Resonances from lattice QCD: Lecture 3

Steve Sharpe
University of Washington

Outline

☑ Lecture 1

- Motivation/Background/Overview

☑ Lecture 2

- Deriving the two-particle quantization condition (QC2)
- Examples of applications

☐ Lecture 3

- Sketch of the derivation of the three-particle quantization condition (QC3)

☐ Lecture 4

- Applications of QC3
- Summary of topics not discussed and open issues

Main references for these lectures

- Briceño, Dudek & Young, "Scattering processes & resonances from LQCD," 1706.06223, RMP 2018
- Hansen & SS, "LQCD & three-particle decays of resonances," 1901.00483, to appear in ARNPS
- Lectures by Dudek, Hansen & Meyer at HMI Institute on "Scattering from the lattice: applications to phenomenology and beyond," May 2018, https://indico.cern.ch/event/690702/
- Lüscher, Commun.Math.Phys. 105 (1986) 153-188; Nucl.Phys. B354 (1991) 531-578 & B364 (1991) 237-251 (foundational papers)
- Kim, Sachrajda & SS [KSS05], hep-lat/0507006, NPB 2015 (direct derivation in QFT of QC2)
- Hansen & SS [HS14, HS15], 1408.5933, PRD14 & 1504.04248, PRD15 (derivation of QC3 in QFT)
- Briceño, Hansen & SS [BHS17], 1701.07465, PRD17 (including 2↔3 processes in QC3)
- Briceño, Hansen & SS [BHS18], 1803.04169, PRD18 (numerical study of QC3 in isotropic approximation)
- Briceño, Hansen & SS [BHS19], 1810.01429, PRD19 (allowing resonant subprocesses in QC3)
- Blanton, Romero-López & SS [BRS19], 1901.07095, JHEP19 (numerical study of QC3 including d waves)
- Blanton, Briceño, Hansen, Romero-López & SS, in progress, poster at Lattice 2019

Other references for this lecture

- Rubin, R. Sugar & G. Tiktopoulos, PR146 (1966) 1130 (classified divergences in $\mathcal{M}_3$)
- Beane, Detmold & Savage, 0707.1670, PRD07; Tan, 0709.2530, PRA08 (threshold expansion for energies of n particles in a box in QM)
- Polejaeva & Rusetsky, 1203.1241, EPJA12 (3-particle spectrum is determined by $\mathcal{M}_2$ and $\mathcal{M}_3$)
- Briceño & Davoudi, 1212.3398, PRD12 (dimer+particle-based 3-particle formalism)
- Briceño, Hansen, SS & Szczepaniak, 1905.11188 (demonstrated unitarity of HS expression for $\mathcal{M}_3$)
- Jackura, SS, et al., 1905.12007 (relation of HS $\mathcal{K}_{df,3}$ to B-matrix parametrization of $\mathcal{M}_3$)

Outline for Lecture 3

- Overview
- Final result
- Sketch derivation of QC3 in presence of G-parity-like Z_2 symmetry
- Relating the three-particle K matrix ($\mathcal{K}_{\mathrm{df},3}$) to $\mathcal{M}_3$

Overview

Recall motivations

- Understanding resonances with three-particle decay channels
  - Some only decay to three particles, e.g. $\omega \rightarrow \pi\pi\pi$ in an isospin symmetric world
  - Some decay to both two- and three-particle channels:

$$N(1440) \rightarrow N\pi, N\pi\pi \qquad\qquad Z_c(3900) \rightarrow \pi\, J/\psi, D\bar{D}\pi$$

- Predicting electroweak decays to three particles, e.g. $K \rightarrow \pi\pi\pi$
  - Need generalization of Lellouch-Lüscher factors

- Determining three-particle interactions, e.g. NNN

Status of theory?

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Formalism developed, implementation begun. This is what I will focus on

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Formalism not yet developed

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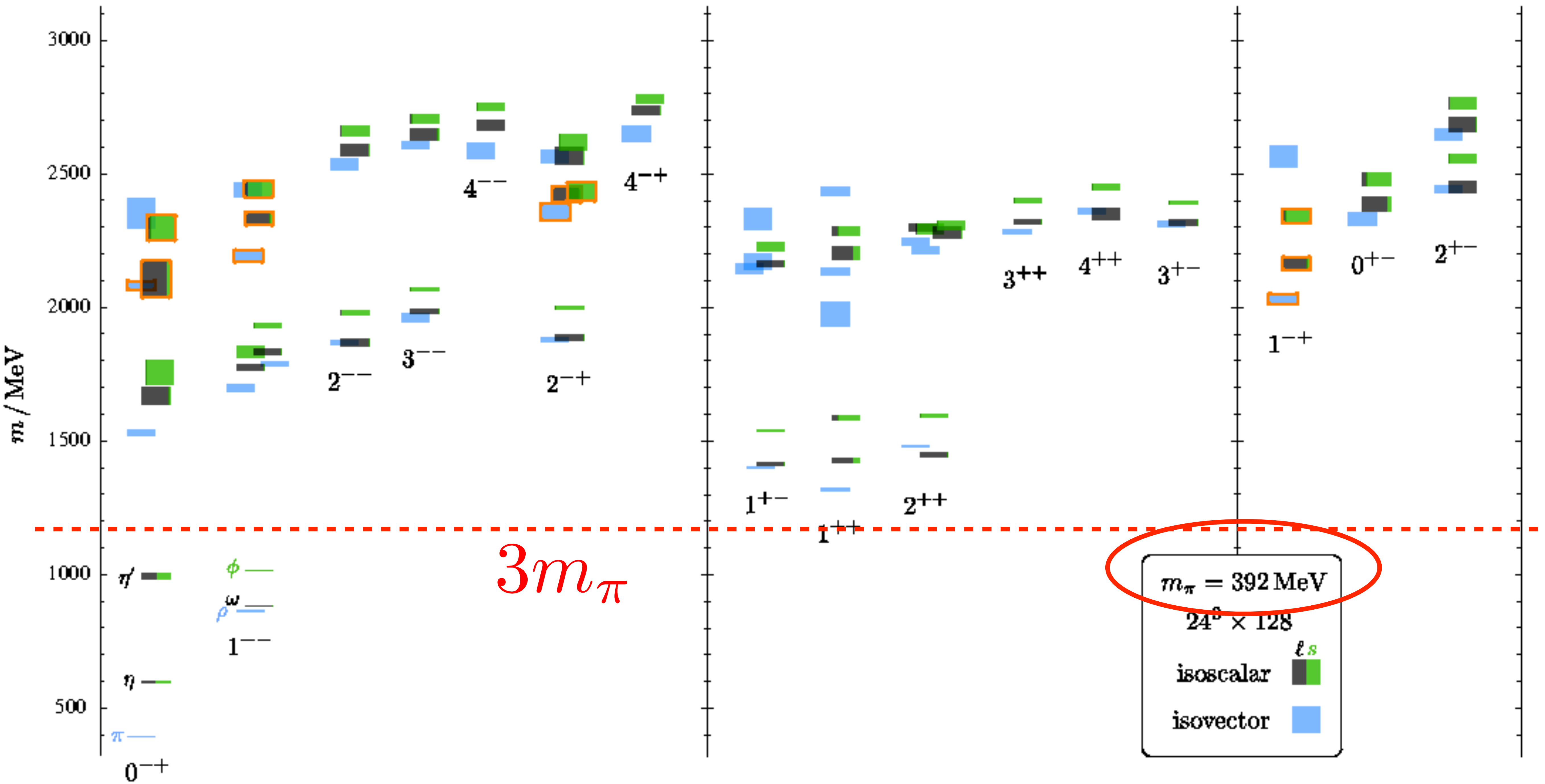
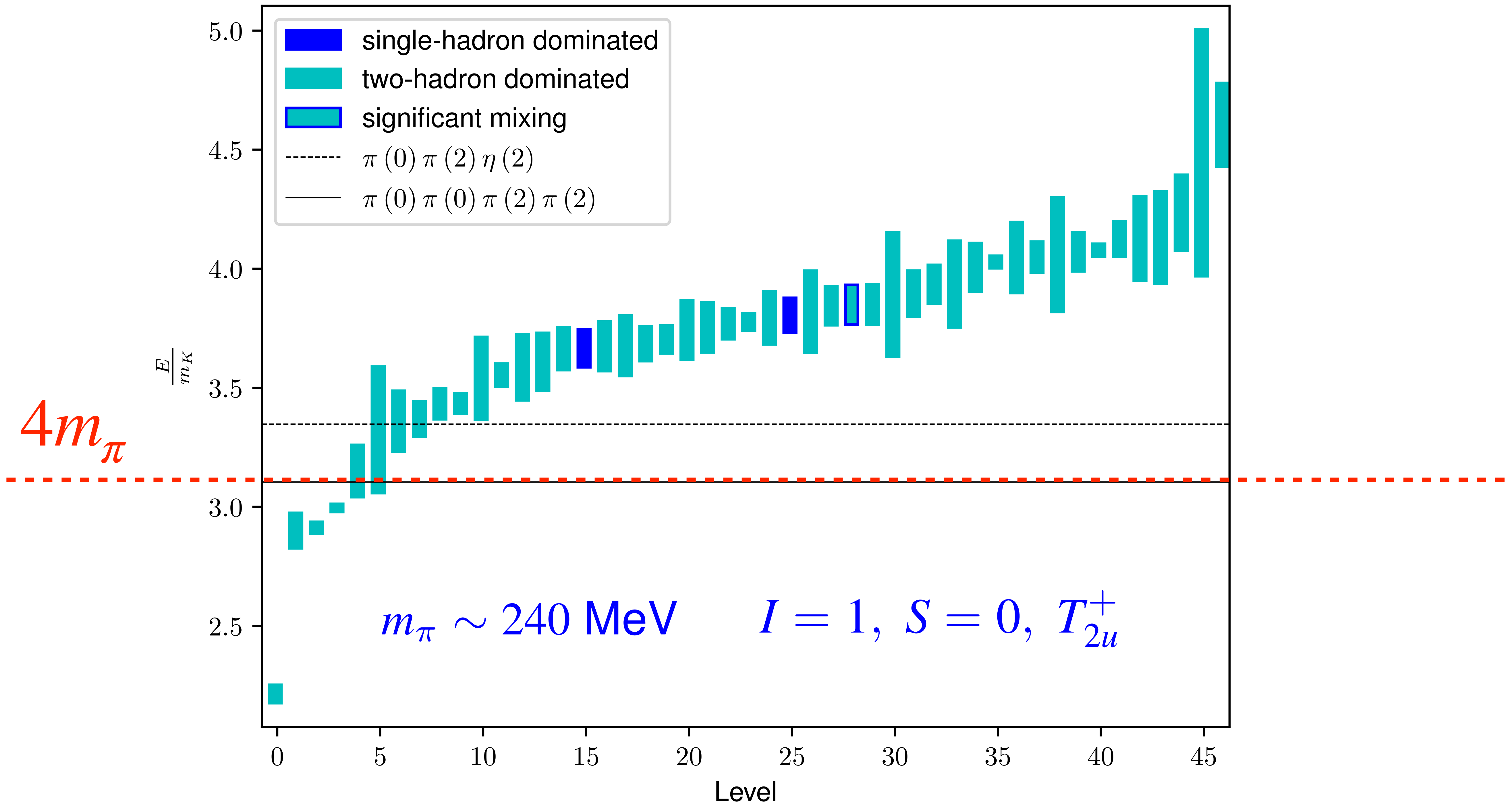
Spin not yet included in formalism

LQCD spectrum already includes 3⁺-particle states



[Dudek, Edwards, Guo & C.Thomas [HadSpec], arXiv:1309.2608]

LQCD spectrum already includes 3+-particle states



Slide from seminar by Colin Morningstar, Munich, 10/18

LQCD spectrum already includes 3^+-particle states

Two- and three-pion finite-volume spectra at maximal isospin from lattice QCD

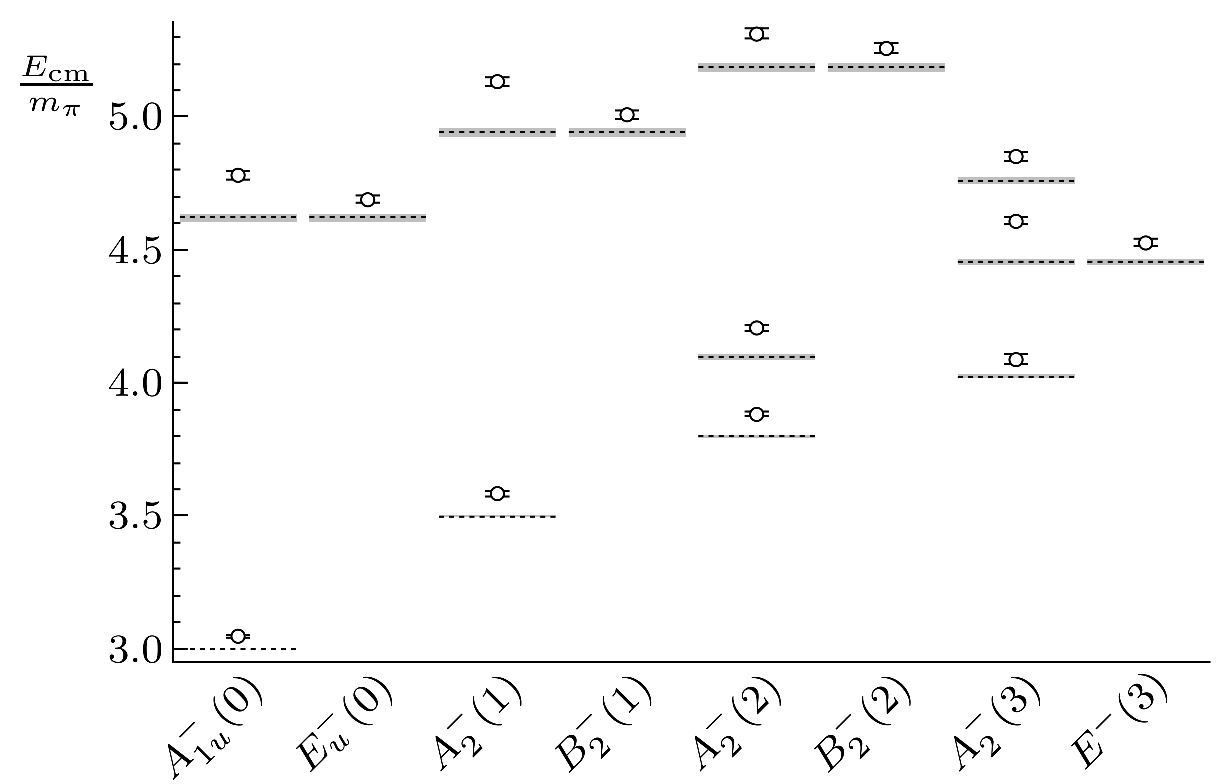
[arXiv:1905.04277]

Ben Hörz*
Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

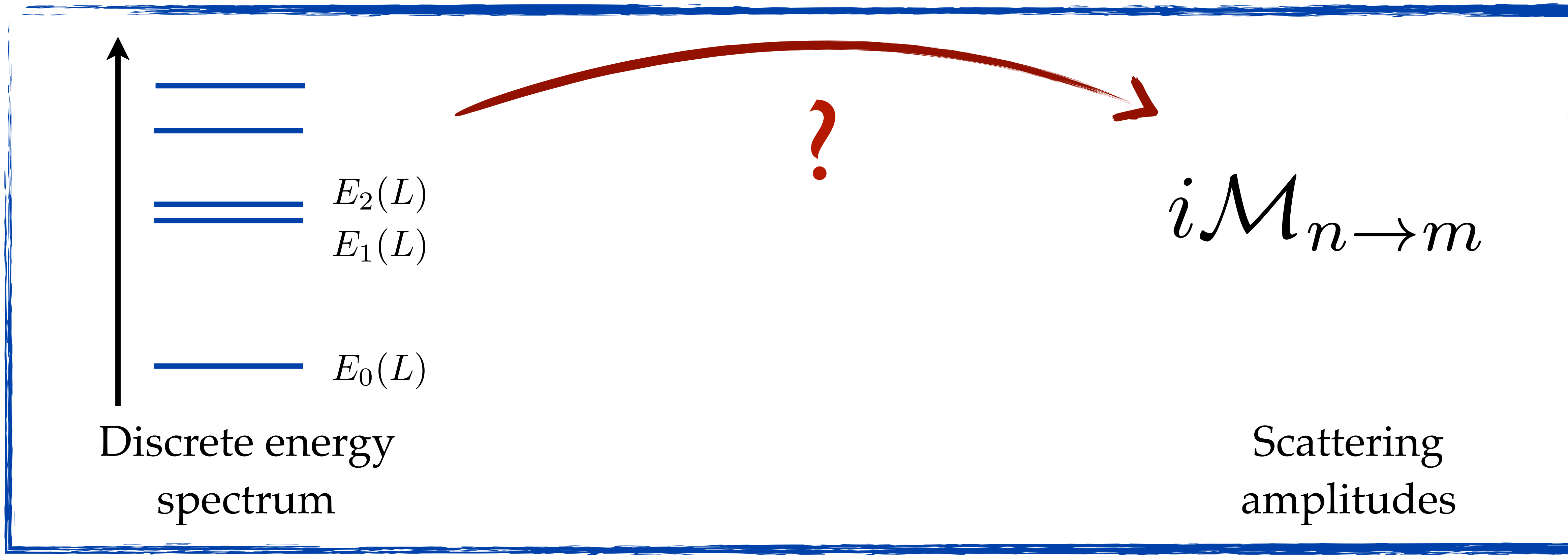
Andrew Hanlon†
Helmholtz-Institut Mainz, Johannes Gutenberg-Universität, 55099 Mainz, Germany

(Dated: May 13, 2019)

We present the three-pion spectrum with maximum isospin in a finite volume determined from lattice QCD, including, for the first time, excited states across various irreducible representations at zero and nonzero total momentum, in addition to the ground states in these channels. The required correlation functions, from which the spectrum is extracted, are computed using a newly implemented algorithm which reduces the number of operations, and hence speeds up the computation by more than an order of magnitude. The results for the $I = 3$ three-pion and the $I = 2$ two-pion spectrum are publicly available, including all correlations, and can be used to test the available three-particle finite-volume approaches to extracting three-pion interactions.

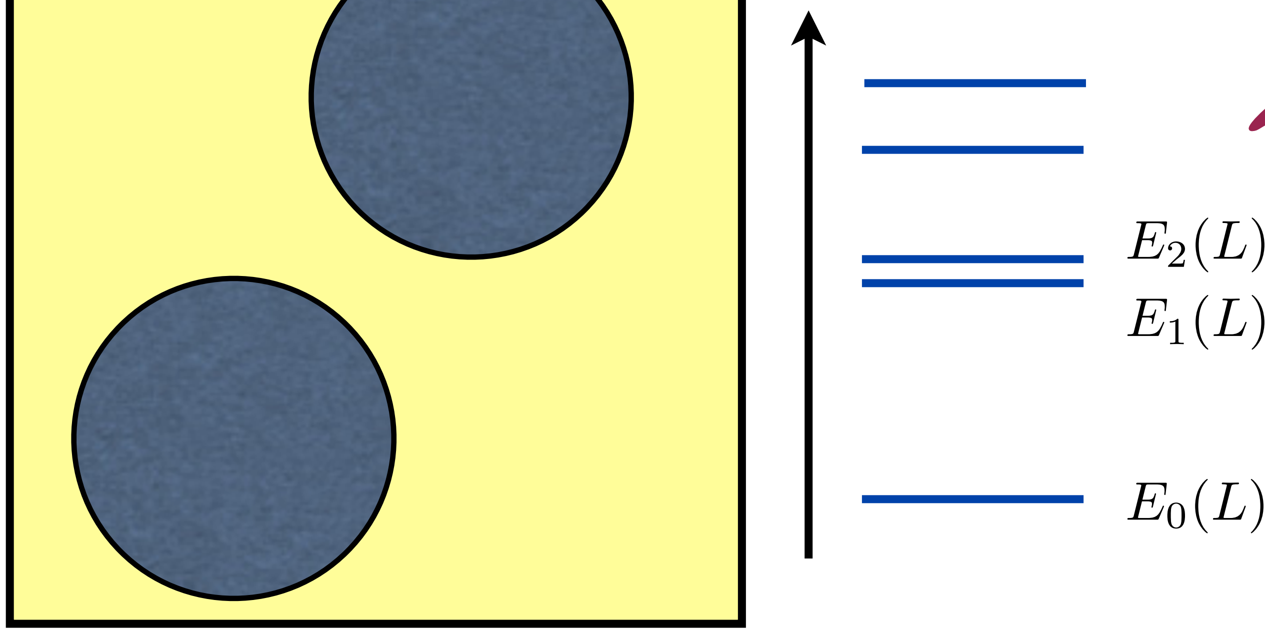


Problem in finite-volume QFT



No Z_2 symmetry

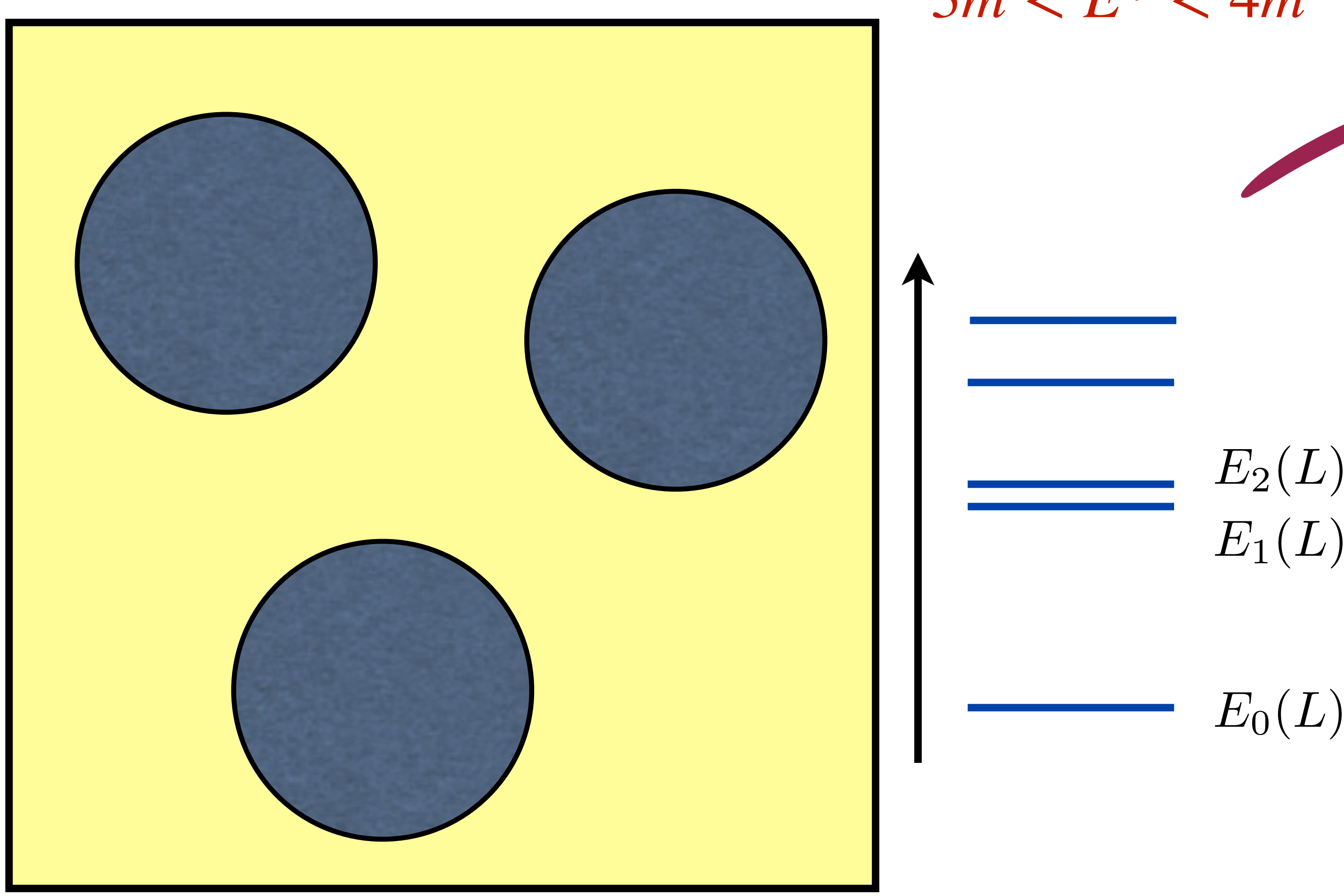
$E^* < 3m$



QC2

$\mathcal{M}_{2\to 2}$

$3m < E^* < 4m$



QC3

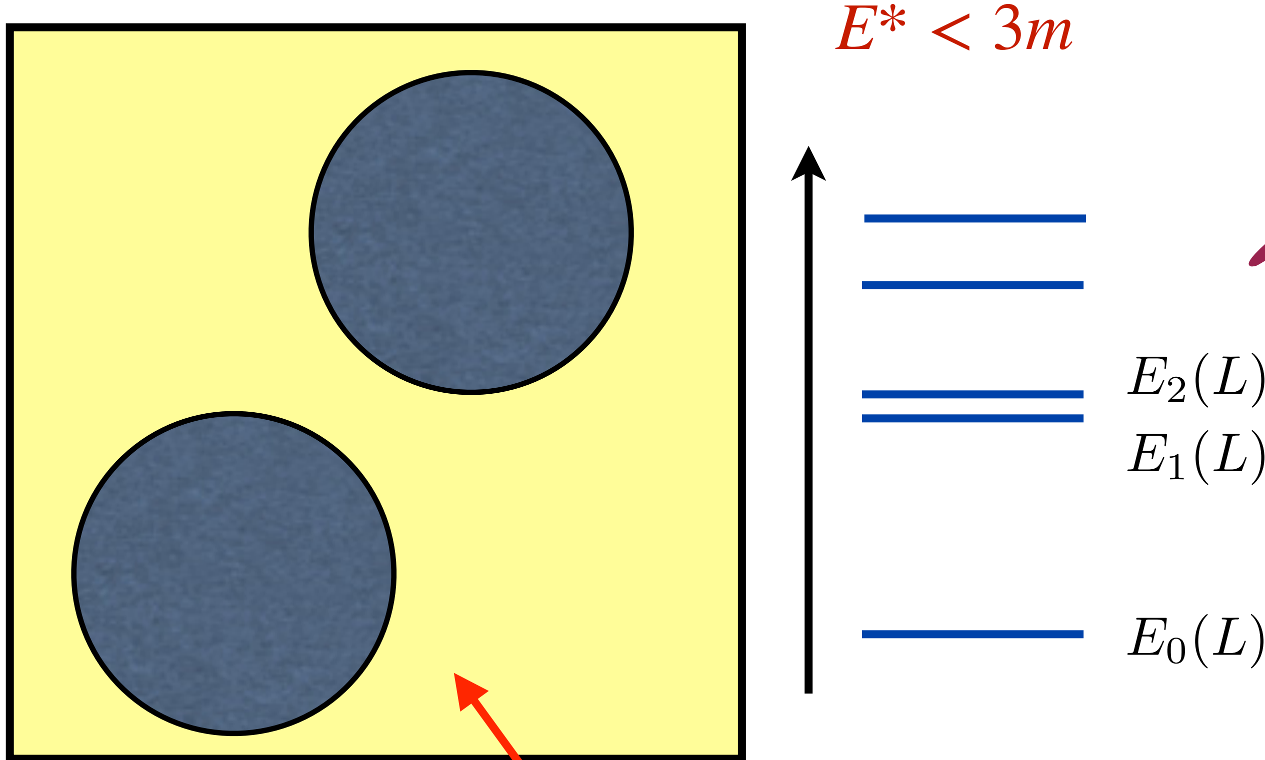
$\mathcal{M}_{2\to 3}$

& $\mathcal{M}_{3\to 2}$

& $\mathcal{M}_{3\to 3}$

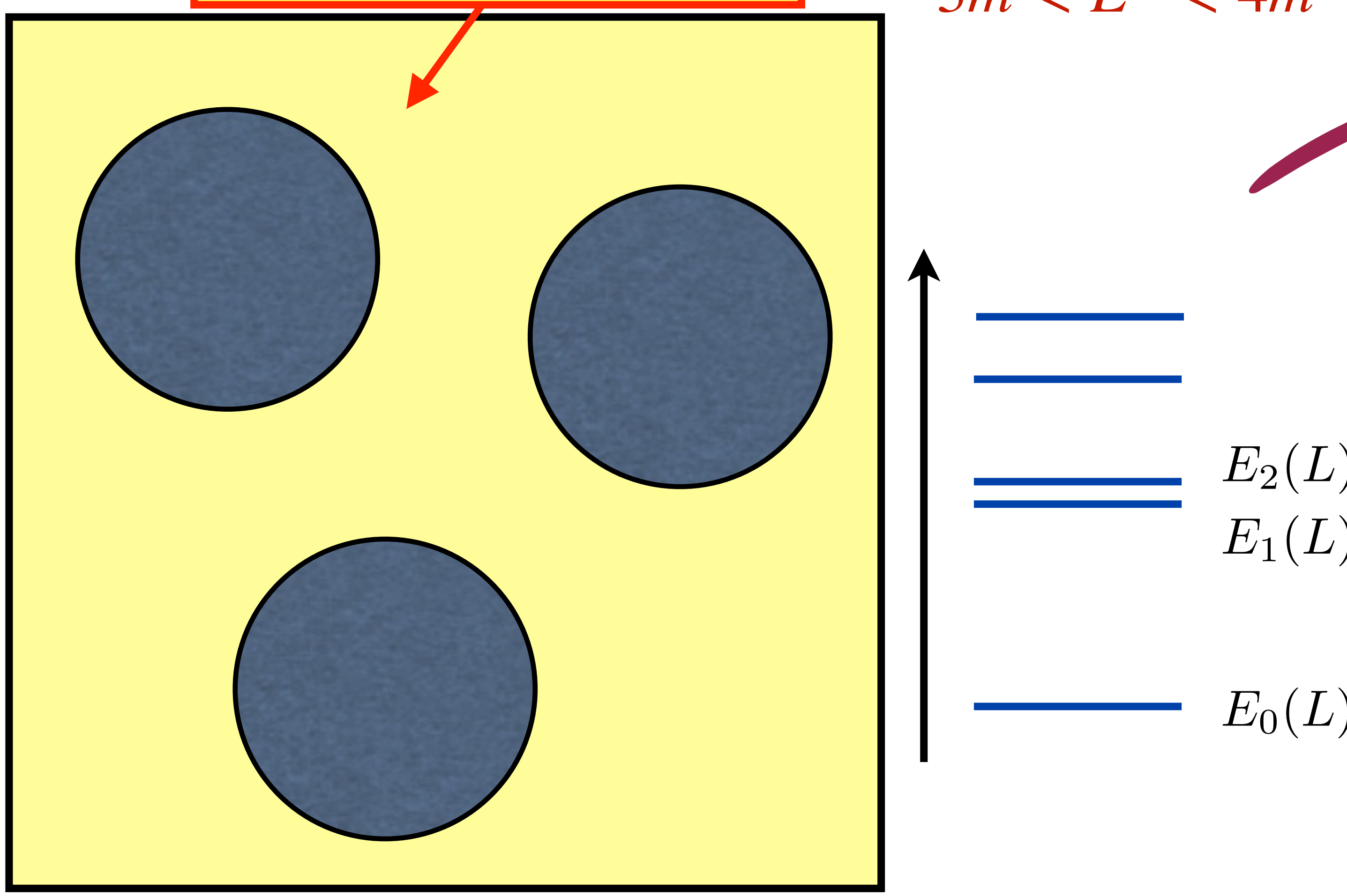
No Z_2 symmetry

$E^* < 3m$



Part of single spectrum!

$3m < E^* < 4m$



QC2

$\mathcal{M}_{2\to 2}$

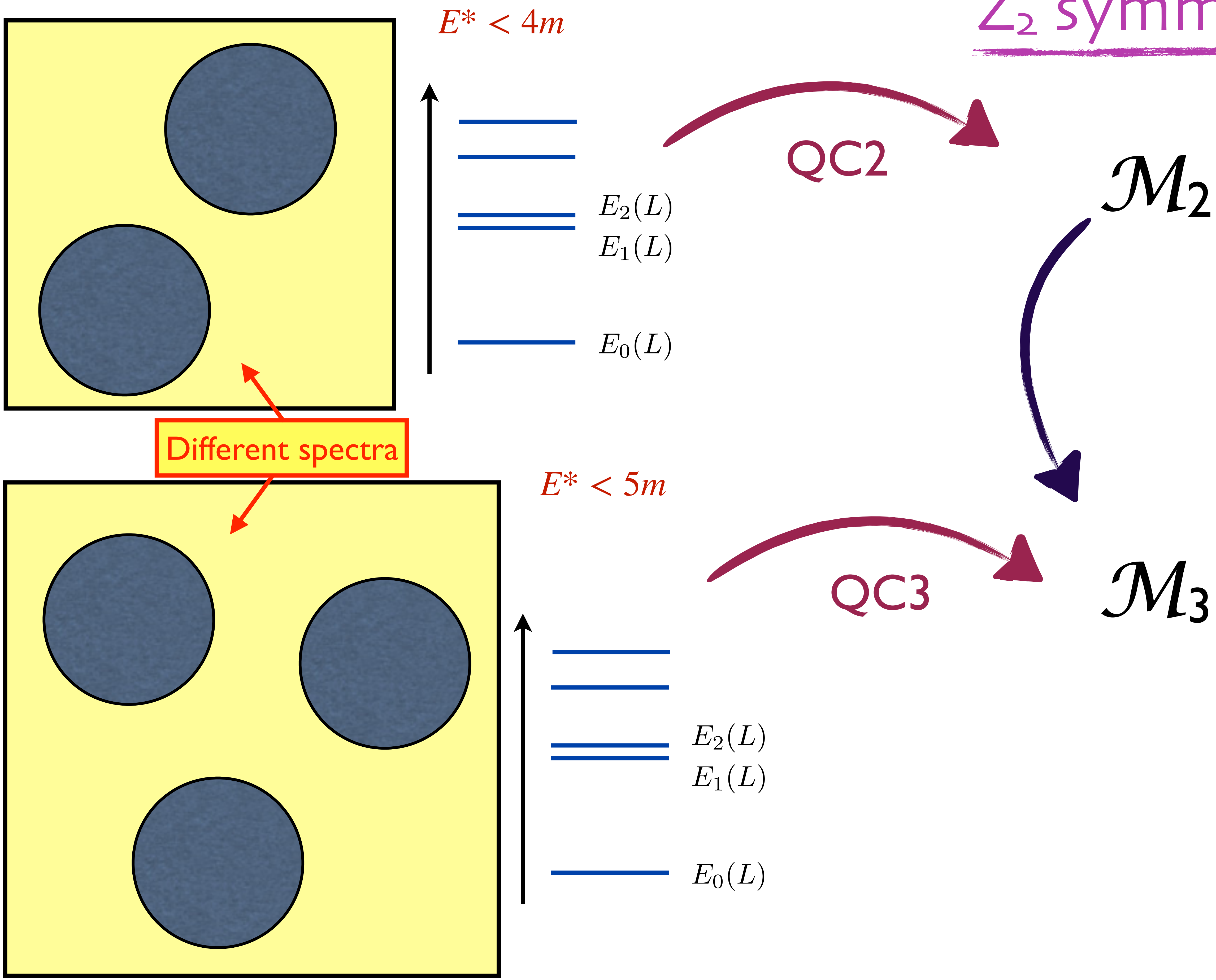
QC3

$\mathcal{M}_{2\to 3}$

& $\mathcal{M}_{3\to 2}$

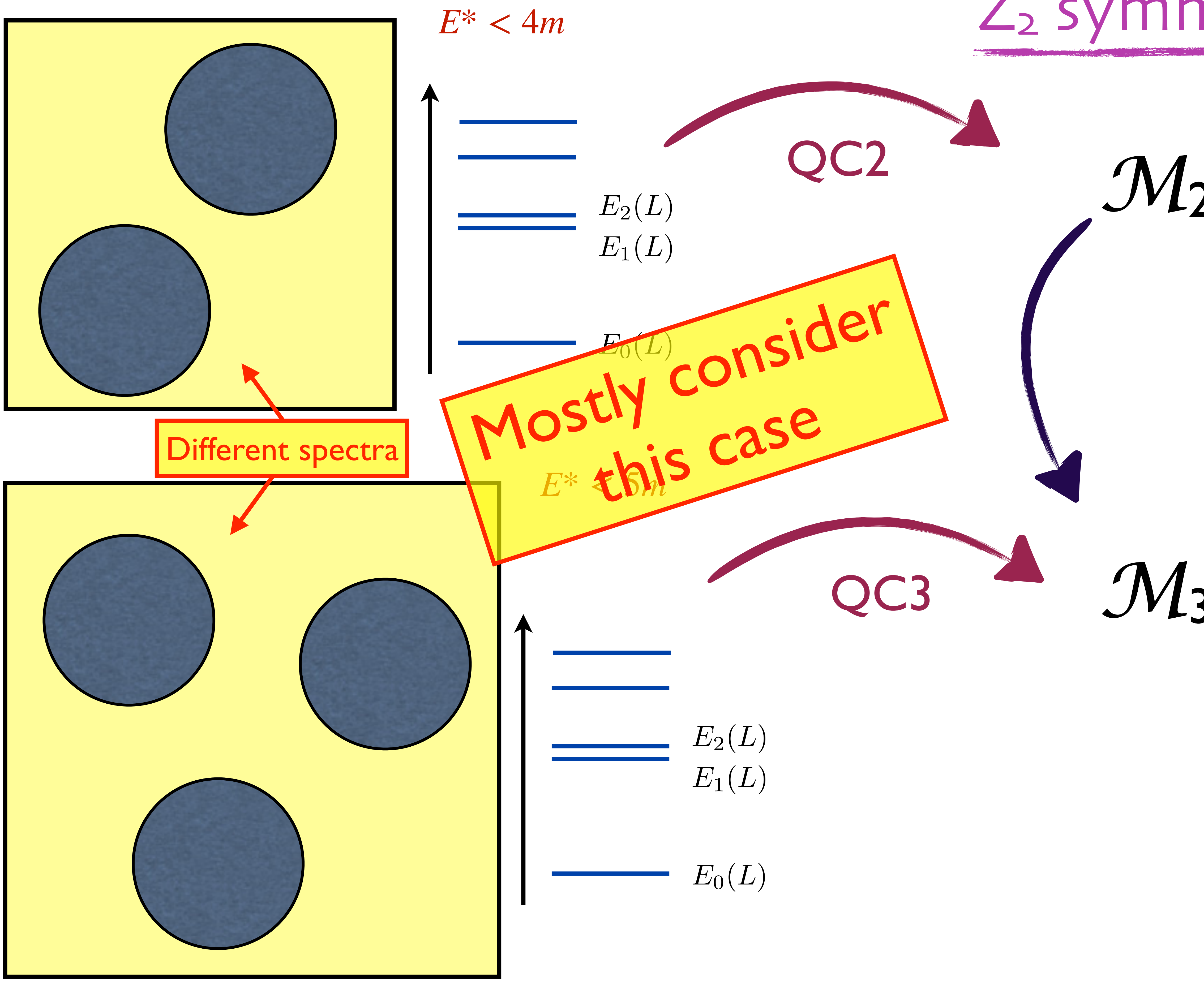
& $\mathcal{M}_{3\to 3}$

Z_2 symmetry



$E^* < 4m$
$E_2(L)$
$E_1(L)$
$E_0(L)$
QC2
$\mathcal{M}_2$
Different spectra
$E^* < 5m$
$E_2(L)$
$E_1(L)$
$E_0(L)$
QC3
$\mathcal{M}_3$

Z_2 symmetry



$E^* < 4m$
$E_2(L)$
$E_1(L)$
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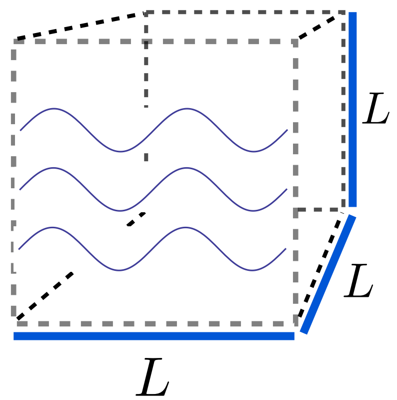
QC2
$\mathcal{M}_2$
Different spectra
Mostly consider this case
$E^* < 5m$
QC3
$\mathcal{M}_3$
$E_2(L)$
$E_1(L)$
$E_0(L)$

Complication: 2-step method

2 & 3 particle spectrum from LQCD



Quantization conditions

$$\text{QC2}: \ \det\left[F^{-1} + \mathcal{K}_2\right] = 0$$

$$\text{QC3}: \ \det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

Intermediate, unphysical scattering quantity

Integral equations in infinite volume

Scattering amplitudes

$$\mathcal{M}_{2\to3}, \ \mathcal{M}_{3\to2}, \ \mathcal{M}_{3\to3}$$

Final result for QC3 (assuming Z_2 symmetry)

QC2

$$\boxed{\det\left[F_{\mathrm{PV}}(E,\vec{P},L)^{-1}+\mathcal{K}_2(E^*)\right]=0}$$

- Total momentum (E, **P**)
- Matrix indices are $l,\ m$
- F_{PV} is a finite-volume geometric function
- $\mathcal{K}_2$ is a physical infinite-volume amplitude, which is real and has no threshold cusps
- $\mathcal{K}_2$ is algebraically related to $\mathcal{M}_2$

$$\frac{1}{\mathcal{M}_2^{(\ell)}}\equiv\frac{1}{\mathcal{K}_2^{(\ell)}}-i\rho$$

QC2 $\longrightarrow$ QC3 [HS14]

$$\det\left[F_{\mathrm{PV}}(E,\vec{P},L)^{-1}+\mathcal{K}_2(E^*)\right]=0 \quad\longrightarrow\quad \det\left[F_3(E,\vec{P},L)^{-1}+\mathcal{K}_{\mathrm{df},3}(E^*)\right]=0$$

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- Total momentum (E, **P**)
- Matrix indices are $\boldsymbol{k},\ l,\ m$
- F_3 depends on geometric functions (F_{PV} and G) and also on $\mathcal{K}_2$
  - F_3 is known if first solve QC2
- $\mathcal{K}_{\mathrm{df},3}$ is a physical infinite-volume 3-particle amplitude, which is real and has no threshold cusps
- It is cutoff dependent and thus unphysical
- It is related to $\mathcal{M}_3$ via integral equations [HS15]

Matrix indices

- All quantities are (infinite-dimensional) matrices, e.g. $(F_3)_{\mathbf{k}lm;\mathbf{p}l'm'}$, with indices

[finite volume "spectator" momentum: $\mathbf{k}=2\pi\mathbf{n}/L$] x [2-particle CM angular momentum: l,m]

Describes three on-shell particles with total energy-momentum (E, **P**)

- For large **k** (at fixed E, L), the other two particles are below threshold
- Must include such configurations, by analytic continuation, up to a cut-off at k~m [Polejaeva & Rusetsky, `12]

F_3 collects 2-particle interactions

$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\frac{1}{\mathcal{K}_2^{-1} + F + G}F\right]$$

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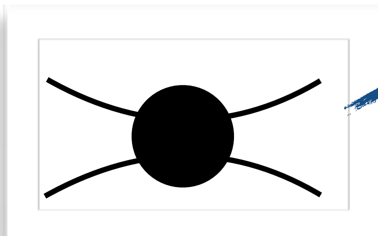

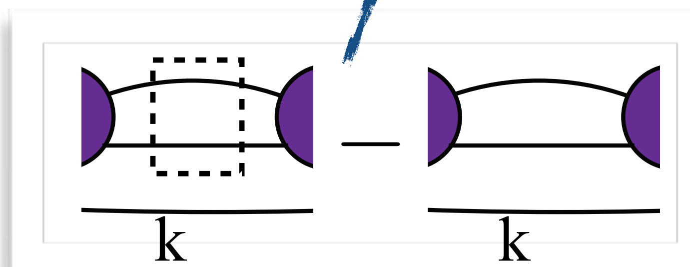

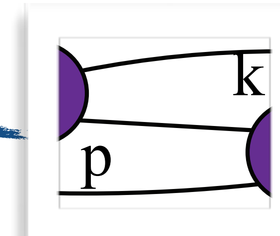

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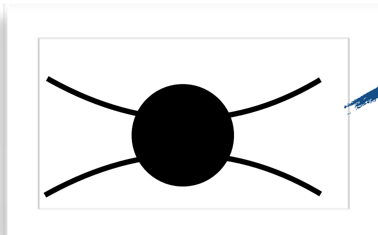
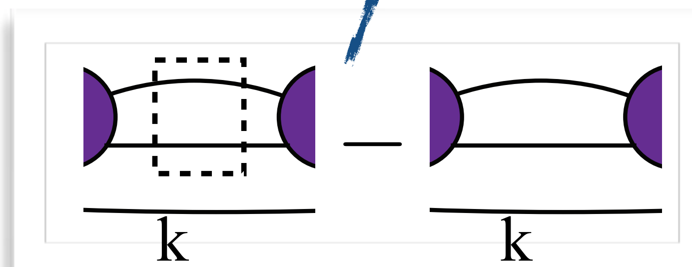
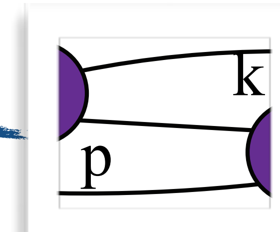
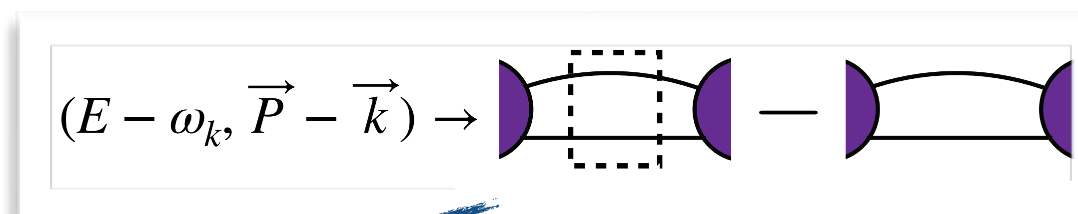






- F & G are known geometrical functions, containing cutoff function H



$$F_{p\ell'm';k\ell m} = \delta_{pk}\, H(\vec{k})\, F_{\mathrm{PV},\ell'm';\ell m}(E-\omega_k, \vec{P}-\vec{k}, L)$$

$$G_{p\ell'm';k\ell m} = \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell'm'}(\hat{k}^*)H(\vec{p})H(\vec{k})Y^*_{\ell m}(\hat{p}^*)}{(P-k-p)^2-m^2}\left(\frac{p^*}{q_k^*}\right)^{\ell}\frac{1}{2\omega_k L^3}$$

Relativistic form introduced in [BHS17]

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Relativistic form introduced in [BHS17]

$$F_{\mathrm{PV};\ell'm';\ell m}(E,\vec{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \mathrm{PV}\int\frac{d^3k}{(2\pi)^3}\right)\frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k})}$$

Relativistic form equivalent up to exponentially-suppressed terms

$$\mathcal{Y}_{\ell m}(\vec{k}^*) = \sqrt{4\pi}\left(\frac{k^*}{q^*}\right)^{\ell} Y_{\ell m}(\hat{k}^*)$$

Divergence-free K matrix

$$\det\left[F_3(E,\overrightarrow{P},L)^{-1}+\mathscr{K}_{\mathrm{df},3}(E^*)\right]=0$$

What is this? A quasi-local divergence-free 3-particle interaction

Divergence-free K matrix

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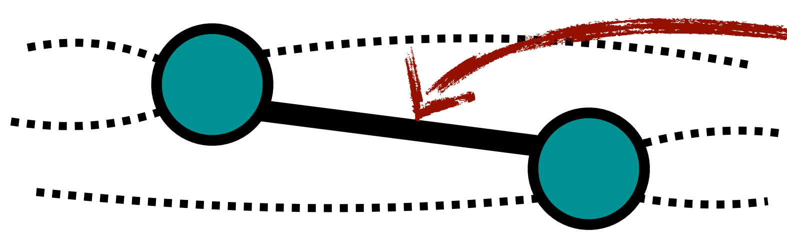
What is this? A quasi-local divergence-free 3-particle interaction

<u>Three-to-three amplitude has kinematic singularities</u>

$i\mathcal{M}_{3\to3}\equiv$ **fully connected correlator with six external legs amputated and projected on shell**

$=$  $+\cdots$

Certain external momenta put this on-shell!

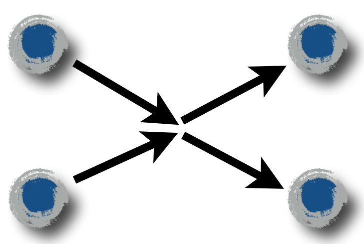
[Artwork from Hansen, HMI lectures]

- To have a nonsingular (divergence-free) quantity, need to subtract pole

Divergence-free K matrix

- $\mathcal{K}_{df,3}$ has the same symmetries as $\mathcal{M}_3$: relativistic invariance, particle interchange, T-reversal

$\mathcal{M}_2$, $\mathcal{K}_2$

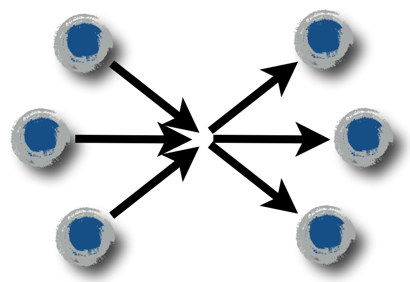


12 momentum components

-10 Poincaré generators

2 degrees of freedom
$s=E^{*2}$ + θ

$\mathcal{M}_3$, $\mathcal{K}_{df,3}$



18 momentum components

-10 Poincaré generators

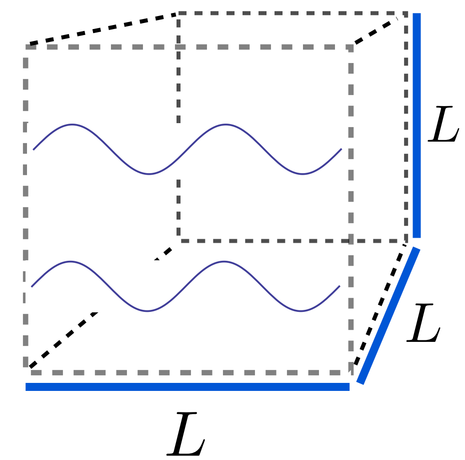
8 degrees of freedom
$s=E^{*2}$ + 7 "angles"

- Need more parameters to describe $\mathcal{K}_{df,3}$ than $\mathcal{K}_2$ (will be discussed in lecture 4)
- Why $\mathcal{K}_2$ and $\mathcal{K}_{df,3}$ appear in QC3, rather than $\mathcal{M}_2$ and $\mathcal{M}_{df,3}$, will be explained shortly

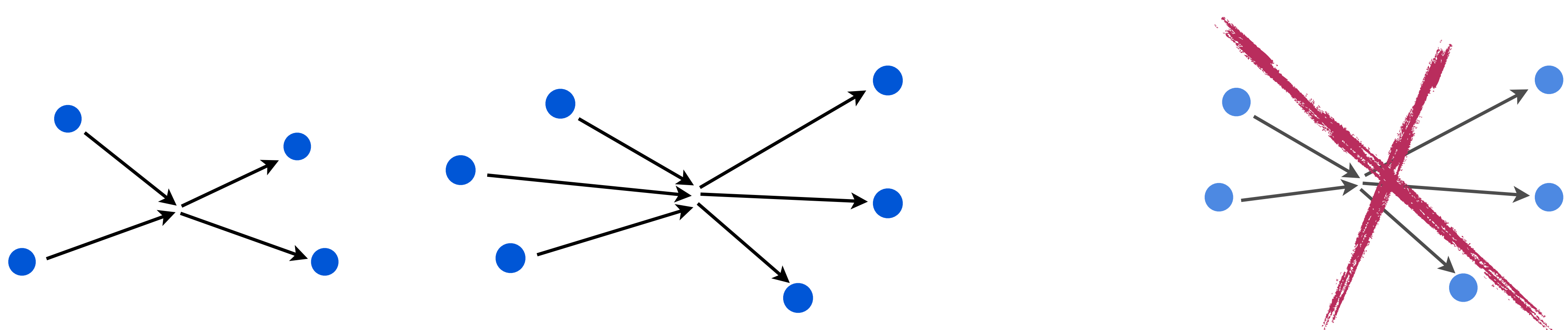
Sketch of derivation of QC3

Set-up

- Work in continuum (assume that LQCD can control discretization errors)



- Cubic box of size L with periodic BC, and infinite (Minkowski) time
  - Spatial loops are sums: $\frac{1}{L^3}\sum_{\vec{k}}$ $\vec{k} = \frac{2\pi}{L}\vec{n}$
- Consider identical particles with physical mass m, interacting *arbitrarily* except for a Z_2 (G-parity-like) symmetry
  - Only vertices are 2→2, 2→4, 3→3, 3→1, 3→5, 5→7, etc.
  - Even & odd particle-number sectors decouple



Methodology

CM energy is $E^*=\sqrt{(E^2-P^2)}$

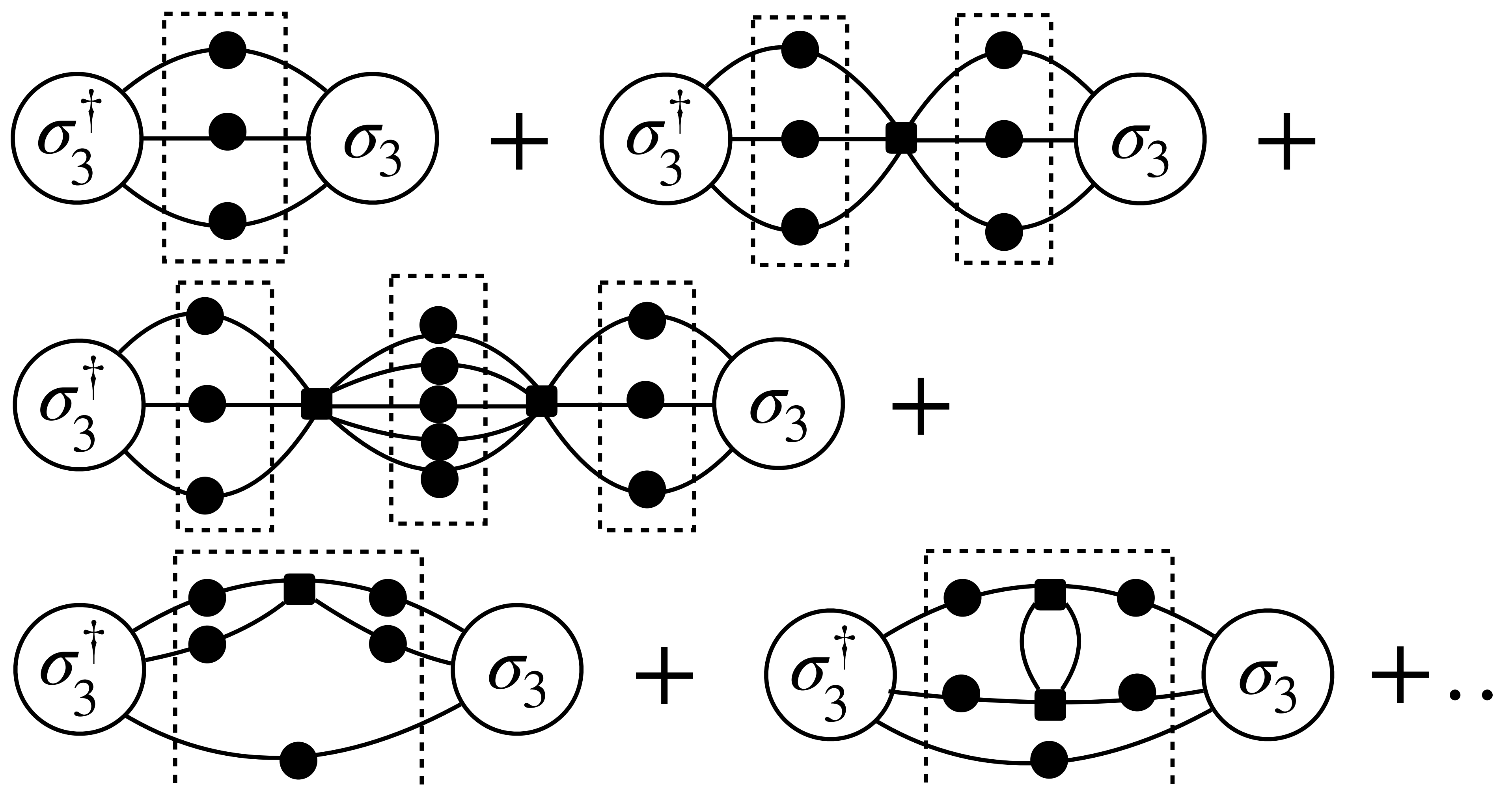
- Calculate (for some $\mathbf{P}=2\pi\mathbf{n_P}/L$)

$$C_L(E,\vec{P}) \equiv \int_L d^4x\, e^{iEt-i\vec{P}\cdot\vec{x}} \langle\Omega|\,T\left\{\sigma_3^\dagger(x)\sigma_3(0)\right\}|\Omega\rangle_L$$

- Poles in C_L occur at energies of finite-volume spectrum

- Here $\sigma_3 \sim \pi^3$

$C_L(E,\vec{P}) =$



Key step 1

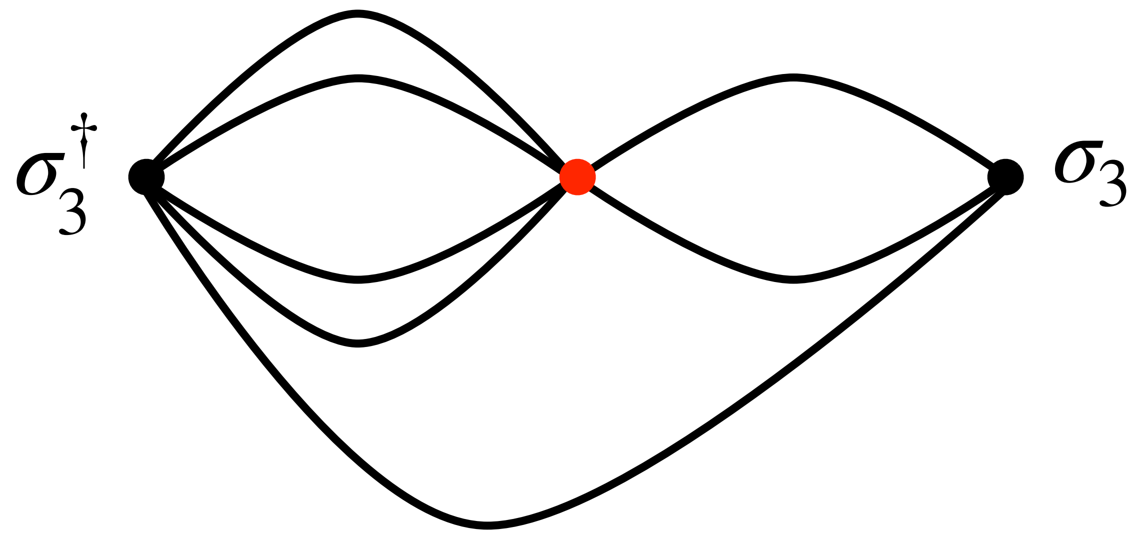
- Replace loop sums with integrals where possible
  - Drop exponentially suppressed terms ($\sim e^{-ML}$, $e^{-(ML)^2}$, etc.) while keeping power-law dependence

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \cancel{\sum_{\vec{l}\neq\vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})}$$

Exp. suppressed if g(k) is smooth
and scale of derivatives of g is ~1/M

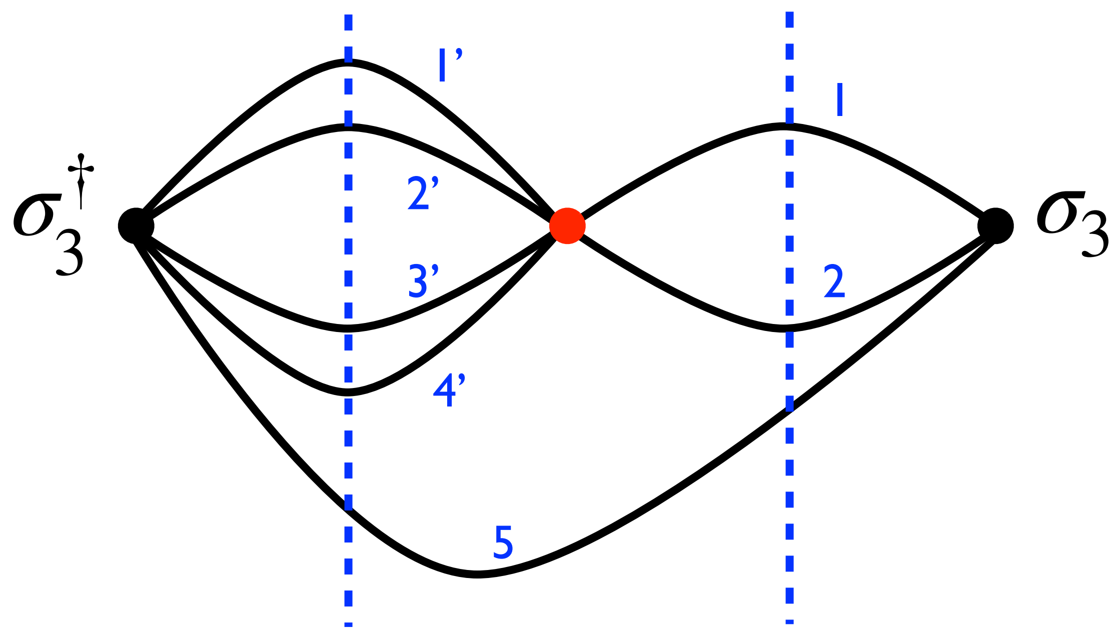
Key step 3

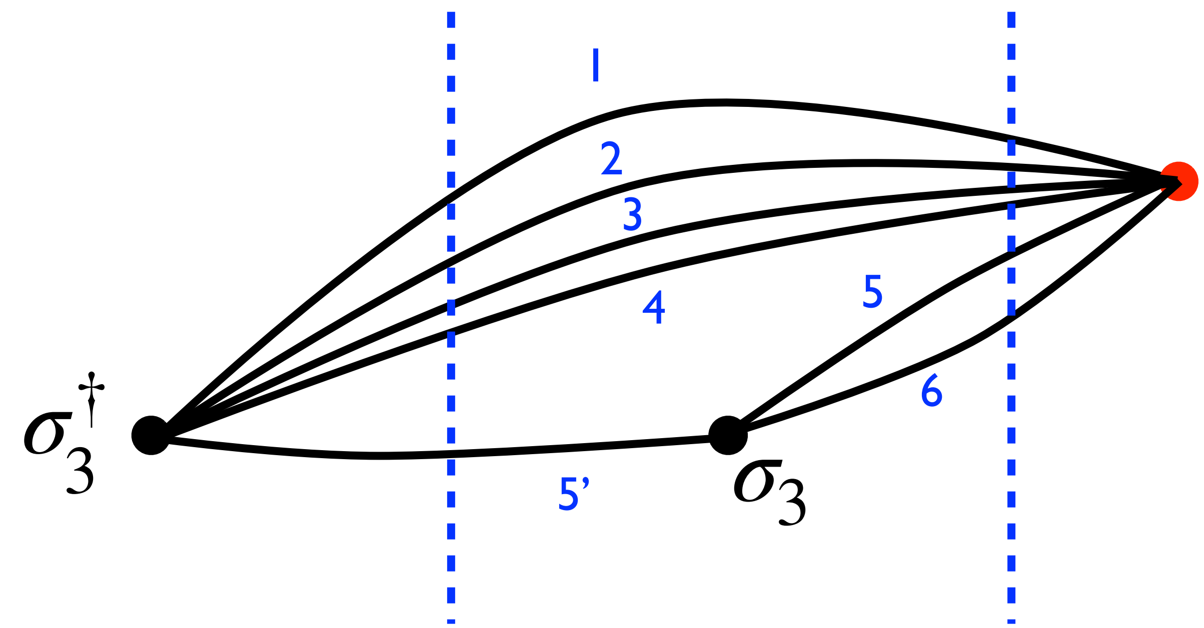
- Use time-order PT to identify potential singularities
- Example



Key step 3

- 2 out of 6 time orderings:



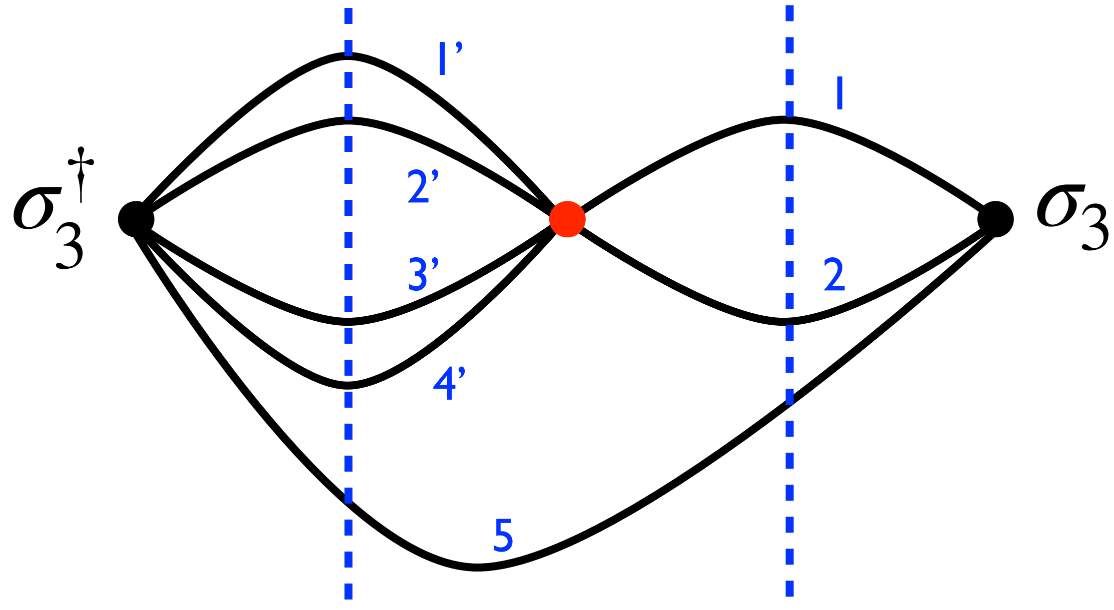


$$\frac{1}{E-\omega_1'-\omega_2'-\omega_3'-\omega_4'-\omega_5} \qquad \frac{1}{E-\omega_1-\omega_2-\omega_5} \qquad \frac{1}{E-\omega_1-\omega_2-\omega_3-\omega_4-\omega_5'} \qquad \frac{1}{\sum_{j=1,6}\omega_j}$$

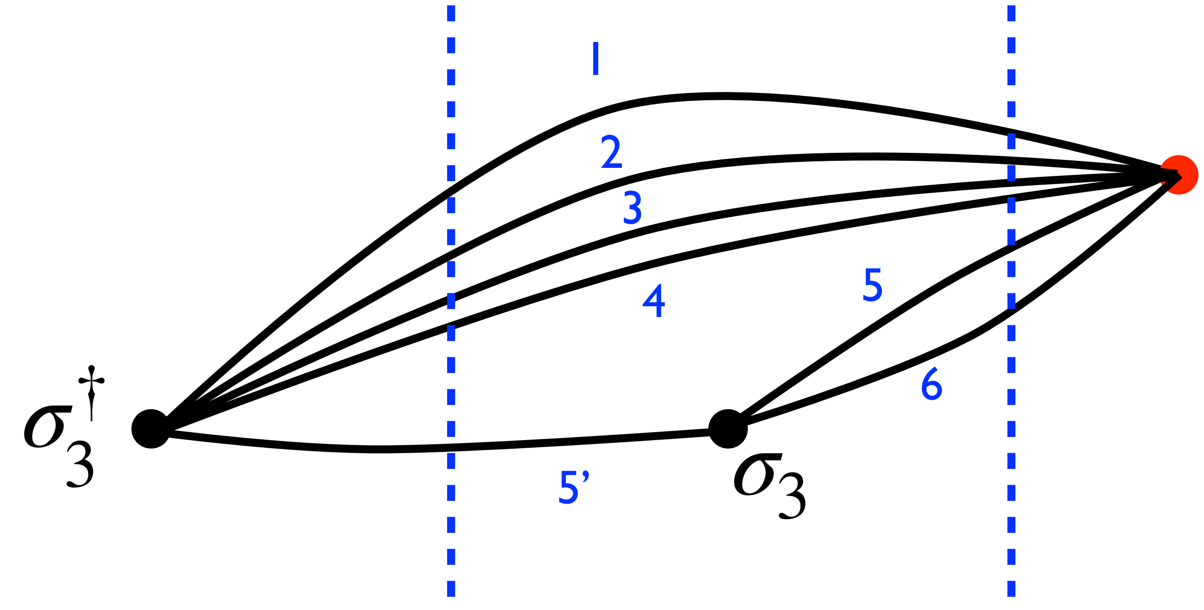
On-shell energy $\omega_j = \sqrt{\vec{k}_j^2 + M^2}$

Key step 3

- 2 out of 6 time orderings:



$$\frac{1}{E-\omega_1'-\omega_2'-\omega_3'-\omega_4'-\omega_5} \qquad \frac{1}{E-\omega_1-\omega_2-\omega_5}$$

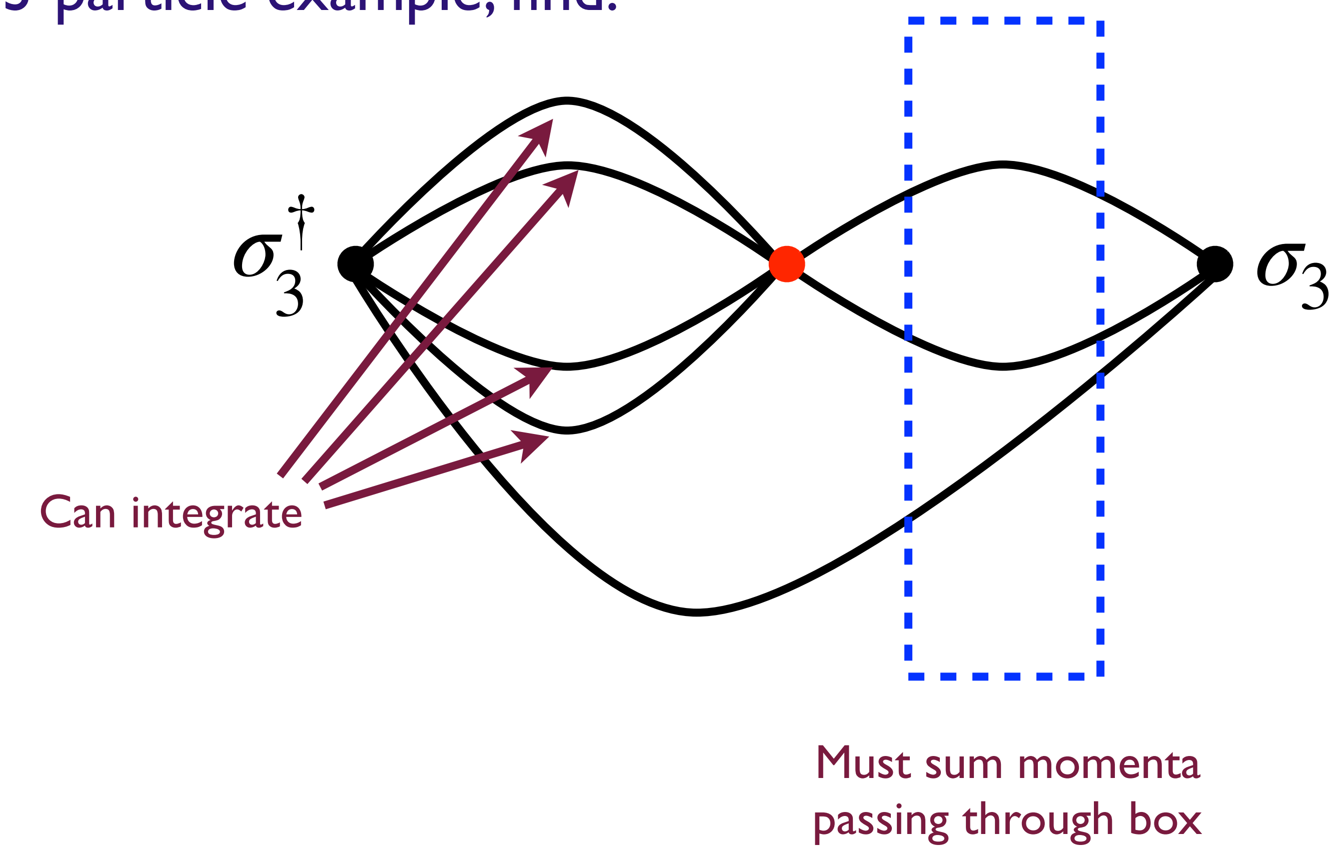


$$\frac{1}{E-\omega_1-\omega_2-\omega_3-\omega_4-\omega_5'} \qquad \frac{1}{\sum_{j=1,6}\omega_j}$$

- If restrict M < E*< 5M then only 3-particle "cuts" have singularities, and these occur only when all three particles to go on-shell

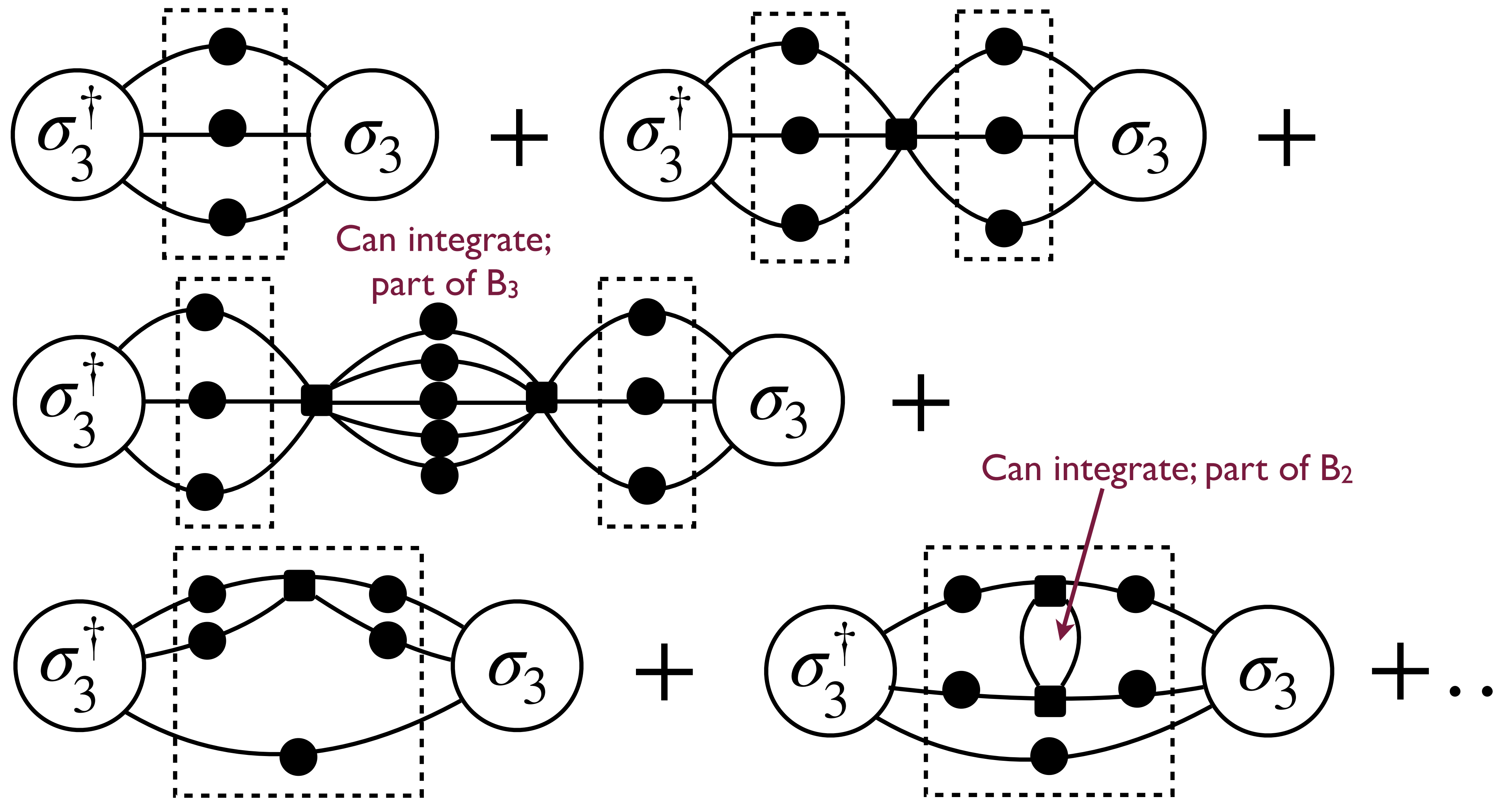
Combining key steps 1 & 3

- For each diagram, determine which momenta must be summed, and which can be integrated
- In our 3-particle example, find:



Combining key steps 1 & 3

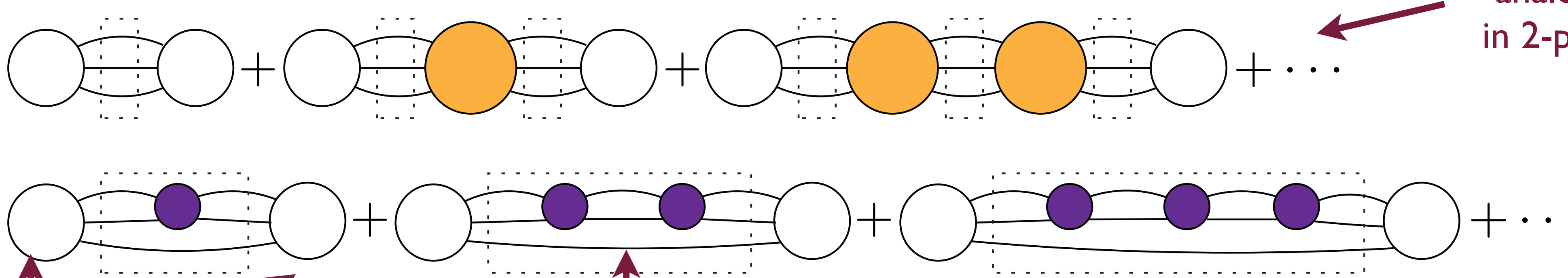
$C_L(E, \overrightarrow{P}) =$



- This leads to the "skeleton expansion"

Skeleton expansion

$C_L(E, \vec{P}) =$



analog of bubble sum
in 2-particle correlator

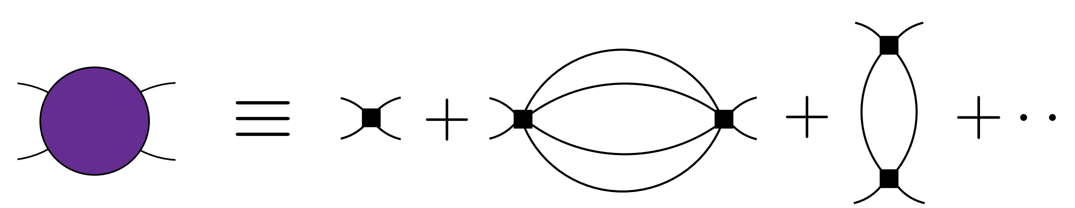
matrix elements
of σ3 and h.c.

lines are here
fully-dressed propagators

new feature: diagrams
with spectator

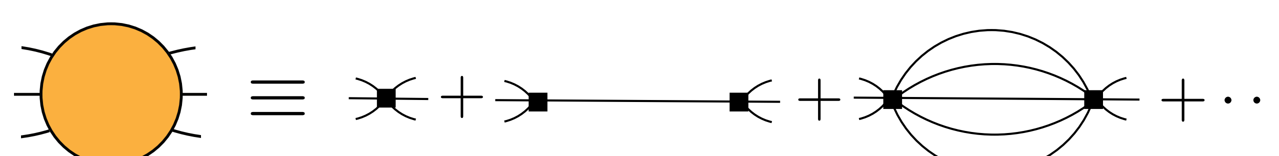
Bethe-Salpeter
kernels, with all
loops integrated,
so infinite-volume
quantities

$B_2 =$



as in derivation of QC2

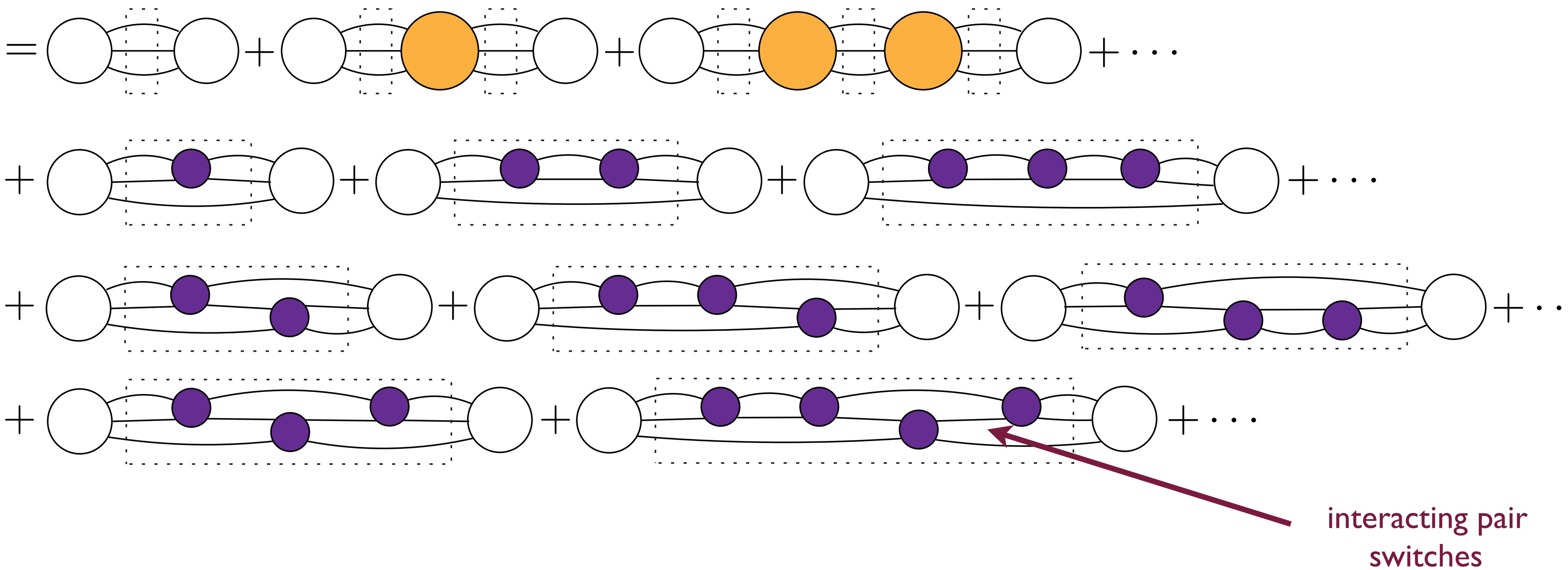
$B_3 =$



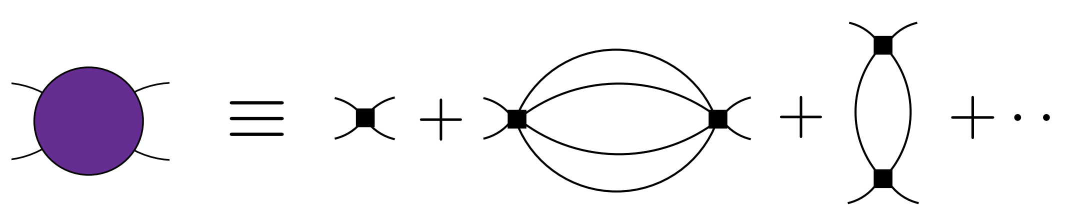
new B-S kernel,
3-particle irreducible
in s channel

Skeleton expansion

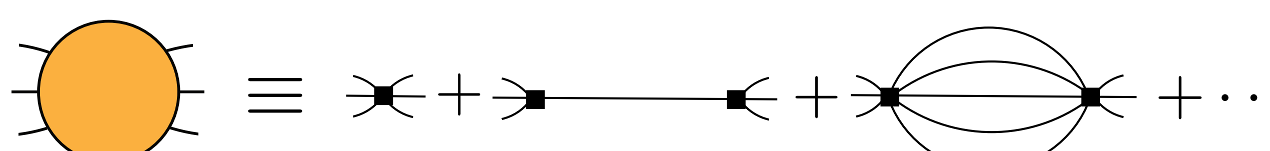
$C_L(E, \vec{P}) =$



$B_2 =$

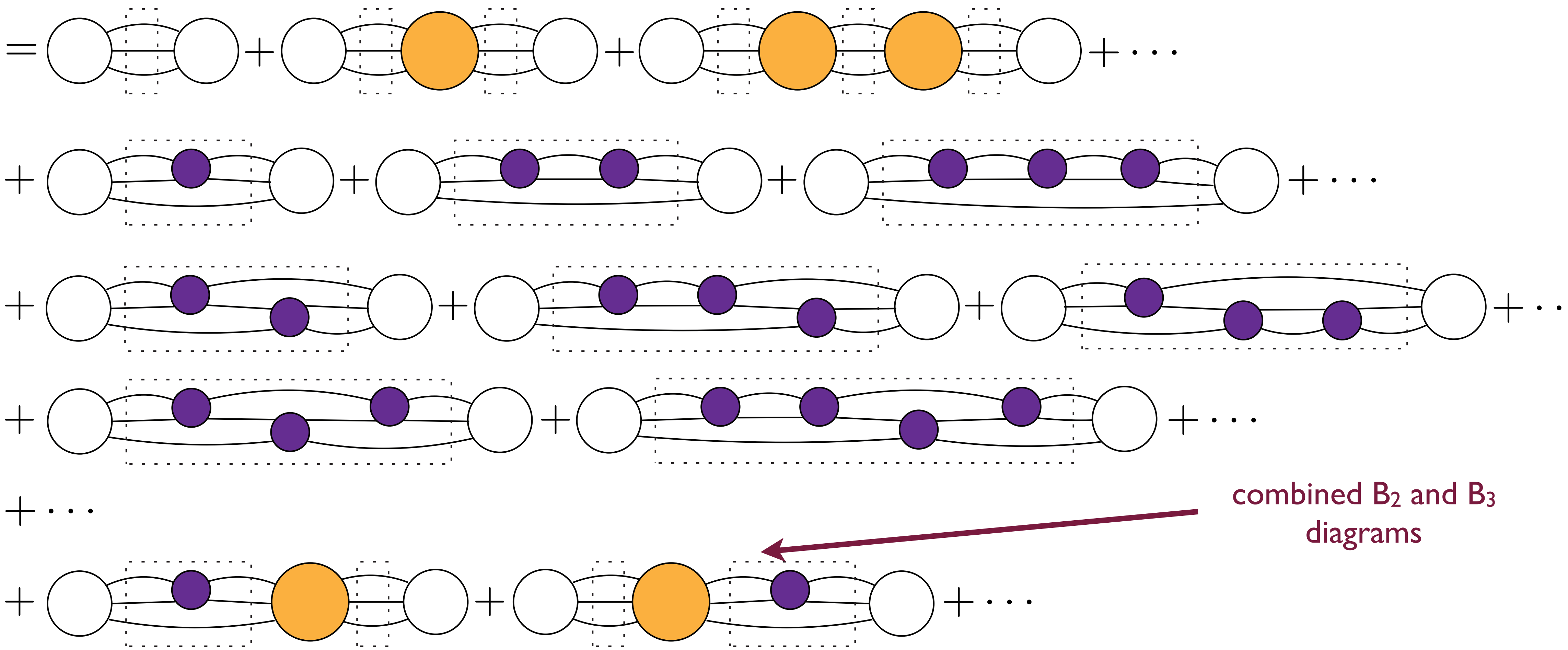


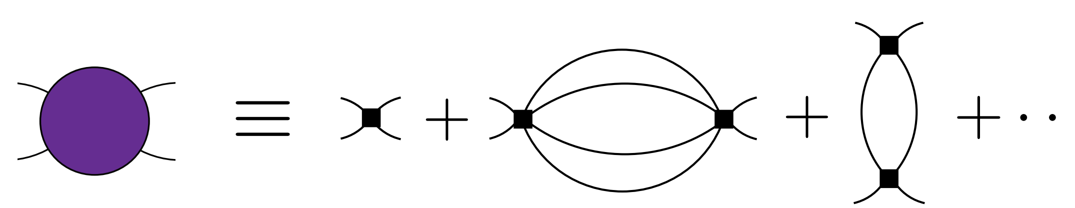
$B_3 =$

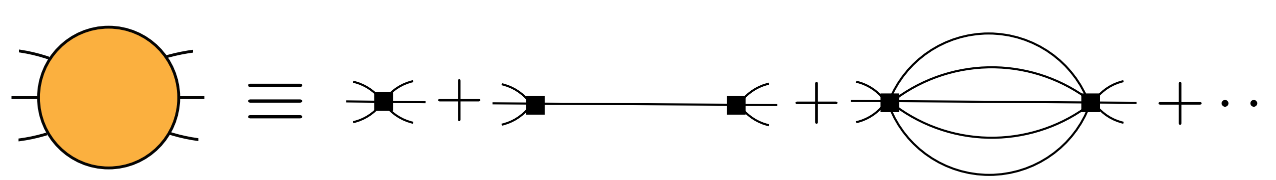


Skeleton expansion

$C_L(E,\vec{P}) =$

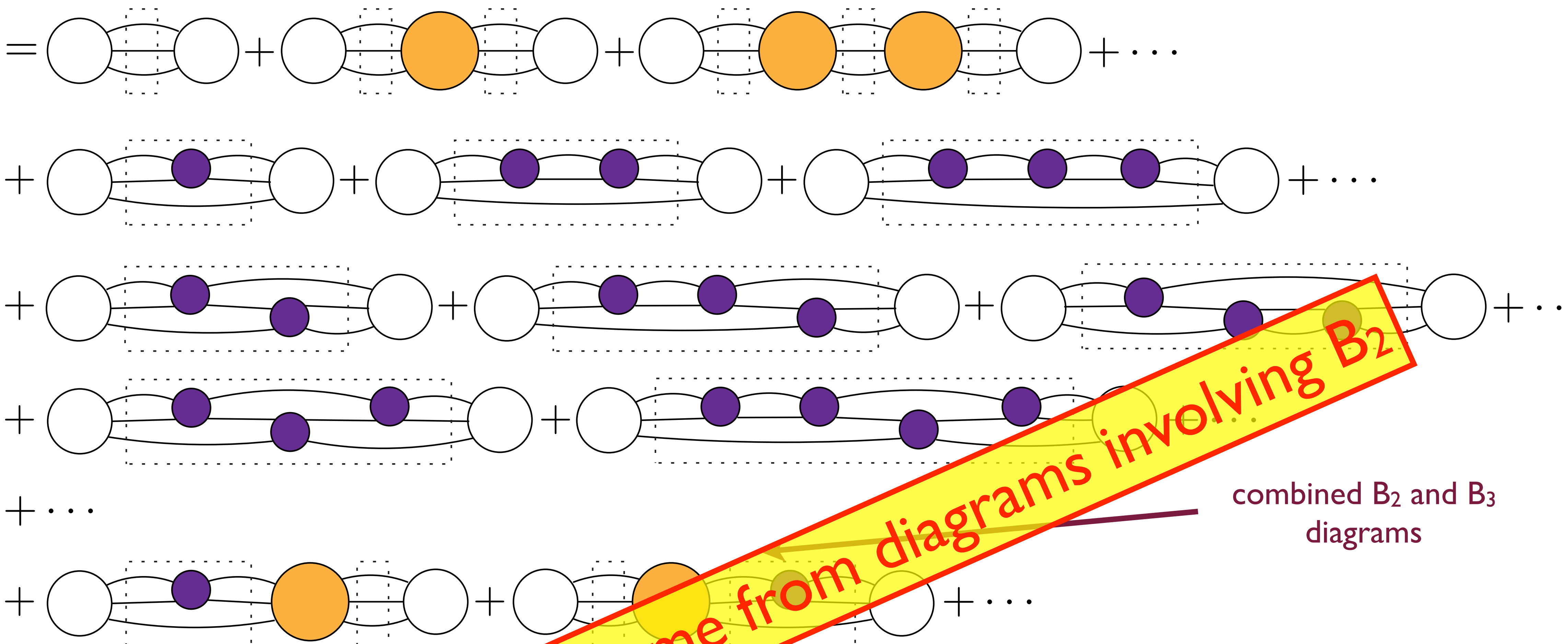


$B_2 =$ 

$B_3 =$ 

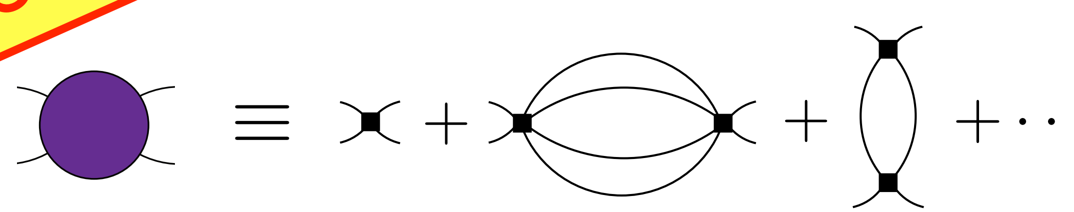
Skeleton expansion

$C_L(E,\vec{P}) =$

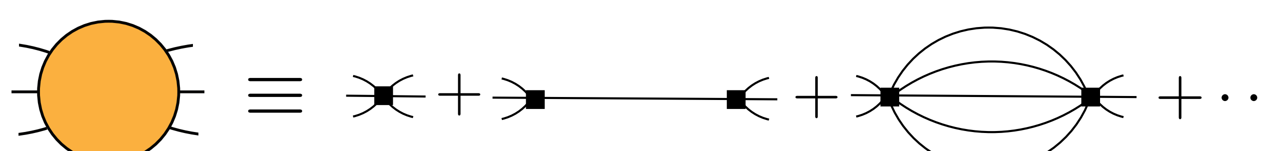


All the complications come from diagrams involving B_2

$B_2 =$

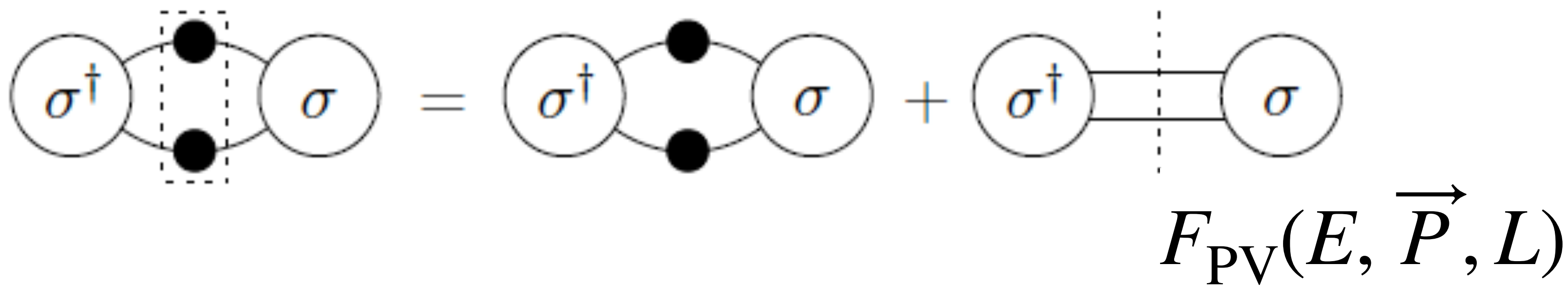


$B_3 =$

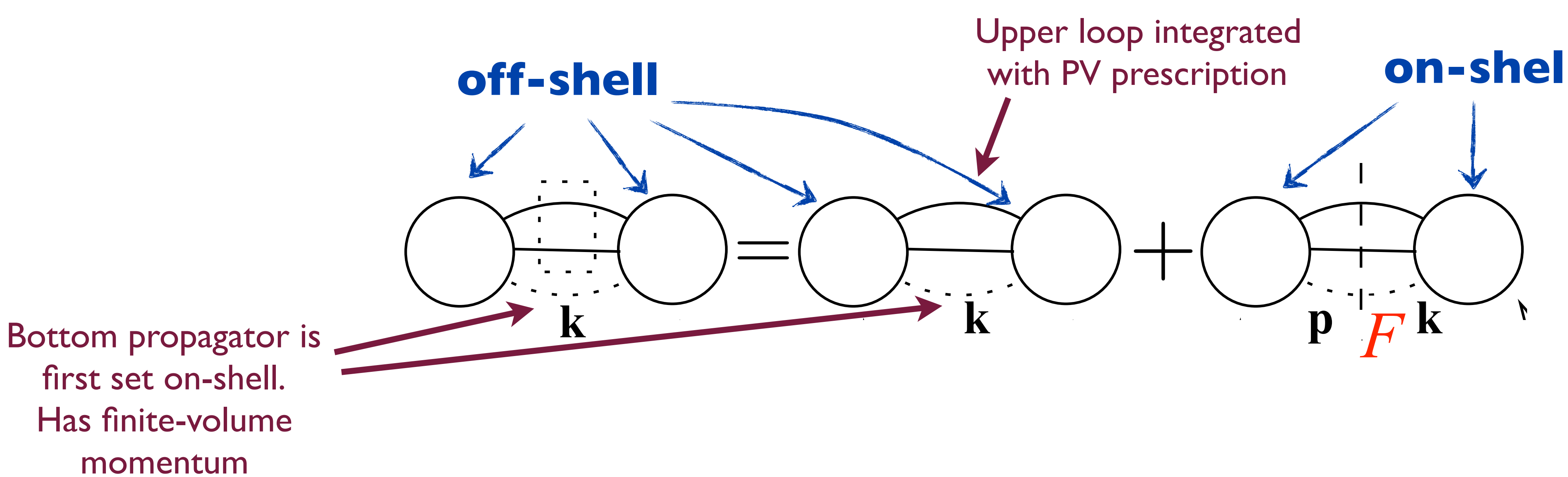


Key step 2A

- Recall key step 2 in derivation of QC2 (using PV prescription)



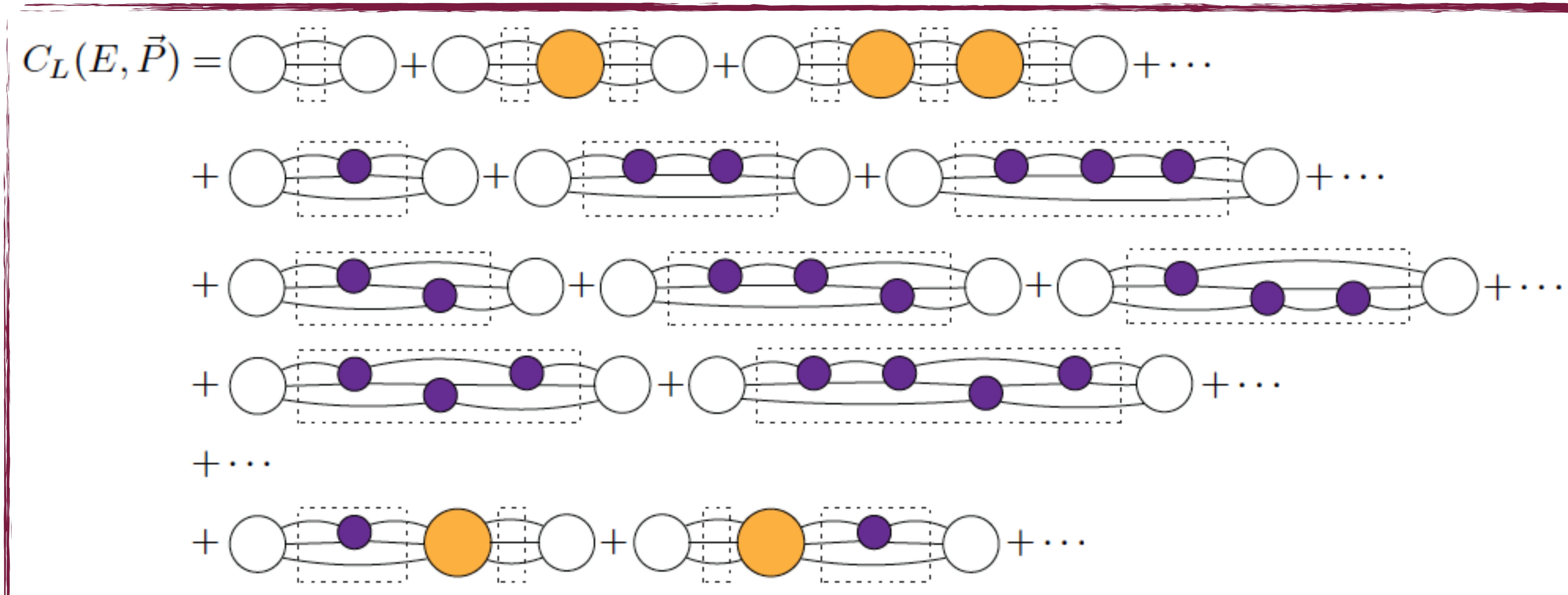
- Essentially unchanged here, except have additional label **k** for spectator



$$F_{p\ell'm';k\ell m} = \delta_{pk}\, H(\vec{k})\, F_{{\rm PV},\ell'm';\ell m}(E-\omega_k, \vec{P}-\vec{k}, L)$$

Key issue 4: dealing with cusps

- Want to replace sums with integrals + F-cuts for each 3-particle int. state
- Presence of cusps forces us to use the PV prescription
- Only an issue where cuts adjacent to B_2s



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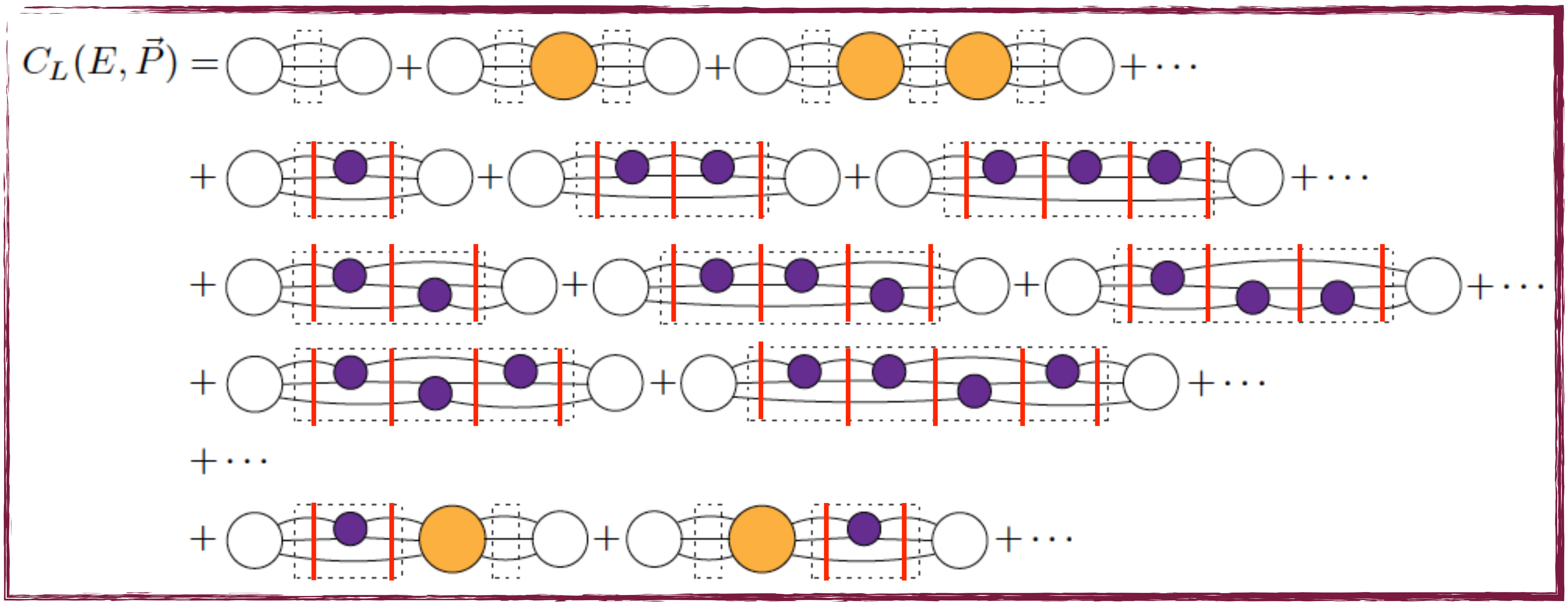
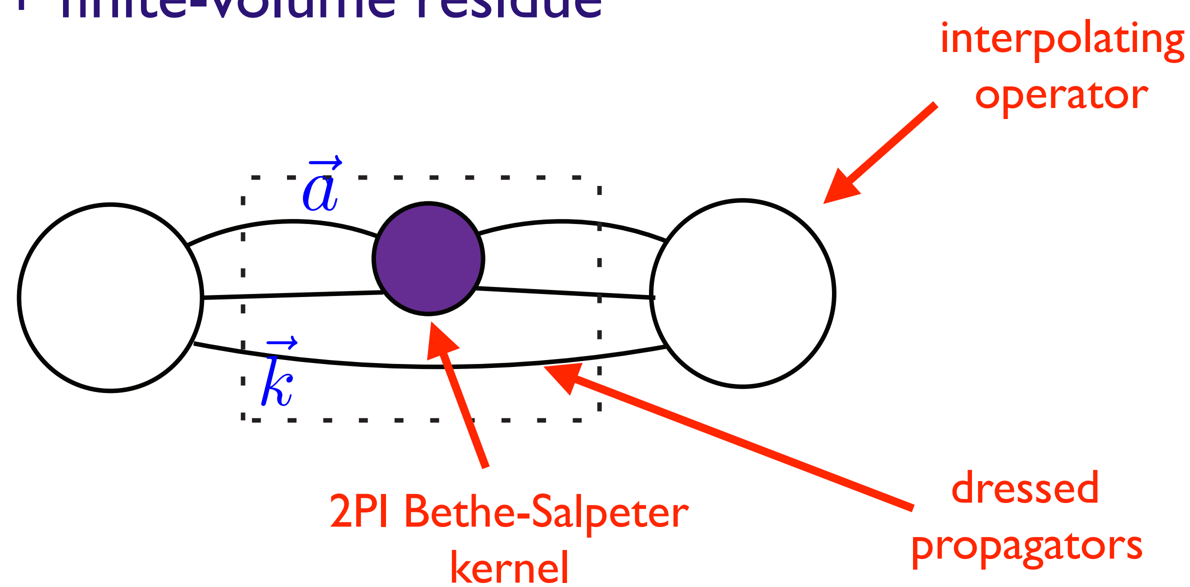


Cusp analysis (1)

- Aim: replace sums with integrals + finite-volume residue
- E.g.

$$(E,\vec{P}) \longrightarrow \quad \frac{1}{L^6}\sum_{\vec{k}}\sum_{\vec{a}}$$



- Can replace sums with integrals for smooth, non-singular parts of summand
- Singular part of left-hand 3-particle intermediate state

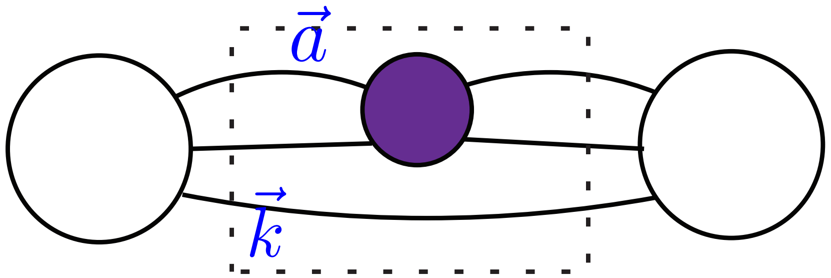
smooth functions

$$\frac{1}{L^6}\sum_{\vec{k}}\sum_{\vec{a}} \frac{A(\vec{k},\vec{a})B(\vec{k},\vec{a})}{E-\omega_k-\omega_a-\omega_{ka}}$$

denominator vanishes on-shell

$\sqrt{\vec{k}^2+m^2}$ $\qquad$ $\sqrt{\vec{a}^2+m^2}$ $\qquad$ $\sqrt{(\vec{P}-\vec{k}-\vec{a})^2+m^2}$

Cusp analysis (2)



$$\frac{1}{L^6}\sum_{\vec{k}}\sum_{\vec{a}} \frac{A(\vec{k},\vec{a})B(\vec{k},\vec{a})}{E-\omega_k-\omega_a-\omega_{ka}}$$

Step 1: treat sum over **a**

$$\frac{1}{L^3}\sum_{\vec{a}} \longrightarrow \int_{\vec{a}} + \underline{\Big(\frac{1}{L^3}\sum_{\vec{a}} - \int_{\vec{a}}\Big)}$$

Step 2: treat sum over **k**

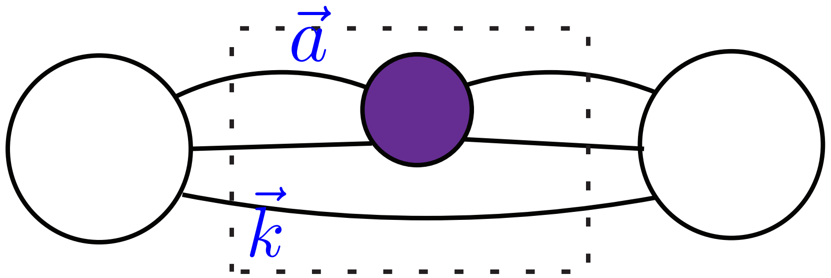
Difference gives zeta-function F with A & B projected on shell [Lüscher,...]

F has multiple singularities, so leave **k** summed for F-term

- Want to replace sum over **k** with integral for $\int_{\vec{a}}$ term
- Only possible if integral over **a** gives smooth function
- iε prescription leads to cusps at threshold
- $\Rightarrow$ sum-integral ~$1/L^4$ [Polejaeva & Rusetsky]
- Requires use of a PV prescription

Result: $$\frac{1}{L^6}\sum_{\vec{k}}\sum_{\vec{a}} = \int_{\vec{k}}\int_{\vec{a}} + \sum_{\vec{k}} \text{“F term”}$$

Cusp analysis (2)



$$\frac{1}{L^6}\sum_{\vec{k}}\sum_{\vec{a}}\frac{A(\vec{k},\vec{a})B(\vec{k},\vec{a})}{E-\omega_k-\omega_a-\omega_{ka}}$$

Step 1: treat sum over **a**

$$\frac{1}{L^3}\sum_{\vec{a}} \longrightarrow \int_{\vec{a}} + \left(\frac{1}{L^3}\sum_{\vec{a}} - \int_{\vec{a}}\right)$$

Difference gives zeta-function F with A & B projected on shell [Lüscher,...]

F has multiple singularities, so leave **k** summed for F-term

This is the origin of the extra index

Step 2: treat sum over **k**

- Want to replace sum over **k** with integral for $\int_{\vec{a}}$ term
- Only possible if integral over **a** gives smooth function
- iε prescription leads to cusps at threshold
- ⇒ sum-integral ~$1/L^4$ [Polejaeva & Rusetsky]
- Requires use of a PV prescription

Result:

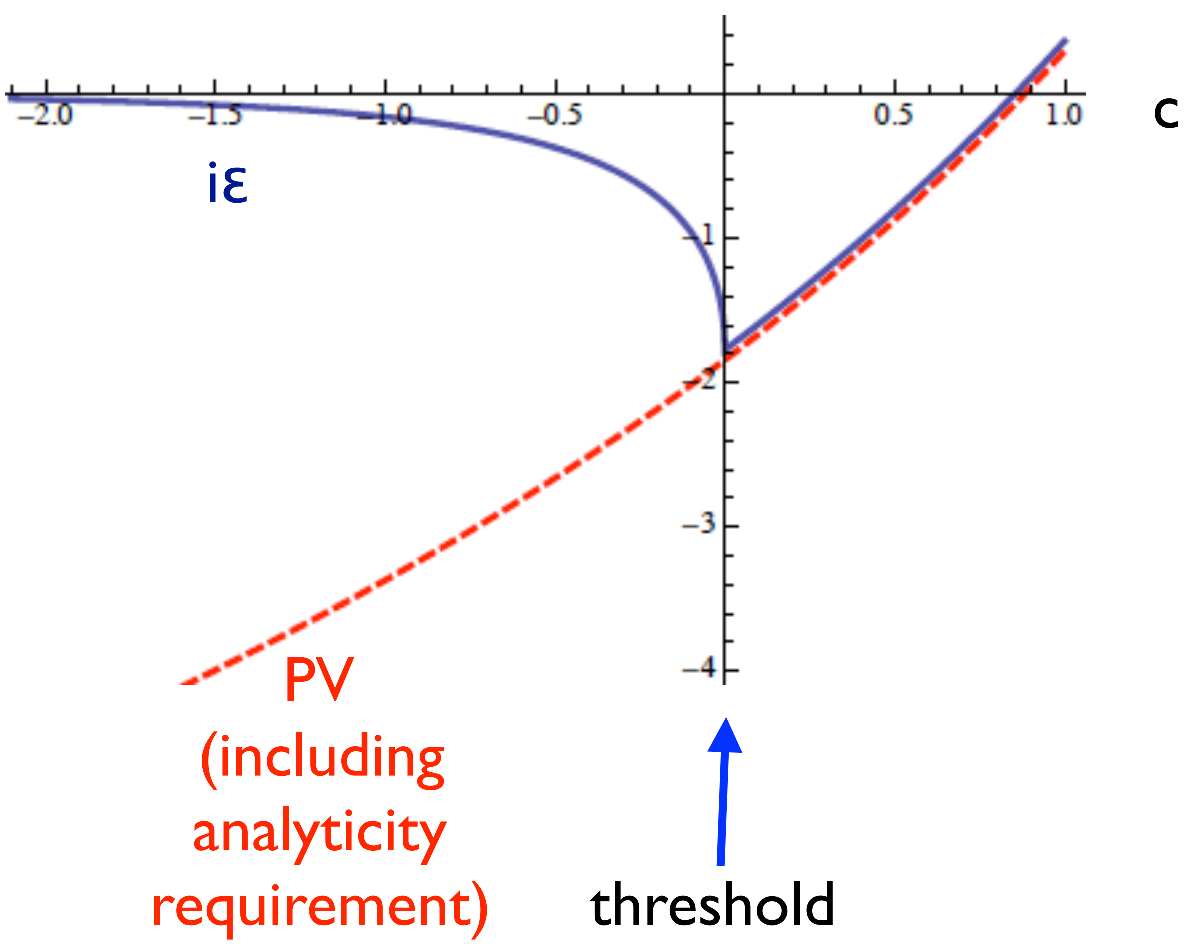
$$\frac{1}{L^6}\sum_{\vec{k}}\sum_{\vec{a}} = \int_{\vec{k}}\int_{\vec{a}} + \sum_{\vec{k}} \text{“F term”}$$

Cusp analysis (3)

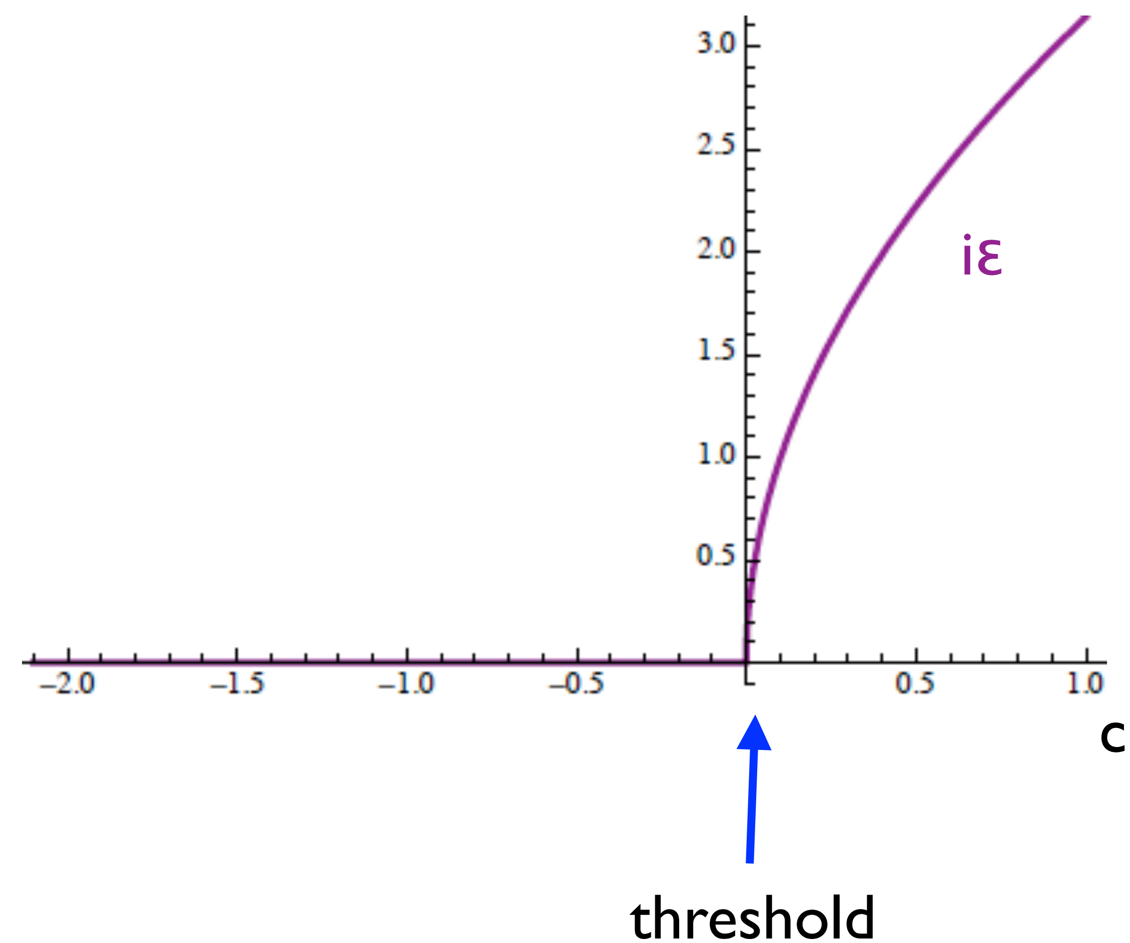
- Simple example: $\int_{\vec{a}} \frac{A(\vec{k},\vec{a})B(\vec{k},\vec{a})}{E-\omega_k-\omega_a-\omega_{ka}} \longrightarrow f(c) = \int_0^\infty dx \frac{\sqrt{x}e^{-(x-c)}}{c-x}$

$x \sim (a^*)^2$

Re f(c)



Im f(c)



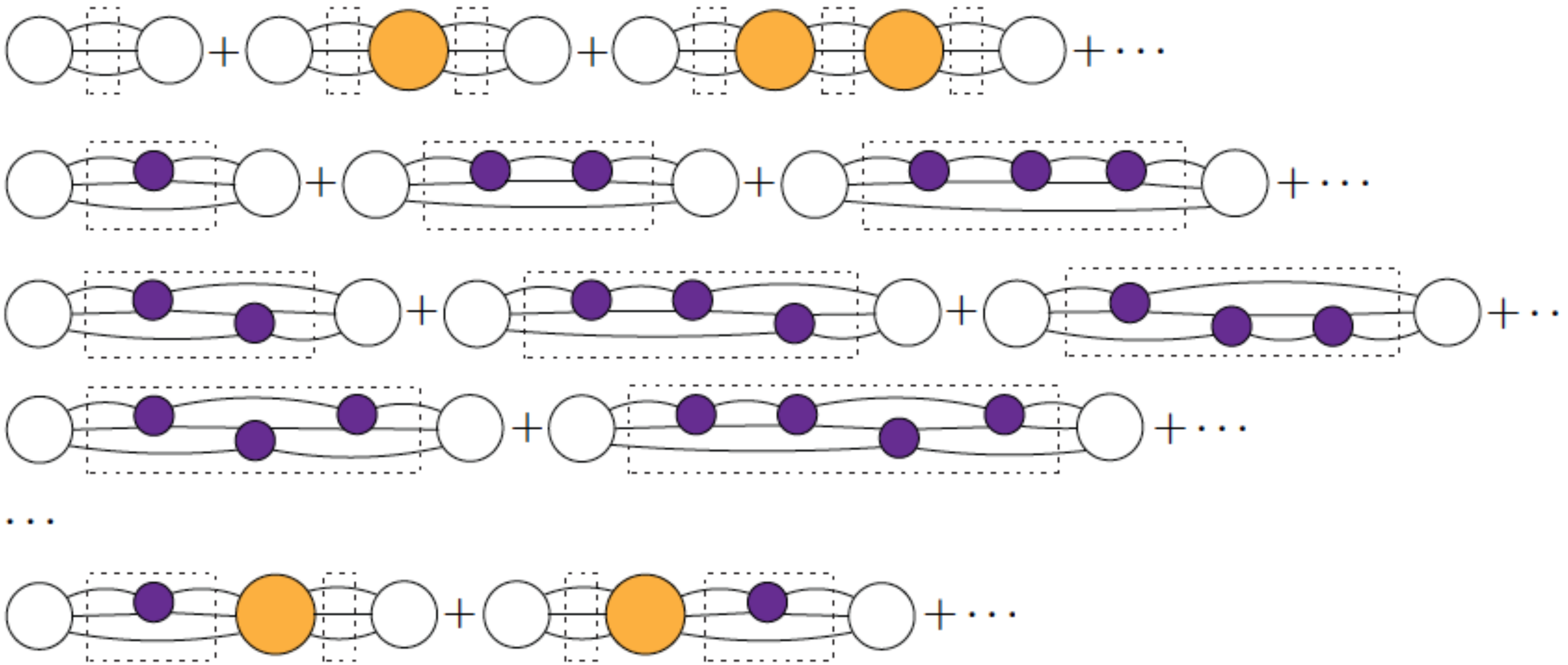
- Far below threshold, our PV smoothly turns back into iε due to cutoff function H

Cusp analysis (4)

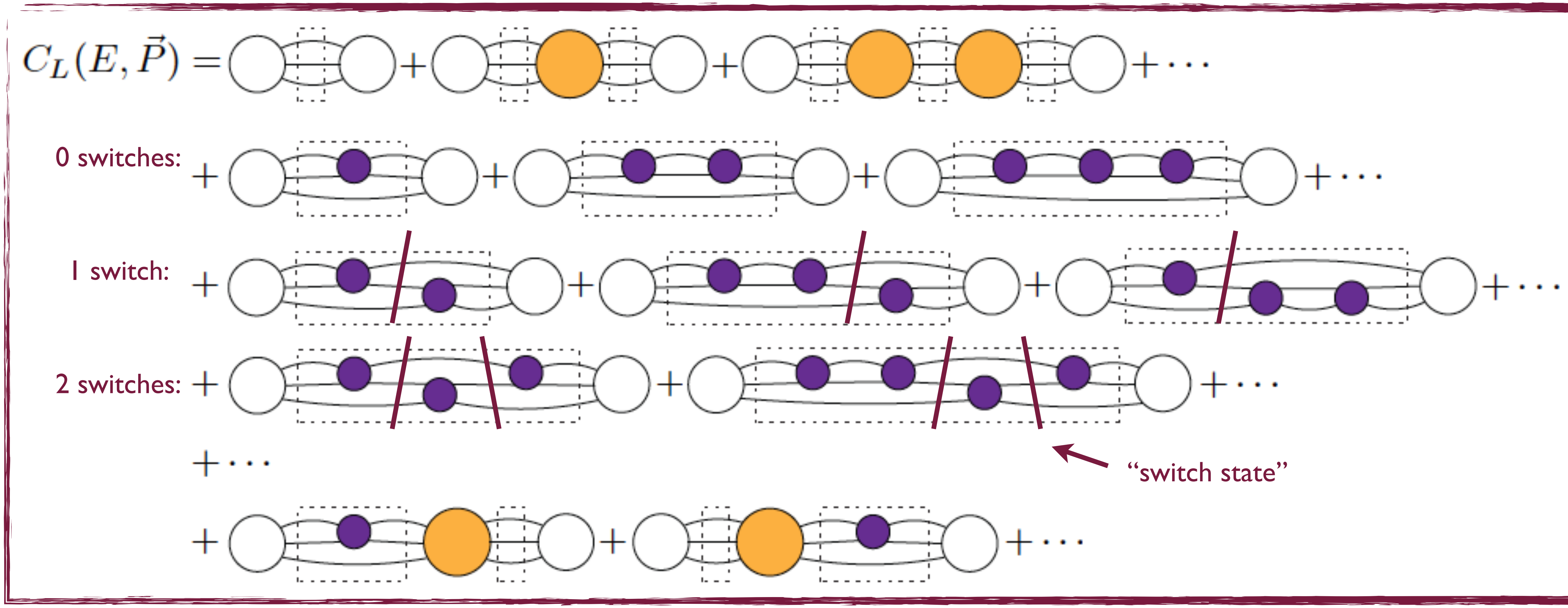
- Bottom line: must use PV prescription for all loops
- This is why $\mathcal{K}_2$ appears in QC3, rather than $\mathcal{M}_2$
- It is also why QC3 contains a three-particle K matrix (which is real)
- $\mathcal{K}_2$ is standard above threshold, and is given below by analytic continuation (so there is no cusp)
- Far below threshold (k~m), our $\mathcal{K}_2$ turns smoothly into $\mathcal{M}_2$

Key issue 5: dealing with "switches"

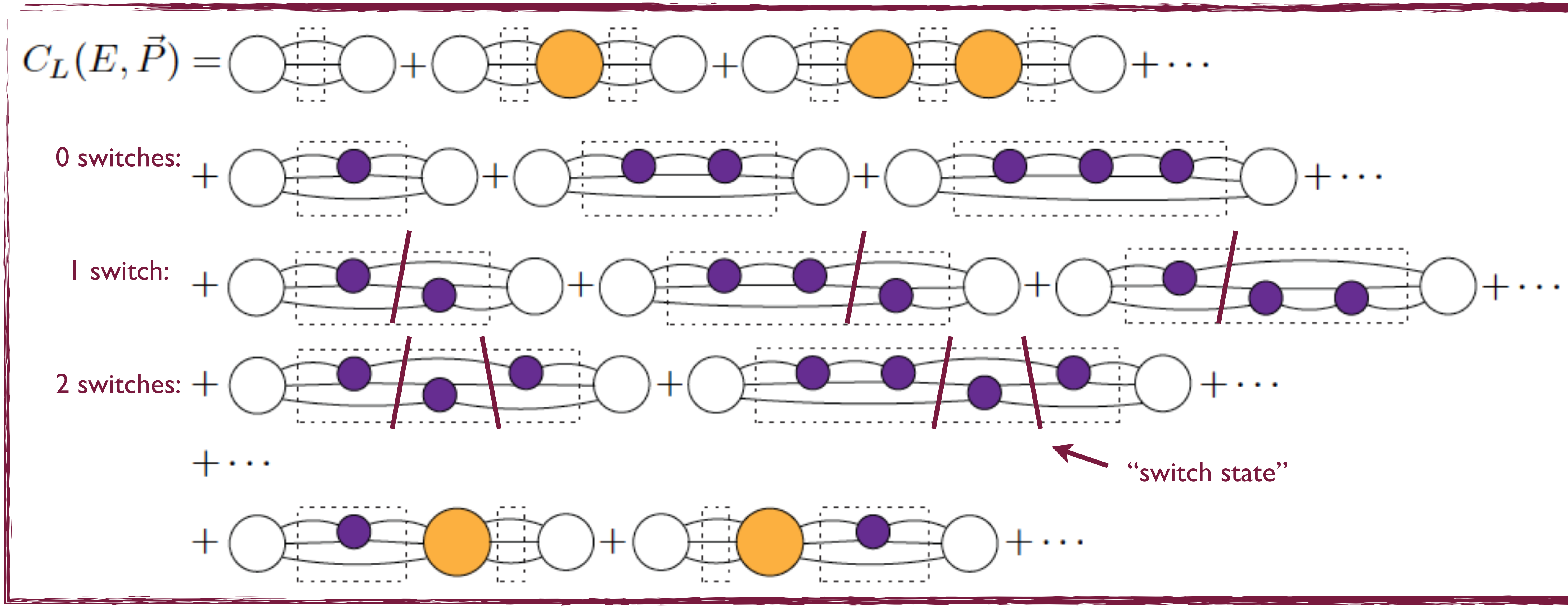
$$C_L(E, \vec{P}) =$$



Key issue 5: dealing with "switches"

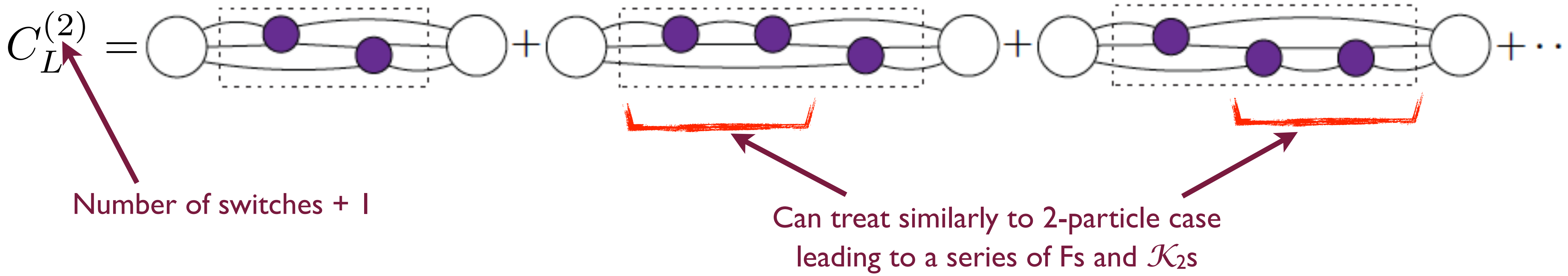


Key issue 5: dealing with "switches"

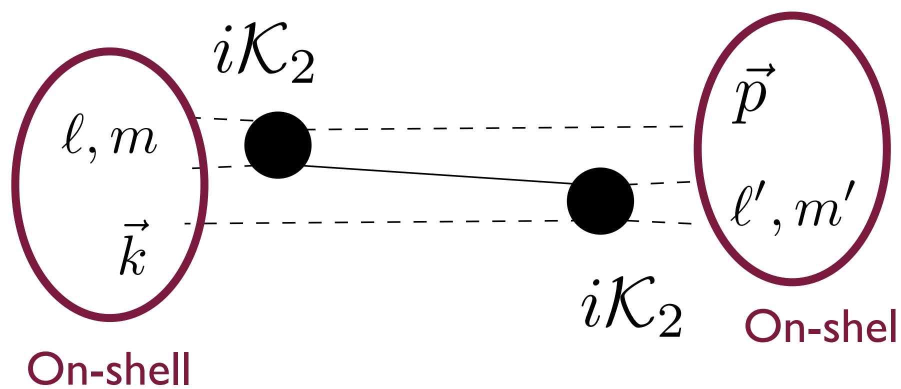


- With cusps removed, no-switch diagrams can be summed as for 2-particle case
- "Switches" present a new challenge

One-switch diagrams

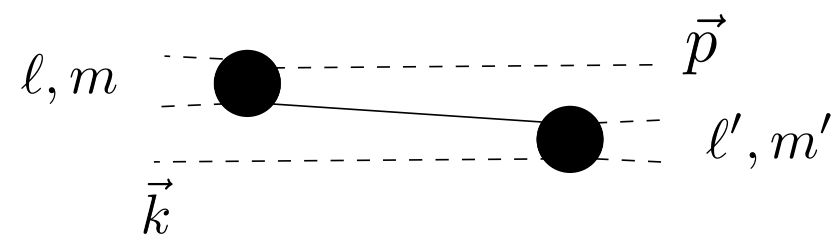


- End up with L-dependent part of $C_L^{(2)}$ having at its core:

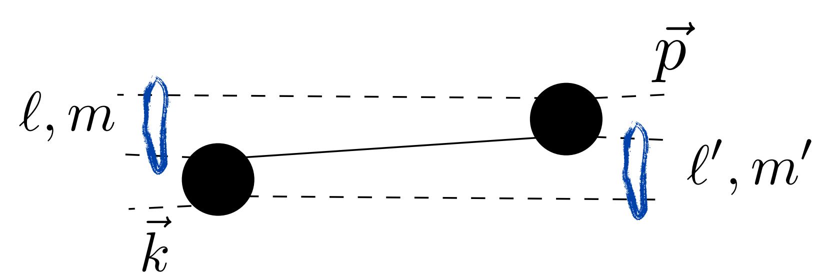


- This is our first contribution to the infinite-volume 3 particle scattering amplitude

One-switch problem



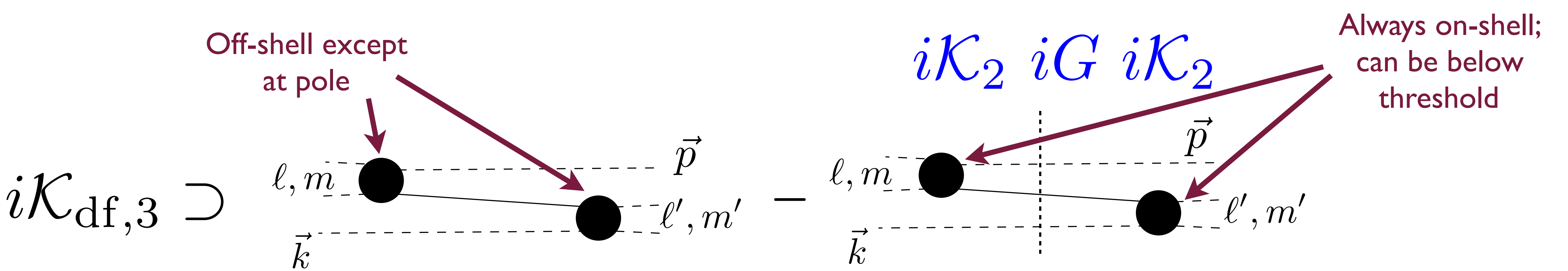
- Amplitude is **singular** for some choices of **k, p** in physical regime
    - Propagator goes on shell if top two (and thus bottom two) scatter elastically
- Not a problem per se, but leads to difficulties when amplitude is symmetrized
    - Occurs when include three-switch contributions



- Singularity implies that decomposition in $Y_{l,m}$ will not converge uniformly
    - Cannot usefully truncate angular momentum expansion

One-switch solution

- Define divergence-free amplitude by subtracting singular part
  - Utility of subtraction noted in [Rubin, Sugar & Tiktopoulos, '66]



Cutoff dependence enters here

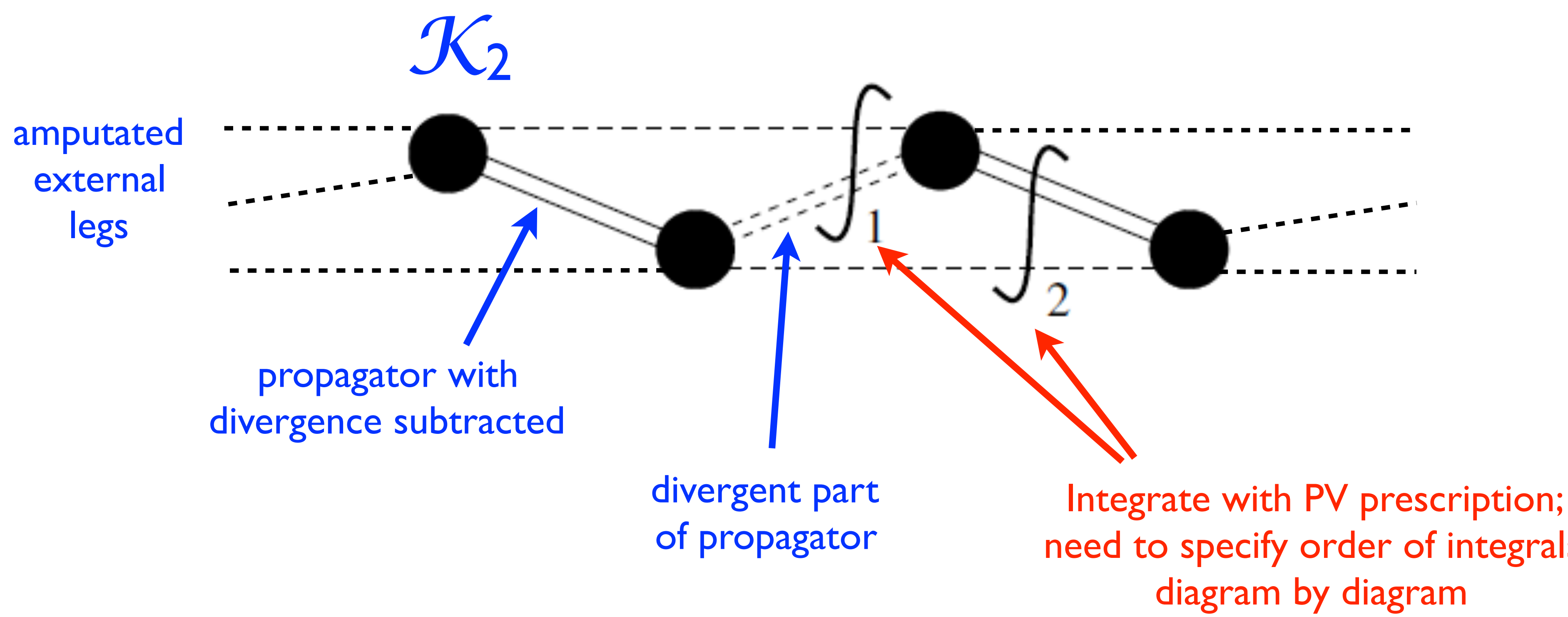
$$G_{p\ell'm';k\ell m} = \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell'm'}(\hat{k}^*) H(\vec{p}) H(\vec{k}) Y^*_{\ell m}(\hat{p}^*)}{(P-k-p)^2 - m^2} \left(\frac{p^*}{q_k^*}\right)^{\ell} \frac{1}{2\omega_k L^3}$$

- Key point: $\mathcal{K}_{df,3}$ is local and its expansion in harmonics can be truncated
- Subtracted term must be added back---leads to G contributions to F_3
- Can extend divergence-free definition to any number of switches
- Higher-order terms involve loops for which cutoff is essential

Key issue 6: symmetry breaking

- Our analysis breaks particle interchange symmetry
  - Top two particles treated differently from spectator
  - Leads to very complicated definition for $\mathcal{K}_{\mathrm{df},3}$, e.g.

$i\mathcal{K}_{\mathrm{df},3} \supset$



Key issue 6: symmetry breaking

- Definition of $\mathcal{K}_{\mathrm{df},3}$ is constructive:
  - Sum all Feynman diagrams contributing to $\mathcal{M}_3$
  - Use PV prescription, plus a (well-defined) set of rules for ordering integrals
  - Subtract leading divergent parts
  - Apply a set of (completely specified) extra factors ("decorations") to ensure external symmetrization 😩

Key issue 6: symmetry breaking

- Definition of $\mathcal{K}_{\mathrm{df},3}$ is constructive:
  - Sum all Feynman diagrams contributing to $\mathcal{M}_3$
  - Use PV prescription, plus a (well-defined) set of rules for ordering integrals
  - Subtract leading divergent parts
  - Apply a set of (completely specified) extra factors ("decorations") to ensure external symmetrization 😩
- $\mathcal{K}_{\mathrm{df},3}$ is a real, divergence-free infinite-volume quantity, smooth aside from possible dynamical poles arising from 3-particle resonances (just like $\mathcal{K}_2$) ☺️
- But it is cut-off dependent, and has an ugly construction 😞
- It can, however, be related (in infinite volume) to $\mathcal{M}_3$—our next topic! 😀
- This relation shows that $\mathcal{K}_{\mathrm{df},3}$ is not as ugly as we first thought, as it has the same symmetries as $\mathcal{M}_3$ 😃

Now do a lot of manipulations… [HS14]

our derivation. For this reason the three-particle case is fundamentally different. After much investigation, we found it most convenient to require that iB_3 only contain connected diagrams and thus display all pairwise scatterings explicitly.

Finally, in our skeleton expansion all kernels and interpolating functions are connected by fully dressed propagators,

$$\Delta(q) \equiv \int d^4x e^{iq\cdot x} \langle 0|\mathrm{T}\phi(x)\phi(0)|0\rangle. \tag{51}$$

Here $\phi(x)$ is a one-particle interpolating field defined with on shell renormalization such that

$$\lim_{q^0\to\omega_q} \Delta(q)[(q^2-m^2)/i] = 1. \tag{52}$$

Since we are working with fully dressed propagators, we do not include self-energy contributions explicitly in our skeleton expansion. We use infinite-volume fully dressed propagators throughout, which is justified because the self-energy graphs do not contain on shell intermediate states.

In summary, the skeleton expansion of Fig. 4 displays explicitly all the intermediate states that can go on shell and give rise to power-law corrections. All intermediate states which cannot go on shell are included in the infinite-volume two-to-two and three-to-three Bethe-Salpeter kernels.

In the remaining subsections, we work through the different classes of diagrams appearing in this expansion. First, in Sec. IV A, we sum diagrams containing only iB_2 kernels on the same pair of propagators (second line of Fig. 4). Then, in Secs. IV B and IV C, we sum diagrams with, respectively, one or two changes in the pair that is being scattered (third and fourth lines of Fig. 4). At this stage, we can extend the pattern and sum all diagrams built from iB_2 kernels with any number of changes in the scattered pair. This is done in Sec. IV D. Incorporating three-to-three insertions at this point is relatively easy, and is done in Sec. IV E, leading to the final result for C_L given in Eq. (42).

As we proceed we identify the diagrams contributing to $\mathcal{K}_2$ and $\mathcal{K}_{\mathrm{df},3}$, as well as A, A' and C_∞. The precise definitions of these infinite-volume quantities will thus emerge step by step.

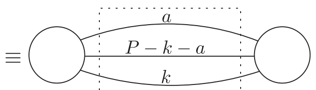
$C_L^{(0)} \equiv$ 

FIG. 6. Finite-volume correlator diagram with no kernel insertions.

A. Two-to-two insertions: no switches

In this section we sum the diagrams of Figs. 6–7. Each diagram contains only B_2 insertions, all of which scatter the same pair of propagators. We separate the diagram with no B_2 insertions, labeled $C_L^{(0)}$ (Fig. 6), from the sum of diagrams with one or more insertions, denoted $C_L^{(1)}$ (Fig. 7). We refer to these diagrams as having no switches, meaning that the pair that is scattered does not change. This designation anticipates subsequent sections in which we sum diagrams with one or more switches in the scattered pair.

An important check on the calculation of this subsection is obtained by noting that the no-switch diagrams are the complete set appearing in a theory of two different particle types, with one of the types noninteracting. This is the case provided that the correlator is constructed with fields that interpolate one free particle and two interacting particles. Thus the result for $C_L^{(0)} + C_L^{(1)}$ must be that for the full finite-volume correlator in the two-plus-spectator theory. This check is discussed below.

We begin our detailed calculation by determining the finite-volume residue of the no-insertion diagram of Fig. 6. This diagram represents the expression[20]

$$C_L^{(0)} \equiv \frac{1}{6}\frac{1}{L^6}\sum_{\vec k,\vec a}\int_{a^0}\int_{k^0}\sigma(k,a)\Delta(k)\Delta(a) \times \Delta(P-k-a)\sigma^\dagger(k,a), \tag{53}$$

where $\int_{k^0} \equiv \int dk^0/(2\pi)$, etc., and the $1/6$ is the symmetry factor. We stress that the Δs are fully dressed propagators, with the normalization given in Eq. (52).

We first evaluate the a^0 and k^0 integrals using contour integration, wrapping both contours in the lower half of the respective complex planes. Each contour encircles a one-particle pole ($a^0 = \omega_a - i\epsilon$ and $k^0 = \omega_k - i\epsilon$) as well as three-particle (and higher) poles from excited-state contributions to the propagators. The result of integration may thus be written

$$C_L^{(0)} = \frac{1}{6}\frac{1}{L^6}\sum_{\vec k,\vec a}\left[\frac{\sigma([\omega_k,\vec k],[\omega_a,\vec a])\Delta(P-k-a)\sigma^\dagger([\omega_k,\vec k],[\omega_a,\vec a])}{2\omega_k 2\omega_a} + \mathcal{R}(\vec k,\vec a)\right], \tag{54}$$

[20]In the remainder of this article we drop tildes on the Fourier-transformed interpolating operators, $\tilde\sigma(k,a)$ and $\tilde\sigma^\dagger(k,a)$, since we no longer use the position-space forms.

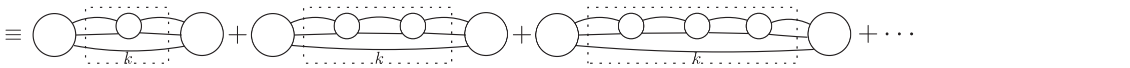
$C_L^{(1)} \equiv$ 

FIG. 7. Finite-volume correlator diagrams containing only two-to-two insertions with no change in the scattered pair.

where $\mathcal{R}(\vec k,\vec a)$ is the contribution from excited-state poles. Here k and a appearing in $\Delta(P-k-a)$ are now understood as on shell four vectors, a fact that we have made explicit in the arguments of σ and $\sigma^\dagger$. We next note that $\Delta(P-k-a)$ can be split into its one-particle pole plus a remainder:

$$\Delta(P-k-a) = \frac{i}{2\omega_{ka}(E-\omega_k-\omega_a-\omega_{ka})} + r(\vec k,\vec a). \tag{55}$$

Substituting Eq. (55) into Eq. (54) gives

$$C_L^{(0)} = \frac{1}{6}\frac{1}{L^6}\sum_{\vec k,\vec a}\left[\frac{i\sigma([\omega_k,\vec k],[\omega_a,\vec a])\sigma^\dagger([\omega_k,\vec k],[\omega_a,\vec a])}{2\omega_k 2\omega_a 2\omega_{ka}(E-\omega_k-\omega_a-\omega_{ka})} + \mathcal{R}'(\vec k,\vec a)\right], \tag{56}$$

where $\mathcal{R}'$ is the sum of $\mathcal{R}$ and the term containing r. This grouping is convenient because $\mathcal{R}'(\vec k,\vec a)$ is a smooth function of $\vec k$ and $\vec a$ for our range of E, since we have explicitly pulled out the three-particle singularity. Indeed, we are free to further adjust the separation between first and second terms, as long as the latter remains smooth. For the following development we need to include the damping function $H(\vec k)$ in the singular term. We recall that $H(\vec k)$, defined in Eqs. (27)–(28), is a smooth function which equals unity when the other two particles (those with momenta a and $P-k-a$) are kinematically allowed to be on shell (for the given values of E, $\vec P$, and $\vec k$). In particular, if we multiply the singular term by $1 = H(\vec k) + [1-H(\vec k)]$, then the $1-H(\vec k)$ term cancels the singularity, leading to a smooth function that can be added to $\mathcal{R}'$ to obtain a new residue $\mathcal{R}''$:

$$C_L^{(0)} = \frac{1}{6}\frac{1}{L^6}\sum_{\vec k,\vec a}\left[\frac{i\sigma([\omega_k,\vec k],[\omega_a,\vec a])\sigma^\dagger([\omega_k,\vec k],[\omega_a,\vec a])H(\vec k)}{2\omega_k 2\omega_a 2\omega_{ka}(E-\omega_k-\omega_a-\omega_{ka})} + \mathcal{R}''(\vec k,\vec a)\right]. \tag{57}$$

At this stage we want to rewrite $C_L^{(0)}$ as an infinite-volume (L-independent) quantity plus a remainder. Infinite-volume quantities differ only in that loop momenta are integrated rather than summed. We can thus pull out the infinite-volume object by replacing each sum with an integral plus a sum-integral difference. We stress that integrals, unlike sums, require a pole prescription. We are free to use any prescription we like, and it turns out to be most convenient to make a nonstandard choice which we call the $\widetilde{\mathrm{PV}}$ prescription. This is defined in the present context as follows[21]:

$$\begin{aligned}\frac{1}{2}\widetilde{\mathrm{PV}}\int_{\vec a}&\frac{i\sigma([\omega_k,\vec k],[\omega_a,\vec a])\sigma^\dagger([\omega_k,\vec k],[\omega_a,\vec a])H(\vec k)}{2\omega_a 2\omega_{ka}(E-\omega_k-\omega_a-\omega_{ka})}\\ &\equiv \frac{1}{2}\int_{\vec a}\frac{i\sigma([\omega_k,\vec k],[\omega_a,\vec a])\sigma^\dagger([\omega_k,\vec k],[\omega_a,\vec a])H(\vec k)}{2\omega_a 2\omega_{ka}(E-\omega_k-\omega_a-\omega_{ka}+i\epsilon)}\\ &\quad - \sigma^*_{\ell',m'}(\vec k)i\rho_{\ell',m';\ell,m}(\vec k)\sigma^{\dagger *}_{\ell,m}(\vec k),\end{aligned} \tag{59}$$

where ρ was introduced in Eq. (25) above.

To complete the definition we need to explain the meanings of the on shell quantities $\sigma^*_{\ell',m'}(\vec k)$ and $\sigma^{\dagger *}_{\ell,m}(\vec k)$. Similar quantities will appear many times below so we give here a detailed description. First recall that $(\omega_a^*,\vec a^*)$ is the four vector obtained by boosting $(\omega_a,\vec a)$ with velocity $\vec\beta_k = -(\vec P-\vec k)/(E-\omega_k)$. This boost is only physical if $E^*_{2,k} > 0$, a constraint which is guaranteed to be satisfied by the presence of $H(\vec k)$ in Eq. (59). We now change variables from $\vec a$ to $\vec a^*$ and define

$$\sigma^*(\vec k,\vec a^*) \equiv \sigma([\omega_k,\vec k],[\omega_a,\vec a]), \tag{60}$$

and similarly for $\sigma^\dagger$. The left-hand side exemplifies our general notation that, if the momentum argument is a three vector, e.g. $\vec k$, then the momentum is on shell, e.g. $k^0 = \omega_k$. If the argument is a four momentum, e.g. k, then it is, in general, off shell. Here we include a superscript $*$ on σ to indicate that it is strictly a different function from that appearing in say Eq. (57), since it depends on different coordinates (in particular on momenta defined in different frames). Next we decompose σ^* and $\sigma^{\dagger *}$ into spherical harmonics in the CM frame

[21]In the definition of $\widetilde{\mathrm{PV}}$ we are using σ and $\sigma^\dagger$ which are continuous functions of $\vec a$ and $\vec k$. Since these were originally defined only for discrete finite-volume momenta, this requires a continuation of the original functions. We require only that the continuation is smooth and slowly varying. More precisely we demand

$$\left[\frac{1}{L^3}\sum_{\vec a} - \int_{\vec a}\right]\sigma([\omega_k,\vec k],[\omega_a,\vec a]) = \mathcal{O}(e^{-mL}). \tag{58}$$

Now do a lot of manipulations… [HS14]

$$\sigma^*(\vec{k},\vec{a}^*) \equiv \sqrt{4\pi}Y_{\ell,m}(\hat{a}^*)\sigma^*_{\ell,m}(\vec{k},a^*) \tag{61}$$

$$\sigma^{\dagger *}(\vec{k},\vec{a}^*) \equiv \sqrt{4\pi}Y^*_{\ell,m}(\hat{a}^*)\sigma^{\dagger *}_{\ell,m}(\vec{k},a^*), \tag{62}$$

where there is an implicit sum over ℓ and m. Our convention, used throughout, is that the quantities to the left of the three-particle cut are decomposed using $Y_{\ell,m}$s while those to the right use the complex conjugate harmonics. Finally, with the starred quantities in hand we can define on shell restrictions. As explained in the introduction, $P-k-a$ is only on shell if $a^* = q^*_k$, so we define

$$\sigma^*_{\ell,m}(\vec{k}) \equiv \sigma^*_{\ell,m}(\vec{k},q^*_k), \qquad \sigma^{\dagger *}_{\ell,m}(\vec{k}) \equiv \sigma^{\dagger *}_{\ell,m}(\vec{k},q^*_k). \tag{63}$$

These are the quantities appearing in the ρ term in Eq. (59). If $E^*_{2,k} < 2m$, then the $\vec{a}, \vec{b}_{ka}$ pair is below threshold, and $\sigma^*_{\ell,m}$ and $\sigma^{\dagger *}_{\ell,m}$ must be obtained by analytic continuation from above threshold.

The reason for using this rather elaborate pole prescription is that we want the integral over $\vec{a}$ to produce a smooth function of $\vec{k}$. This allows the sum over $\vec{k}$ to be replaced by an integral. If we were to instead use the $i\epsilon$ prescription, then the resulting function of $\vec{k}$ would have a unitary cusp at $E^*_{2,k} = 2m$. This observation leads us to consider a principal-value pole prescription instead. We note that ρ is defined so that, for $E^*_{2,k} > 2m$, Eq. (59) simply gives the standard principal-value prescription. It turns out that this choice gives a smooth function of $\vec{k}$, provided that one uses analytic continuation to extend from $E^*_{2,k} > 2m$ to $E^*_{2,k} < 2m$. This is accomplished by our subthreshold definition of ρ, which is then smoothly turned off by the function $H(\vec{k})$. A derivation of the smoothness property is given in Appendix B. We stress that the $\widetilde{\mathrm{PV}}$ prescription is always defined relative to a spectator momentum, here $\vec{k}$.

A slightly more general form of the $\widetilde{\mathrm{PV}}$ prescription is instructive and will be useful below. For any two-particle four momentum P_2 for which the only kinematically allowed cut involves two particles, we can write

$$\begin{aligned}\widetilde{\mathrm{PV}}\int_a A(P_2,a)B(P_2,a)\Delta(a)\Delta(P_2-a) &= \int_a A(P_2,a)B(P_2,a)\Delta(a)\Delta(P_2-a) \\ &\quad - 2iJ(P_2^2/[4m^2])\tilde{\rho}(P_2) \\ &\quad \times \left[\int_{\hat{a}^*} A^*(P_2,\vec{a}^*)B^*(P_2,\vec{a}^*)\right]\Bigg|_{a^*=\sqrt{P_2^2/4-m^2}}.\end{aligned} \tag{64}$$

Here A and B are smooth, nonsingular functions of their arguments. The quantities A^* and B^* are defined in a similar way to σ^* above, e.g. $A^*(P_2,\vec{a}^*) = A(P_2,[\omega_a,\vec{a}])$, where the boost to the two-particle CM has velocity $-\vec{P}_2/P_2^0$. The function J, defined in Eq. (29), ensures that this boost is well defined.[22] Finally, the angular integral is normalized such that $\int_{\hat{a}^*} 1 = 1$. The form (64) makes clear that the prescription can be defined for four-momentum integrals (and not just three-momentum integrals) and that its dependence on external momenta enters entirely through P_2. We have also used the angular independence of ρ to rewrite the subtraction term as an angular average in the CM frame. The two functions A and B could be combined into one, but are left separate since in our applications we always have separate functions to the left and right of the cut.

Returning to the main argument, we now substitute

$$\frac{1}{L^3}\sum_{\vec{a}} = \widetilde{\mathrm{PV}}\int_{\vec{a}} + \left[\frac{1}{L^3}\sum_{\vec{a}} - \widetilde{\mathrm{PV}}\int_{\vec{a}}\right] \tag{65}$$

into Eq. (57) to reach

$$\begin{aligned}C_L^{(0)} &= \frac{1}{6}\frac{1}{L^3}\sum_{\vec{k}}\widetilde{\mathrm{PV}}\int_{\vec{a}}\left[\frac{i\sigma([\omega_k,\vec{k}],[\omega_a,\vec{a}])\sigma^\dagger([\omega_k,\vec{k}],[\omega_a,\vec{a}])H(\vec{k})}{2\omega_k 2\omega_a 2\omega_{ka}(E-\omega_k-\omega_a-\omega_{ka})} + \mathcal{R}''(\vec{k},\vec{a})\right] \\ &\quad + \frac{1}{6}\frac{1}{L^3}\sum_{\vec{k}}\left[\frac{1}{L^3}\sum_{\vec{a}} - \widetilde{\mathrm{PV}}\int_{\vec{a}}\right]\frac{i\sigma([\omega_k,\vec{k}],[\omega_a,\vec{a}])\sigma^\dagger([\omega_k,\vec{k}],[\omega_a,\vec{a}])H(\vec{k})}{2\omega_k 2\omega_a 2\omega_{ka}(E-\omega_k-\omega_a-\omega_{ka})}.\end{aligned} \tag{66}$$

Note that the sum-integral-difference operator annihilates $\mathcal{R}''(\vec{k},\vec{a})$ up to exponentially suppressed terms. As already noted, we can replace the sum over $\vec{k}$ with an integral in the first term, resulting in the infinite-volume quantity

$$C_\infty^{(0)} \equiv \frac{1}{6}\int_{\vec{k}}\widetilde{\mathrm{PV}}\int_{\vec{a}}\left[\frac{i\sigma([\omega_k,\vec{k}],[\omega_a,\vec{a}])\sigma^\dagger([\omega_k,\vec{k}],[\omega_a,\vec{a}])H(\vec{k})}{2\omega_k 2\omega_a 2\omega_{ka}(E-\omega_k-\omega_a-\omega_{ka})} + \mathcal{R}''(\vec{k},\vec{a})\right]. \tag{67}$$

Note that no pole prescription is required for the $\vec{k}$ integral.

[22] Here J is playing the role of $H(\vec{k}) = J(P_2^2/[4m^2])$ in Eq. (59).

The second term in Eq. (66) is then the finite-volume residue. First we note that we can multiply the summand/integrand by $H(\vec{a})H(\vec{b}_{ka})$, since the remainder cancels the pole and thus has vanishing sum-integral difference. Next we use the identity for sum-integral differences presented in Eq. (A1) of Appendix A. This is based on an extension of the work of Ref. [17] to include the effects of subthreshold momenta and the $\widetilde{\mathrm{PV}}$ prescription. The essence of the identity is that the sum-integral difference picks out the on shell residue of the singularity multiplied by a kinematic function. In more detail the identity makes use of the analytic properties of $\sigma^*_{\ell,m}(\vec{k},a^*)$ and $\sigma^{\dagger *}_{\ell,m}(\vec{k},a^*)$, the functions defined in Eqs. (60)–(62) above. The result is that

$$C_L^{(0)} = C_\infty^{(0)} + \frac{1}{L^3}\sum_{\vec{k}}\frac{1}{6\omega_k}\sigma^*_{\ell',m'}(\vec{k})iF_{\ell',m';\ell,m}(\vec{k})\sigma^{\dagger *}_{\ell,m}(\vec{k}), \tag{68}$$

$$= C_\infty^{(0)} + \sigma^*_{k',\ell',m'}\frac{1}{6\omega_k L^3}iF_{k',\ell',m';k,\ell,m}\sigma^{\dagger *}_{k,\ell,m}, \tag{69}$$

where the finite-volume kinematical function F is defined in Eqs. (22)–(24), and

$$\sigma^*_{k,\ell,m} \equiv \sigma^*_{\ell,m}(\vec{k}), \qquad \sigma^{\dagger *}_{k,\ell,m} \equiv \sigma^{\dagger *}_{\ell,m}(\vec{k}) \quad \text{for } \vec{k} \in (2\pi/L)\mathbb{Z}^3 \tag{70}$$

are the restrictions of the on shell functions to finite-volume momenta. All indices in Eq. (69) are understood to be summed, including k and k' which are summed over the allowed values of finite-volume momenta. This index structure appears repeatedly in our derivation, and from now on we leave indices implicit. Indeed, using the matrix notation introduced in Sec. II, we can write the final result compactly as

$$C_L^{(0)} = C_\infty^{(0)} + \sigma^*\frac{iF}{6\omega L^3}\sigma^{\dagger *}. \tag{71}$$

This is the main result of this subsection.

Our treatment of the three-particle cut will be reused repeatedly in the following, except that σ and $\sigma^\dagger$ will be replaced by other smooth functions of the momenta. Since no properties of σ and $\sigma^\dagger$ other than smoothness were used in the derivation of Eq. (71), the result generalizes immediately. It is useful to have a diagrammatic version, and this is given in Fig. 8. The key feature of the result is that the finite-volume residue depends only on on shell restrictions of the quantities appearing on either side of the cut (analytically continued below threshold as needed).

Before considering diagrams containing two-to-two insertions, we take stock of the impact of using the nonstandard $\widetilde{\mathrm{PV}}$ pole prescription. First we relate $C_\infty^{(0)}$

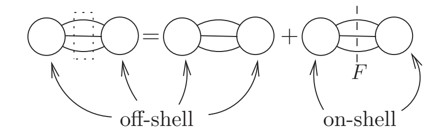


FIG. 8. Diagrammatic representation of Eq. (71).

[defined in Eq. (67)] to the conventional infinite-volume form which uses the $i\epsilon$ prescription. The latter is

$$\begin{aligned}C_\infty^{(0),i\epsilon} &\equiv \frac{1}{6}\int_{\vec{k},\vec{a}}\left[\frac{i\sigma([\omega_k,\vec{k}],[\omega_a,\vec{a}])\sigma^\dagger([\omega_k,\vec{k}],[\omega_a,\vec{a}])H(\vec{k})}{2\omega_k 2\omega_a 2\omega_{ka}(E-\omega_k-\omega_a-\omega_{ka}+i\epsilon)} + \mathcal{R}''(\vec{k},\vec{a})\right], \end{aligned}\tag{72}$$

$$= \frac{1}{6}\int_{k,a}\sigma(k,a)\Delta(k)\Delta(a)\Delta(P-k-a)\sigma^\dagger(k,a), \tag{73}$$

where $\int_k \equiv \int d^4k/(2\pi)^4$, etc., indicate integrals over four-momenta. To obtain the second line, which is the standard expression for the Feynman diagram, we have reversed the steps leading from Eq. (53) to (57). It then follows from the definition of the $\widetilde{\mathrm{PV}}$ prescription, Eq. (59), that

$$C_\infty^{(0)} = C_\infty^{(0),i\epsilon} - \int_{\vec{k}}\sigma^*(\vec{k})\frac{i\rho(\vec{k})}{6\omega_k}\sigma^{\dagger *}(\vec{k}). \tag{74}$$

This relation is similar in form to Eq. (71), with the "F cut" being replaced by a "ρ cut." The key point for present purposes is that the ρ-cut term in Eq. (74) does not introduce poles as a function of E. This follows from noting that ρ is a finite function of $(E,\vec{P})$ and $\vec{k}$, which has a finite range of support in the latter.

We can also determine the form of the finite-volume correction if we use the $i\epsilon$ prescription throughout, including in F [see Eq. (24) above]. This connects our result to earlier work on two-particle quantization conditions, e.g. Ref. [17], where $F^{i\epsilon}$ was used. Defining

$$F^{i\epsilon}_{k',\ell',m';k,\ell,m} \equiv \delta_{k',k}F^{i\epsilon}_{\ell',m';\ell,m}(\vec{k}), \tag{75}$$

it follows from Eq. (22) that

$$F_{k',\ell',m';k,\ell,m} = F^{i\epsilon}_{k',\ell',m';k,\ell,m} + \delta_{k',k}\rho_{\ell',m';\ell,m}(\vec{k}). \tag{76}$$

Combining the results above we then find

$$\begin{aligned}C_L^{(0)} &= C_\infty^{(0),i\epsilon} + \sigma\frac{iF^{i\epsilon}}{6\omega L^3}\sigma^\dagger \\ &\quad + \left[\frac{1}{L^3}\sum_{\vec{k}} - \int_{\vec{k}}\right]\sigma^*(\vec{k})\frac{i\rho(\vec{k})}{6\omega_k}\sigma^{\dagger *}(\vec{k}).\end{aligned} \tag{77}$$

Now do a lot of manipulations… [HS14]

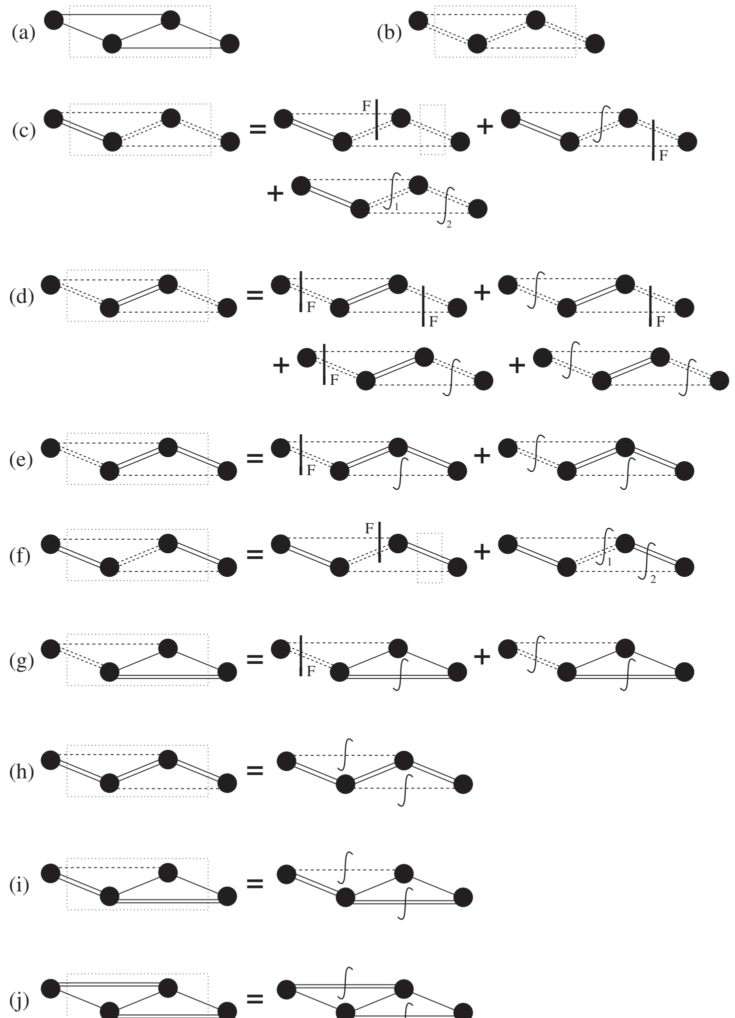


FIG. 18. Decomposition of $\mathcal{K}_{3,L}^{(4,u,u)}$. All external propagators are dropped, and the notation of Figs. 12 and 16 is used. (a) $\mathcal{K}_{3,L}^{(4,u,u)}$ itself [see Eq. (171)]; (b) the most singular term (with three singular propagators); (c) and (d): terms with two singular propagators and their decompositions; (e), (f), and (g): terms with one singular propagator and their decompositions; (h), (i), and (j): nonsingular terms. Terms in the decompositions are always ordered from most to least singular. The treatment of loop momenta is indicated explicitly: they are either summed (dashed box), integrated (integral sign) or the sum-minus-integral identity is used (factor of F). Where the order of integrals matters it is shown explicitly.

obtain the usual F term plus integral. The former gives rise to another contribution to $i\mathcal{K}_2 iF 2i\mathcal{K}_{\mathrm{df},3}^{(3,s,u)}$, while the latter contributes to $i\mathcal{K}_{\mathrm{df},3}^{(4,u,u)}$. Analogous results hold for the reflection of Fig. 18(e).

The diagram of Fig. 18(f) leads to a new effect. Here we can use the sum-integral identity either on $\vec{q}_1$ or $\vec{q}_2$. Our convention (as above) is to work from left to right when there is such a choice. This gives the singly singular term

$$\text{(f) [singly singular]} = 2i\mathcal{K}_{\mathrm{df},3}^{(2,s,u)} \frac{iF}{2\omega L^3} i\mathcal{K}_{\mathrm{df},3}^{(2,u,u)}, \tag{205}$$

where our convention has led to the (s) being on the left side of the F, rather than on the right. The nonsingular term contributes to $i\mathcal{K}_{\mathrm{df},3}^{(4,u,u)}$. Here our convention leads to a definite (left to right) ordering of the $\widetilde{\mathrm{PV}}$ integrals.

Another new feature of the $n = 4$ analysis is the appearance of singular contributions in which one of the

q_j^0 integrals does not circle the particle pole. The corresponding diagrams are Fig. 18(g) and its reflection. The decomposition exactly follows that of Fig. 18(e).

Finally, we reach the completely nonsingular contributions, where sums can be immediately converted to integrals. There are four such diagrams, Fig. 18(h), its reflection, Fig. 18(i), and Fig. 18(j). These all contribute to $i\mathcal{K}_{\mathrm{df},3}^{(4,u,u)}$.

Adding all contributions we find the total result

$$i\mathcal{K}_{3,L}^{(4,u,u)} = i\mathcal{K}_2 iG i\mathcal{K}_2 iG i\mathcal{K}_2[iG2\omega L^3]i\mathcal{K}_2 + i\mathcal{K}_2 iG[2\omega L^3]i\mathcal{K}_2 \frac{iF}{2\omega L^3} 2i\mathcal{K}_{\mathrm{df},3}^{(2,s,u)} + 2i\mathcal{K}_{\mathrm{df},3}^{(2,u,s)} \frac{iF}{2\omega L^3} i\mathcal{K}_2 iG[2\omega L^3]i\mathcal{K}_2 \\ + i\mathcal{K}_2 iF 4i\mathcal{K}_{\mathrm{df},3}^{(2,s,s)} iF i\mathcal{K}_2 + 2i\mathcal{K}_{\mathrm{df},3}^{(3,u,s)} iF i\mathcal{K}_2 + i\mathcal{K}_2 iF 2i\mathcal{K}_{\mathrm{df},3}^{(3,s,u)} + 2i\mathcal{K}_{\mathrm{df},3}^{(2,u,s)} \frac{iF}{2\omega L^3} i\mathcal{K}_{\mathrm{df},3}^{(2,u,u)} + i\mathcal{K}_{\mathrm{df},3}^{(4,u,u)}, \tag{206}$$

where we have ordered terms in decreasing strength of divergence. The only aspect of this result not explained above is that contributions combine properly to give the quantities $\mathcal{K}_{\mathrm{df},3}^{(3,u,s)}$ and $\mathcal{K}_{\mathrm{df},3}^{(3,s,u)}$ in the fifth and sixth terms, respectively. For example, the $\mathcal{K}_{\mathrm{df},3}^{(3,u,s)}$ term receives the required four contributions (see Fig. 16) from diagrams (c), (d), and the reflections of (e) and (g). One can demonstrate that the correct contributions occur in all cases by observing that (i) the result (206) provides a complete classification of possible divergence structures and (ii) that expanding out each term in (206) leads to a unique set of contributions each of which is necessarily present in the decomposition of $\mathcal{K}_{3,L}^{(4,u,u)}$. Finally, we note that the nonsingular term in Eq. (206), $\mathcal{K}_{\mathrm{df},3}^{(4,u,u)}$, is simply defined as the sum of contributions from all the diagrams in Fig. 18 (plus appropriate reflections) that contain only loop integrals.

We are now ready to explain the result for general $i\mathcal{K}_{3,L}^{(n,u,u)}$. What arises are sequences alternating between one of the $\mathcal{K}$s,

$$i\mathcal{K}_2,\ i\mathcal{K}_{\mathrm{df},3}^{(j,u,u)},\ 2i\mathcal{K}_{\mathrm{df},3}^{(j,s,u)},\ 2i\mathcal{K}_{\mathrm{df},3}^{(j,u,s)} \quad \text{and} \quad 4i\mathcal{K}_{\mathrm{df},3}^{(j,s,s)}, \tag{207}$$

and one of

$$\frac{iF}{2\omega L^3} \quad \text{and} \quad iG. \tag{208}$$

All possible combinations should be included, subject to the following rules:

(i) The number of switches must add up to n. This number is given by the total number of Fs and Gs plus the number of switches in the $\mathcal{K}_{\mathrm{df},3}$s.
(ii) There must be a $\mathcal{K}_2$ or $\mathcal{K}_{\mathrm{df},3}$ on both ends.
(iii) Each $\mathcal{K}_{\mathrm{df},3}$ must have F on both sides unless external. This is because the loop momenta next to a $\mathcal{K}_{\mathrm{df},3}$ have only one singular propagator in their summands and so the sum-integral identity can be used. This implies, given the rules above, that each G must have a $\mathcal{K}_2$ (and not a $\mathcal{K}_{\mathrm{df},3}$) on both sides.
(iv) Fs must have a $\mathcal{K}_{\mathrm{df},3}$ on at least one side, or, equivalently, Fs always appear on one side or other of a $\mathcal{K}_{\mathrm{df},3}$. This is because one cannot use the sum-integral identity in the middle of a sequence of singular propagators, since each loop sum runs over two singularities. The identity can only be used at the end of the sequence, and only then if it terminates with the nonsingular part of a propagator. An example of this rule is that Fig. 18(b) cannot be decomposed using the sum-integral identity, whereas Fig. 18(c) can at the left-hand end. A consequence of this rule is that the only long sequences involving $\mathcal{K}_2$ have the form $\ldots i\mathcal{K}_2 iG i\mathcal{K}_2 iG i\mathcal{K}_2 \ldots$. These correspond to diagrams with sequences of singular propagators.
(v) In a sequence of the form $\ldots i\mathcal{K}_2 iG i\mathcal{K}_2 iG i\mathcal{K}_2 \ldots$ the rightmost G is multiplied on the right by $[2\omega L^3]$. This arises from keeping track of on shell propagators.
(vi) The right-hand superscript of each $\mathcal{K}_{\mathrm{df},3}$ is (s) unless it is external, when it is a (u). Examples are the third, fifth, and seventh terms in the expression (206) for $\mathcal{K}_{3,L}^{(4,u,u)}$.
(vii) The middle superscript of each $\mathcal{K}_{\mathrm{df},3}$ is (s) unless it is either external or it appears to the right of another $\mathcal{K}_{\mathrm{df},3}$, separated by a single F, in which cases it is a (u). The difference from the previous rule arises due to our left-to-right convention of dealing with loop momenta. An example of the new exception is given by the penultimate term in Eq. (206).

A simple consequence of these rules is that the most divergent contribution to $i\mathcal{K}_{3,L}^{(n,u,u)}$ is

$$i\mathcal{K}_2(iGi\mathcal{K}_2)^{n-2} iG[2\omega L^3]i\mathcal{K}_2. \tag{209}$$

Similarly, sequences having this form (but with smaller values of n) can appear both connecting the ends to factors of $\mathcal{K}_{\mathrm{df},3}$, or between such factors.

Now do a lot of manipulations… [HS14]

$$C_{L,2} = C_{\infty,2} + iA_2' \frac{1}{1+F\mathcal{K}_2} FA_2. \tag{252}$$

The subscripts "2" on A, A', and C indicate that these are the two-particle end caps and correlator, while F is defined in Eq. (22) (although here we drop the spectator-momentum argument).

What we now show is that there are poles in A_2, A_2', and $C_{\infty,2}$, but these cancel in $C_{L,2}$. To see this we use the freedom to arbitrarily choose the interpolating functions σ and $\sigma^\dagger$ without affecting the position of poles in $C_{L,2}$. Specifically, we set both σ and $\sigma^\dagger$ equal to the two-particle Bethe-Salpeter kernel iB_2, which, we recall, is a smooth nonsingular function. One then finds that

$$C_{\infty,2} = i\mathcal{K}_2 - iB_2 \quad \text{and} \quad A_2 = A_2' = i\mathcal{K}_2. \tag{253}$$

Inserting these results into Eq. (252) we find that (for this choice of end caps)

$$\begin{aligned} C_{L,2} &= -iB_2 + i\mathcal{K}_2 + i\mathcal{K}_2 \frac{1}{1 - iFi\mathcal{K}_2} iFi\mathcal{K}_2 \\ &= -iB_2 + \frac{i}{\mathcal{K}_2^{-1} + F}. \end{aligned} \tag{254}$$

From Eqs. (253) and (254) we draw two conclusions. First, A_2, A_2', and $C_{\infty,2}$ have poles whenever $\mathcal{K}_2$ diverges. Such poles occur, for a given angular momentum, when $\delta_\ell = \pi/2 \bmod \pi$. Thus, using the $\widetilde{\mathrm{PV}}$ prescription, there are, in general, poles in A_2, A_2', and $C_{\infty,2}$. Second, these poles cancel in $C_{L,2}$, as shown by the second form in Eq. (254), which is clearly finite when $\mathcal{K}_2$ diverges.

We suspect that a similar result holds for the three-particle analysis, but have not yet been able to demonstrate this. Thus, in the three-particle case we must rely for now on the intuitive argument given above.

V. CONCLUSIONS AND OUTLOOK

In this work we have presented and derived a three-particle quantization condition relating the finite-volume spectrum to two-to-two and three-to-three infinite-volume scattering quantities. This condition separates the dependence on the volume into kinematic quantities, as was achieved previously for two particles.

There are two new features of the result compared to the two-particle case. First, the three-particle scattering quantity entering the quantization condition has the physical on shell divergences removed. The resulting divergence-free quantity is thus spatially localized. This is crucial for any practical application of the formalism since it allows for the partial-wave expansion to be truncated. Indeed, it is difficult to imagine a quantization condition involving the three-particle scattering amplitude itself, given that the latter is divergent for certain physical momenta.

The second feature is that the three-particle scattering quantity is nonstandard—it is not simply related to the (divergence-free part) of the physical scattering amplitude. This is because it is defined using the $\widetilde{\mathrm{PV}}$ pole prescription, and also because of the decorations explained in Sec. IV E. We strongly suspect, however, that a relation to the physical amplitude exists. In particular, we know from Ref. [13] that the finite-volume spectrum in a nonrelativistic theory can be determined solely in terms of physical amplitudes, and the same is true in the approximations adopted in Ref. [14]. We are actively investigating this issue.

The three-particle quantization condition involves a determinant over a larger space than that required for two particles. Nevertheless, as explained in Secs. III, because the three-particle quantity that enters has a uniformly convergent partial-wave expansion, one can make a consistent truncation of the quantization condition so that it involves only a finite number of parameters. This opens the way to practical application of the formalism.

We have provided in this paper two mild consistency checks on the formalism—that it correctly reproduces the known results if one particle is noninteracting (see Sec. IV A), and that the number of solutions to the quantization condition in the isotropic approximation is as expected (see Appendix C). We have also worked out a more detailed check by comparing our result close to the three-particle threshold $E^* \approx 3m$ to those obtained using nonrelativistic quantum mechanics [27,28]. Here one has an expansion in powers of $1/L$, and we have checked that the results agree for the first four nontrivial orders. This provides, in particular, a nontrivial check of the form of F_3, Eq. (19), and allows us to relate $\mathcal{K}_{\mathrm{df},3}$ to physical quantities in the nonrelativistic limit. We will present this analysis separately [29].

Two other issues are deferred to future work. First, we would like to understand in detail the relation of our formalism and quantization condition to those obtained in Refs. [13,14]. Second, we plan to test the formalism using simple models for the scattering amplitudes, in order to ascertain how best to use it in practice.

ACKNOWLEDGMENTS

We thank Raúl Briceño, Zohreh Davoudi, and Akaki Rusetsky for discussions. This work was supported in part by the U.S. Department of Energy Grants No. DE-FG02-96ER40956 and No. DE-SC0011637. M. T. H. was supported in part by the Fermilab Fellowship in Theoretical Physics. Fermilab is operated by Fermi Research Alliance, LLC, under Contract No. DE-AC02-07CH11359 with the United States Department of Energy.

APPENDIX A: SUM-MINUS-INTEGRAL IDENTITY

In this appendix we derive the sum-minus-integral identity that plays a central role in the main text. This identity is

...and obtain the final answer

[HS14]

$$C_\infty^{[B_2,\rho]} \equiv \sigma D_C^{[B_2,\rho]}\sigma^\dagger,$$
$$A'^{[B_2,\rho]} \equiv \sigma D_{A'}^{[B_2,\rho]}, \quad \text{and}$$
$$A^{[B_2,\rho]} \equiv D_A^{[B_2,\rho]}\sigma^\dagger. \tag{241}$$

These are infinite-volume integral operators defined implicitly by the work of previous subsections. This allows us to write Eq. (239) as

$$C_L^{[B_2]} = \sigma\{D_C^{[B_2,\rho]} + D_{A'}^{[B_2,\rho]}\mathcal{Z}D_A^{[B_2,\rho]}\}\sigma^\dagger. \tag{242}$$

The reason for using this notation is that it works also for segments of diagrams involving B_3s at the ends. Thus, for example, a segment of the finite-volume correlator between two B_3s can be written

$$\ldots B_3\{D_C^{[B_2,\rho]} + D_{A'}^{[B_2,\rho]}\mathcal{Z}D_A^{[B_2,\rho]}\}B_3\cdots. \tag{243}$$

The key point is that the same decoration operators appear as in (242).

We can now write down the result for the full finite-volume correlator

$$C_L = \sigma\{D_C^{[B_2,\rho]} + D_{A'}^{[B_2,\rho]}\mathcal{Z}D_A^{[B_2,\rho]}\}\sigma^\dagger + \sigma\{D_C^{[B_2,\rho]} + D_{A'}^{[B_2,\rho]}\mathcal{Z}D_A^{[B_2,\rho]}\}iB_3\{D_C^{[B_2,\rho]} + D_{A'}^{[B_2,\rho]}\mathcal{Z}D_A^{[B_2,\rho]}\}\sigma^\dagger$$
$$+ \sigma\{D_C^{[B_2,\rho]} + D_{A'}^{[B_2,\rho]}\mathcal{Z}D_A^{[B_2,\rho]}\}iB_3\{D_C^{[B_2,\rho]} + D_{A'}^{[B_2,\rho]}\mathcal{Z}D_A^{[B_2,\rho]}\}iB_3\{D_C^{[B_2,\rho]} + D_{A'}^{[B_2,\rho]}\mathcal{Z}D_A^{[B_2,\rho]}\}\sigma^\dagger + \cdots \tag{244}$$

As in the previous subsection, this can be reorganized into the form

$$C_L = C_\infty + \sum_{n=0}^{\infty} A'[\mathcal{Z}iB_3^{[B_2,\rho]}]^n\mathcal{Z}A \tag{245}$$

where

$$iB_3^{[B_2,\rho]} = \sum_{n=0}^{\infty} D_A^{[B_2,\rho]}[iB_3D_C^{[B_2,\rho]}]^n iB_3D_{A'}^{[B_2,\rho]}, \tag{246}$$

$$A' = \sum_{n=0}^{\infty}\sigma[D_C^{[B_2,\rho]}iB_3]^n D_{A'}^{[B_2,\rho]}, \tag{247}$$

$$A = \sum_{n=0}^{\infty} D_A^{[B_2,\rho]}[iB_3D_C^{[B_2,\rho]}]^n\sigma^\dagger, \tag{248}$$

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The latter three equations give the final forms of the end caps and the infinite-volume correlator, now including all factors of B_3.

We can now sum the geometric series in Eq. (245) and perform some simple algebraic manipulations to bring the result to its final form,

$$C_L = C_\infty + A'\frac{1}{1+F_3\mathcal{K}_{\mathrm{df},3}}iF_3A, \tag{250}$$

where

$$\mathcal{K}_{\mathrm{df},3} \equiv \mathcal{K}_{\mathrm{df},3}^{[B_2,\rho]} + B_3^{[B_2,\rho]} \tag{251}$$

is the full divergence-free three-to-three amplitude. Thus we have obtained our claimed result, Eq. (42), from which follows the quantization condition Eq. (18).

We close our derivation by returning to an issue raised in the introduction to this section, namely the possibility of poles in A, A', and C_∞. We argue that, while such poles can be present, they cannot contribute to the finite-volume spectrum, i.e. they do not lead to poles in C_L. Only solutions to the quantization condition (18) lead to poles in C_L.

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We can now write down the result for the full finite-volume correlator

$$C_L = \sigma\{D_C^{[B_2,\rho]} + D_{A'}^{[B_2,\rho]} \mathcal{Z} D_A^{[B_2,\rho]}\}\sigma^\dagger + \sigma\{D_C^{[B_2,\rho]} + D_{A'}^{[B_2,\rho]} \mathcal{Z} D_A^{[B_2,\rho]}\} iB_3 \{D_C^{[B_2,\rho]} + D_{A'}^{[B_2,\rho]} \mathcal{Z} D_A^{[B_2,\rho]}\}\sigma^\dagger$$
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$$C_L = C_\infty + iA_3'\frac{1}{1+F_3\mathscr{K}_{\mathrm{df},3}}F_3A_3$$

Infinite-volume quantities—cannot contain finite-volume poles

Thus QC3 is as stated earlier:

$$\det\left[F_3(E,\vec{P},L)^{-1} + \mathscr{K}_{\mathrm{df},3}(E^*)\right] = 0$$

Simpler derivation in recent review:
[Hansen & SS, 1901.00483]

Understanding the result

$$C_L - C_\infty = iA'_3 \left(F_3 - F_3 \mathscr{K}_{\mathrm{df},3} F_3 + F_3 \mathscr{K}_{\mathrm{df},3} F_3 \mathscr{K}_{\mathrm{df},3} F_3 - \ldots\right) A_3$$

Smooth, symmetric,
real infinite-volume
endcap
(details irrelevant)

Smooth, symmetric,
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quasi-local 3-particle
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All volume-dependence enters through F_3

Understanding the result

$$C_L - C_\infty = iA_3' \left(F_3 - F_3 \mathcal{K}_{\mathrm{df},3} F_3 + F_3 \mathcal{K}_{\mathrm{df},3} F_3 \mathcal{K}_{\mathrm{df},3} F_3 - \ldots\right) A_3$$

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Smooth, symmetric, real, infinite-volume amplitude: quasi-local 3-particle interaction

Smooth, symmetric, real infinite-volume endcap (details irrelevant)

All volume-dependence enters through F_3

$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\frac{1}{\mathcal{K}_2^{-1} + F + G}F\right]$$

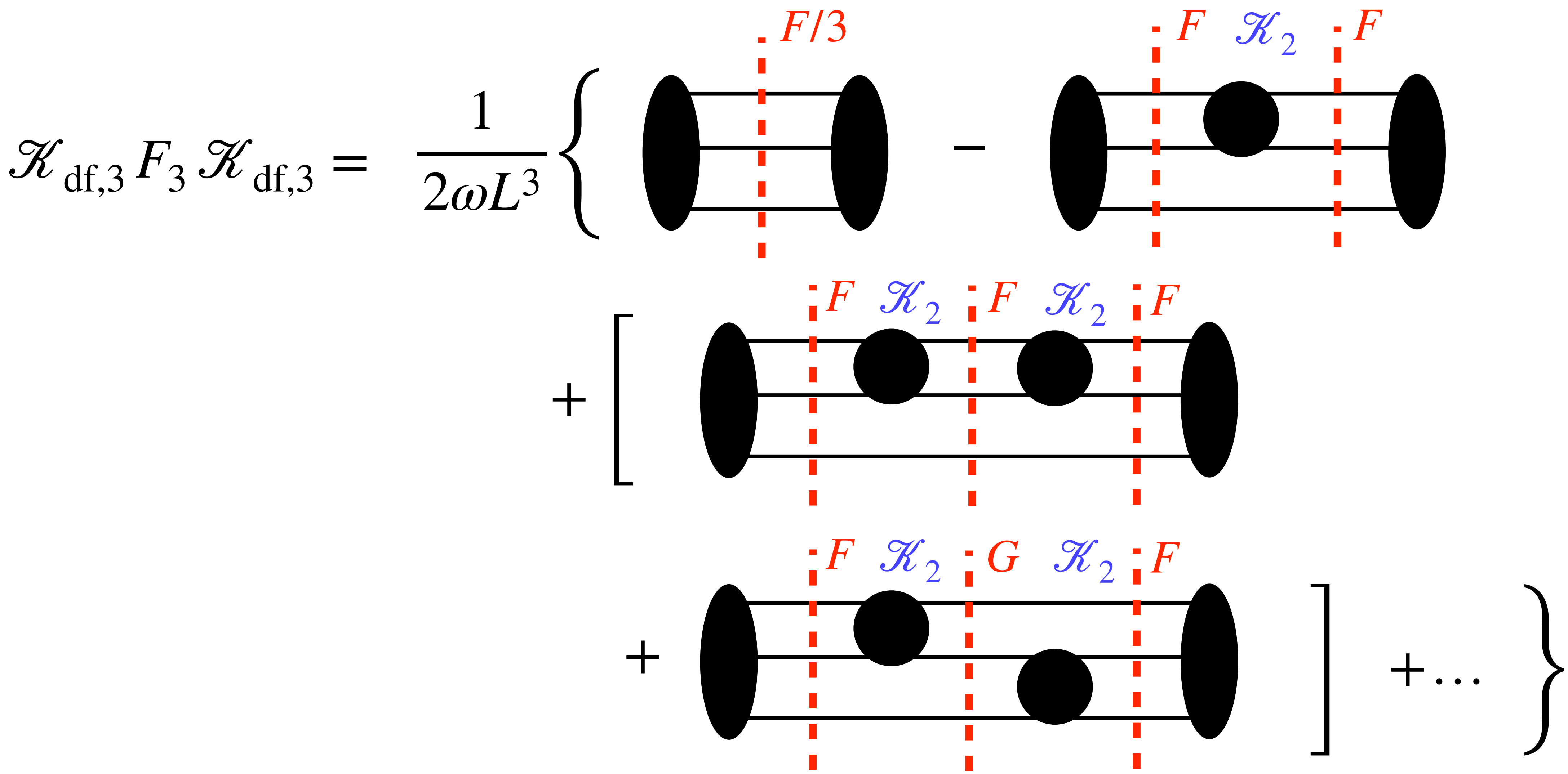
$$= \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\mathcal{K}_2\frac{1}{1 + (F+G)\mathcal{K}_2}F\right]$$

Another geometric series with alternating $\mathcal{K}_2$s and (F+G)s

Role of F_3

$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\mathcal{K}_2\frac{1}{1+(F+G)\mathcal{K}_2}F\right]$$

- Always lies between symmetric, infinite-volume objects, e.g.



- Sums up effects of 2→2 scattering with potentially on shell cuts between

Phew!

Phew!


Derivation is complicated, but final result is fairly simple and intuitive!

An old restriction & a new solution

- Original derivation of [HS14] requires that $\mathcal{K}_2$ be nonsingular, because singularities lead to additional (uncontrolled) finite-volume effects
  - This rules out two-particle bound states (dimers) & resonances
    - Physical $\mathcal{K}_2$ does not have dimer poles, but our modified version does
  - Major restriction on the application of the formalism, since most resonances with 3-particle decays have two-particle subchannel resonances, e.g.
    - $a_2(1320) \to \rho\pi \to 3\pi, \quad N(1440) \to \Delta\pi \to N\pi\pi, \quad Z_c(3900) \to \bar{D}D^* \to \bar{D}D\pi$

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- First solution: extend formalism by taking account of such singularities explicitly [Briceño, Hansen & SS, 2018]
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- New solution: modify PV prescription so that modified $\mathcal{K}_2$ entering QC3 does not have singularities [Blanton, Briceño, Hansen, Romero-López & SS, in progress]
  - Simple to implement; will show first results in final lecture

Infinite volume relation between $\mathcal{K}_{df,3}$ & $\mathcal{M}_3$

[HS15]

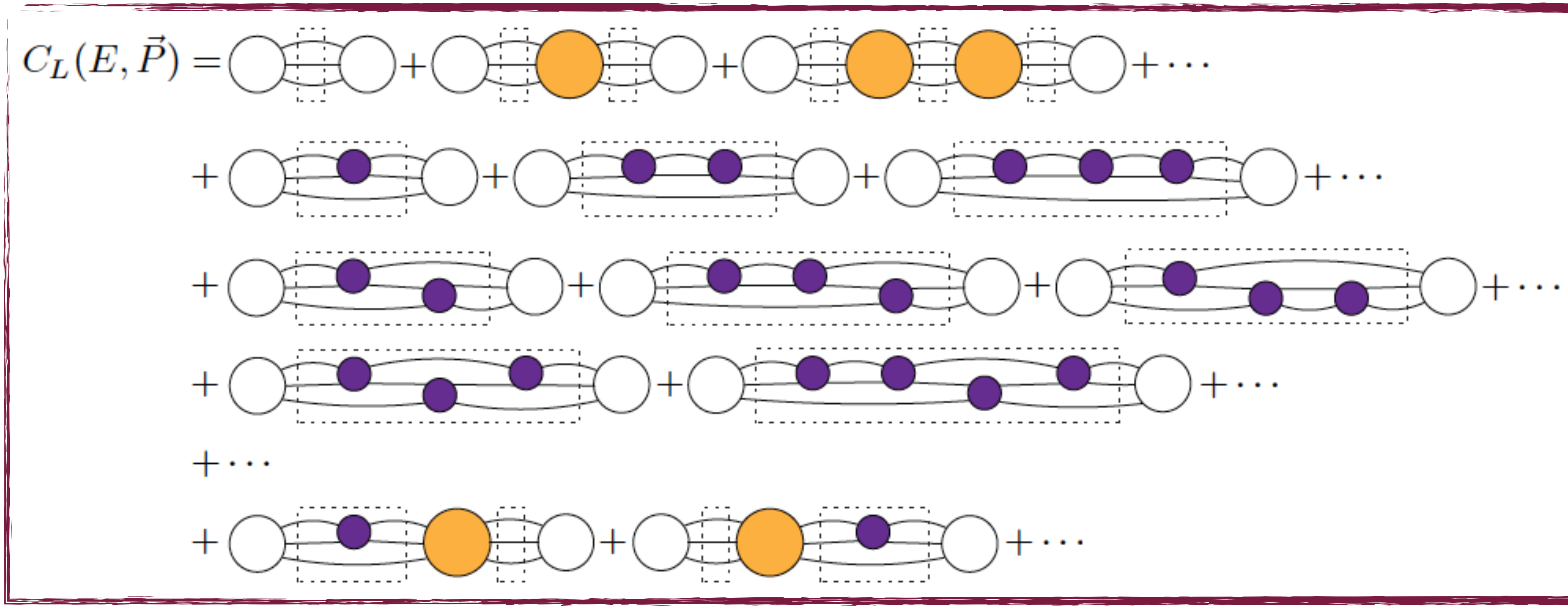
The issue

- Three particle quantization condition depends on $\mathcal{K}_{\mathrm{df},3}$ rather than the three particle scattering amplitude $\mathcal{M}_3$
- $\mathcal{K}_{\mathrm{df},3}$ is an infinite-volume quasi-local 3-particle amplitude, but is unphysical
  - Has a very complicated, unwieldy definition
  - Depends on the cut-off function H
  - It was forced on us by the analysis
- To complete the formalism we must relate $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

The method

- Define a "finite volume scattering amplitude" $\mathcal{M}_{L,3}$ which goes over to $\mathcal{M}_3$ in an (appropriately taken) $L \to \infty$ limit
- Relate $\mathcal{M}_{L,3}$ to $\mathcal{K}_{df,3}$ at finite volume—which turns out to require only a small generalization of the methods used to derive the quantization condition
- Take $L \to \infty$, obtaining nested integral equations

Modifying C_L to obtain $\mathcal{M}_{L,3}$



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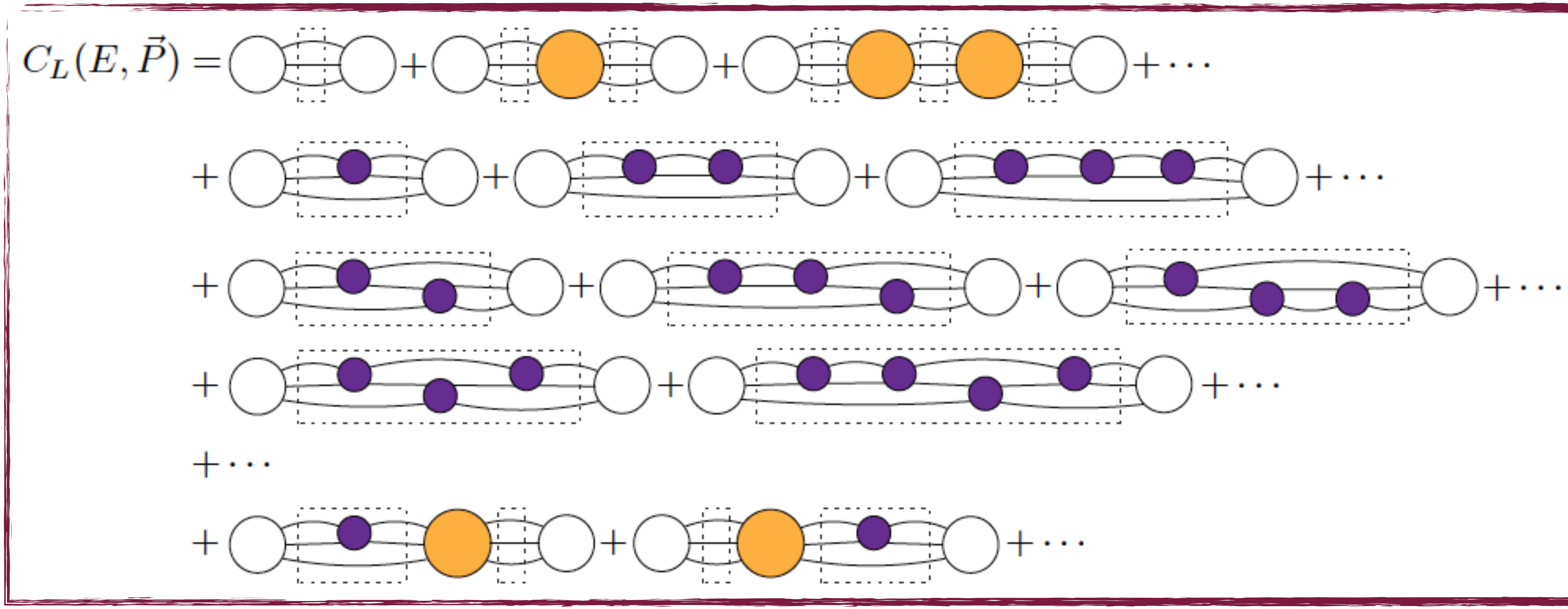
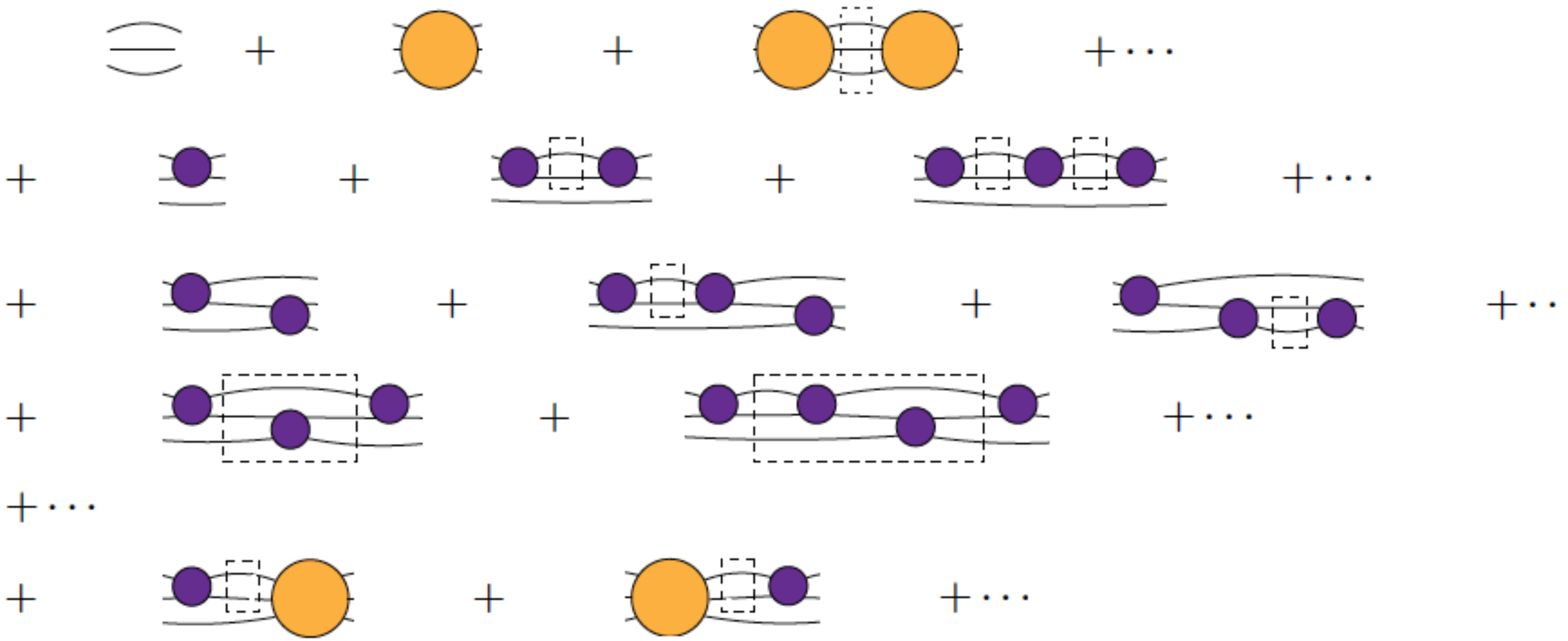


Step 1: "amputate"

Modifying C_L to obtain $\mathcal{M}_{L,3}$



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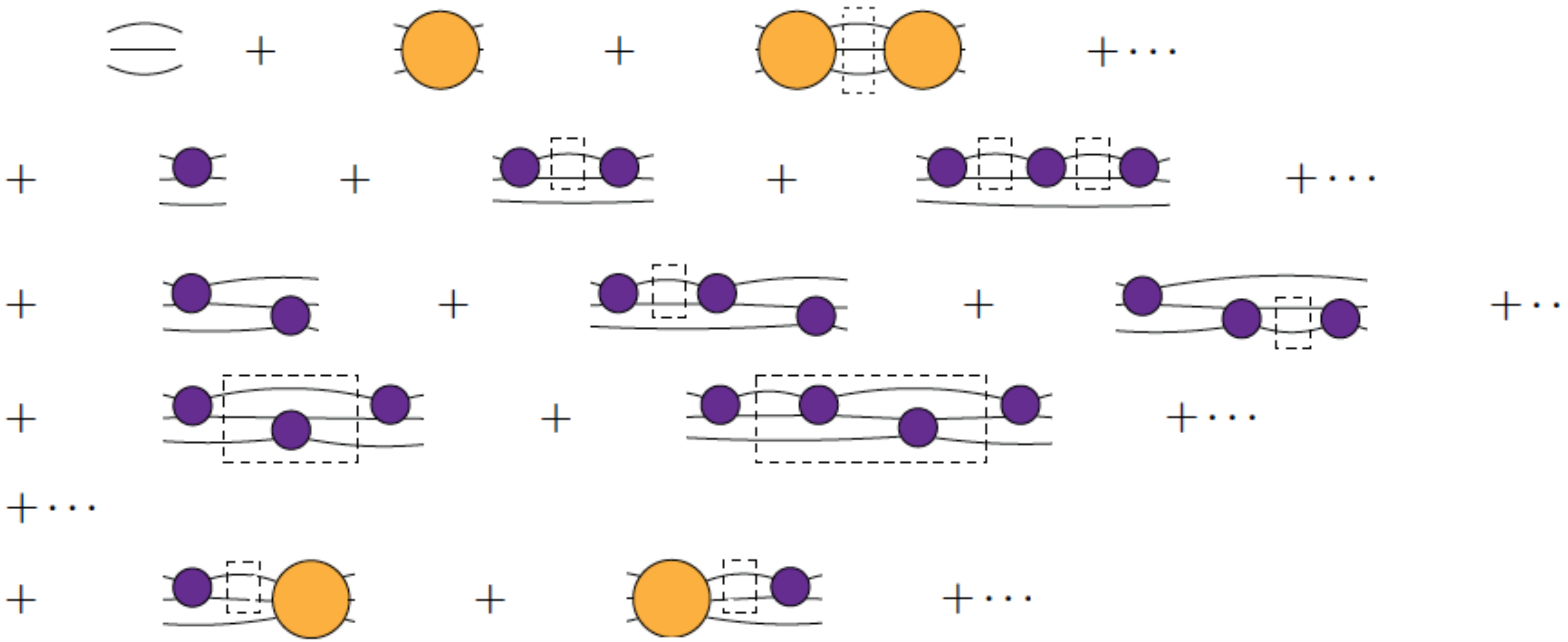

Step 2: Drop disconnected diagrams

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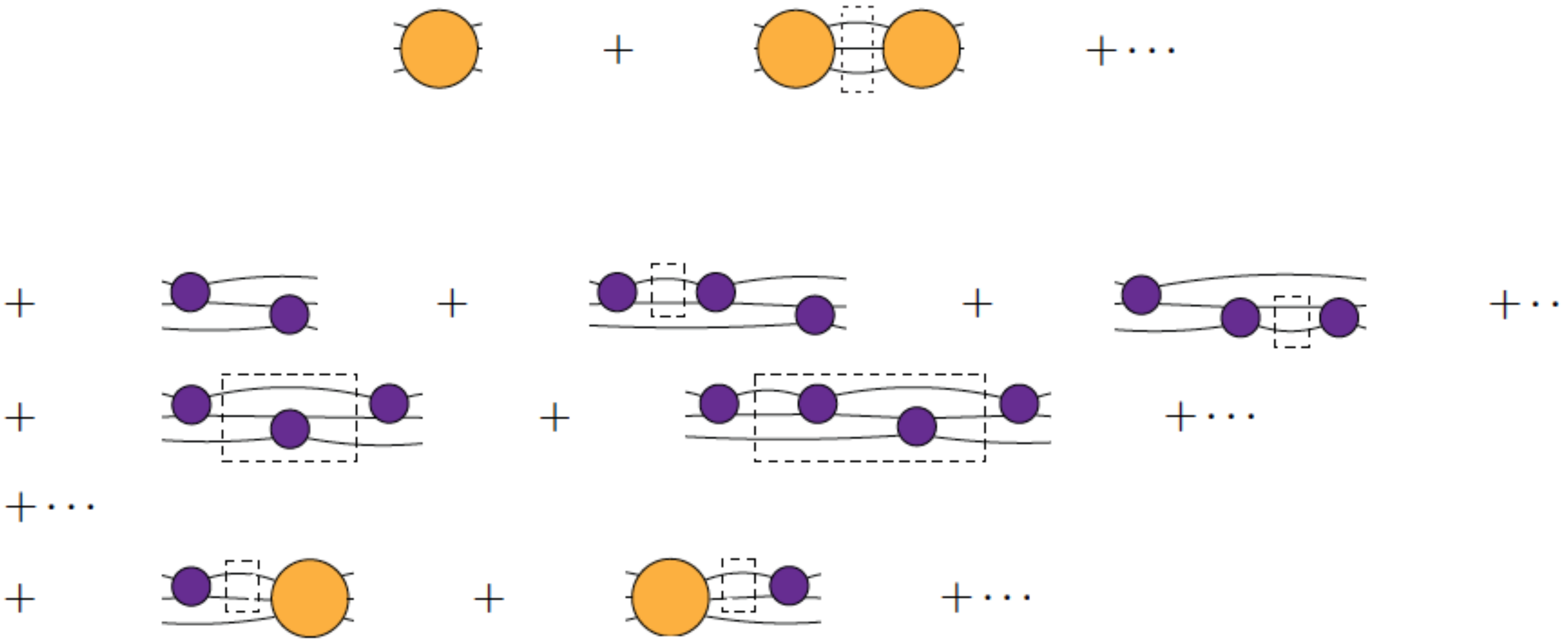

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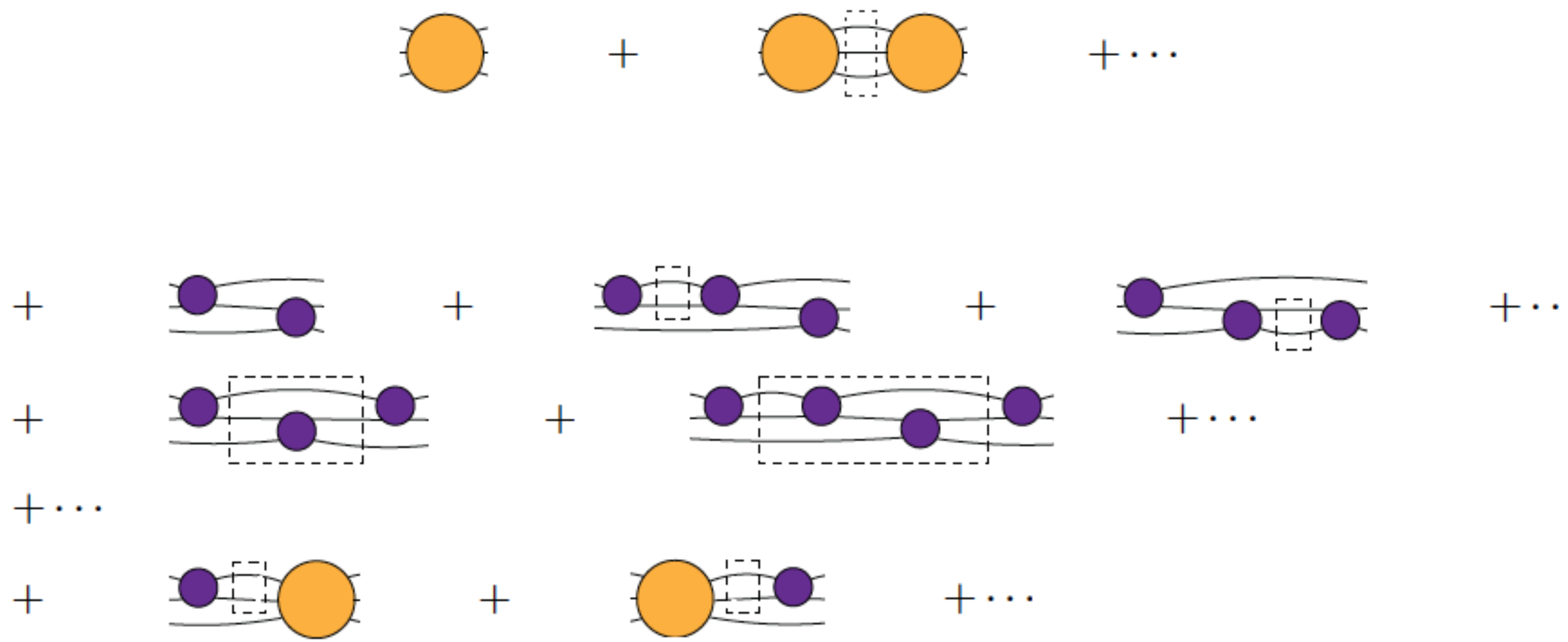
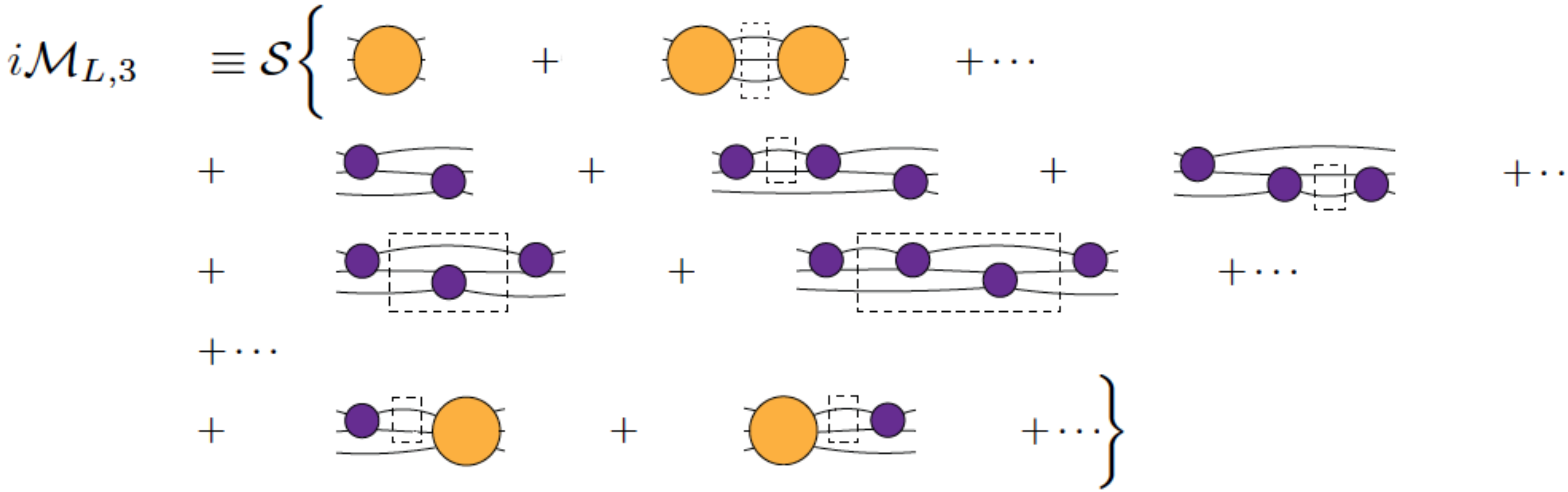


Step 3: Symmetrize

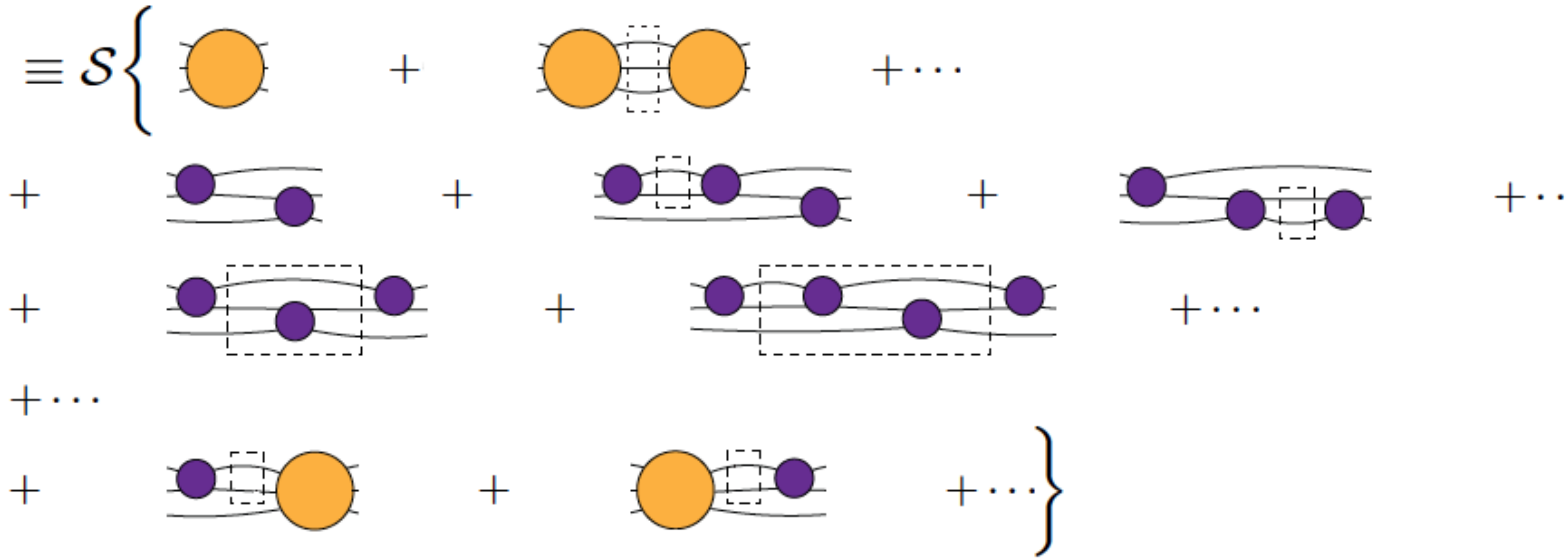
Modifying C_L to obtain $\mathcal{M}_{L,3}$



Step 3: Symmetrize

Modifying C_L to obtain $\mathcal{M}_{L,3}$

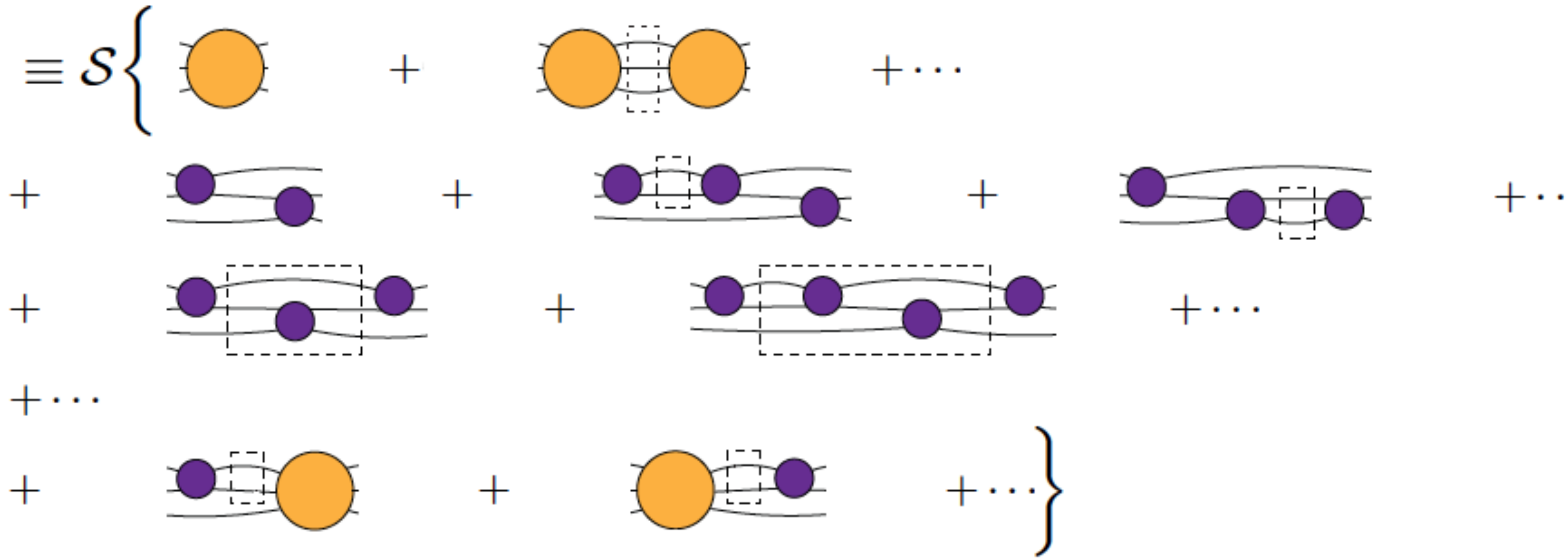
$$i\mathcal{M}_{L,3} \equiv$$



Allows one to obtain $\mathcal{M}_{L,3}$ from expression from C_L

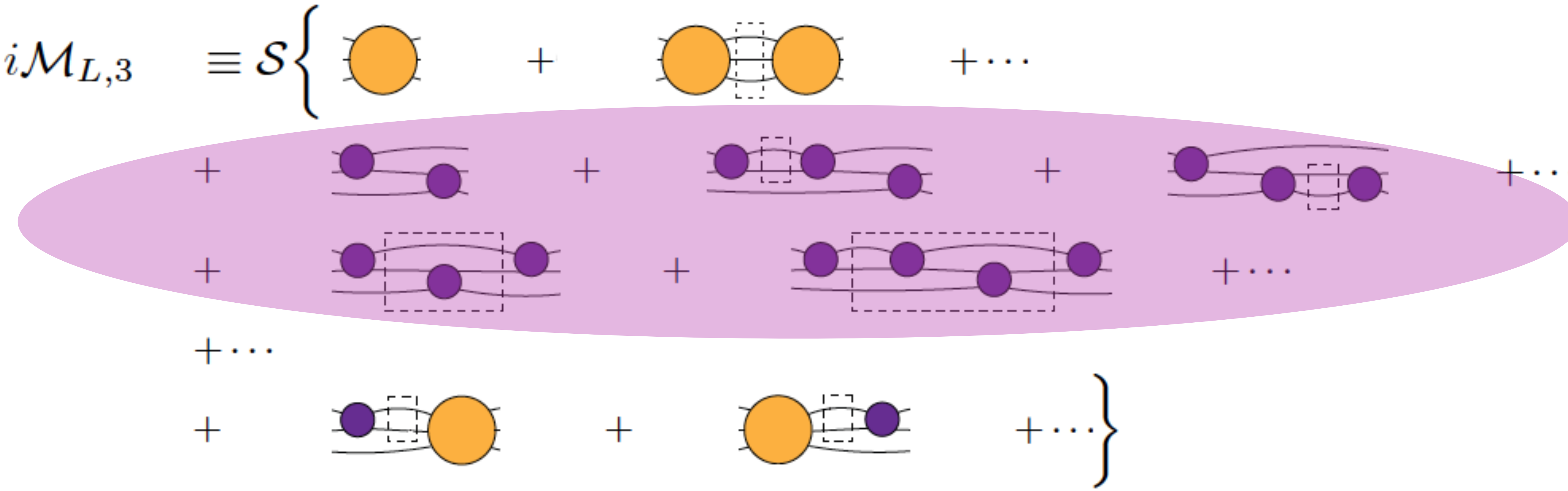
$\mathcal{M}_{L,3}$ in terms of $\mathcal{K}_{df,3}$

$$i\mathcal{M}_{L,3} \equiv \mathcal{S}\Big\{$$



$$C_L = C_\infty + iA'_3 \frac{1}{1+F_3\mathscr{K}_{\mathrm{df},3}} F_3 A_3 \quad \Rightarrow \quad \mathscr{M}_{L,3} = \mathscr{D}_L + \mathscr{S}\left[\mathscr{L}_L \mathscr{K}_{\mathrm{df},3} \frac{1}{1+F_3\mathscr{K}_{\mathrm{df},3}} \mathscr{L}_L^\dagger\right]$$

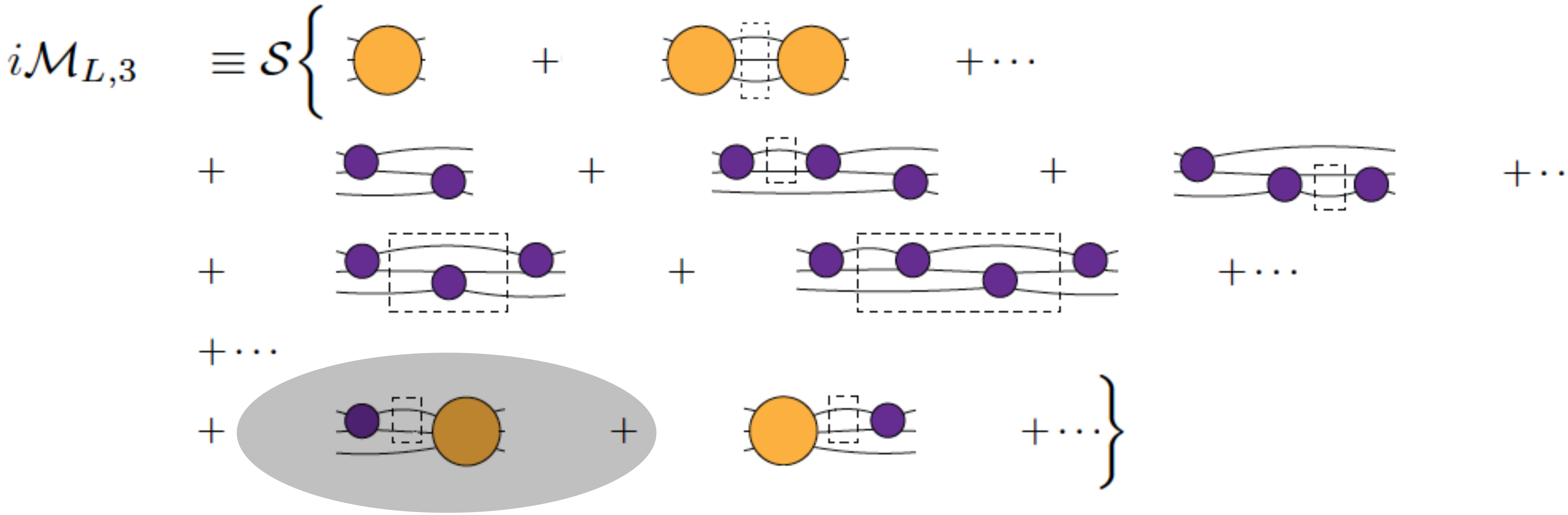
$\mathcal{M}_{L,3}$ in terms of $\mathcal{K}_{df,3}$



$$C_L = C_\infty + iA_3' \frac{1}{1 + F_3 \mathcal{K}_{\mathrm{df},3}} F_3 A_3 \quad \Rightarrow \quad \mathcal{M}_{L,3} = \mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \mathcal{K}_{\mathrm{df},3} \frac{1}{1 + F_3 \mathcal{K}_{\mathrm{df},3}} \mathcal{L}_L^\dagger \right]$$

$$\mathcal{D}_L = -\,\mathcal{S}\left[\frac{1}{1 + \mathcal{M}_{L,2} G} \mathcal{M}_{L,2} G \mathcal{M}_{L,2} (2\omega L^3) \right]$$

$\mathcal{M}_{L,3}$ in terms of $\mathcal{K}_{df,3}$



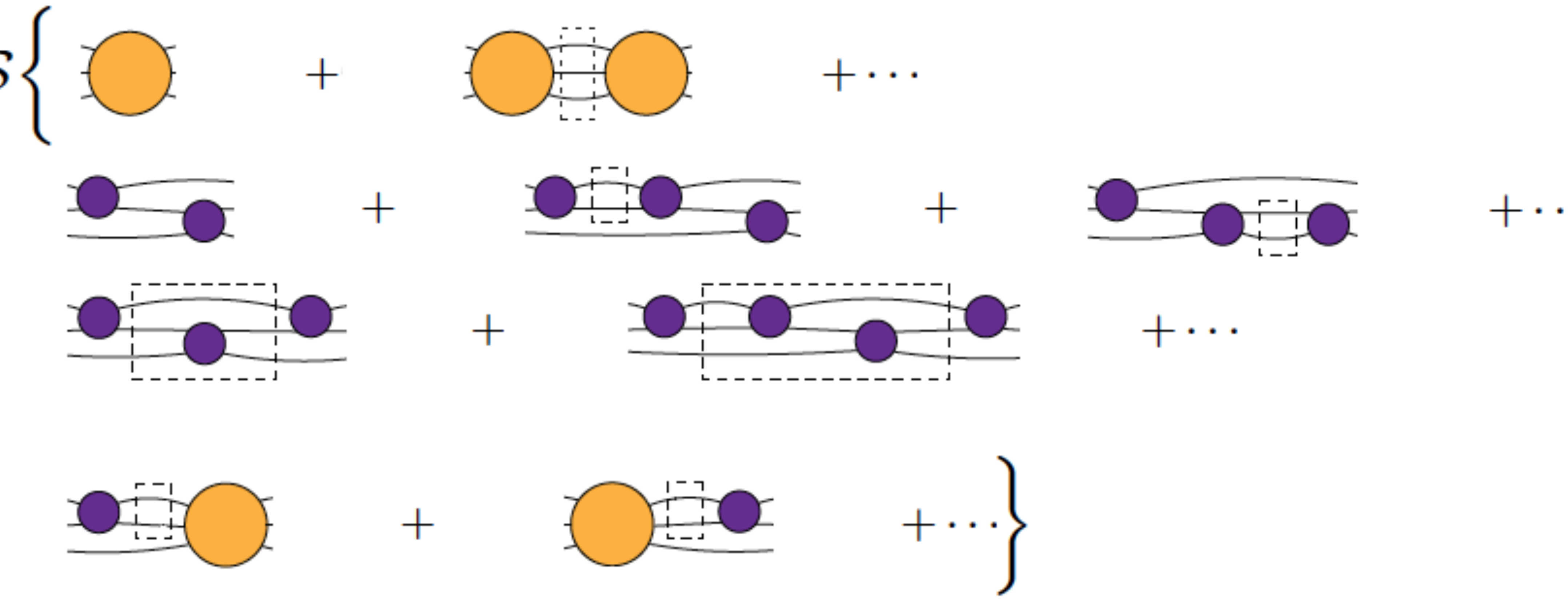
$$C_L = C_\infty + iA_3' \frac{1}{1+F_3\mathcal{K}_{\mathrm{df},3}} F_3 A_3 \quad \Rightarrow \quad \mathcal{M}_{L,3} = \mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \mathcal{K}_{\mathrm{df},3}\frac{1}{1+F_3\mathcal{K}_{\mathrm{df},3}}\mathcal{L}_L^\dagger\right]$$

$$\mathcal{D}_L = -\mathcal{S}\left[\frac{1}{1+\mathcal{M}_{L,2}G}\mathcal{M}_{L,2}G\mathcal{M}_{L,2}(2\omega L^3)\right]$$

$$\mathcal{L}_L = \frac{1}{3} - \frac{1}{1+\mathcal{M}_{L,2}G}\mathcal{M}_{L,2}F$$

$\mathcal{M}_{L,3}$ in terms of $\mathcal{K}_{df,3}$

$i\mathcal{M}_{L,3} \equiv \mathcal{S}$



$$C_L = C_\infty + iA_3' \frac{1}{1 + F_3 \mathcal{K}_{\mathrm{df},3}} F_3 A_3 \quad \Rightarrow \quad \mathcal{M}_{L,3} = \mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \mathcal{K}_{\mathrm{df},3} \frac{1}{1 + F_3 \mathcal{K}_{\mathrm{df},3}} \mathcal{L}_L^\dagger\right]$$

$$\mathcal{D}_L = -\,\mathcal{S}\left[\frac{1}{1+\mathcal{M}_{L,2}G}\mathcal{M}_{L,2}G\mathcal{M}_{L,2}(2\omega L^3)\right] \qquad \mathcal{L}_L = \frac{1}{3} - \frac{1}{1+\mathcal{M}_{L,2}G}\mathcal{M}_{L,2}F$$

$$\mathcal{M}_{L,2} = \mathcal{K}_2 \frac{1}{1 + F\mathcal{K}_2}$$

Finite-volume 2-particle scattering amplitude: appears in 2-particle C_L

$\mathcal{M}_{L,3}$ in terms of $\mathcal{K}_{df,3}$

$$C_L = C_\infty + iA_3' \frac{1}{1+F_3 \mathscr{K}_{\text{df},3}} F_3 A_3 \quad \Rightarrow \quad \mathscr{M}_{L,3} = \mathscr{D}_L + \mathscr{S}\left[\mathscr{L}_L \mathscr{K}_{\text{df},3} \frac{1}{1+F_3 \mathscr{K}_{\text{df},3}} \mathscr{L}_L^\dagger\right]$$

$$\mathscr{D}_L = -\mathscr{S}\left[\frac{1}{1+\mathscr{M}_{L,2}G} \mathscr{M}_{L,2} G \mathscr{M}_{L,2}(2\omega L^3)\right] \qquad \mathscr{L}_L = \frac{1}{3} - \frac{1}{1+\mathscr{M}_{L,2}G} \mathscr{M}_{L,2} F$$

$$\mathscr{M}_{L,2} = \mathscr{K}_2 \frac{1}{1+F\mathscr{K}_2}$$

- Key point: the ***same (ugly)*** $\mathcal{K}_{df,3}$ appears in $\mathcal{M}_{L,3}$ as in C_L
- Indeed, can use $\mathcal{M}_{L,3}$ to derive QC3

Final step: taking L→∞

$$\mathcal{M}_{L,3} = \mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \mathcal{K}_{\mathrm{df},3} \frac{1}{1 + F_3 \mathcal{K}_{\mathrm{df},3}} \mathcal{L}_L^\dagger\right]$$

$$\mathcal{D}_L = -\mathcal{S}\left[\frac{1}{1+\mathcal{M}_{L,2}G}\mathcal{M}_{L,2}G\mathcal{M}_{L,2}(2\omega L^3)\right]$$

$$\mathcal{L}_L = \frac{1}{3} - \frac{1}{1+\mathcal{M}_{L,2}G}\mathcal{M}_{L,2}F$$

$$\mathcal{M}_{L,2} = \mathcal{K}_2 \frac{1}{1 + F\mathcal{K}_2}$$

$$F_3 = \frac{F}{2\omega L^3}\left[\frac{1}{3} - \frac{1}{1+\mathcal{M}_{L,2}G}\mathcal{M}_{L,2}F\right]$$

- All equations involve matrices with indices *k, l, m*

Spectator momentum
$\mathbf{k} = 2\,\mathbf{n}\,\pi / L$
Summed over $\mathbf{n}$

l, m already infinite-volume variables

Final step: taking L→∞

$$\mathscr{M}_{L,3} = \mathscr{D}_L + \mathcal{S}\left[\mathscr{L}_L \mathscr{K}_{\mathrm{df},3} \frac{1}{1 + F_3 \mathscr{K}_{\mathrm{df},3}} \mathscr{L}_L^{\dagger}\right]$$

$$\mathscr{D}_L = -\,\mathcal{S}\left[\frac{1}{1+\mathscr{M}_{L,2}G}\mathscr{M}_{L,2}G\mathscr{M}_{L,2}(2\omega L^3)\right]$$

$$\mathscr{L}_L = \frac{1}{3} - \frac{1}{1+\mathscr{M}_{L,2}G}\mathscr{M}_{L,2}F$$

$$\mathscr{M}_{L,2} = \mathscr{K}_2 \frac{1}{1 + F\mathscr{K}_2}$$

$$F_3 = \frac{F}{2\omega L^3}\left[\frac{1}{3} - \frac{1}{1+\mathscr{M}_{L,2}G}\mathscr{M}_{L,2}F\right]$$

- L→∞: Sums over momenta → integrals (+ now irrelevant 1/L terms!)
- Must introduce pole prescription for sums to avoid singularities

$$\mathscr{M}_3 = \lim_{L\to\infty}\Big|_{i\epsilon}\mathscr{M}_{L,3}$$

Final result: nested integral equations

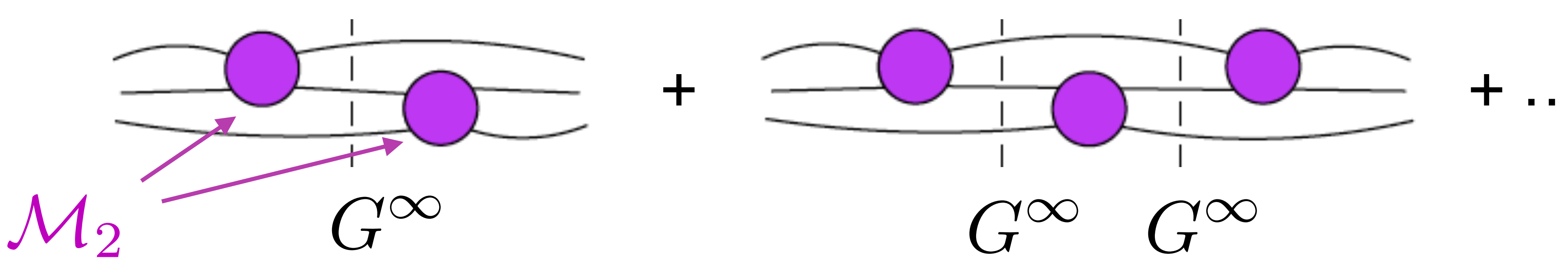
(1) Obtain $L\to\infty$ limit of $\mathcal{D}_L$ $\qquad \mathscr{D} = \mathscr{S}\{\mathscr{D}^{(u,u)}\}$

$$\mathscr{D}^{(u,u)}(\vec{p},\vec{k}) = -\mathscr{M}_2(\vec{p})G^\infty(\vec{p},\vec{k})\mathscr{M}_2(\vec{k}) - \int_s \frac{1}{2\omega_s}\mathscr{M}_2(\vec{p})G^\infty(\vec{p},\vec{x})\mathscr{D}^{(u,u)}(\vec{s},\vec{k})$$

$$G_{p\ell'm';k\ell m} = \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell'm'}(\hat{k}^*)H(\vec{p})H(\vec{k})Y^*_{\ell m}(\hat{p}^*)}{(P-k-p)^2-m^2+i\epsilon}\left(\frac{p^*}{q_k^*}\right)^{\ell}$$

N.B. iε prescription

- Quantities are still matrices in *l,m* space
- Presence of cut-off function means that integrals have finite range
- $\mathcal{D}^{(u,u)}$ sums geometric series which gives physical divergences in $\mathcal{M}_3$



Final result: nested integral equations

(2) Sum geometric series involving $\mathcal{K}_{\text{df},3}$

$$i\mathcal{T}(\vec{p},\vec{k}) = i\mathcal{K}_{\text{df},3}(\vec{p},\vec{k}) + \int_s\int_r i\mathcal{K}_{\text{df},3}(\vec{p},\vec{s})\frac{i\rho(\vec{s})}{2\omega_s}i\mathcal{L}^{(u,u)}(\vec{s},\vec{r})i\mathcal{T}(\vec{r},\vec{k})\,,$$

$$\mathcal{L}^{(u,u)}(\vec{p},\vec{k}) = \left(\frac{1}{3} + i\mathcal{M}_2(\vec{p})i\rho(\vec{p})\right)(2\pi)^3\delta^3(\vec{p}-\vec{k}) + i\mathcal{D}^{(u,u)}(\vec{p},\vec{k})\frac{i\rho(\vec{k})}{2\omega_k}\,,$$

- $\rho(\mathbf{k})$ is a phase space factor (analytically continued when below threshold)
- Requires $\mathcal{D}^{(u,u)}$ and $\mathcal{M}_2$
- Corresponds to summing the core geometric series, i.e.

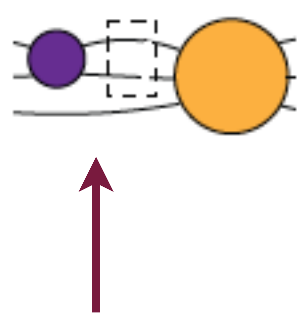
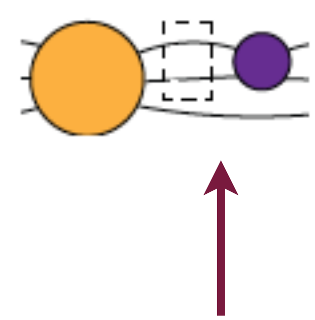
$$\mathcal{K}_{\text{df},3}\frac{1}{1+F_3\mathcal{K}_{\text{df},3}}$$

Final result: nested integral equations

(3) Add in effects of external 2→2 scattering:

$$\underbrace{\mathcal{M}_3(\vec{p},\vec{k}) - \mathcal{S}\left\{\mathcal{D}^{(u,u)}(\vec{p},\vec{k})\right\}}_{\mathcal{M}_{df,3}} = -\mathcal{S}\left\{\int_s \int_r \mathcal{L}^{(u,u)}(\vec{p},\vec{s})\mathcal{T}(\vec{s},\vec{r})\mathcal{R}^{(u,u)}(\vec{r},\vec{k})\right\}$$

- Sums geometric series on "outside" of $\mathcal{K}_{df,3}$'s

$\lim_{L\to\infty}\Big\{$  $+$  $+\cdots \Big\}$

- Can also formally invert and determine $\mathcal{K}_{df,3}$ given $\mathcal{M}_3$ and $\mathcal{M}_2$
  - This is how one demonstrates the symmetry properties of $\mathcal{K}_{df,3}$

Comments on "K to $\mathcal{M}$" relation

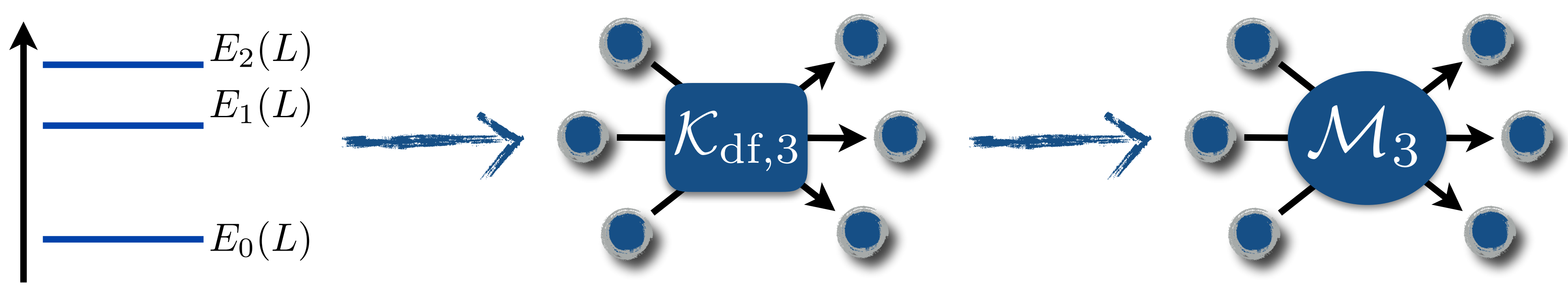
- Integral equations are similar to those arising when solving Dyson-Schwinger eqs
  - Easy to solve for $E^* \lesssim 3m$, where no pole prescription is needed
- Result provides a parametrization of $\mathcal{M}_3$ in terms of a real K matrix
  - May be useful for amplitude analyses of experimental data (e.g. from JLab)
  - Very recently we checked explicitly that the parametrization leads to a unitary $\mathcal{M}_3$, as expected [Briceño, Hansen, SS & Szczepaniak, 2019]
  - Also determined the relation between $\mathcal{K}_{df,3}$ and the corresponding quantity ("B matrix") appearing in another parametrization of $\mathcal{M}_3$ used by JPAC (Joint Physics Analysis Center @ JLab/Indiana) [Jackura, …,SS, *et al.*, 2019]

Summary of Lecture 3

Summary

- QC3 for identical scalars with G-parity-like Z_2 symmetry [HS14,HS15]
  - Subchannel resonances allowed by modifying PV prescription [BBHRS, in progress]

$$\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$



Thank you!
Questions?