Effective Field Theories for lattice QCD: Lecture 2

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Outline of Lectures

1. Overview & Introduction to continuum chiral perturbation theory (ChPT)
2. Continuum ChPT continued: Power counting; adding sources; illustrative results; SU(2) ChPT with a heavy strange quark; finite volume effects
3. Including discretization effects in ChPT using Symanzik's effective theory
4. Partially quenched ChPT and applications, including a discussion of whether m_u=0 is meaningful

Outline of lecture 2

- Power counting & sources in ChPT (carried over from lecture 1)
- Examples of results from continuum ChPT
  - Focus on those with lattice applications
- SU(2) vs SU(3) ChPT in the presence of kaons
- ChPT at finite volume
  - "p regime" (large M_π L)
  - "ε regime" (small M_π L)

Exercises for lectures 1 & 2

Available on the course web page as a separate file

1. Calculate the next-to-leading order (NLO) expression for M_π^2 in SU(3) χPT in the isospin limit ($m_u = m_d = m_\ell$). Include both analytic and non-analytic (chiral logarithmic) contributions.

2. Show that, in the chiral limit ($M \to 0$), the amplitude for "pion" scattering in SU(3) is given by

$$\mathcal{A}(\pi_a\pi_b \to \pi_c\pi_d) = \frac{1}{6f^2}\left[f_{abc}f_{cde}(t-u) + f_{ace}f_{bde}(s-u) + f_{ade}f_{bce}(s-t)\right].$$

   Here $a, b, \ldots$ label the flavor of the pion: $\pi = \sum_a \pi_a T_a$, with T_a the SU(3) generators satisfying

$$\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}, \qquad [T_a, T_b] = i f_{abc} T_c.$$

3. The Cayley-Hamilton theorem implies that a 3×3 matrix A satisfies

$$A^3 - A^2\mathrm{tr}A + A\frac{1}{2}\left[\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)\right] = \mathbf{1}\det(A).$$

   Use this to show that $\sum_{\mu,\nu}\mathrm{tr}(L_\mu L_\nu L_\mu L_\nu)$ is not an independent term in $\mathcal{L}^{(4)}$ for SU(3).

4. Using the 2×2 version of the Cayley-Hamilton theorem, or otherwise, show how several terms in the SU(3) NLO chiral Lagrangian presented in the lectures collapse into single terms in the SU(2) version.

5. The general form of the NLO mesonic chiral Lagrangian $\mathcal{L}^{(4)}$ is missing one term. Find it and give a field redefinition that removes this term at NLO.

1. Determine the form of the mass terms in $\mathcal{L}_K^{(2)}$, i.e. the second-order part of the heavy-kaon SU(2) chiral Lagrangian (see slide 35 of lecture 2). You will need to decorate the quark mass matrix with appropriate factors of ξ and $\xi^\dagger$ in order to convert it to an object that transforms explicitly with the $SU(2)_V$ matrix V.

2. Calculate the next-to-leading order (NLO) expression for f_K using heavy-kaon SU(2) χPT in the isospin limit ($m_u = m_d = m_\ell$). Include both analytic and non-analytic (chiral logarithmic) contributions. (You will need to use the set-up described on slide 36 of lecture 2 to define the axial current.)

Additional references for lecture 2

- E. Scholtz *et al.*, "Determination of SU(2) ChPT LECs from 2+1 flavor staggered lattice simulations," arXiv:1301.7557
- A. Roessl, "Pion-kaon scattering near the threshold in chiral SU(2) perturbation theory", Nucl. Phys. B 555, 507 (1999) [arXiv:hep-ph/9904230]
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- J. Flynn & C. Sachrajda, "SU(2) chiral perturbation theory for K_{l3} decay amplitudes," [RBC/UKQCD], Nucl. Phys. B812 (2009) 64, [arXiv:0809.1229 (hep-ph)]
- J. Bijnens & A. Celis, "$K\to\pi\pi$ decays in SU(2) ChPT," Phys. Lett. B680 (2009) 466 [arXiv:0906.0302 (hep-ph)]
- G. Colangelo *et al.*, "On the factorization of the chiral logarithms in the pion form factor," arXiv:1208.0498 [hep-ph]
- M. Luscher, "Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories: 1. Stable Particle States," Comm. Math. Phys. 104 (1986) 177
- G. Colangelo, S. Durr & C. Haefeli, "Finite volume effects for meson masses and decay constants," Nucl. Phys. B721 (2005) 136 [hep-lat/0503014]
- J. Gasser & H. Leutwyler, "Thermodynamics of chiral symmetry", Phys. Lett. B188, 477 (1987)
- J. Gasser & H. Leutwyler, "Light quarks at low temperatures", Phys. Lett. B184, 83 (1987)
- J. Gasser & H. Leutwyler, "Spontaneously Broken Symmetries: Effective Lagrangians at Finite Volume", Nucl. Phys. B307 (1988) 763
- L. Giusti *et al.*, "Low energy couplings of QCD from current correlators near the chiral limit," JHEP 04 (2004) 013 [hep-lat/0402002]
- P. Damgaard *et al.*, "A new method for determining f_π on the lattice," Phys. Rev. D72 (2005) 091501 [hep-lat/0508029]
- P. Damgaard *et al.*, "The microsopic density of the QCD Dirac operator,", Nucl. Phys. B547 (1999) 305

Recap

$$\Sigma = \exp(2i\pi/f) \qquad \pi \equiv \pi^a T^a$$

$$\Sigma \quad \to \quad U_L \Sigma U_R^\dagger\,, \qquad \Sigma^\dagger \to U_R \Sigma^\dagger U_L^\dagger\,,$$

$$\begin{aligned}
\mathcal{L}^{(2)} &= \frac{f^2}{4}\mathrm{tr}\left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right) - \frac{f^2 B_0}{2}\mathrm{tr}(M[\Sigma^\dagger + \Sigma]) \\
&= \mathrm{tr}(\partial_\mu \pi \partial_\mu \pi) + 2B_0\ \mathrm{tr}(M\pi^2) \\
&\quad + \frac{1}{3f^2}\mathrm{tr}([\pi, \partial_\mu \pi][\pi, \partial_\mu \pi]) - \frac{2B_0}{3f^2}\mathrm{tr}(M\pi^4) + O(\pi^6)
\end{aligned}$$

Power-counting in ChPT (M=0)

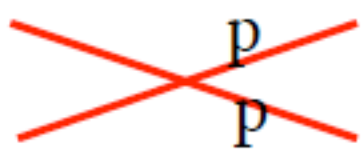
- How can a non-renormalizable theory be predictive?

$$\mathcal{L}^{(2)} \sim f^2 \mathrm{tr}(L_\mu L_\mu) \sim (\partial\pi)^2 + \frac{\pi^2(\partial\pi)^2}{f^2} + \dots$$

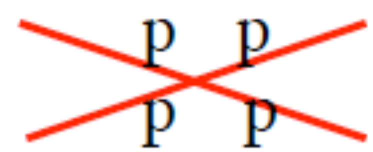
$$\mathcal{L}^{(4)} \sim L_{GL} \mathrm{tr}(L_\mu L_\mu)^2 + \dots \sim L_{GL}\left[\frac{(\partial\pi)^4}{f^4} + \frac{\pi^2(\partial\pi)^4}{f^6}\right]$$

  - L_{GL} are unknown dimensionless Gasser-Leutwyler coeffs

- Consider $\pi\pi$ scattering (with, say, dim. reg. to avoid power divergences):

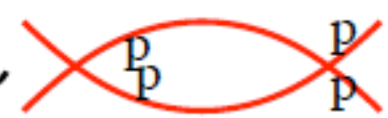
$\mathcal{L}^{(2)}_{\mathrm{tree}}$: 

$\sim \frac{p^2}{f^2}$

$\mathcal{L}^{(4)}_{\mathrm{tree}}$: 

$\sim L_{GL}\left(\frac{p^2}{f^2}\right)^2$

$\mathcal{L}^{(2)}_{1-\mathrm{loop}}$: 

$\sim$ 

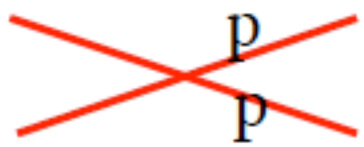
$\sim \left(\frac{p^2}{f^2}\right)^2 \frac{\ln(p^2/\mu^2)}{(4\pi)^2}$

- Straightforward power-counting exercise (counting factors of f) $\Rightarrow$ have expansion in p^2/f^2 up to logs
  - LO: $\mathcal{L}^{(2)}_{\mathrm{tree}}$ ("trivial" to calculate)
  - NLO: $\mathcal{L}^{(4)}_{\mathrm{tree}} + \mathcal{L}^{(2)}_{1-\mathrm{loop}}$ ("easy" to calculate)
  - NNLO: $\mathcal{L}^{(6)}_{\mathrm{tree}} + \mathcal{L}^{(4)}_{1-\mathrm{loop}} + \mathcal{L}^{(2)}_{2-\mathrm{loop}}$ (hard but done)

- For $M \neq 0$, $p^2/f^2 \longrightarrow (p^2 \text{ or } m^2_{\mathrm{PGB}})/f^2$

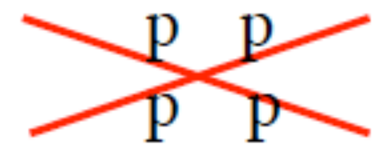
Power-counting in ChPT (M=0)

$\mathcal{L}^{(2)}_{\rm tree}$:



$\sim \frac{p^2}{f^2}$

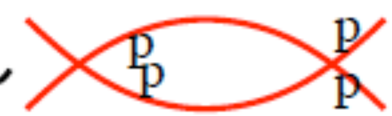
$\mathcal{L}^{(4)}_{\rm tree}$:



$\sim L_{GL} \left(\frac{p^2}{f^2}\right)^2$

$\mathcal{L}^{(2)}_{1-{\rm loop}}$:

$\sim$



$\sim \left(\frac{p^2}{f^2}\right)^2 \frac{\ln(p^2/\mu^2)}{(4\pi)^2}$

- Theory is predictive up to truncation errors:
  - E.g. at LO, $\mathcal{A}(\pi\pi \to \pi\pi)$ predicted in terms of $f (= f_\pi)$, up to errors of relative size p^2/f^2
  - Only a finite number of diagrams and LECs at each order, so can always make predictions
  - Non-analytic behavior ("chiral logs") does not involve new LECs
    - ~ Determined by unitarity (2 particle cut)
  - Loops renormalize LECs: $L_{GL} \to L_{GL}(\mu)$

True expansion parameter?

- LEC's run with μ:
  - $dL_{GL}/d\ln(\mu) \approx 1/(4\pi)^2 \Rightarrow L_{GL}(2\mu) - L_{GL}(\mu) \approx 1/(4\pi)^2$
- So guess ("naive dimensional analysis"):
  - $L_{GL}(\mu \approx m_\rho) \approx 1/(4\pi)^2$
- Works well phenomenologically: $-1 \lesssim L_{GL}(4\pi)^2 \lesssim +1$
- Implies expansion parameter is p^2/Λ_χ^2, with $\Lambda_\chi = 4\pi f$
- For $M \neq 0$, $p^2/\Lambda_\chi^2 \longrightarrow (p^2 \text{ or } m_{\mathrm{PGB}}^2)/\Lambda_\chi^2$

Technical aside: adding sources

- Matrix elements of V_μ, A_μ, S and P are phenomenologically interesting
- Incorporate in QCD using external sources (hermitian matrices)

  $$\mathcal{L}_{\rm QCD} = \overline{Q}_L(i\not{D} - \gamma^\mu l_\mu)Q_L + \overline{Q}_R(i\not{D} - \gamma^\mu r_\mu)Q_R - \overline{Q}_L(s+ip)Q_R - \overline{Q}_R(s-ip)Q_L$$

  - Switched to Minkowski space for the moment
  - s, p not new—rewriting of spurions $M = s + ip$, $M^\dagger = s - ip$
  - Obtain correlation functions in QCD by functional derivatives of $Z_{\rm QCD}(l_\mu, r_\mu, s, p)$
- Basic assumption of χPT: $Z_{\rm QCD}(l_\mu, r_\mu, s, p) = Z_\chi(l_\mu, r_\mu, s, p)$ for $p, m_{\rm PGB} \ll \Lambda_\chi$, up to truncation errors
- Functional derivatives of Z_χ give χPT result for correlation functions

$$\text{e.g. } \frac{\delta}{\delta l_\mu(x)} \frac{\delta}{\delta p(y)} \ln Z_\chi \Big|_{l=r=p=0, s=M} \sim \langle T[L^\mu(x) P(y)] \rangle$$

$$\Rightarrow \qquad f_\pi \propto \langle 0 | L_\mu | \pi \rangle$$

Technical aside: adding sources

- How determine $Z_\chi(l_\mu, r_\mu, s, p)$?
  - Generalize spurion trick to **local** $SU(3)_L \times SU(3)_R$ symmetry
  - $\mathcal{L}_{\rm QCD}$ invariant if l, r_μ transform as gauge fields:
    $l_\mu \to U_L l_\mu U_L^\dagger + i U_L \partial_\mu U_L^\dagger$, $r_\mu \to U_R r_\mu U_R^\dagger + i U_R \partial_\mu U_R^\dagger$
  - s, p transform as before: e.g. $(s+ip) \to U_L (s+ip) U_R^\dagger$
- Z_{QCD} invariant (up to anomalies) $\Rightarrow Z_\chi$ invariant (up to anomalies)
- $\Rightarrow \mathcal{L}_\chi$ invariant [Gasser-Leutwyler]
  - Can be accomplished using covariant derivatives: $\partial_\mu \to D_\mu$
    e.g. $D_\mu \Sigma = \partial_\mu \Sigma - i l_\mu \Sigma + i \Sigma r_\mu \to U_L (D_\mu \Sigma) U_L^\dagger$
  - Normalization of l, r_μ terms fixed
  - Remainder of enumeration as before (except $D_\mu M$ now allowed)
  - Convenient to introduce $\chi = 2B_0(s+ip) = 2B_0 M$
    - In general χ is a matrix source
    - But also use notation $\chi_q = 2B_0 m_q$

LO chiral Lagrangian including sources

- At LO (back to Euclidean space):

$$\mathcal{L}^{(2)} = \frac{f^2}{4}\mathrm{tr}\left(D_\mu \Sigma D_\mu \Sigma^\dagger\right) - \frac{f^2}{4}\mathrm{tr}(\chi\Sigma^\dagger + \Sigma\chi^\dagger)$$

  - ▷ Using $\delta/\delta l_\mu(x)|_{l=r=p=0,s=M}$ can "match" currents with QCD:

$$\overline{Q}_L \gamma_\mu T^a Q_L \simeq (if^2/2)\mathrm{tr}(T^a \Sigma \partial_\mu \Sigma^\dagger) = -(f/2)\partial_\mu \pi^a + \ldots$$

    $\Rightarrow$ at LO, $f = f_\pi \approx 93$ MeV

  - ▷ Using $\delta/\delta s(x)|_{l=r=p=0,s=M}$ can relate condensate to B_0:

$$\overline{Q}Q \simeq -(f^2 B_0/2)\mathrm{tr}(\Sigma + \Sigma^\dagger) = -N_f f^2 B_0 + O(\pi^2)$$

    $\Rightarrow$ at LO, $\langle \overline{q} q\rangle = -f^2 B_0$ [Gell-Mann–Oakes–Renner]

    - Only using lattice can one determine B_0

NLO chiral Lagrangian

- At NLO have 10 LECs and 2 "high-energy coefficients":

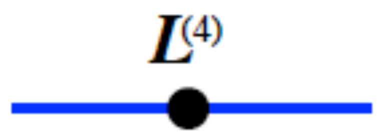
$$\begin{aligned}\mathcal{L}^{(4)} \quad = \quad & -L_1\,\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger)^2 - L_2\,\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger)\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger)\\ & +L_3\,\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger D_\nu\Sigma D_\nu\Sigma^\dagger)\\ & +L_4\,\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{tr}(\chi^\dagger\Sigma+\Sigma^\dagger\chi) + L_5\,\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)[\chi^\dagger\Sigma+\Sigma^\dagger\chi])\\ & -L_6\left[\mathrm{tr}(\chi^\dagger\Sigma+\Sigma^\dagger\chi)\right]^2 - L_7\left[\mathrm{tr}(\chi^\dagger\Sigma-\Sigma^\dagger\chi)\right]^2 - L_8\,\mathrm{tr}(\chi^\dagger\Sigma\chi^\dagger\Sigma+\mathrm{p.c.})\\ & +L_9\,i\mathrm{tr}(L_{\mu\nu}D_\mu\Sigma D_\nu\Sigma^\dagger + p.c.) + L_{10}\,\mathrm{tr}(L_{\mu\nu}\Sigma R_{\mu\nu}\Sigma^\dagger)\\ & +H_1\,\mathrm{tr}(L_{\mu\nu}L_{\mu\nu}+p.c.) + H_2\,\mathrm{tr}(\chi^\dagger\chi)\end{aligned}$$

- L_i are "Gasser-Leutwyler coefficients"
  - Fundamental parameters of QCD, akin to hadron mass ratios
  - A subset can be determined experimentally to good accuracy
  - A different subset is straightforward to determine on the lattice
- $H_{1,2}$ give contact terms in correlation functions
- $L_{\mu\nu} = \partial_\mu\ell_\nu - \partial_\nu\ell_\mu + i[\ell_\mu,\ell_\nu]$
- At NNLO there are 90 LECs and 4 HECs! [Bijnens et al]

Ex. 1: Charged pion mass at NLO

- LO: $m^2_{\mathrm{PGB},0} = (\chi_{q1} + \chi_{q2})/2 = 2B_0(m_{q1} + m_{q2})/2$
- NLO–tree:

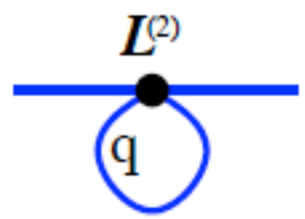
$\delta m^2_{\mathrm{PGB}} \sim$



$\sim \chi L \frac{\chi}{f^2} \sim \chi (16\pi^2 L) \frac{m^2_{\mathrm{PGB},0}}{\Lambda^2_\chi}$

- NLO–loop:

$\delta m^2_{\mathrm{PGB}} \sim$



$\sim \frac{\chi}{f^2} \int_q \frac{1}{q^2 + m^2_{\mathrm{PGB}}} \sim \chi \frac{m^2_{\mathrm{PGB},0}}{\Lambda^2_\chi} \ln\left(\frac{m^2_{\mathrm{PGB},0}}{\mu^2}\right)$

$$\boxed{m^2_{\pi^\pm} = \chi_\ell \left\{ 1 + \frac{8}{f^2}[\underbrace{(2L_8 - L_5)\chi_\ell}_{\text{valence}} + \underbrace{(2L_6 - L_4)(2\chi_\ell + \chi_s)}_{\text{sea}}] + \underbrace{\frac{3L_\pi - L_\eta}{6}}_{\text{logs}} \right\}}$$

$$L_\pi = \frac{m_\pi^2}{\Lambda_\chi^2} \ln\left(\frac{m_\pi^2}{\mu^2}\right), \qquad L_\eta = \frac{m_\eta^2}{\Lambda_\chi^2} \ln\left(\frac{m_\eta^2}{\mu^2}\right)$$

- Chiral logs predicted in terms of $\Lambda_\chi = 4\pi f$

Chiral Lagrangian at NLO

- Terms contributing to charged pion mass

$$\begin{aligned}\mathcal{L}^{(2)} &= \frac{f^2}{4}\mathrm{tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - \frac{f^2 B_0}{2}\mathrm{tr}(M[\Sigma^\dagger+\Sigma]) \\ &= \mathrm{tr}(\partial_\mu\pi\partial_\mu\pi) + 2B_0\,\mathrm{tr}(M\pi^2) \\ &\quad + \frac{1}{3f^2}\mathrm{tr}([\pi,\partial_\mu\pi][\pi,\partial_\mu\pi]) - \frac{2B_0}{3f^2}\mathrm{tr}(M\pi^4) + O(\pi^6)\end{aligned}$$

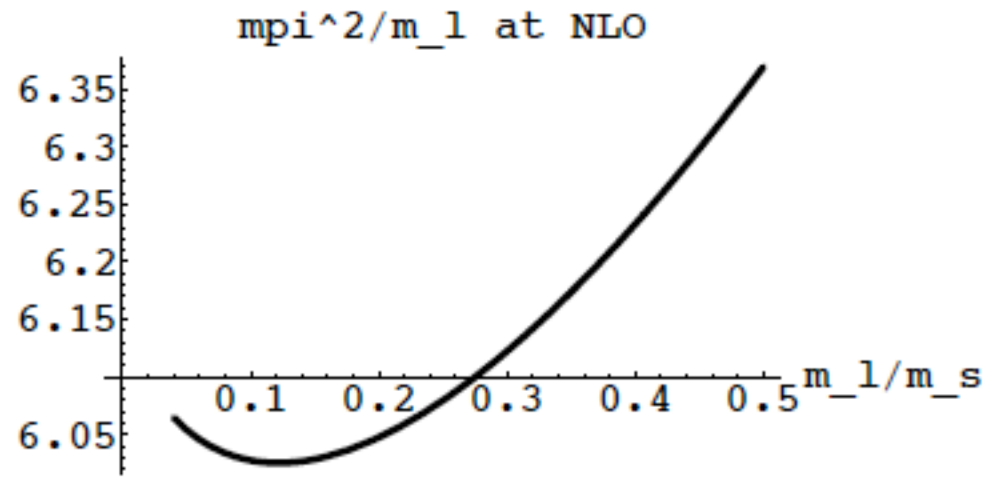
$$\begin{aligned}\mathcal{L}^{(4)} = & -L_1\,\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger)^2 - L_2\,\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger)\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger) \\ & + L_3\,\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger D_\nu\Sigma D_\nu\Sigma^\dagger) \\ & + L_4\,\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{tr}(\chi^\dagger\Sigma+\Sigma^\dagger\chi) + L_5\,\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)[\chi^\dagger\Sigma+\Sigma^\dagger\chi]) \\ & + L_6\left[\mathrm{tr}(\chi^\dagger\Sigma+\Sigma^\dagger\chi)\right]^2 - L_7\left[\mathrm{tr}(\chi^\dagger\Sigma-\Sigma^\dagger\chi)\right]^2 + L_8\,\mathrm{tr}(\chi^\dagger\Sigma\chi^\dagger\Sigma+\mathrm{p.c.}) \\ & + L_9\,i\mathrm{tr}(L_{\mu\nu}D_\mu\Sigma D_\nu\Sigma^\dagger + p.c.) + L_{10}\,\mathrm{tr}(L_{\mu\nu}\Sigma R_{\mu\nu}\Sigma^\dagger) \\ & + H_1\,\mathrm{tr}(L_{\mu\nu}L_{\mu\nu} + p.c.) + H_2\,\mathrm{tr}(\chi^\dagger\chi)\end{aligned}$$

Lessons for lattice (4)

$$m_{\pi^\pm}^2 = \chi_\ell \left\{ 1 + \frac{8}{f^2} [\underbrace{(2L_8 - L_5)\chi_\ell}_{\text{valence}} + \underbrace{(2L_6 - L_4)(2\chi_\ell + \chi_s)}_{\text{sea}}] + \underbrace{\frac{3L_\pi - L_\eta}{6}}_{\text{logs}} \right\}$$

- Non-analytic terms important at small masses

  $m_s = 0.08$ GeV, $f = 0.093$ GeV,
  $L_5 = 1.45 \times 10^{-3}$, $L_8 = 10^{-3}$,
  $L_4 = L_6 = 0$
  [Bijnens, hep-ph/0409068]



- **Must see chiral logs to have convincing results**
- Using PQ simulations allows separation of L_i

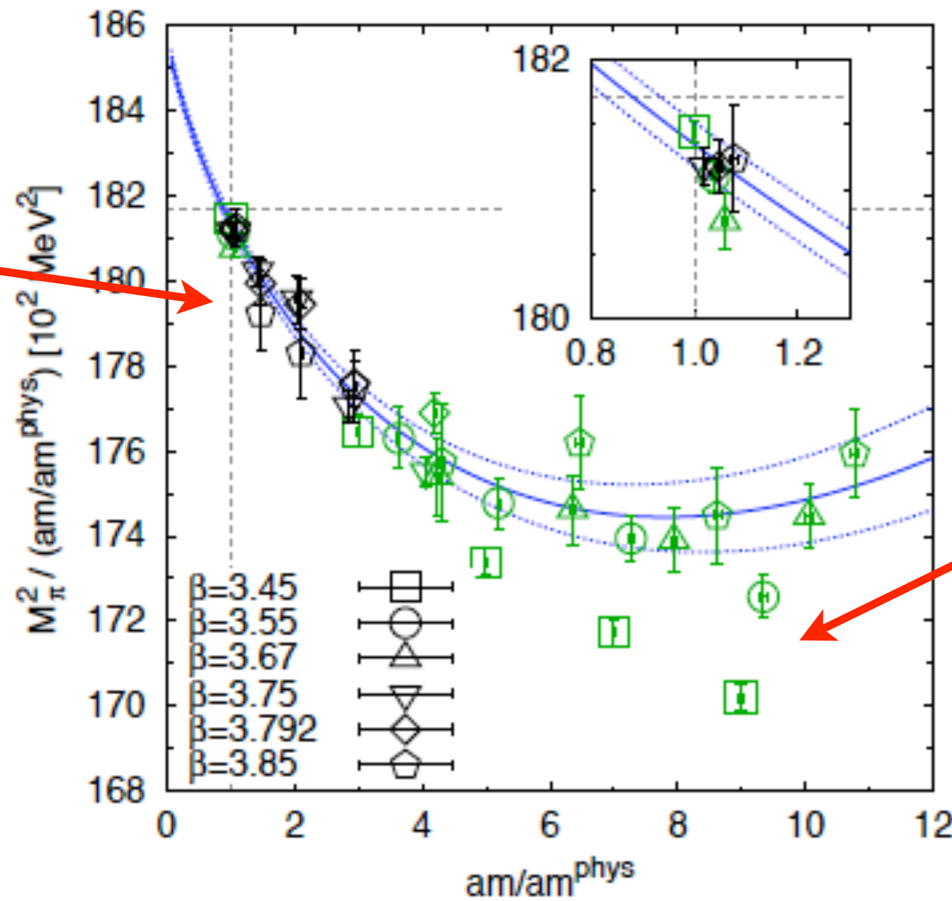
– To distinguish chiral logs from analytic dependence in practice must constrain NNLO terms (quadratic in m_l) to have natural size

Recent example: [Scholtz et al., 1301.7557]

- Determining LECs from lattice fits to NLO ChPT including physical point

Only black points included in fit

m_π^2/m_l



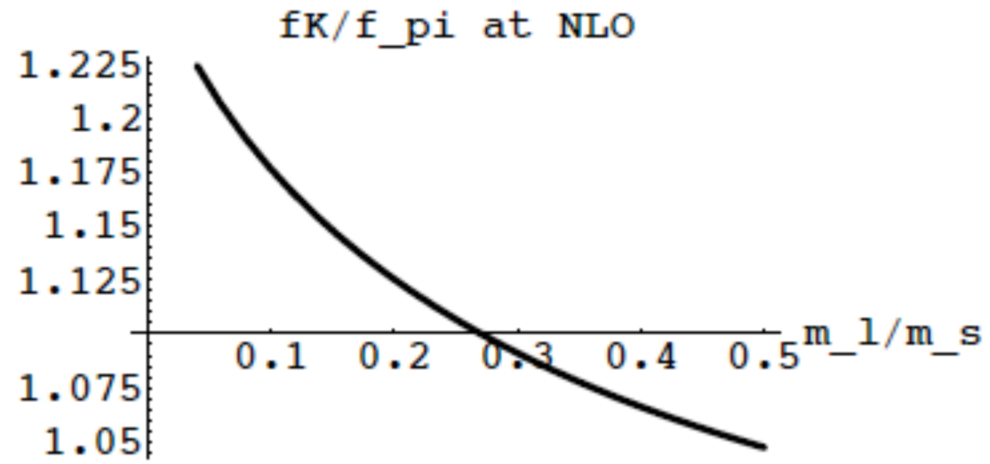
Discretization errors

m_l/m_s 0.2 0.4

Another example of chiral logs

$$\frac{f_K}{f_\pi} = 1 + \frac{2}{f^2}\underbrace{(L_5)(\chi_s - \chi_\ell)}_{\text{valence}} + \underbrace{\frac{5}{8}L_\pi - \frac{1}{4}L_K - \frac{3}{8}L_\eta}_{\text{logs}}$$

- Non-analytic terms important at small masses
  $m_s = 0.08$ GeV, $f = 0.085$ GeV,
  $L_5 = 1.45 \times 10^{-3}$, $L_4 = 0$



- Some quantities have enhanced chiral logs, e.g. $\langle r^2\rangle_\pi \sim \ln(m_\pi^2/\mu^2)$

Status of continuum ChPT for PGBs

- $SU(2)$ χPT complete at NNLO, including electroweak interactions
- Several predictions despite 53 LECs at NNLO (excluding electroweak)!
- Many quantities relevant for lattice simulations, e.g.
  - Pion scattering amplitude
  - Form factors of PGBs (vector and scalar)
  - Semileptonic form factors ($K \to \pi$)
  - B_K, $K \to \pi\pi$

Extended to many more lattice-relevant quantities now

- $SU(3)$ χPT (including electroweak) largely extended to NNLO
- Convergence?. [Bijnens, hep-ph/0401039,hep-ph/0409068]
  - $a_0^0(\pi\pi \to \pi\pi) = \underbrace{0.159}_{\text{LO}} + \underbrace{0.044}_{\text{NLO}} + \underbrace{0.016}_{\text{NNLO}} = 0.219 \pm ?$ c.f. $0.220(5)$ SU(2) ChPT
  - $f_K/f_\pi = \underbrace{1}_{\text{LO}} + \underbrace{0.169}_{\text{NLO}} + \underbrace{0.051}_{\text{NNLO}}$ (fit) SU(3) ChPT
  - But for m^2_{PGB}, NNLO terms larger than NLO SU(3) ChPT

- Overall, SU(2) ChPT converges well at the pion mass, while for SU(3) convergence is quantity-dependent and involves at best a drop of ~1/4 at each order

Extension to heavy sources

- Heavy-light mesons in $1/m_B$ expansion [Wise, Burdman & Donoghue]

$$F_B \sim F_{B,0}(1 + \underbrace{m_\pi^2}_{\text{analytic}} + \underbrace{m_\pi^2 \ln(m_\pi)}_{\text{chiral log}} + \ldots)$$

  - Similar expansion to those for PGB properties
  - Non-analytic terms involve additional coefficient $g_{\pi BB^*}$

- Baryons [Jenkins & Manohar] and Vector mesons [Jenkins et al]

$$M_H \sim M_0 + \underbrace{m_\pi^2}_{\text{analytic}} + \underbrace{g_{\pi HH'}\, m_\pi^3}_{\text{leading loop}} + \underbrace{m_\pi^4 \ln(m_\pi)}_{\text{subleading loop}} + m_\pi^4 + \ldots$$

  - Non-analytic terms involve additional coefficients (e.g. $g_{\pi NN}$)
  - Expansion in powers of m_π/Λ_χ (c.f. $(m_\pi/\Lambda_\chi)^2$ for mesons)
  - $\Rightarrow$ Poorer convergence

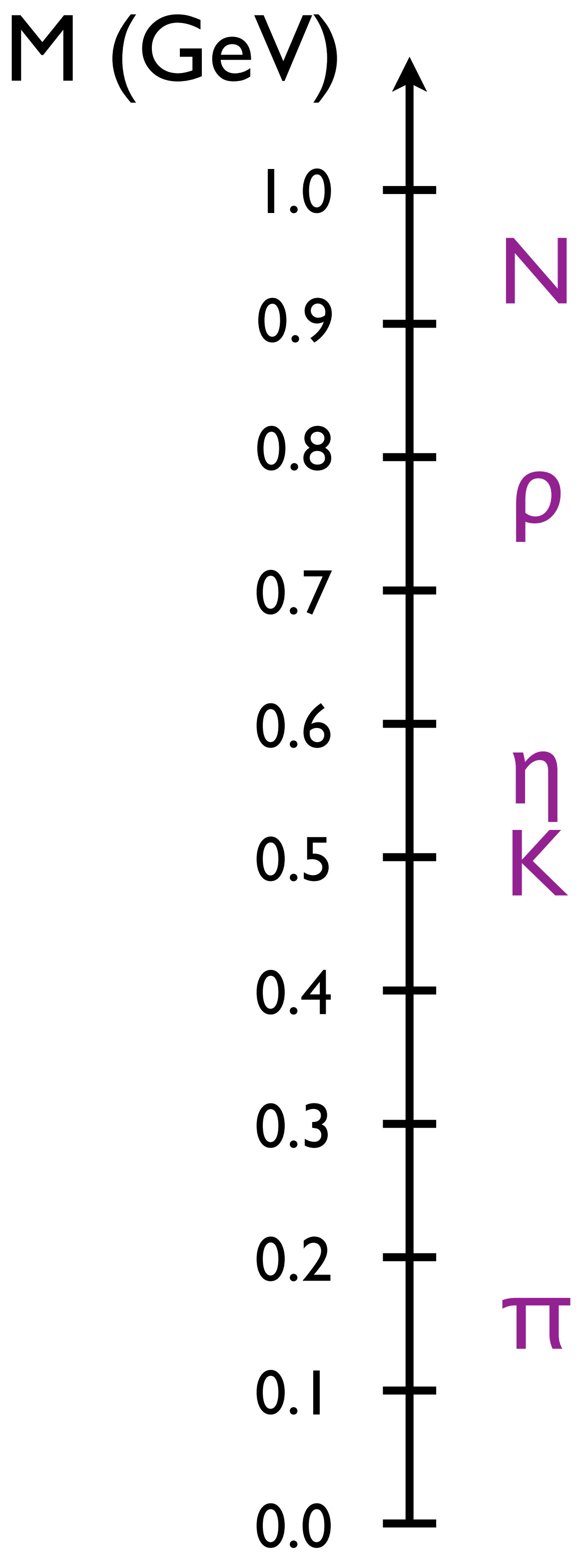
Outline of lecture 2

- Examples of results from continuum ChPT
  - Focus on those with lattice applications
- SU(2) vs SU(3) ChPT in the presence of kaons
- ChPT at finite volume
  - "p regime" (large $M_\pi L$)
  - "ε regime" (small $M_\pi L$)

SU(3) vs SU(2) ChPT

- We have developed ChPT assuming 3 light quarks, so the approximate chiral symmetry is $SU(3)_L \times SU(3)_R$
- However, $SU(2)_L \times SU(2)_R$ is a much better symmetry
  - $m_u/\Lambda_{QCD} \sim m_d/\Lambda_{QCD} \sim 1/100 \ll m_s/\Lambda_{QCD} \sim 1/4$
- Thus we have the option of treating m_s as heavy (and K and η as heavy) with only the pions as the light d.o.f. in the EFT
  - Joint expansion in $m_{u,d}/\Lambda_{QCD}$ and $m_{u,d}/m_s \sim 1/25$ and p_π^2/M_K^2
  - m_s is now a fixed parameter
  - Useful for lattice simulations if $m_s \sim m_{s,phys}$ & $m_{u,d}/m_s \ll 1$ [✶]
- In practice often have wide ranges of m_s and $m_{u,d}$, so can try SU(3) ChPT for a broader range of masses, and SU(2) ChPT for the restricted range [✶]
- Opinions differ as to how useful SU(3) ChPT can be, while SU(2) ChPT is clearly useful in the regime [✶] and now (following [RBC/UKQCD]) very widely used

SU(3) vs SU(2) ChPT



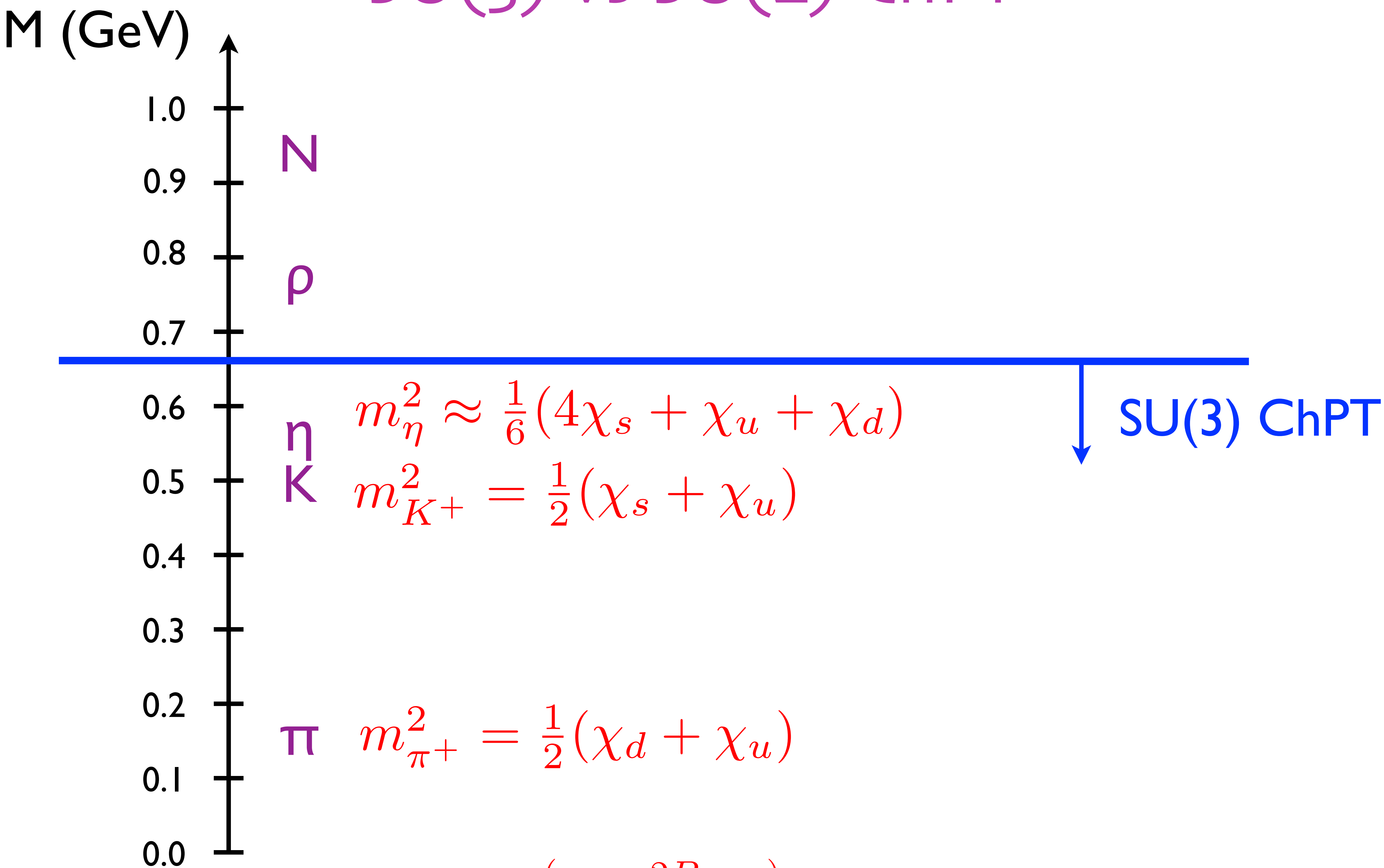
$$m_\eta^2 \approx \frac{1}{6}(4\chi_s + \chi_u + \chi_d)$$

$$m_{K^+}^2 = \frac{1}{2}(\chi_s + \chi_u)$$

$$m_{\pi^+}^2 = \frac{1}{2}(\chi_d + \chi_u)$$

$$(\chi_q = 2B_0 m_q)$$

SU(3) vs SU(2) ChPT



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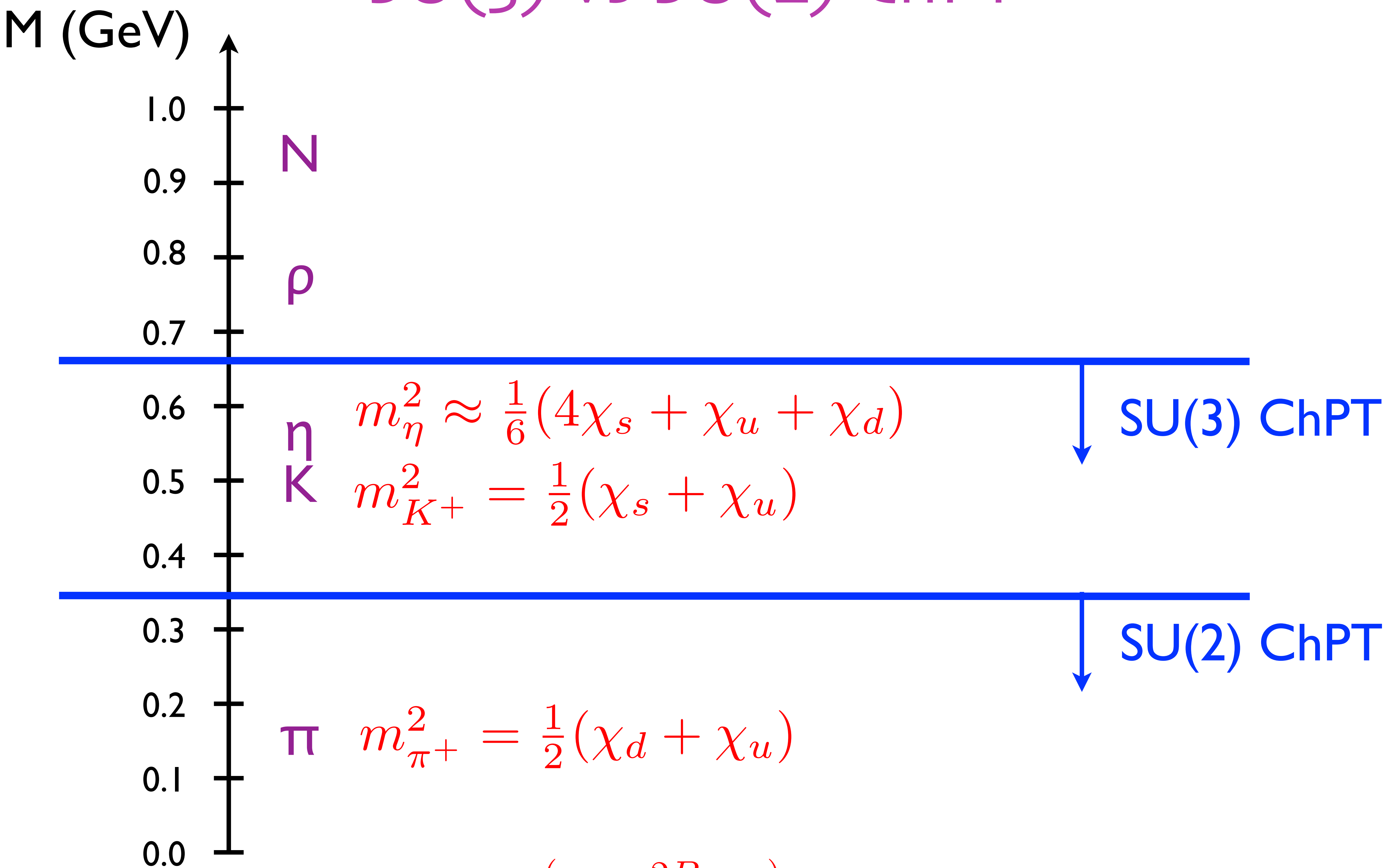

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$$(\chi_q = 2B_0 m_q)$$

SU(2) ChPT for pion properties

- For M_π, f_π, $\pi\pi$ scattering, π charge radius, etc. can develop SU(2) ChPT exactly as we did for SU(3) except that $\Sigma \in SU(2)$
  - Some terms in the NLO Lagrangian are redundant in the SU(2) case, so the number of LECs is reduced
- Alternatively, we can "integrate out" the heavy K & η from SU(3) ChPT results---an approach which we describe as it useful in other examples
- Start with NLO SU(3) results:

$$\boxed{M_{\pi^+}^2 = \chi_\ell \left\{1 + \frac{8}{f^2}\left[(2L_8 - L_5)\,\chi_\ell + (2L_6 - L_4)\,(2\chi_\ell + \chi_s)\right] + \frac{L_\pi}{2} - \frac{L_\eta}{6}\right\}}$$

$$\boxed{f_\pi = f\left\{1 + \frac{4}{f^2}\left[L_5\,\chi_\ell + L_4\,(2\chi_\ell + \chi_s)\right] - L_\pi - \frac{L_K}{2}\right\}}$$

$$\boxed{\chi_s = 2B_0 m_s} \qquad \boxed{L_\pi = \frac{M_\pi^2}{(4\pi f)^2}\log\left(\frac{M_\pi^2}{\mu^2}\right)\,, \quad \text{etc.}}$$

Reducing SU(3) results to SU(2)

$$M_{\pi^+}^2 = 2B_0 m_\ell \left\{1 + \frac{8}{f^2}\left[(2L_8 - L_5)\,\chi_\ell + (2L_6 - L_4)\,(2\chi_\ell + \chi_s)\right] + \frac{L_\pi}{2} - \frac{L_\eta}{6}\right\}$$

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$$M^2_{\pi^+} = 2B_0 m_\ell \left\{1 + \frac{8}{f^2}\left[(2L_8 - L_5)\,\chi_\ell + (2L_6 - L_4)\,(2\chi_\ell + \chi_s)\right] + \frac{L_\pi}{2} - \frac{L_\eta}{6}\right\}$$

Cannot expand in m_s
Must absorb dependence in B_0

$$M^2_{\pi^+} = 2\tilde{B}_0(m_s) m_\ell \left\{1 + \frac{8}{f^2}\left[(2L_8 - L_5)\,\chi_\ell + (2L_6 - L_4)\,2\chi_\ell\right] + \frac{L_\pi}{2} - \frac{L_\eta}{6}\right\}$$

Unknown functions of m_s

$$f_\pi = f_2(m_s)\left\{1 + \frac{4}{f^2}\left[L_5\,\chi_\ell + L_4\,2\chi_\ell\right] - L_\pi - \frac{L_K}{2}\right\}$$

Absorb m_s dependence
into f

$$f_\pi = f\left\{1 + \frac{4}{f^2}\left[L_5\,\chi_\ell + L_4\,(2\chi_\ell + \chi_s)\right] - L_\pi - \frac{L_K}{2}\right\}$$

Reducing SU(3) results to SU(2)

$$M_{\pi^+}^2 = 2\tilde{B}_0(m_s) m_\ell \left\{ 1 + \frac{8}{f^2}\left[(2L_8 - L_5)\,\chi_\ell + (2L_6 - L_4)\,2\chi_\ell \right] + \frac{L_\pi}{2} - \frac{L_\eta}{6} \right\}$$

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Reducing SU(3) results to SU(2)

$$M^2_{\pi^+} = 2\tilde{B}_0(m_s)m_\ell \left\{1 + \frac{8}{f_2(m_s)^2}\left[\tilde{L}_3(m_s)\chi_\ell\right] + \frac{L_\pi}{2} - \frac{L_\eta}{6}\right\}$$

Including higher order terms proportional to $\chi_l\, m_s^n$

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Reducing SU(3) results to SU(2)

$$M^2_{\pi^+} = 2\tilde{B}_0(m_s)m_\ell\left\{1 + \frac{8}{f_2(m_s)^2}\left[\tilde{L}_3(m_s)\chi_\ell\right] + \frac{L_\pi}{2} - \frac{L_\eta}{6}\right\}$$

Last step: $L_\pi = \frac{M_\pi^2}{(4\pi f)^2}\log\left(\frac{M_\pi^2}{\mu^2}\right) \longrightarrow \tilde{L}_\pi = \frac{M_\pi^2}{(4\pi f_2(m_s))^2}\log\left(\frac{M_\pi^2}{\mu^2}\right)$

$$M^2_{\pi^+} = 2\tilde{B}_0(m_s)m_\ell\left\{1 + \frac{8}{f_2(m_s)^2}\left[\tilde{L}_3(m_s)\chi_\ell\right] - \frac{1}{2}\tilde{L}_\pi\right\}$$

$$f_\pi = f_2(m_s)\left\{1 + \frac{4}{f_2(m_s)^2}\left[\tilde{L}_4(m_s)\chi_\ell\right] - \tilde{L}_\pi\right\}$$

$M^2 \log(M)$ analytic away from M=0

Absorb into LECs

$$f_\pi = f_2(m_s)\left\{1 + \frac{4}{f_2(m_s)^2}\left[\tilde{L}_4(m_s)\,\chi_\ell\right] - L_\pi - \frac{L_K}{2}\right\}$$

Reducing SU(3) results to SU(2)

- Final SU(2) ChPT results

$$M_{\pi^+}^2 = 2\tilde{B}_0(m_s)m_\ell \left\{1 + \frac{8}{f_2(m_s)^2}\left[\tilde{L}_3(m_s)\chi_\ell\right] - \frac{1}{2}\tilde{L}_\pi\right\}$$

$$f_\pi = f_2(m_s)\left\{1 + \frac{4}{f_2(m_s)^2}\left[\tilde{L}_4(m_s)\chi_\ell\right] - \tilde{L}_\pi\right\}$$

- SU(2) ChPT results are expansions in χ_l and p_π^2, valid in form for arbitrary, large m_s, with the LECs depending smoothly (analytically) on m_s

- This method/trick gives the correct result, but does not properly derive the factor of $f_2(m_s)$ in the denominator of the chiral logs---for this one can start from the SU(2) chiral Lagrangian, which, at LO, is

$$\mathcal{L}^{(2)}_{SU(2)} = \frac{{f_2}^2}{4}\mathrm{tr}\left(\partial\Sigma_2\partial_\mu{\Sigma_2}^\dagger\right) - \frac{{f_2}^2\tilde{B}_0}{2}\mathrm{tr}\left(M_2[{\Sigma_2}^\dagger + \Sigma_2]\right)$$

- Actually, knowing only the form of the LO SU(2) chiral Lagrangian is sufficient to show that the chiral logs come with $f_2(m_s)$ in the denominators

- The argument given above then gives the correct log coefficients, since it is valid in the regime $m_l << m_s << \Lambda_{QCD}$, where both SU(2) & SU(3) ChPT work

SU(2) ChPT for Kaon properties

- The Kaon is a heavy particle in SU(2) ChPT---a "source" for pion fields
  - Chiral Lagrangian must be extended to include source sources
- We can, however, obtain the results using the same trick as above, so we show these first

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$$M_K^2 = B_0(m_s+m_\ell)\left\{1 + \frac{4}{f^2}\left[(2L_8-L_5)(\chi_\ell+\chi_s) + 2(2L_6-L_4)(2\chi_\ell+\chi_s)\right] + \frac{1}{3}L_\eta\right\}$$

$$\downarrow$$

$$M_K^2 = C_0(m_s)\left\{1 + C_2(m_s)m_\ell\right\}$$

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$$\downarrow$$

$$M_K^2 = C_0(m_s)\left\{1 + C_2(m_s)m_\ell\right\}$$

- No chiral log in this case
- This method loses possible connections between NLO LECs from different processes, which is rarely a concern in practice, since usually do not calculate enough quantities to overconstrain these LECs

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$$\boxed{f_K = f\left\{1 + (LEC)\chi_\ell + (LEC')\chi_s - \frac{3}{8}L_\pi - \frac{3}{4}L_K - \frac{3}{8}L_\eta\right\}}$$

$$\downarrow$$

$$\boxed{f_K = C_1(m_s)\left\{1 + C_3(m_s)m_\ell - \frac{3}{8}\tilde{L}_\pi\right\}}$$

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$$\downarrow$$

$$\boxed{f_K = C_1(m_s)\left\{1 + C_3(m_s)m_\ell - \tfrac{3}{8}\tilde{L}_\pi\right\}}$$

- We have guessed that the chiral log involves $\tilde{L}_\pi = \frac{M_\pi^2}{(4\pi f_2(m_s))^2}\log\left(\frac{M_\pi^2}{\mu^2}\right)$
- To justify this we need to properly develop ChPT for heavy kaons

Heavy Kaon SU(2) ChPT

- Use the methodology of [Callan, Coleman, Wess & Zumino; Roessl]
  - Developed for LQCD (including partial quenching) in [Allton *et al.*, RBC/UKQCD, 2008]
  - Applies also to heavy B-mesons and baryons (not discussed here)
- Key issue is how to incorporate the Kaon field, now that it is not a PGB
  - How does it couple to pions? How does it transform under chiral group?
  - Cannot hope to study K K $\longrightarrow$ ππ, since then $p_\pi \sim M_K$ and need to include terms of all orders
  - Kaons must propagate through diagrams---acting as sources for pions, and not appearing in loops
- Let's begin with a "LH" Kaon field (a fine choice since it couples to kaons)

$$K_L \sim \begin{pmatrix} u_L \bar{s}_R \\ d_L \bar{s}_R \end{pmatrix} \xrightarrow{SU(2)_L \times SU(2)_R} U_L\, K_L$$

- "Leading-order" invariant effective Lagrangian contains no coupling to pions!

$$\mathcal{L}_K^{(0)} = \partial_\mu K_L^\dagger \partial_\mu K_L + \left(M_K^{(0)}\right)^2 K_L^\dagger K_L$$

Heavy Kaon SU(2) ChPT

$$\mathcal{L}_K^{(0)} = \partial_\mu K_L^\dagger \partial_\mu K_L + \left(M_K^{(0)}\right)^2 K_L^\dagger K_L$$

- Problem: not parity invariant!
  - Need a partner field transforming under U_R, related to K_L since we only want a single field per particle

$$\Rightarrow \Sigma^\dagger K_L \longrightarrow U_R \Sigma^\dagger U_L^\dagger U_L K_L = U_R \left(\Sigma^\dagger K_L\right)$$

- Add corresponding "RH" term to L_{eff} to obtain parity invariant result, which now includes coupling to pions because of presence of Σ

$$\mathcal{L}_K^{(0)} = \frac{1}{2}\left\{\partial_\mu K_L \partial_\mu K_L + \partial_\mu (K_L^\dagger \Sigma) \partial_\mu (\Sigma^\dagger K_L)\right\} + \left(M_K^{(0)}\right)^2 K_L^\dagger K_L$$

- Ugly! Can make a simpler, more symmetric form using $\sqrt{\Sigma}$

$$\xi = e^{i\pi(x)/f}\,, \quad \xi^2 = \Sigma\,, \quad K = \xi^\dagger K_L\,, \quad \Sigma^\dagger K_L = \xi^\dagger K$$

$$\mathcal{L}_K^{(0)} = \frac{1}{2}\left\{\partial_\mu (K\xi^\dagger)\partial_\mu(\xi K) + \partial_\mu(K^\dagger \xi)\partial_\mu(\xi^\dagger K)\right\} + \left(M_K^{(0)}\right)^2 K^\dagger K$$

Heavy Kaon SU(2) ChPT

$$\mathcal{L}_K^{(0)} = \frac{1}{2}\left\{\partial_\mu(K\xi^\dagger)\partial_\mu(\xi K) + \partial_\mu(K^\dagger\xi)\partial_\mu(\xi^\dagger K)\right\} + \left(M_K^{(0)}\right)^2 K^\dagger K$$

- Can simplify further, by defining

$$V_\mu = \tfrac{1}{2}\left(\xi\partial_\mu\xi^\dagger + \xi^\dagger\partial_\mu\xi\right), \quad A_\mu = \tfrac{i}{2}\left(\xi\partial_\mu\xi^\dagger - \xi^\dagger\partial_\mu\xi\right), \quad D_\mu K = \partial_\mu K + V_\mu K$$

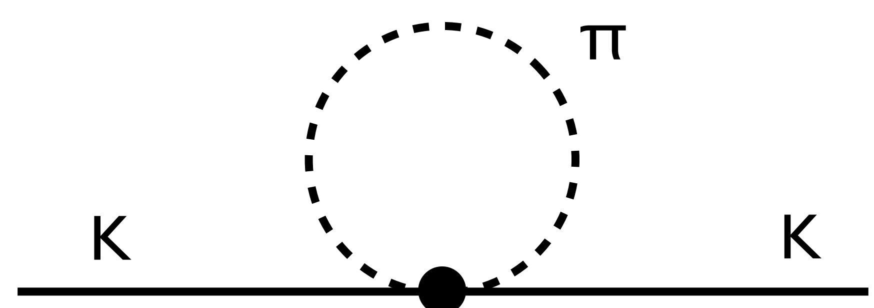
- Then one finds (algebra)

$$\mathcal{L}_K^{(0)} = (D_\mu K)^\dagger D_\mu K + \left(M_K^{(0)}\right)^2 K^\dagger K + \underbrace{\frac{1}{2}\mathrm{tr}(A_\mu A_\mu)K^\dagger K}_{\text{usually put in } \mathcal{L}_K^{(2)}}$$

- Contains $KK\pi\pi$ couplings (since $V_\mu \sim \pi\partial_\mu\pi/f$ & $A_\mu \sim \partial_\mu\pi/f$) but no $KK\pi$ (parity)

NNLO!
(No NLO log,
as found above)

$\delta M_K^2 \sim$



$\sim \dfrac{M_\pi^4}{f^2 M_K^2}\log\left(\dfrac{M_\pi}{\mu}\right) M_K^2$

Only $\mathrm{tr}(A_\mu A_\mu)K^\dagger K$ contributes

Transformation of ξ=√Σ

$$\xi = e^{i\pi(x)/f}\,, \quad \xi^2 = \Sigma\,, \quad K = \xi^\dagger K_L\,, \quad \Sigma^\dagger K_L = \xi^\dagger K$$

- Σ parametrizes the vacuum manifold (coset space) $SU(2)_L$ X $SU(2)_R$ / $SU(2)_V$

$$\Sigma = 1 \xrightarrow{SU(2)_L\times SU(2)_R} U_L U_R^\dagger = U_L' U_R'^\dagger \ \text{ if } \ U_L' = U_L V \ \&\ U_R' = U_R V \quad (\text{with } \ V \in SU(3)_V)$$

- Choose representative of equivalence class so that $U_L = U_R^\dagger \equiv \xi \ \Rightarrow \ \Sigma = \xi^2$

- Can then determine how ξ transforms

$$\Sigma = \xi^2 \longrightarrow U_L\,\xi\,\xi\,U_R^\dagger = \underline{(U_L\,\xi\,V^\dagger)}\underline{(V\,\xi\,U_R^\dagger)}$$

Choose V so these are equal!

$$\Rightarrow \quad \boxed{\xi \longrightarrow U_L\,\xi\,V^\dagger = V\,\xi\,U_R^\dagger}$$

$V=V(U_L,U_R,\xi)$ [don't need explicitly!]

- Extend to other quantities

$$K = \xi^\dagger K_L \longrightarrow V\xi^\dagger U_L^\dagger U_L K = VK, \quad D_\mu K \longrightarrow V D_\mu K, \quad A_\mu \to V A_\mu V^\dagger, \ldots$$

Building Kaon ChPT using V

- Simplest method to determine Kaon chiral Lagrangian uses only $SU(2)_V$ transformation properties

$$K \longrightarrow VK, \quad D_\mu K \longrightarrow V D_\mu K, \quad A_\mu \longrightarrow V A_\mu V^\dagger$$

$$\mathcal{L}_K^{(0)} = (D_\mu K)^\dagger D_\mu K + \left(M_K^{(0)}\right)^2 K^\dagger K$$

$$\mathcal{L}_K^{(2)} = A_1 \mathrm{tr}(A_\mu A_\mu) K^\dagger K + A_2 \mathrm{tr}(A_\mu A_\nu)(D_\mu K)^\dagger D_\nu K + A_2' \mathrm{tr}(A_\mu A_\mu)(D_\nu K)^\dagger D_\nu K + \text{mass terms}$$

$$\mathcal{L}_\pi^{(2)} = \frac{f^2}{4} \mathrm{tr}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) - \frac{f^2}{4} \mathrm{tr}(\chi \Sigma^\dagger + \Sigma \chi^\dagger)$$

- Here $M_K^{(0)}$, $f=f_2$ (2 flavor decay constant in limit $m_u=m_d=0$) & A_j are new LECs
  - Pion comes in as π/f_2, implying that chiral logs contain $1/f_2$ factors
- SU(2) power counting: $p_\pi^2 \sim M_\pi^2 \ll M_K^2$
  - At LO, can have any even number of D_μ K's, but can reduce to above form using equations of motion
  - Can equivalently treat kaon as a static source (as in heavy-meson ChPT)---remove M_K from Lagrangian
  - Do not, however, have heavy-quark spin symmetry ($M_{K^*} \gg M_K$), so do not need to include the K^*

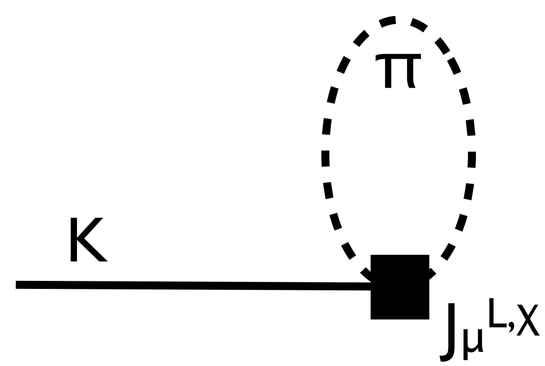
Operators in Kaon ChPT

- To determine chiral expansion for f_K, B_K, $K \longrightarrow \pi$ form factors, *etc.*, need to map quark-level operators into the EFT
  - For f_π this can be done using sources and local chiral symmetry, but not for a general operator
  - Same issue arises for operators other than currents in "normal" ChPT
- Match chiral transformation properties---conveniently done with spurions
- Example: $f_K \propto \langle K|\bar{q}_L \gamma_\mu s_L|0\rangle$ with $\bar{q}_L = (\bar{u}_L, \bar{d}_L)$ and L an arbitrary choice
  - Introduce spurion H_L: Since $\bar{q}_L \to \bar{q}_L U_L^\dagger$, $J_\mu^L = \bar{q}_L H_L \gamma_\mu s_L$ is invariant if $H_L \to U_L H_L$
  - Form all invariant vectors with correct strangeness and one derivative

$$J_\mu^{L,X} = C_1 (D_\mu K)^\dagger \xi^\dagger H_L + C_2 K^\dagger A_\mu \xi^\dagger H_L + \text{higher order}$$

  - Set H_L=(1,0) to get, say, K^+ current and evaluate matrix element

$$f_K = -2C_1 \left(1 + C_u \frac{m_u}{f_2^2} + C_d \frac{m_d}{f_2^2} - \frac{3}{8}\tilde{L}_\pi\right)$$



Agrees with result obtained above by reduction from SU(3)

Not related to f_2

Combinations of NLO LECs

Denominator in chiral log now shown to contain f_2

Final comments on SU(2) ChPT

- Results available for many quantities & widely used in lattice chiral extrapolations
- Can use trick of SU(3) reduction + $f \to f_2$ for most (all?) quantities
  - Gives simple method for obtaining SU(2) staggered ChPT results [SWME collab.]
- Extended to processes involving "hard" external pions, e.g. $K \to \pi$ [Flynn & Sachrajda], pion form factors with $s \gg M_\pi^2$ [Bijnens et al.]
  - Only soft part of internal loops leads to chiral logarithms, so can be reliably calculated
  - Simple "factorized" form of the 1- and 2-loop results fails at 3-loops [Colangelo et al.]
- Should one use SU(2) or SU(3) results in practice for chiral extrapolations?
  - SU(2) is safer but often requires dropping some data points
  - SU(3) uses more data, but must include parametrized high-order (NNNLO) terms to get good fits
  - No general answer $\Rightarrow$ either stick with SU(2) or try both & compare

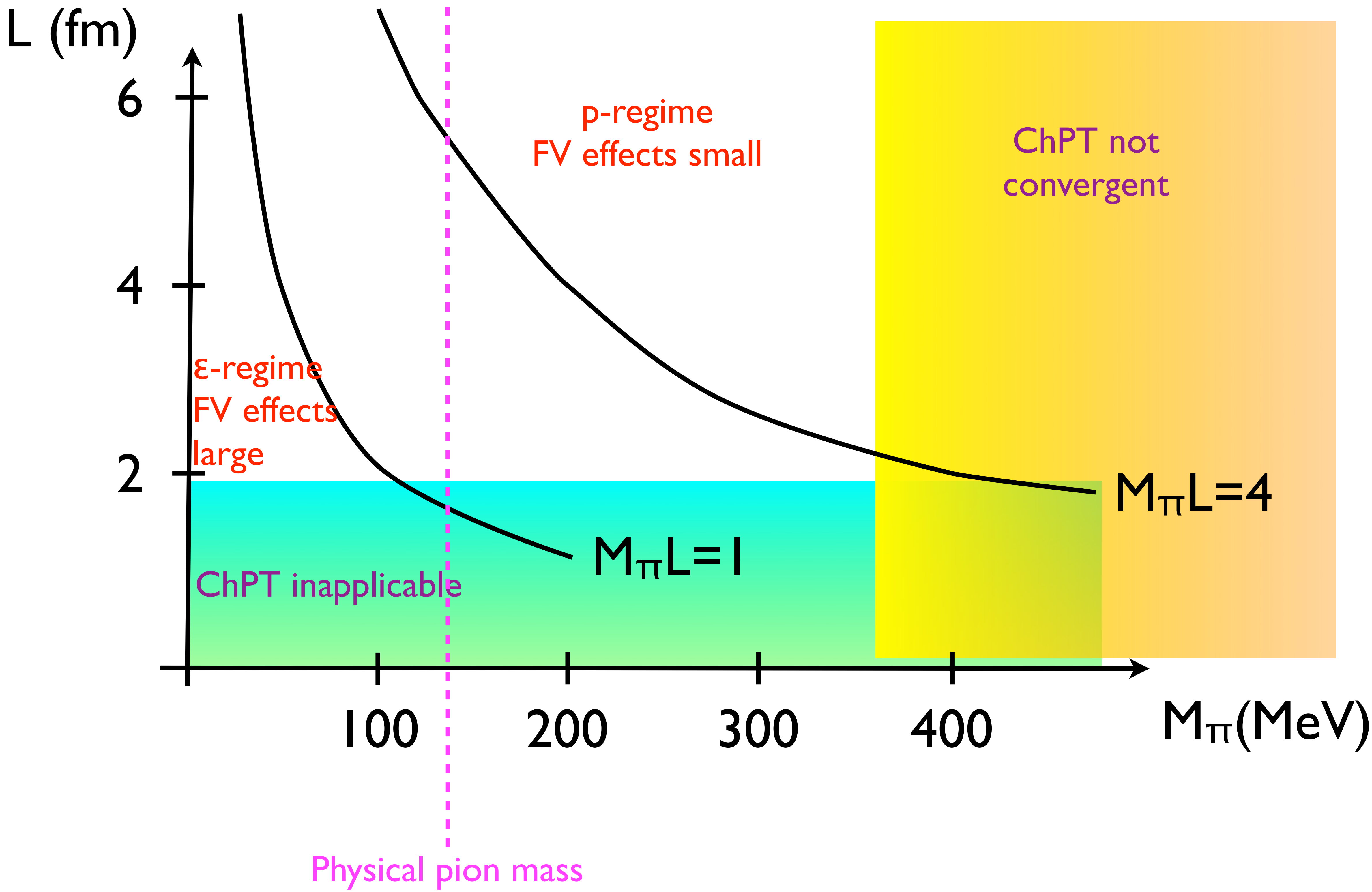
Outline of lecture 2

- Examples of results from continuum ChPT
  - Focus on those with lattice applications
- SU(2) vs SU(3) ChPT in the presence of kaons
- ChPT at finite volume
  - "p regime" (large M_π L)
  - "ε regime" (small M_π L)

ChPT in finite volume (FV)

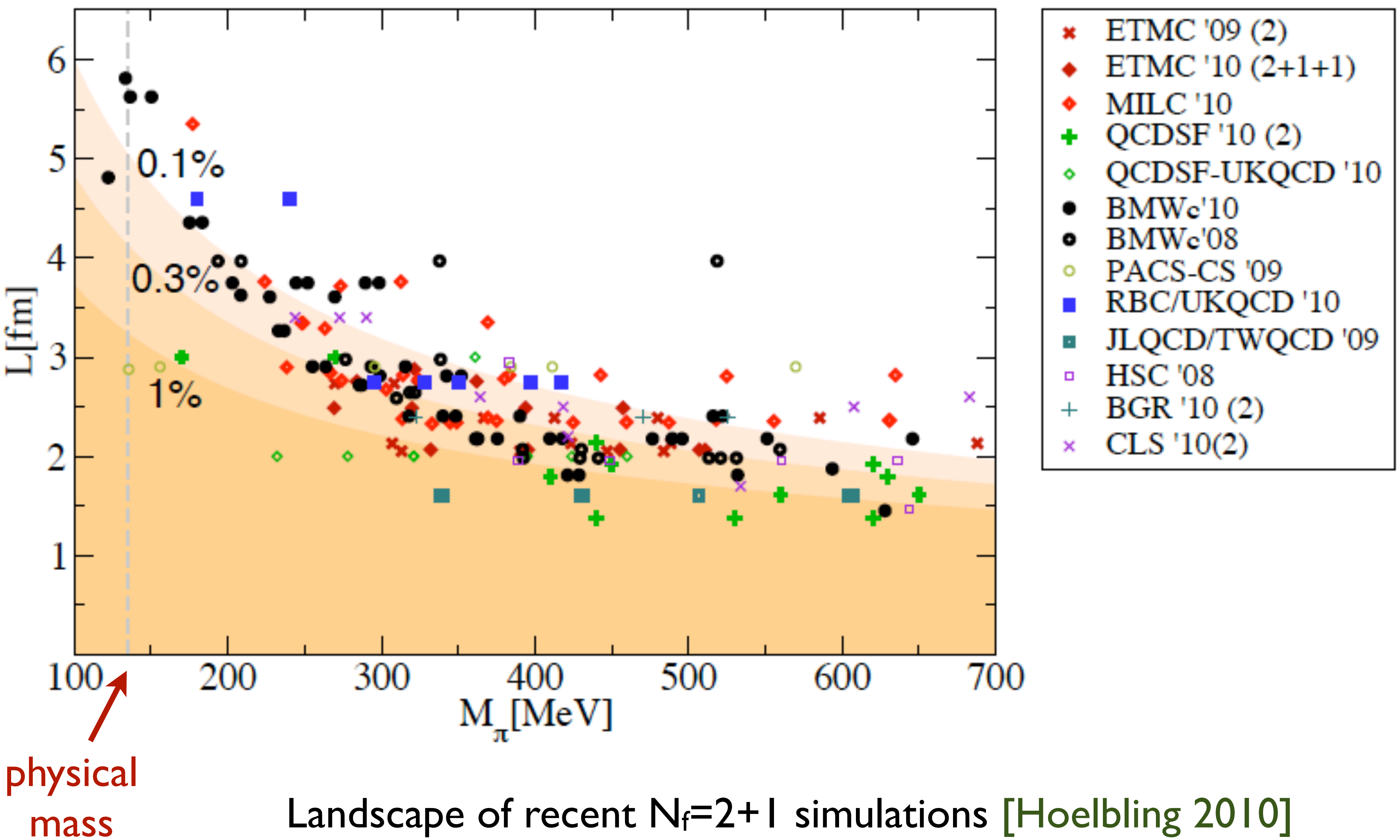
- Simulations done in finite volume: $V = L^3 \times L_t$ [typically L_t=(2-4)L]
- Boundary conditions on quarks (e.g. periodic in space, antiperiodic in time) translate into BC on mesons
  - Usually periodic in all directions, but can also twist: $\pi(L) = e^{i\theta}\,\pi(0)$
  - Can view as working on infinite volume with all external fields having periodic images
- Key EFT result: as long as $L \gg 1/\Lambda_{QCD}$ (in practice $L \gtrsim 2$ fm) then ChPT Lagrangian is unchanged---only PGB propagators "feel" the finite volume
  - True at finite T (i.e finite L_t) because Hamiltonian is independent of T
  - Extend to other directions using Euclidean invariance [Gasser & Leutwyler, 88]
  - Intuitively reasonable: ChPT vertices arise from integrating out heavy particles (ρ, N, ...) so have size ~1/(1 GeV)~ 0.2 fm $\ll$ L
  - Size of finite-volume effects on propagators determined by ML ⇒ FV effects dominated by pions
- ChPT breaks down for $L \lesssim 1/\Lambda_{QCD}$ where enter "femto-universe" in which there is no confinement & can study using (partially) perturbative techniques

ChPT in finite volume: landscape



Volumes of recent simulations

- Percentages are more sophisticated estimate of FV corrections
  - ➡ Most simulations well within p-regime
  - ➡ ε-regime simulations not shown on plot



Landscape of recent N_f=2+1 simulations [Hoelbling 2010]

p-regime (large volumes)

- B.C. imply quantization of momenta: for periodic BC $\vec{p} = \frac{2\pi}{L}\vec{n}$
- Relation between FV and infinite volume pion propagators (taking $L_t \gg L$)

$$G_L(\vec{p} = \tfrac{2\pi}{L}\vec{n}, t_E) = G_\infty(\vec{p}, t_E) = \tfrac{1}{2E} e^{-E|t_E|} \qquad (E = \sqrt{\vec{p}^2 + M^2})$$

No tree-level FV effects

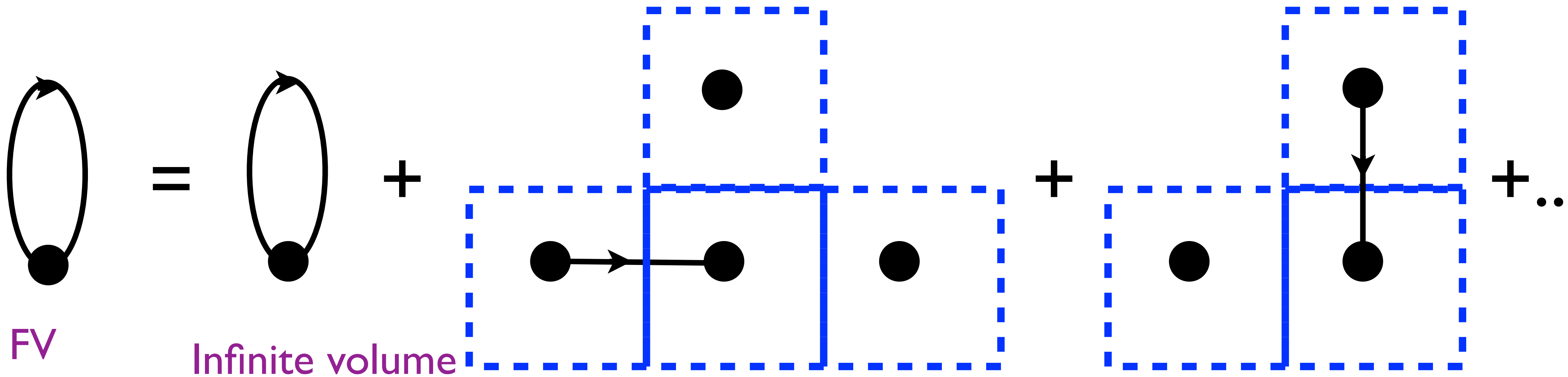
$$G_L(\vec{x}, t_E) = \sum_{\vec{m}} G_\infty(\vec{x} + \vec{m}L, t_E)$$

image sum

$$G_\infty(x) = \frac{M^2}{4\pi^2}\frac{K_1(z)}{z} \qquad (z = M|x|)$$

$$\sim \frac{M^2}{4\pi^2}\sqrt{\frac{\pi}{2z^3}}e^{-z}$$

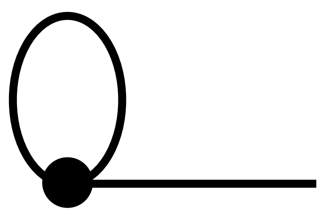
exponential fall-off with ML



p-regime (large volumes)

- Substitute G_L for G_∞ in loops $\Rightarrow$ only chiral logs are effected
- At NLO, most loops are tadpoles, and can just read off FV shift with no work

$$f_\pi = f_2(m_s)\left\{1 + \frac{4}{f_2(m_s)^2}\left[\tilde{L}_4(m_s)\chi_\ell\right] - \tilde{L}_\pi\right\}$$



$\Rightarrow$

$$\frac{1}{f_2^2}G_\infty(0)_{\text{d.r.}} = \frac{1}{f_2^2}\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+M_\pi^2} = \tilde{L}_\pi + \text{analytic}, \quad \tilde{L}_\pi = \frac{M_\pi^2}{(4\pi f_2)^2}\log\left(\frac{M_\pi^2}{\mu^2}\right)$$

$$\frac{f_\pi(L)-f_\pi}{f_\pi} = -\frac{1}{f_2^2}\sum_{\vec{m}\neq\vec{0}} G_\infty(\vec{m}L, 0)$$

FV shift from non-zero images

- One-loop FV shifts routinely included in all chiral fits
- Shifts can be substantial for small $M_\pi L$
- Two loop/partial all orders calculations for M_π and f_π indicate that NLO ChPT result is only trustworthy as indicator of size of FV effects [Lüscher; Colangelo, Durr & Haefli]

- Rule of thumb: require $M_\pi L \gtrsim 4$ for sub-percent level FV effects

Power-counting and ε-regime

- In p-regime: $m \sim M_\pi^2 \sim p^2 \sim 1/L^2$ (usually with $L_t \gg L$ though not essential)
- In ε-regime, consider boxes with $L_t \approx L$ (we'll take $L_t = L$ for simplicity)
- Reduce m so that $m \sim M_\pi^2 \sim 1/L^4$ while keeping $p^2 \sim 1/L^2$ (with $L \gtrsim 2$ fm always)
  - Organize power-counting using $\varepsilon \sim 1/L$: $p^2 \sim \varepsilon^2$, $m \sim \varepsilon^4$
  - Zero-mode (p=0) of propagator is enhanced relative to $p \neq 0$ modes

$$G_L(x) = \sum_p \frac{1}{L^4(p^2+M^2)} e^{ipx}$$

Zero-mode ~ ε^0 while non-zero-modes ~ ε^2

- Must treat zero-mode non-perturbatively [Gasser & Leutwyler, 87, 88]
- Natural context is SU(2) ChPT, since only pion has large FV effects
  - We will assume isospin symmetry, with common mass m, for simplicity

Zero-mode integral

$$Z_\chi = \int [D\Sigma] \exp\left\{-\tfrac{f^2}{4}\int_V \left[\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger) - \mathrm{tr}(\chi^\dagger\Sigma + \chi\Sigma^\dagger)\right] + \ldots\right\} \qquad [\chi = 2B_0(s+ip)]$$

- Pull out zero-mode U: $\Sigma = U e^{2i\pi(x)/f}$ with $\int_V \pi(x) = 0$
- Keeping only U for now: $Z_\chi \approx \int dU \exp\left\{\frac{f^2}{4}L^4\, 2B_0 \mathrm{tr}\left[(s-ip)U + (s+ip)U^\dagger\right]\right\}$
- Setting $s=m_q\sim 1/(f^2 B_0 L^4)$ & $p=0$, we see that U integral is not restricted to lie near U=1, but ranges over entire group [here SU(2)]
- Need Haar measure: $U = e^{i\theta\vec{n}\cdot\vec{\tau}} \quad\Rightarrow\quad dU = \frac{1}{2\pi^2}\sin^2\theta\, d\theta\, d\Omega(\vec{n})$
- Evaluate Z_χ as function of $z = L^4 f^2\, 2B_0 m = L^4 f^2 M_\pi^2$

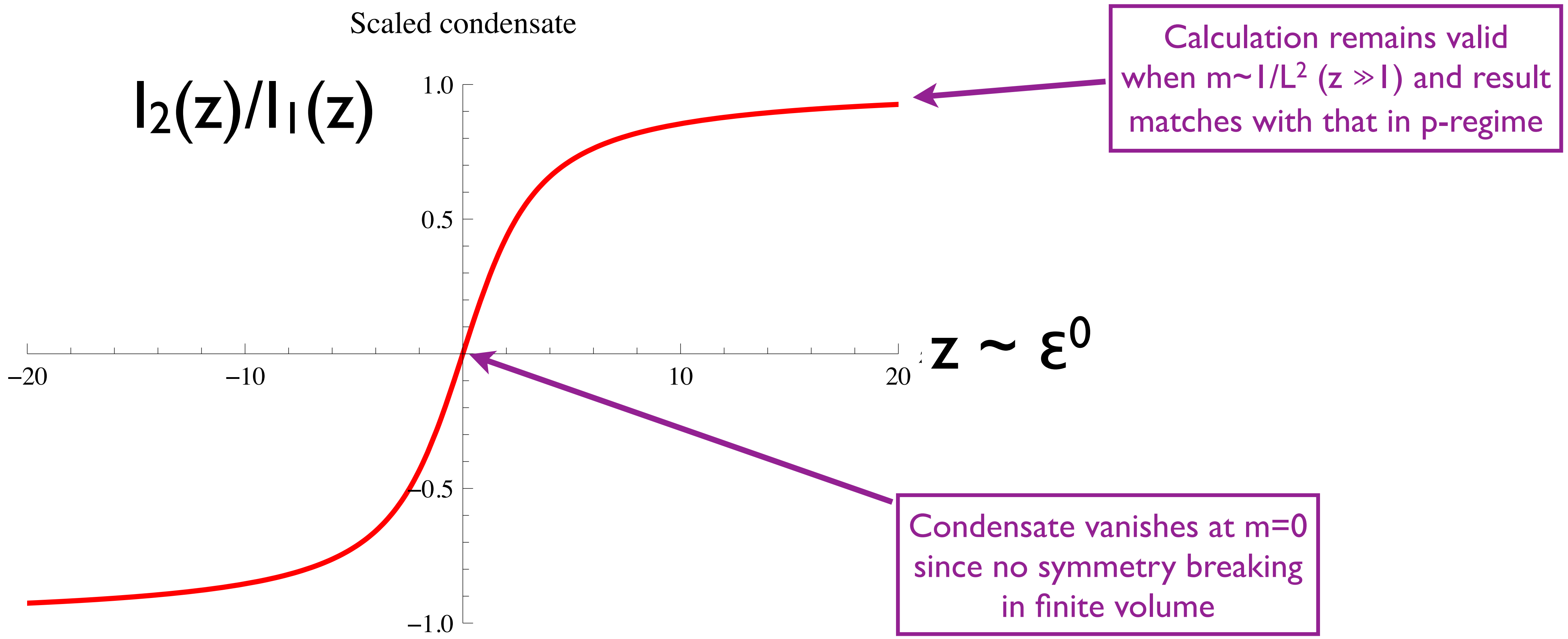
$$Z_\chi \approx \int dU \exp\left\{\tfrac{z}{4}\mathrm{tr}\left[(U+U^\dagger\right]\right\} = \tfrac{2}{\pi}\int_0^\pi d\theta \sin^2\theta e^{z\cos\theta} = \frac{2I_1(z)}{z} = 1 + \frac{z^2}{8} + \ldots$$

- Evaluate condensate using "source" m and find $O(\varepsilon^0)$ contribution

$$-\langle\bar{q}q\rangle = \frac{1}{L^4 Z_\chi}\frac{\delta Z_\chi}{\delta s} = \frac{1}{L^4}\frac{d\log Z_\chi}{dm} = 2B_0 f^2 \frac{d\log Z_\chi}{dz} = 2B_0 f^2 \frac{I_2(z)}{I_1(z)}$$

Condensate in ε-regime

$$-\langle \bar{q} q \rangle = 2 B_0 f^2 \frac{I_2(z)}{I_1(z)}, \qquad (z = L^4 f^2 2 B_0 m)$$



1-parameter prediction allows determination of condensate $f^2 B_0$!

Justifying ε-regime power counting

$$Z_\chi = \int [D\Sigma] \exp\left\{ -\tfrac{f^2}{4} \int_V \left[\mathrm{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) + \mathrm{tr}(\chi^\dagger \Sigma + \chi \Sigma^\dagger) \right] + \ldots \right\} \qquad [\chi = 2B_0(s+ip)]$$

$$\Sigma = U e^{2i\pi(x)/f} \quad \text{with} \quad \int_V \pi(x) = 0$$

- Measure factorizes: $[D\Sigma] = dU[d\pi(x)]$
- Lagrangian simplifies (check!). Kinetic term maintains usual form

$$\frac{f^2}{4}\mathrm{tr}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) = \frac{f^2}{4}\mathrm{tr}(\partial_\mu e^{2i\pi/f}\partial_\mu e^{-2i\pi/f}) = \overbrace{\mathrm{tr}(\partial_\mu \pi \partial_\mu \pi)}^{\varepsilon^0=\text{NLO}} + \overbrace{O(\partial^2 \pi^4/f^2)}^{\varepsilon^2=\text{NLO}}$$

  - In momentum space, action ~ L^4 p^2 ~ L^2 so fluctuations are small as in p-regime: $\pi_p \sim 1/L \sim \varepsilon$
  - Leading term gives m-independent correction to Z_χ and thus does not change the LO prediction for the condensate

$$\frac{f^2 B_0 m}{2}\mathrm{tr}(\Sigma + \Sigma^\dagger) = \underline{\frac{f^2 B_0 m}{2}\mathrm{tr}(U + U^\dagger)} - \underbrace{B_0 m \mathrm{tr}([U + U^\dagger]\pi^2)}_{\varepsilon^2=\text{NLO}} + \ldots$$

Leads to ε^0=LO, as described above

- Power-counting differs from p-regime with terms containing m~ε^4 moving to higher order ⇒ less LECs at each order ⇒ easier to determine

A few ε-regime applications

- Determination of f (decay constant in chiral limit) from two-point correlator of left-handed current [Giusti et al., 2004]
- Using partial quenching, can show that low-lying ("microscopic") eigenvalues of Dirac operator ($\lambda L^4 f^2 B_0 \sim 1$) are described by random matrix theory, with a calculable distribution depending on $f^2 B_0$ [Damgaard et al., 1998]
- Introducing imaginary isospin chemical potential, distribution of eigenvalues depends also on f [Damgaard et al., 2005]
- ......

Summary of continuum ChPT for LQCD

- Provides forms for extrapolating in quark masses and box size
- SU(2) ChPT useful in general; utility of SU(3) ChPT more quantity-dependent
- Straightforward to obtain NLO expressions; some NNLO known
- ε-regime provides an unphysical regime well suited to extracting certain LECs