (both requiring $L \to \infty$)
  - Used in practice by [Farchioni, 0710.4454]
Relation to PQQCD

- Other (closely related) ways of picking out $m = 0$
  - Vanishing of topological susceptibility, which is defined using PQ correlators [Giusti et al, hep-lat/0402027; Lüscher, hep-lat/0404034]
  - $1/m$ divergences in certain finite volume PQ correlation functions [Bernard et al, 0711.0696]

- **CONCLUSION**: If $m = 0$ is ambiguous, then the PQ extension of $N_f = 1$ QCD does not have a universal continuum limit
  - For $m = 0$ the PQ pions are massless but $m_\eta$, etc. are regularization dependent

- Same argument would apply to other $N_f$ if one of the quark masses vanishes

- These results seem to me to imply that, if $m = 0$ is ambiguous, PQQCD is ill-defined in general (even when $m \neq 0$), and thus that extrapolations using PQxPT are invalid!
Consequences for rooting

- Staggered fermion simulations use the “$\text{det}^{1/4}$” trick to remove extra tastes
- $\text{det}([D+m]^4)^{1/4} = \text{det}(D+m)$ is trivial (assuming $m>0$)
- $D_{\text{stag}} + m \rightarrow [D+m]^4$ only in continuum limit
- Using $\text{det}(D_{\text{stag}}+m)^{1/4}$ leads to an unphysical theory for $a \neq 0$
- Key question: Do the unphysical features vanish when $a \rightarrow 0$?
- Variety of analytic arguments (with assumptions) and numerics suggest YES
- If rooting staggered fermions are in the correct universality class, then they necessarily give PQQCD in the continuum limit (e.g. for one staggered fermion, end up with 4 valence and 1 sea quark)
- If PQ theories are ill-defined, so is this continuum limit, and thus so are rooted staggered fermions
Summary of $m_u=0$ part

- There are several related theoretical issues

1. Is $m_u=0$ ambiguous?

2. Is $m=0$ ambiguous in the $N_f=1$ theory?

3. Are PQ theories well defined in the continuum limit?

4. Does rooted staggered LQCD have the correct continuum limit?

5. Does $N_f=1$ QCD have a non-zero (Banks-Casher) density of microscopic ($\lambda \sim 1/V$) eigenvalues?

6. Does $m_u=0$ solve the strong CP problem?
Summary of $m_u=0$ part

- There are several related theoretical issues
  
  1. Is $m_u=0$ ambiguous?  \textbf{NO}
  
  2. Is $m=0$ ambiguous in the $N_f=1$ theory?  \textbf{NO}
  
  3. Are PQ theories well defined in the continuum limit?  \textbf{YES}
  
  4. Does rooted staggered LQCD have the correct continuum limit?  \textbf{NO}
  
  5. Does $N_f=1$ QCD have a non-zero (Banks-Casher) density of microscopic ($\lambda\sim1/V$) eigenvalues?  \textbf{NO}
  
  6. Does $m_u=0$ solve the strong CP problem?
Summary of $m_u=0$ part

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5. Does $N_f=1$ QCD have a non-zero (Banks-Casher) density of microscopic ($\lambda \sim 1/V$) eigenvalues?
6. Does $m_u=0$ solve the strong CP problem?

For #4, I think that the main issues lie elsewhere, & that the answer is “very likely” (another lecture)
Summary of $m_u=0$ part

- There are several related theoretical issues
  1. Is $m_u=0$ ambiguous?
  2. Is $m=0$ ambiguous in the $N_f=1$ theory?
  3. Are PQ theories well defined in the continuum limit?
  4. Does rooted staggered LQCD have the correct continuum limit?
  5. Does $N_f=1$QCD have a non-zero (Banks-Casher) density of microscopic ($\lambda \sim 1/V$) eigenvalues?
  6. Does $m_u=0$ solve the strong CP problem?

- These issues deserve further study, including by numerical simulations
- Key issue is whether hadron mass ratios are unambiguous in continuum limit

I have argued

- NO
- NO
- YES
Kaplan-Manohar ambiguity


- Ambiguity in determination of quark mass ratios from comparison of ChPT with experiment
  - Unrelated to fact that cannot determine masses themselves because they are not RG invariant
- Chiral Lagrangian is constructed using symmetries alone
- $M$ and $(M^\dagger)^{-1}\det(M)$ transform identically under $SU(3)_L \times SU(3)_R$
- Chiral Lagrangian invariant under $m_u \rightarrow m_u + \alpha m_d m_s$, $m_d \rightarrow m_d + \alpha m_s m_u$, $m_s \rightarrow m_s + \alpha m_u m_d$, as long as change LECs appropriately
- Cannot determine whether $m_u=0$ using ChPT
- However, QCD is NOT invariant under Kaplan-Manohar transformation, so it does not prevent determination of $m_u$ using LQCD
- Similarity of form to ‘t Hooft vertex due to underlying chiral symmetry
Solving the strong CP problem?

- Full QCD Lagrangian includes $\theta F \tilde{F}$ term which violates CP
- Formally, can rotate into mass matrix because of axial anomaly & bring entire phase onto $m_u$
  $$M = \text{diag}(m_u e^{i\theta}, m_d, m_s)$$
- $|\theta\text{-bar}| \approx 10^{-10}$ to agree with bounds on electric dipole moments
- Could have avoided, apparently, with $m_u = 0$ (not, in fact, true in nature)
- Theoretically, could $m_u = 0$ have worked? If $m_u$ ambiguous, clearly not
- [Srednicki: hep-ph/0503051] notes that additive mass renormalization only affects $\text{Re}(m_u)$: if $\text{Im}(m_u) = 0$ at any scale, then true at all scales
- More generally, solve strong CP problem if $\text{Im}[\det(M)] = 0$ at any scale
- Another solution is the axion (make $\theta$ dynamical)---does this work?
Rooted staggered theory has spurious, unphysical cuts in pion scattering amplitude.
Spurious cuts?

- Rooted staggered theory has spurious, unphysical cuts in pion scattering amplitude.

- Answer (obtained using rSChPT): Unphysical cuts are present for $a \neq 0$ but have discontinuities ("strengths") which vanish like $a^2$.

- Also, if one wanted to study the $m_u=0$ issue with staggered fermions, one must take the $a\rightarrow 0$ limit before $m_u\rightarrow 0$ (otherwise, e.g., the condensate will vanish).

- Numerical checks of these properties in Schwinger model by [Durr & Hoelbling].

- Related issues arise in scalar two-point correlator where unphysical cuts lead to negative contributions that vanish like $a^2$, and which have been observed and found to be consistent with rSChPT by the [MILC collaboration].