Resonances from lattice QCD: Lecture 2

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Outline

☑ Lecture 1

- Motivation/Background/Overview

☐ Lecture 2

- Overview of problem
- Deriving the two-particle quantization condition (QC2)
- Examples of applications

☐ Lecture 3

- Sketch of the derivation of the three-particle quantization condition (QC3)

☐ Lecture 4

- Applications of QC3
- Summary of topics not discussed and open issues

Main references for these lectures

- Briceño, Dudek & Young, "Scattering processes & resonances from LQCD," 1706.06223, RMP 2018
- Hansen & SS, "LQCD & three-particle decays of resonances," 1901.00483, to appear in ARNPS
- Lectures by Dudek, Hansen & Meyer at HMI Institute on "Scattering from the lattice: applications to phenomenology and beyond," May 2018, https://indico.cern.ch/event/690702/
- Lüscher, Commun.Math.Phys. 105 (1986) 153-188; Nucl.Phys. B354 (1991) 531-578 & B364 (1991) 237-251 (foundational papers)
- Kim, Sachrajda & SS [KSS05], hep-lat/0507006, NPB 2015 (direct derivation in QFT of QC2)
- Hansen & SS [HS14, HS15], 1408.5933, PRD14 & 1504.04248, PRD15 (derivation of QC3 in QFT)
- Briceño, Hansen & SS [BHS17], 1701.07465, PRD17 (including 2↔3 processes in QC3)
- Briceño, Hansen & SS [BHS18], 1803.04169, PRD18 (numerical study of QC3 in isotropic approximation)
- Briceño, Hansen & SS [BHS19], 1810.01429, PRD19 (allowing resonant subprocesses in QC3)
- Blanton, Romero-López & SS [BRS19], 1901.07095, JHEP19 (numerical study of QC3 including d waves)
- Blanton, Briceño, Hansen, Romero-López & SS, in progress, poster at Lattice 2019

Other references for this lecture

- Rummukainen & Gottlieb, hep-lat/9503028, Nucl. Phys B 1995 (generalized QC2 to moving frames)
- Lellouch & Lüscher, hep-lat/0003023, Comm.Math.Phys 01 ($K \to \pi\pi$ amplitude from FV matrix element)
- He, Feng & Liu, hep-lat/0504019, JHEP05 (multiple-channel generalization of QC2 in QM)
- Lage, Meissner & Rusetsky, 0905.0069, PLB09 (multiple-channel generalization of QC2 in NREFT)
- Meyer, 1105.1892, PRL11 (method for timeline pion form factor)
- Hansen & SS, 1204.0826 , PRD12 (multiple-channel generalization of QC2 and LL in QFT)
- Briceño & Davoudi, 1204.1110 , PRD12 (multiple-channel generalization of QC2 in QFT)
- Briceño, 1401.3312 [Bric14], PRD14 (QC2 with particles of arbitrary spin)
- Agadjanov, Bernard, Meissner & Rusetsky, 1405.3476 , NPB14 (method for photoproduction of Δ)
- Briceño, Hansen & Walker-Loud, 1406.5965, PRD15 (general derivation of LL factor)
- Briceño & Hansen, 1502.04314 , PRD15 (LL for arbitrary spin)
- Briceño & Hansen, 1509.08507, PRD15 ; Baroni, Briceño, Hansen & Ortega-Gama, 1812.10504 (EM form factor of the ρ)

HALQCD method

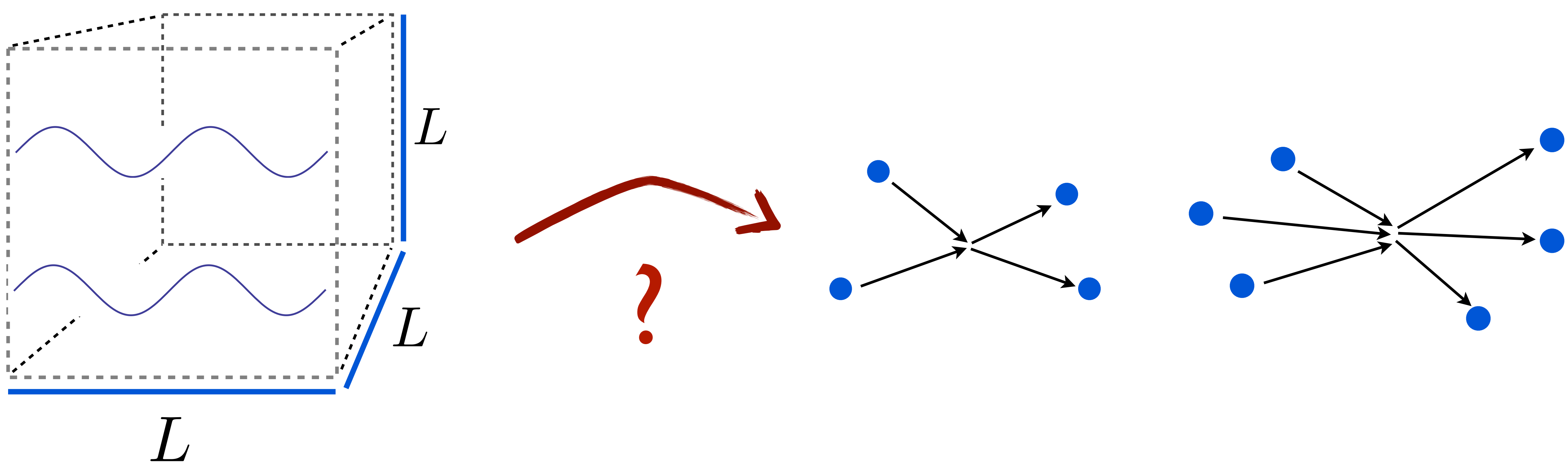
[Aoki, Hatsuda & Ishii, 0909.5585; Ishii *et al.*, 1203.3642, PLB 2012; … ; Aoki lectures]

- I will describe the "Lüscher approach" in these lectures
- There is an alternative approach, introduced by the HALQCD collaboration [S. Aoki et al.], which uses the Bethe-Salpeter wave-function calculated with LQCD to determine a two-particle "potential" from which one can determine scattering amplitudes and bound-state energies
- It is a fully relativistic method (like that I describe)
- It is potentially more powerful than the Lüscher approach, but in practice requires, to date, certain assumptions (truncation of derivative expansion)
- It has been widely applied to two-baryon systems, where the Lüscher approach has challenges due to poor signal/noise
- For meson resonances applications of the Lüscher approach are more advanced

Overview of problem

The fundamental issue

- Lattice QCD can calculate energy levels of multiple particle systems in a box
- How are these related to scattering amplitudes?



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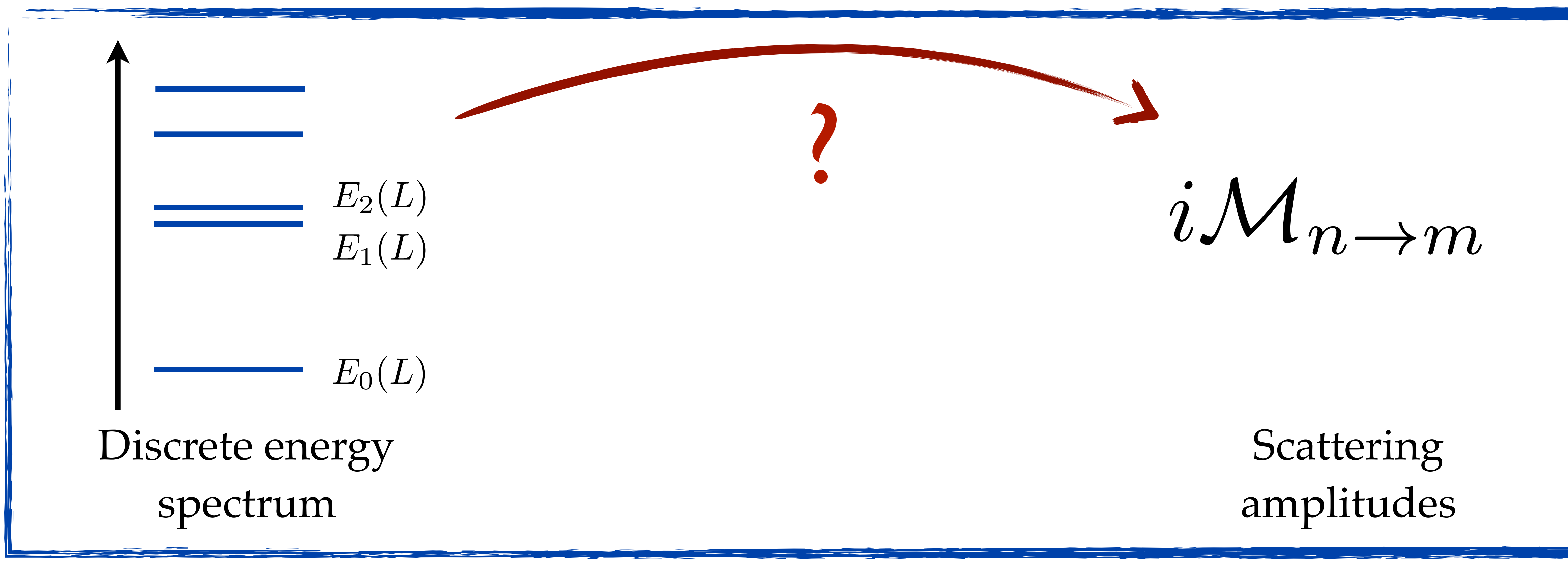
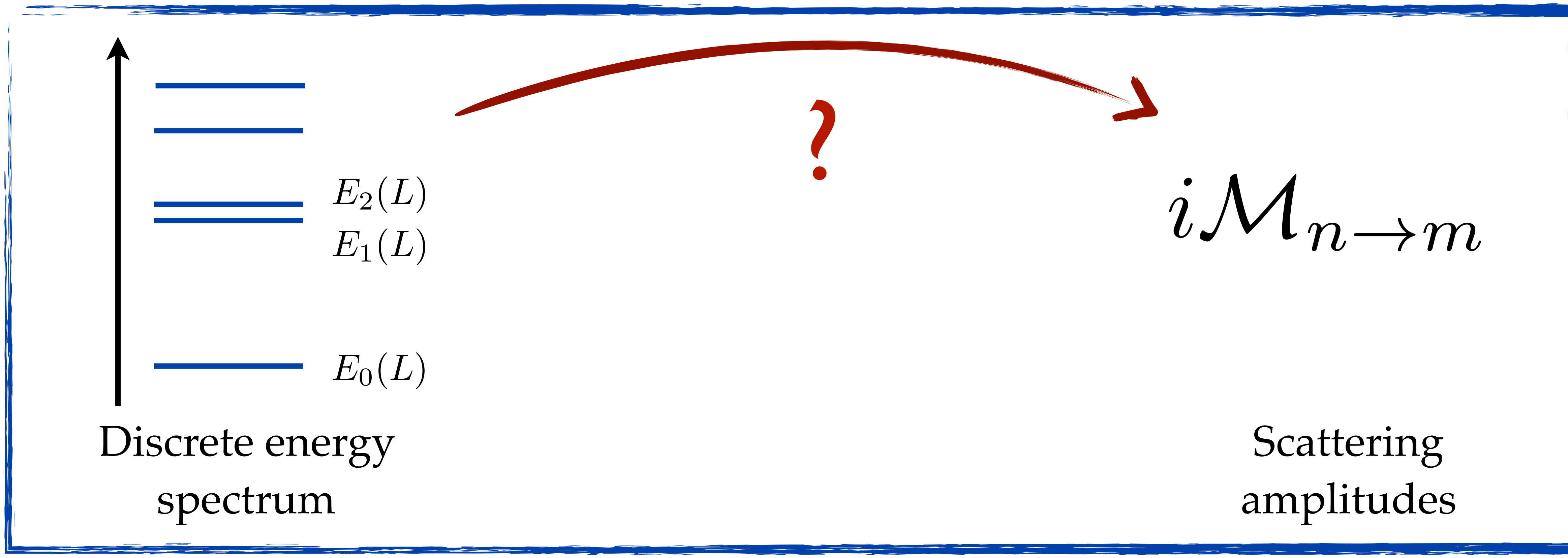
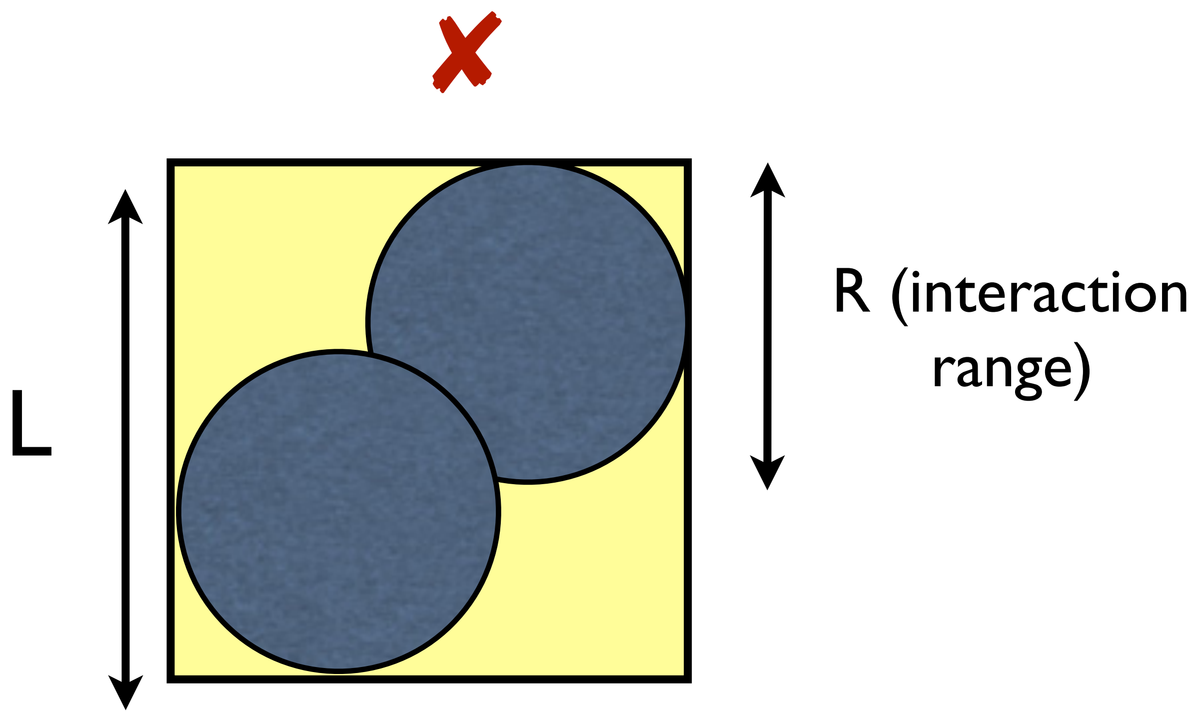


Problem in finite-volume QFT

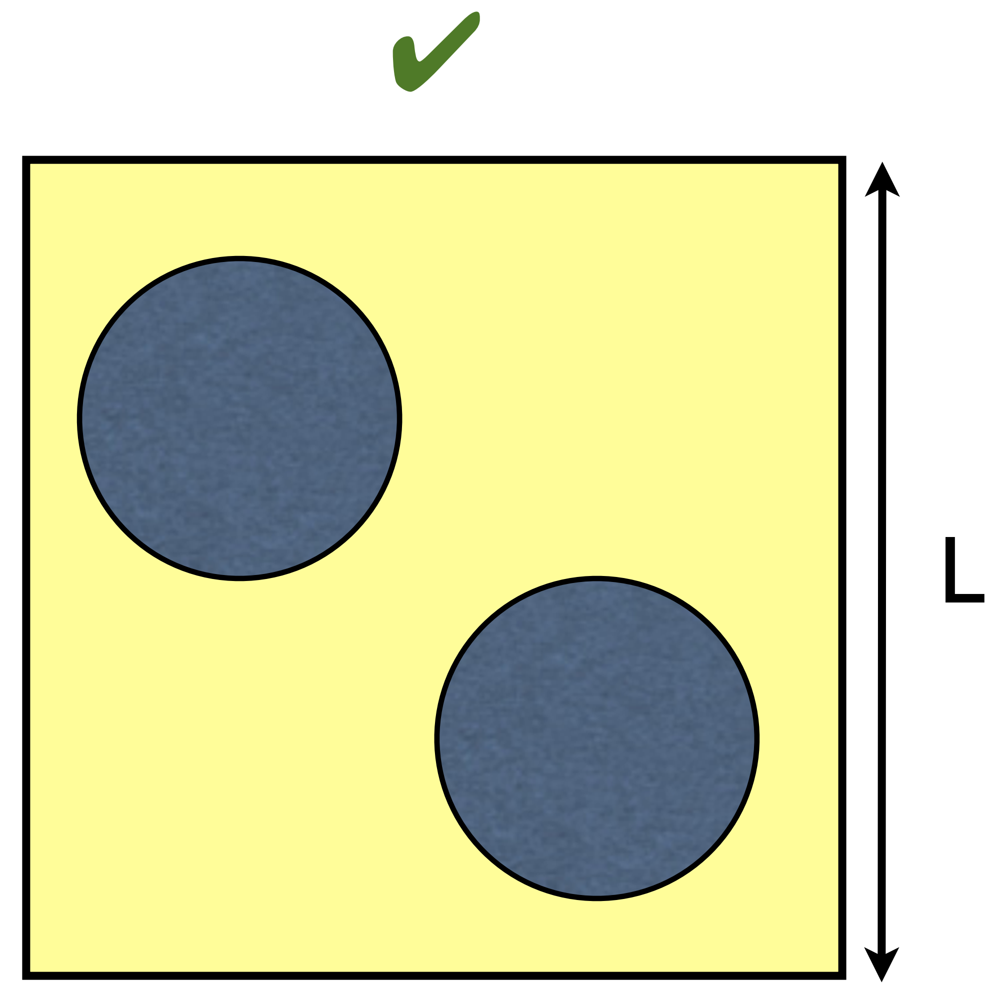


- We assume that LQCD has “done its job” and determined the spectrum
- Thus the problem becomes one in continuum, finite-volume QFT
- Note that the spectrum in finite volume IS PHYSICAL—it is just not directly experimentally observable

When is spectrum related to scattering amplitudes in QM?



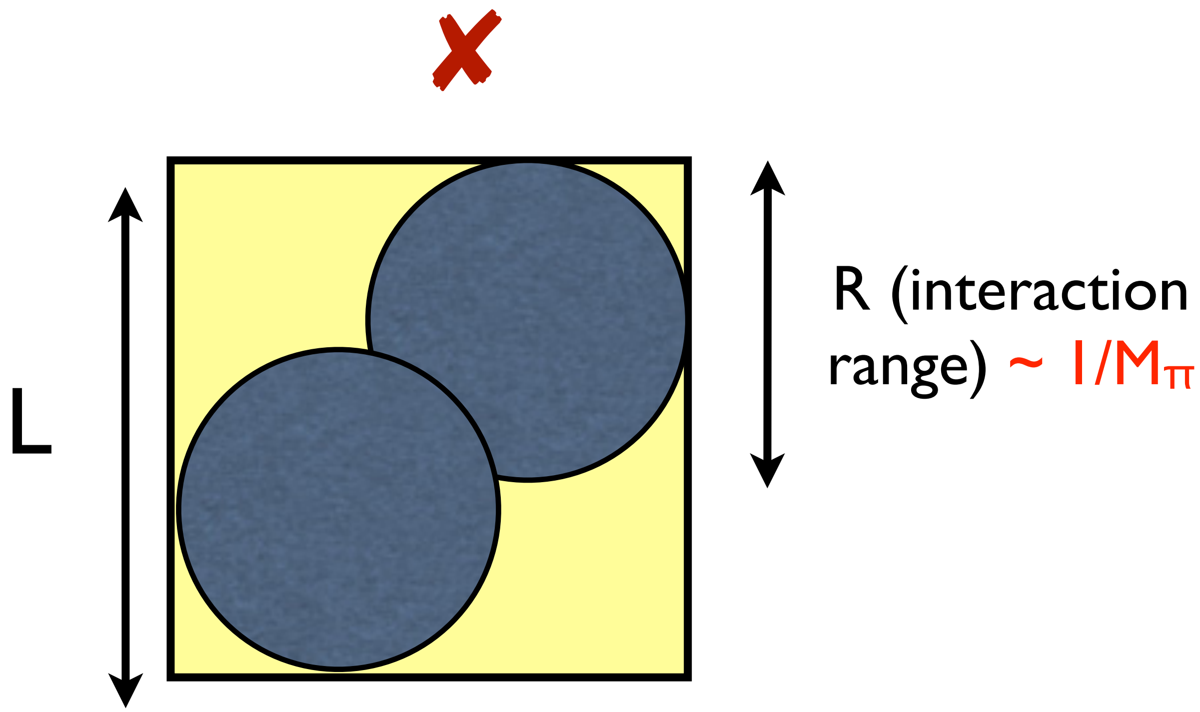
L<2R
No “outside” region.
Spectrum NOT related to scatt. amps.
Depends on finite-density properties



L>2R
There is an “outside” region.
Spectrum IS related to scatt. amps.

[Lüscher]

When is spectrum related to scattering amplitudes **in QCD**?



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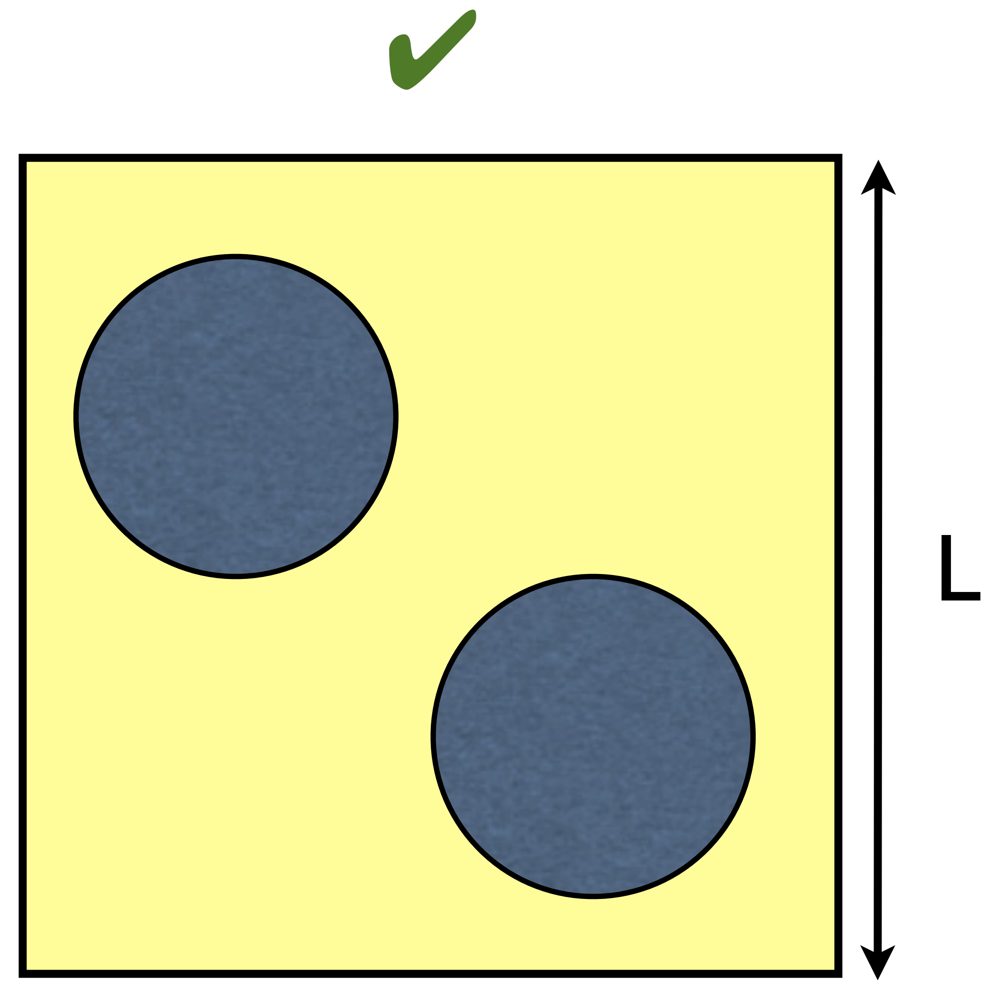


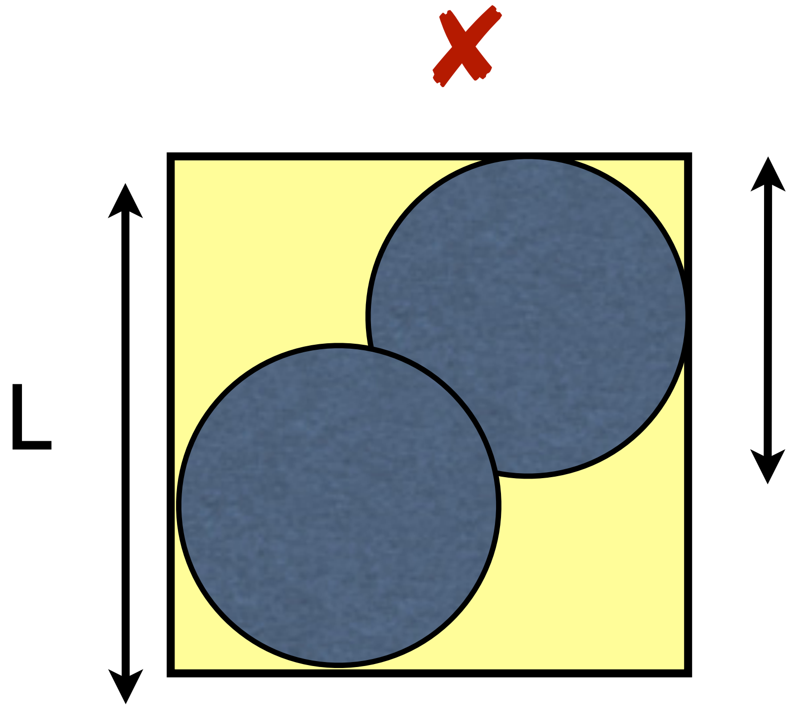
L>2R
There is an “outside” region.
Spectrum IS related to scatt. amps.
up to corrections proportional to

$$e^{-M_\pi L}$$

arising from tail of interaction

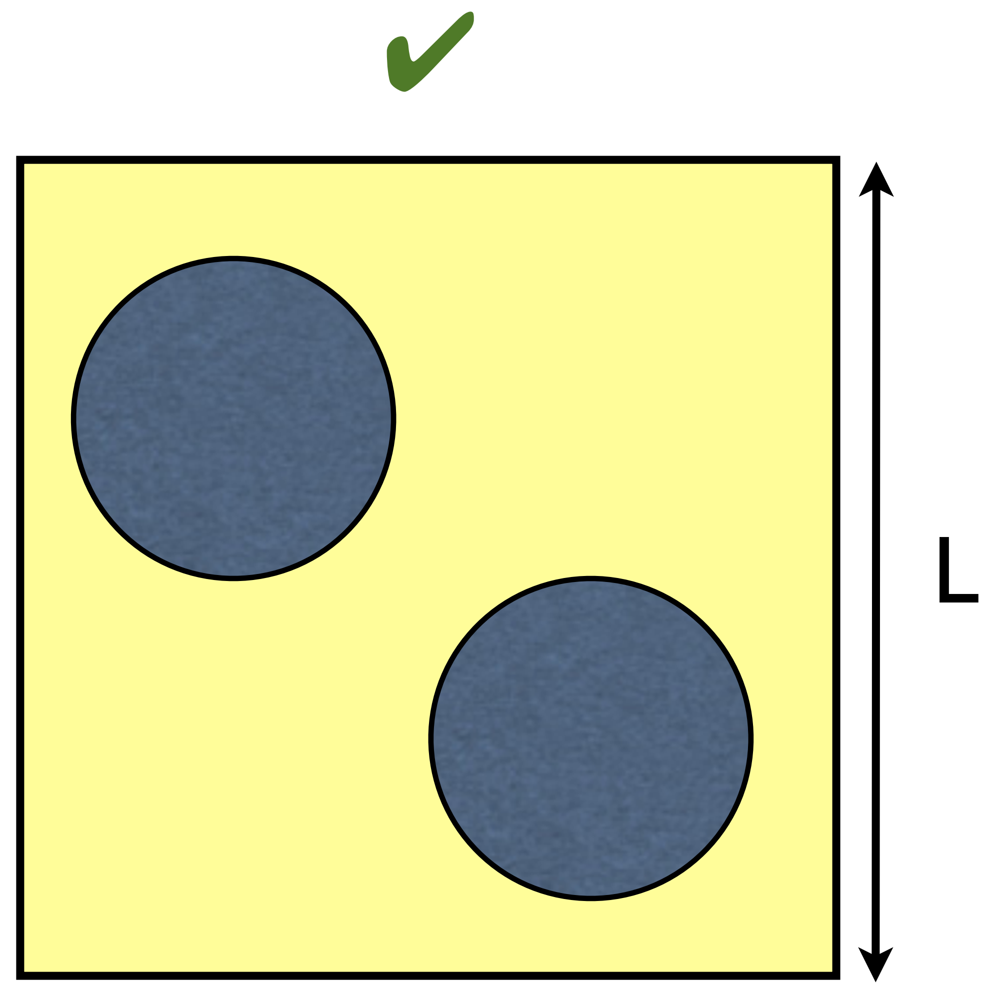
[Lüscher]

When is spectrum related to scattering amplitudes **in QCD**?



R (interaction range) ~ $1/M_\pi$

L<2R
No "outside" region.
Spectrum NOT related to scatt. amps.
Depends on finite-density properties



L>2R
There is an "outside" region.
Spectrum IS related to scatt. amps.
up to corrections proportional to

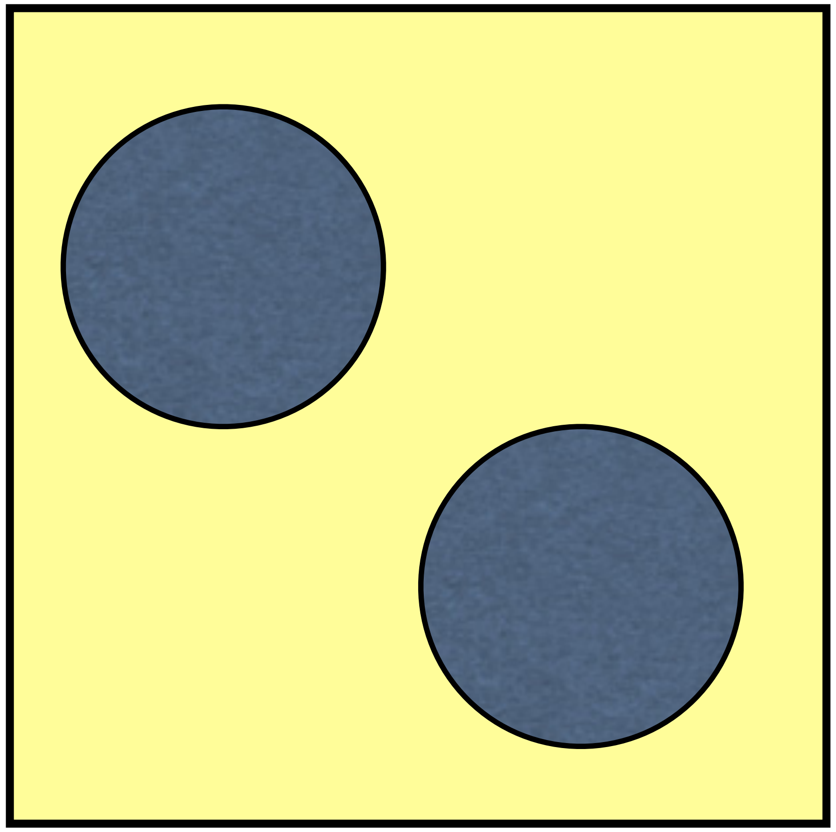
$$e^{-M_\pi L}$$

arising from tail of interaction

[Lüscher]

We ignore such exponentially-suppressed corrections throughout:
If $M_\pi L$=4 / 5 / 6, exp($-M_\pi L$)~2 / 0.7 / 0.2%

Aside on "two-particle states" in QFT



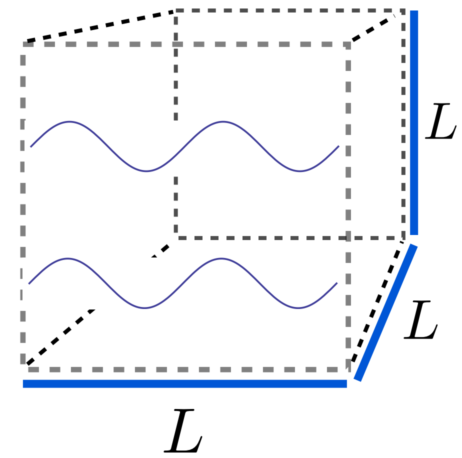
- I talk loosely about "two-particle finite-volume states"
- But in QFT all possible states appear that are consistent with the chosen quantum numbers
- We often impose a Z_2 symmetry decoupling even- and odd-particle-number states (cf. G parity for pions)
  - In this case there are states with 2, 4, 6, … particles
- Similar comments hold for "three-particle states"

Deriving the two-particle QC

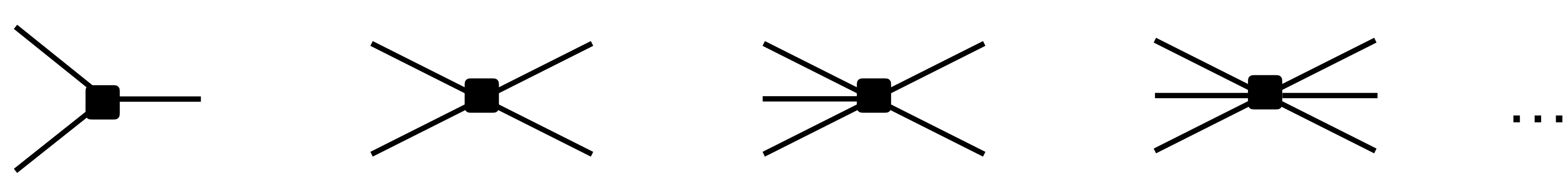
Following the method of [KSS05]

Set-up

- Work in continuum (assume that LQCD can control discretization errors)



- Cubic box of size L with periodic BC, and infinite (*Minkowski*) time
  - Spatial loops are sums: $\frac{1}{L^3}\sum_{\vec{k}}$ $\qquad \vec{k} = \frac{2\pi}{L}\vec{n}$
  - Can easily generalize to other geometries and BC
- Consider identical particles with physical mass m (think of pions), interacting *arbitrarily*—a generic (relativistic) effective field theory (RFT)
  - Work to all orders in perturbation theory with no assumptions about the size of coupling constants
  - Generalizations are known for nonidentical particles [Many authors] and to particles with spin [Bric14]



Methodology

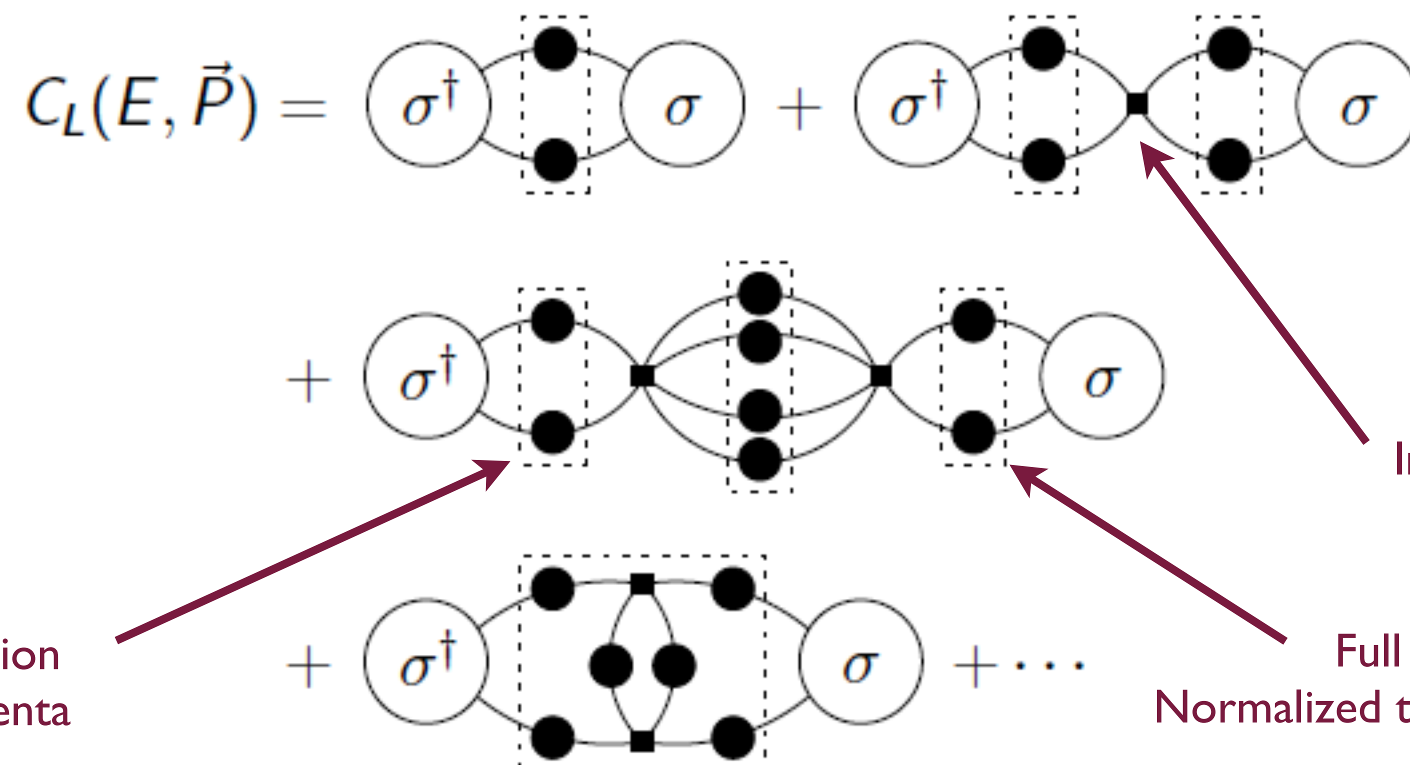
CM energy is $E^*=\sqrt{(E^2-P^2)}$

- Calculate (for some $\mathbf{P}=2\pi\mathbf{n_P}/L$)

$$C_L(E,\vec{P}) \equiv \int_L d^4x\, e^{iEt-i\vec{P}\cdot\vec{x}}\langle\Omega|\,T\left\{\sigma^\dagger(x)\sigma(0)\right\}|\Omega\rangle_L$$

- $\sigma \sim \pi^2$, e.g. $\sigma(\vec{x},t) = \int_L d^3y\, \pi(\vec{x}+\vec{y},t)\pi(\vec{x}-\vec{y},t)e^{-i\vec{k}\cdot\vec{y}}$ $\qquad \pi(x) = \bar{u}(x)\gamma_5 d(x)$

- Poles in C_L occur at energies of finite-volume spectrum [Exercise]



Infinite-volume vertices

Boxes indicated summation over finite-volume momenta

Full propagators Normalized to unit residue at pole

Here I have assumed no odd-legged vertices—-not necessary for subsequent arguments, but used in 3-particle case

Key step 1

- Replace loop sums with integrals where possible (using Poisson summation formula)
  - Drop exponentially suppressed terms ($\sim e^{-ML}$, $e^{-(ML)^2}$, etc.) while keeping power-law dependence

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l}\neq\vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})$$

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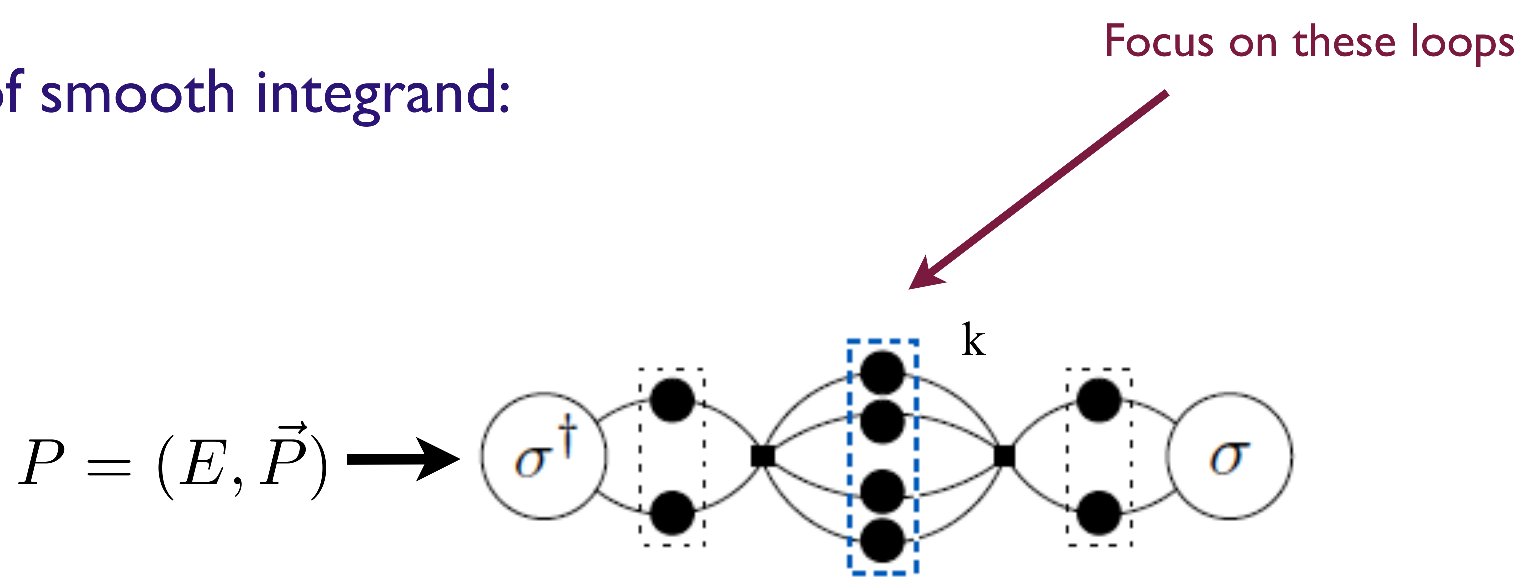
- Example of smooth integrand:



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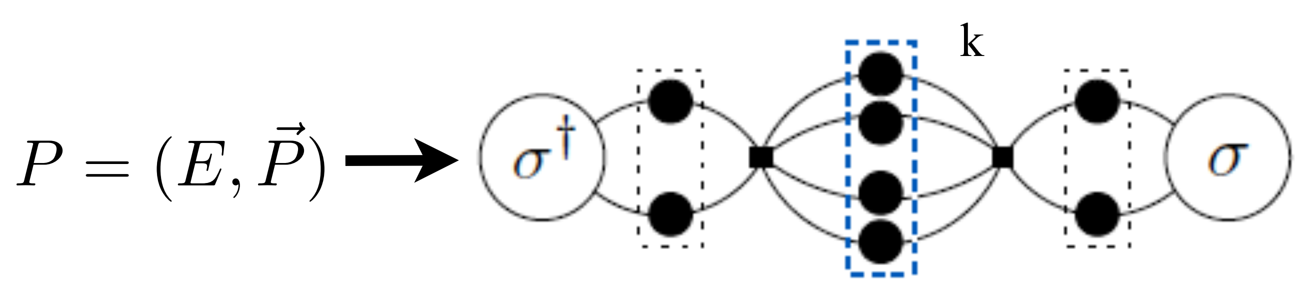
Exp. suppressed if g(k) is smooth
and scale of derivatives of g is ~1/M

- Example of smooth integrand:

Focus on these loops



Key step 2

- Use "sum=integral + [sum-integral]" if integrand has pole, and use identity [KSS]

$$\frac{1}{2}\left(\int\frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}}-\int\frac{d^4k}{(2\pi)^4}\right)f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

symmetry factor

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symmetry factor

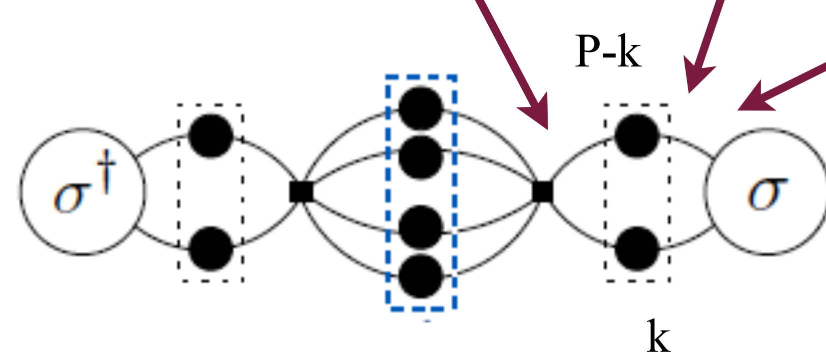
- Example of pole:

Focus on this loop

f is left-hand part of integrand

g is right-hand part of integrand

$P = (E, \vec{P}) \longrightarrow$



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$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*)\mathcal{F}\ (q^*, q^{*'}) g^*(\hat{q}^{*'}) \quad + \text{exp. suppressed}$$

symmetry factor

q* is relative momentum of pair on left in CM

Kinematic function

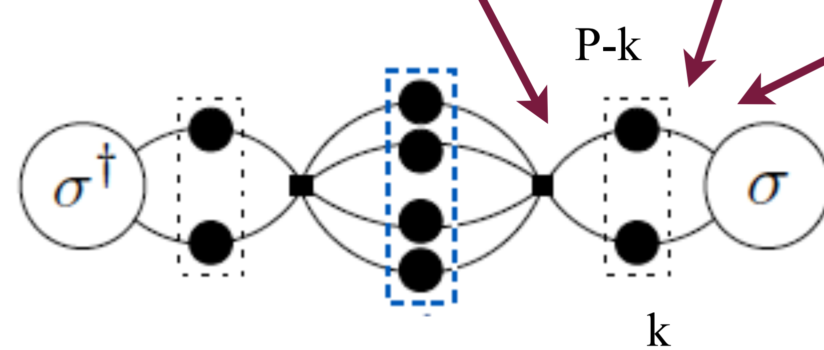
f & g evaluated for ON-SHELL momenta
Depend only on direction in CM

- Example of pole:

Focus on this loop

f is left-hand part of integrand

g is right-hand part of integrand

$P = (E, \vec{P}) \longrightarrow$ 

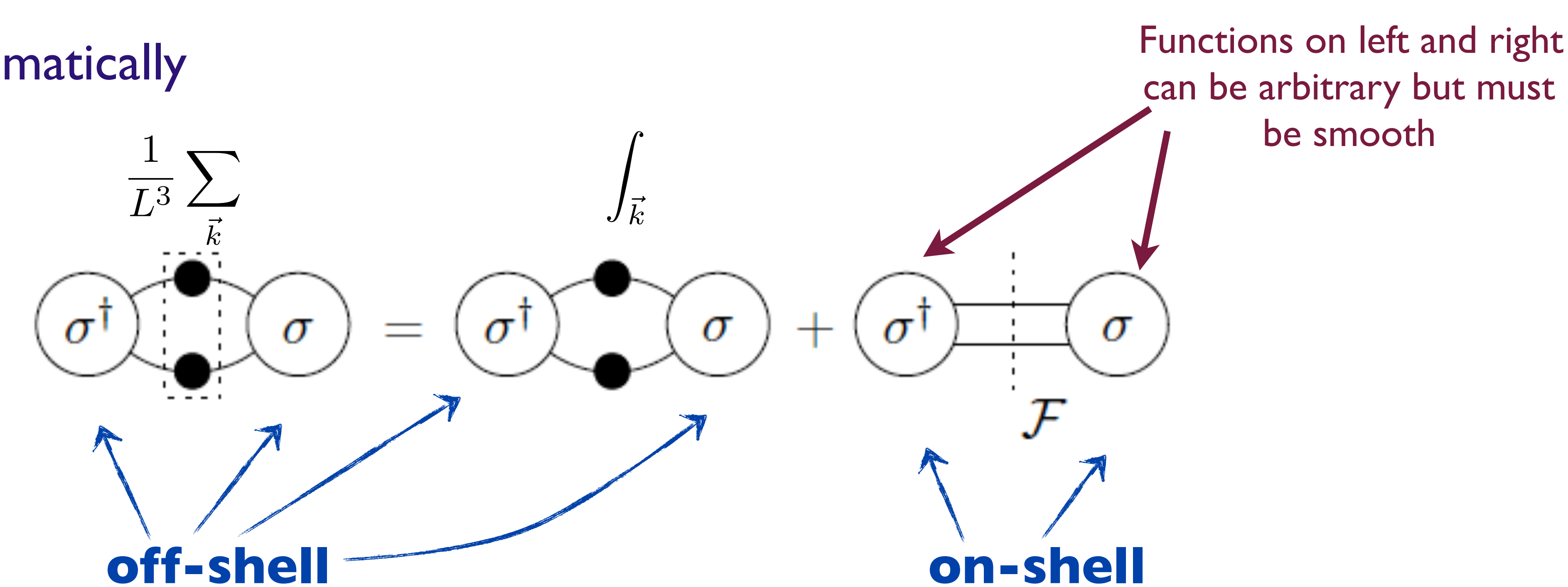
Key step 2

- Use "sum=integral + [sum-integral]" where integrand has pole, with [KSS]

$$\left(\int\frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec k} - \int\frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

$$= \int d\Omega_{q^*}\, d\Omega_{q^{*\prime}}\, f^*(\hat q^*)\mathcal{F}\ (q^*,q^{*\prime})g^*(\hat q^{*\prime})$$

- Diagrammatically



A new type of "cut"

Variant of key step 2

- For generalization to 3 particles will use a PV prescription instead of iε

$$\frac{1}{2}\left(\int\frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \overset{\text{PV}}{\int}\frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+\cancel{i\epsilon}}\frac{1}{(P-k)^2-m^2+\cancel{i\epsilon}}g(k)$$

$$= \int d\Omega_{q^*}d\Omega_{q^{*\prime}}\, f^*(\hat{q}^*)\mathcal{F}_{\text{PV}}(q^*,q^{*\prime})g^*(\hat{q}^{*\prime})$$

- Key properties of $\mathcal{F}_{\text{PV}}$: real and no unitary cusp at threshold
- These properties are important for the derivation of three-particle QC

More detail on key step 2 [HS14]

$$\frac{1}{2}\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m_j^2+i\epsilon}\frac{1}{(P-k)^2-m_j^2+i\epsilon}g(k)$$

$$= \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right)\frac{f(\vec{k}^*)g(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k}+i\epsilon)} + \mathcal{O}(e^{-mL})$$

Smooth UV regulator
Equals unity on shell

Time integrals set k on shell
k* is on-shell k boosted to CM

$$= \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right) f_{\ell'm'}\frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k}+i\epsilon)} g_{\ell m} + \mathcal{O}(e^{-mL})$$

Decompose f & g into spherical harmonics, and evaluate with P-k on shell

$$\mathcal{Y}_{\ell m}(\vec{k}^*) = \sqrt{4\pi}\left(\frac{k^*}{q^*}\right)^{\ell} Y_{\ell m}(\hat{k}^*)$$

$$q^* = \sqrt{E^{*2}/4 - m^2}$$

$$\equiv f_{\ell'm'}\, F_{\ell'm';\ell m}(E,\vec{P},L)\, g_{\ell m}$$

More convenient to use this matrix form

- Thus power-law volume dependence enters through geometrical function:

$$\boxed{F_{\ell'm';\ell m}(E,\vec{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right)\frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k}+i\epsilon)}}$$

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- Similarly, the PV version is

$$F_{\mathrm{PV};\ell'm';\ell m}(E,\vec{P},L)=\frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}}-\mathrm{PV}\int\frac{d^3k}{(2\pi)^3}\right)\frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k})}$$

$$=F_{\ell'm';\ell m}(E,\vec{P},L)-i\delta_{\ell'\ell}\,\delta_{m'm}\frac{q^*}{16\pi E^*}$$

x=q*L/(2π)

$$\propto\left(\frac{2\pi}{L}\right)^{1+\ell+\ell'}\mathcal{Z}_{\ell',m';\ell,m}(x^2,\mathbf{P})$$

"Lüscher zeta function"

Kinematic functions

$Z_{4,0}$ & $Z_{6,0}$ for **P=0** [Luu & Savage, `11]

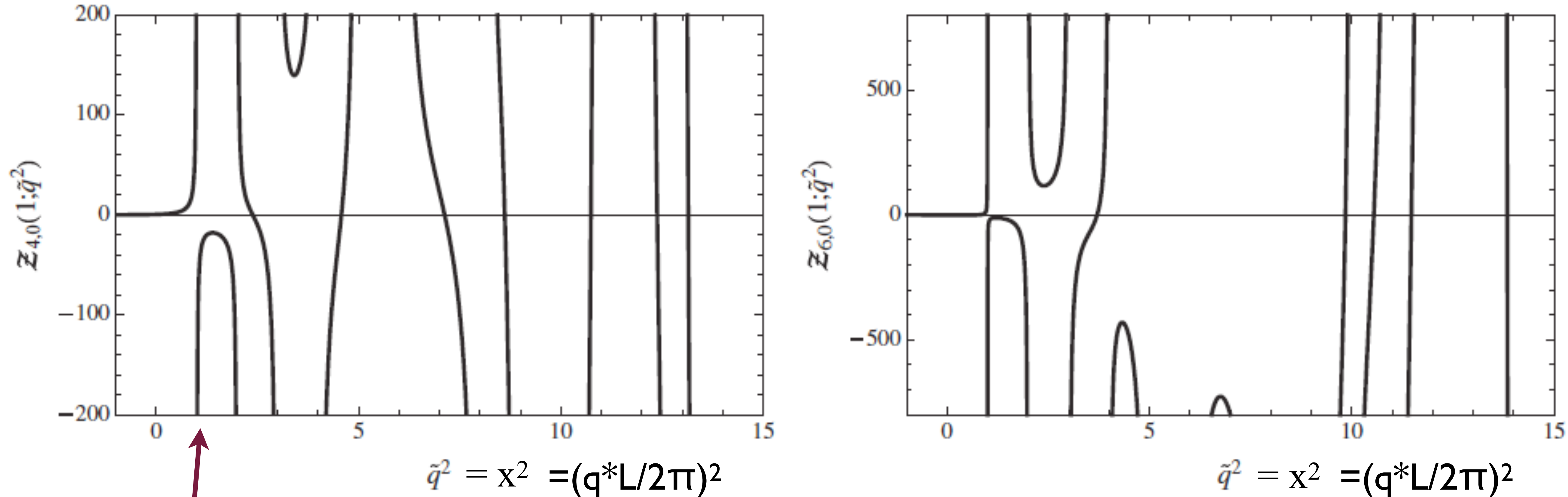


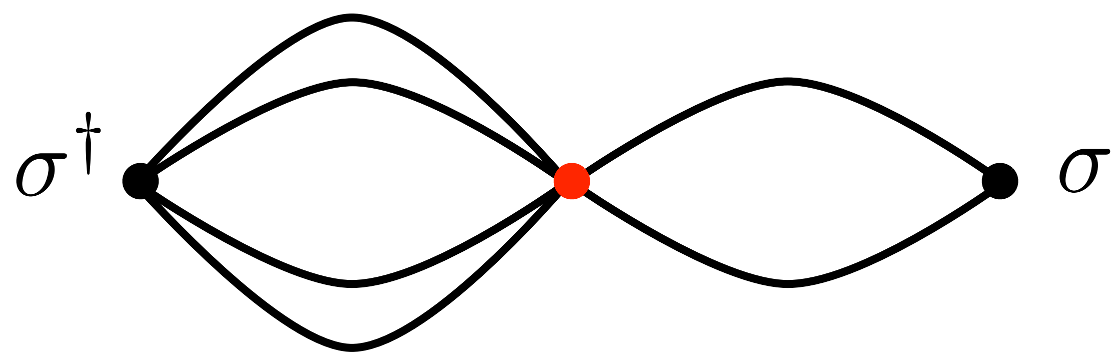
FIG. 29. The functions $Z_{4,0}(1;\tilde{q}^2)$ (left panel) and $Z_{6,0}(1;\tilde{q}^2)$ (right panel).

Divergences occur for values of E equal to the energy of two free particles in the box [Exercise: why no divergence at x=0?]

Example:
$\mathbf{n}_1 = -\mathbf{n}_2 = (0,0,1)$
$\Rightarrow q^* = 2\pi/L \Rightarrow x=1$

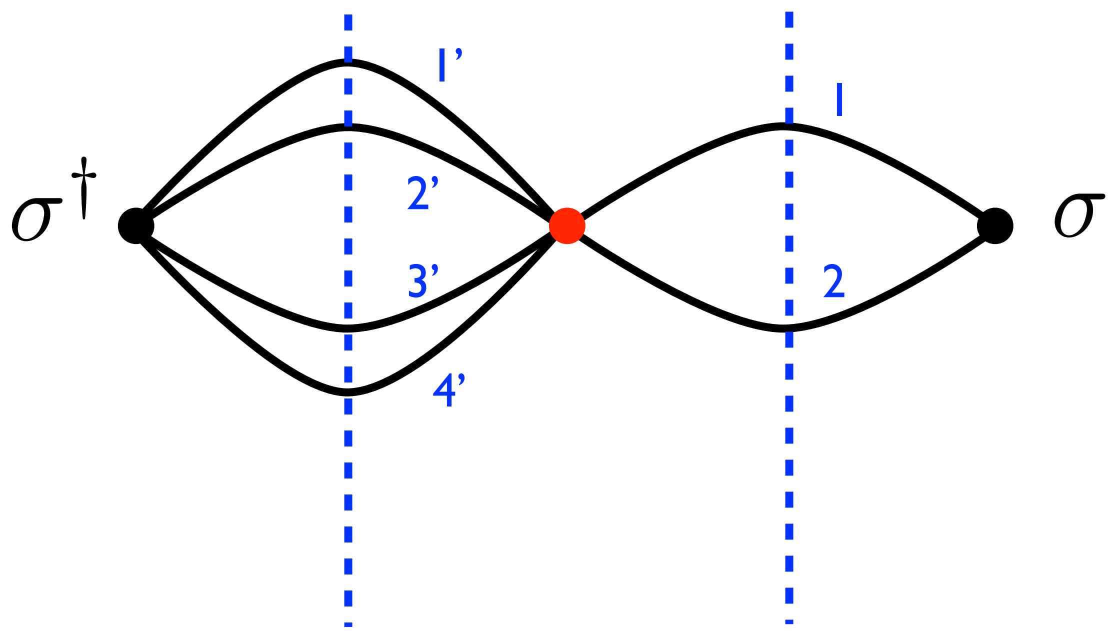
Key step 3

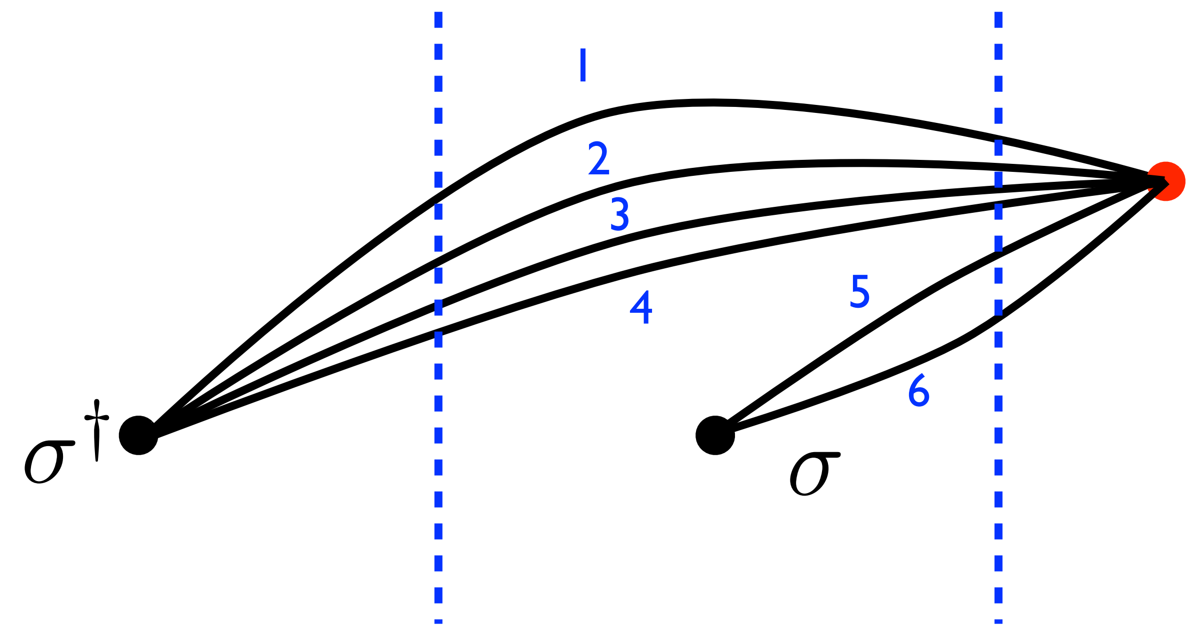
- Identify potential singularities using time-ordered PT (i.e. do k_0 integrals)
- Example (again assuming only even-legged vertices)



Key step 3

- 2 out of 6 time orderings:



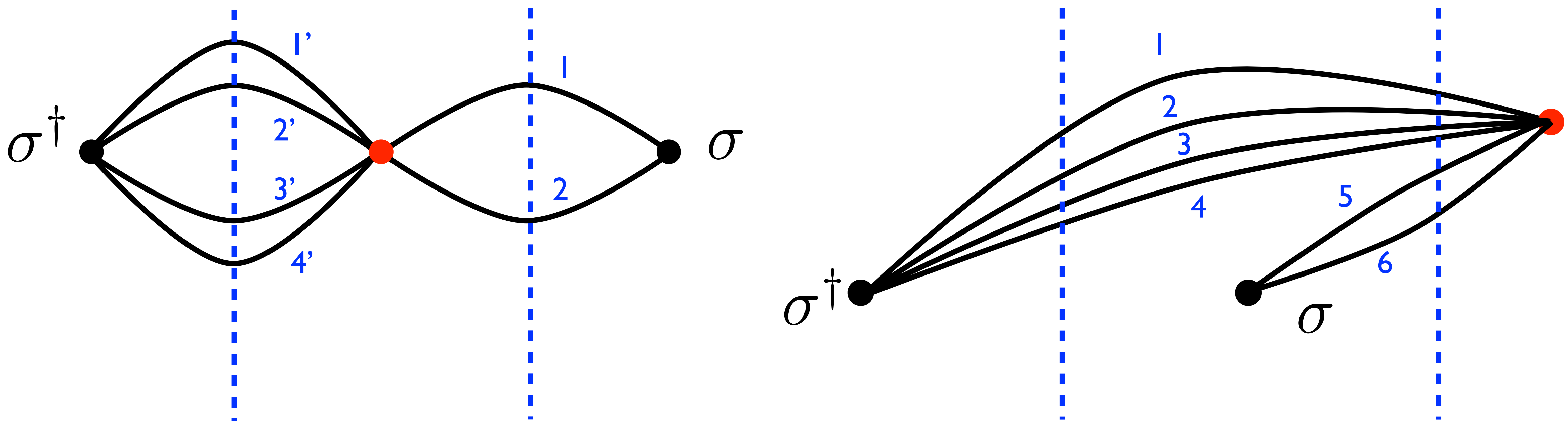


$$\frac{1}{E-\omega_1'-\omega_2'-\omega_3'-\omega_4'} \qquad \frac{1}{E-\omega_1-\omega_2} \qquad \frac{1}{E-\omega_1-\omega_2-\omega_3-\omega_4} \qquad \frac{1}{\sum_{j=1,6}\omega_j}$$

On-shell energy $\omega_j = \sqrt{\vec{k}_j^2 + M^2}$

Key step 3

- 2 out of 6 time orderings:

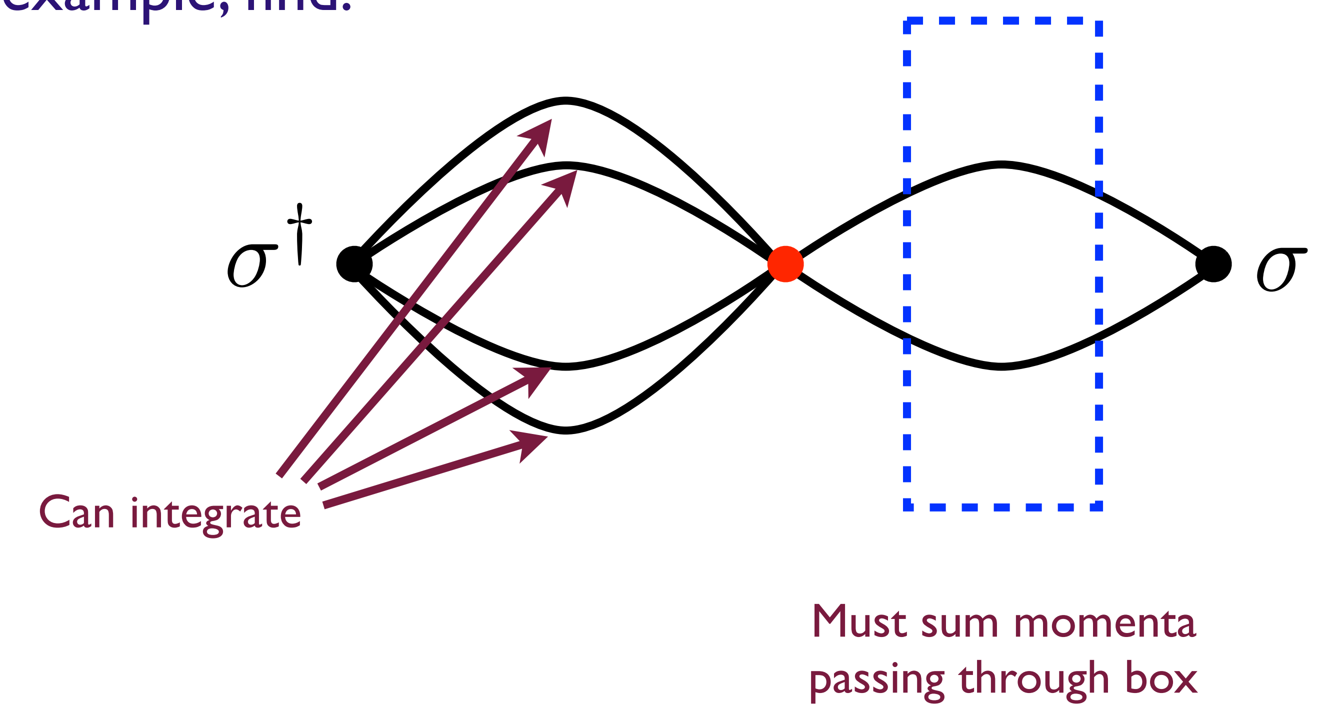


$$\frac{1}{E-\omega_1'-\omega_2'-\omega_3'-\omega_4'} \qquad \frac{1}{E-\omega_1-\omega_2} \qquad \frac{1}{E-\omega_1-\omega_2-\omega_3-\omega_4} \qquad \frac{1}{\sum_{j=1,6}\omega_j}$$

- If restrict $0 < E^* < 4M$ ($M < E^* < 3M$ if have odd-legged vertices) then only 2-particle "cuts" have singularities, and these occur only when both particles go simultaneously on shell

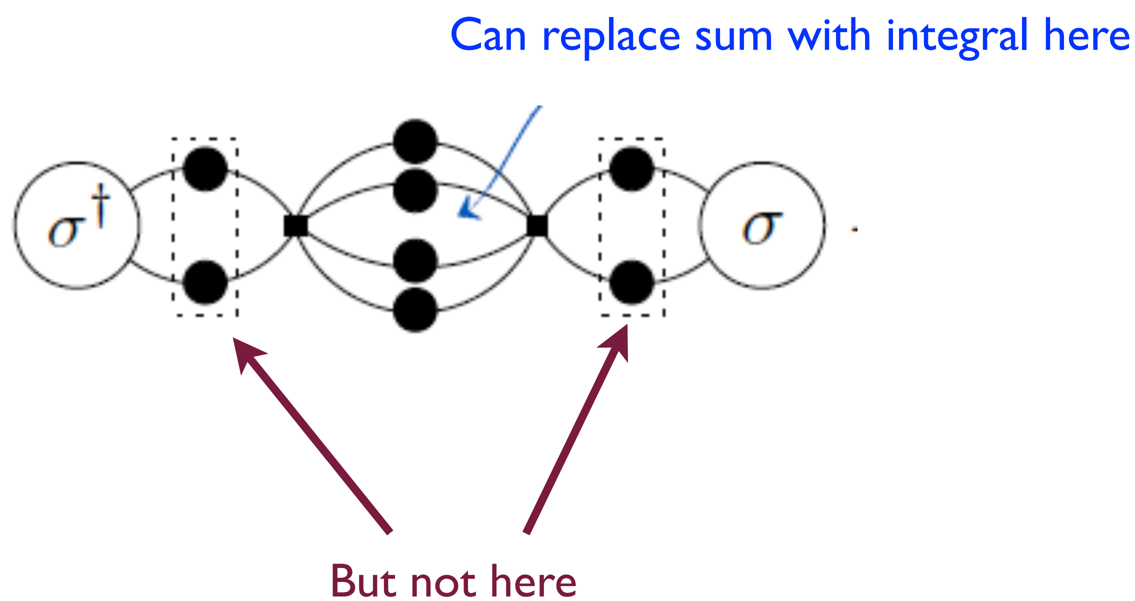
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated
- In our example, find:



Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated
- Another example:



Combining key steps 1-3

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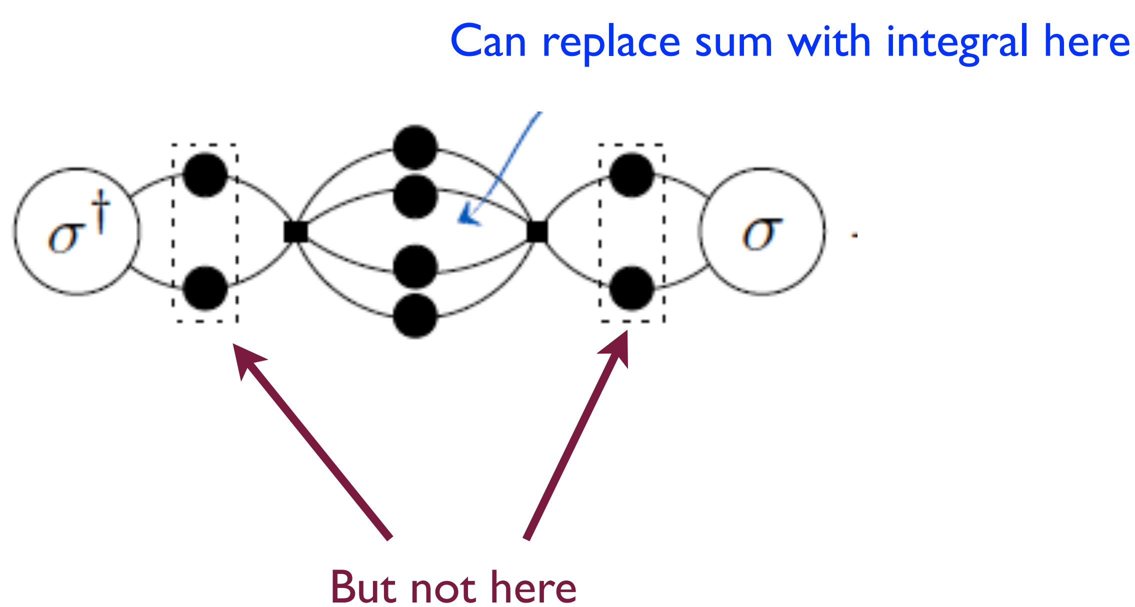
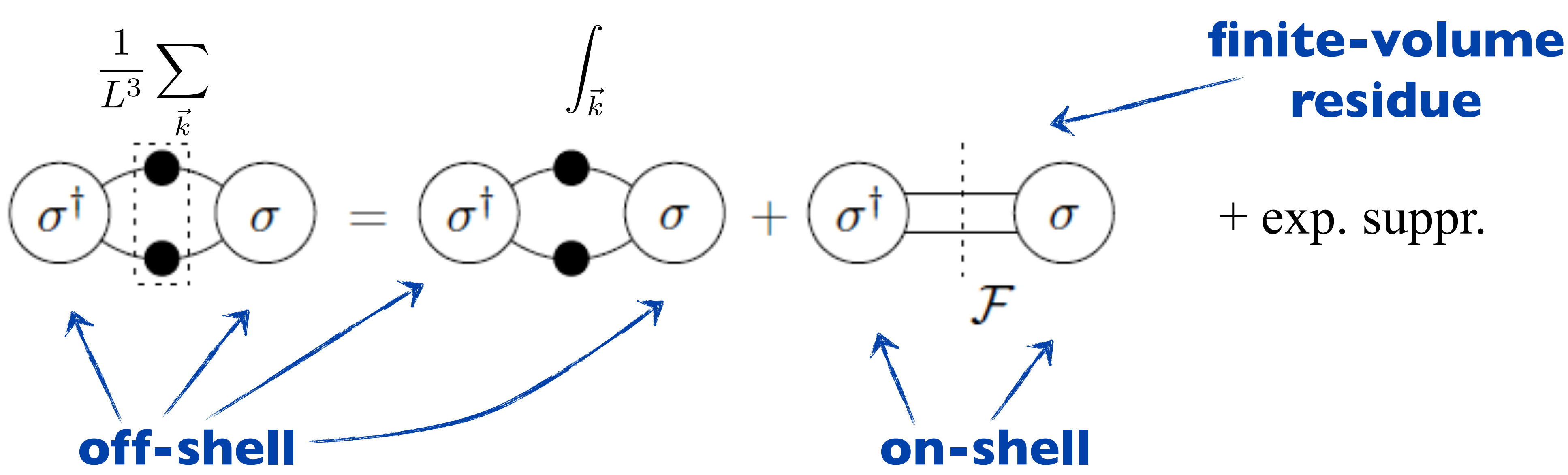


- Then repeatedly use sum=integral + "sum-integral" to simplify

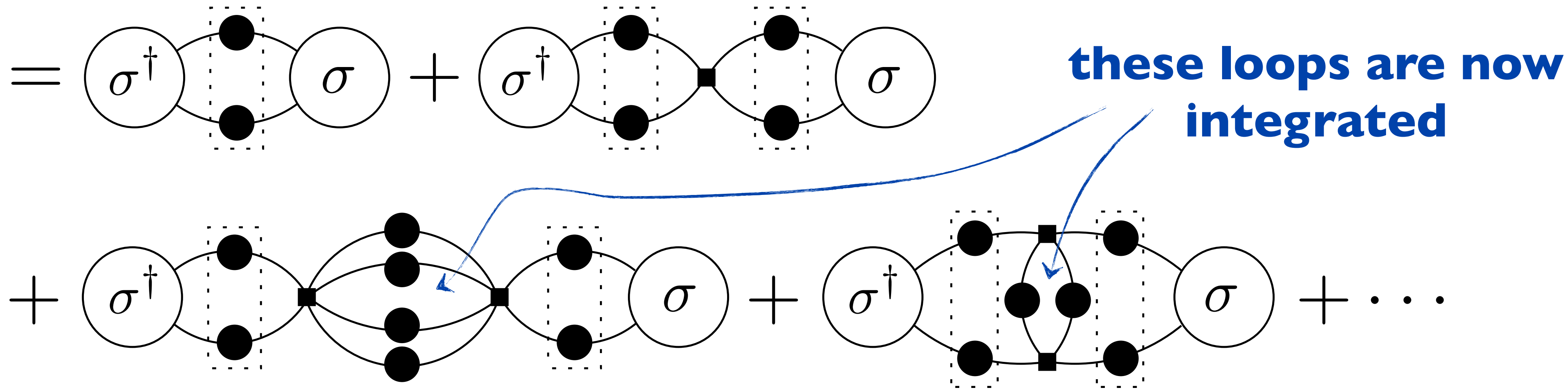
Summary: the key "move"



A new type of "cut"

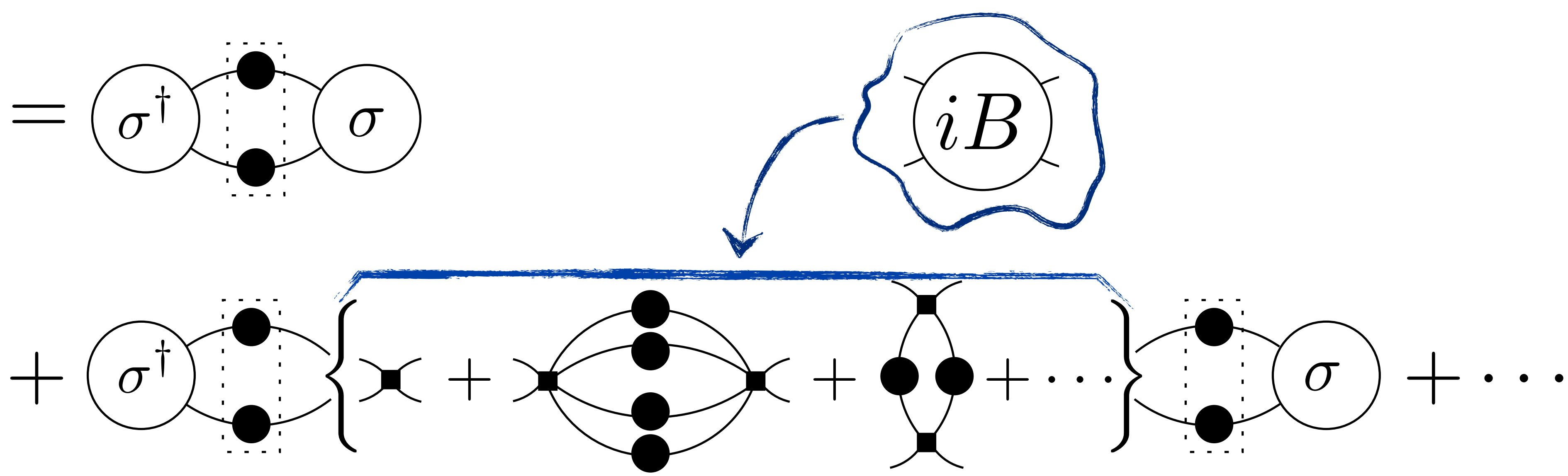
- Apply previous analysis to 2-particle correlator ($0 < E^* < 4M$)

$C_L(E,\vec{P}) =$



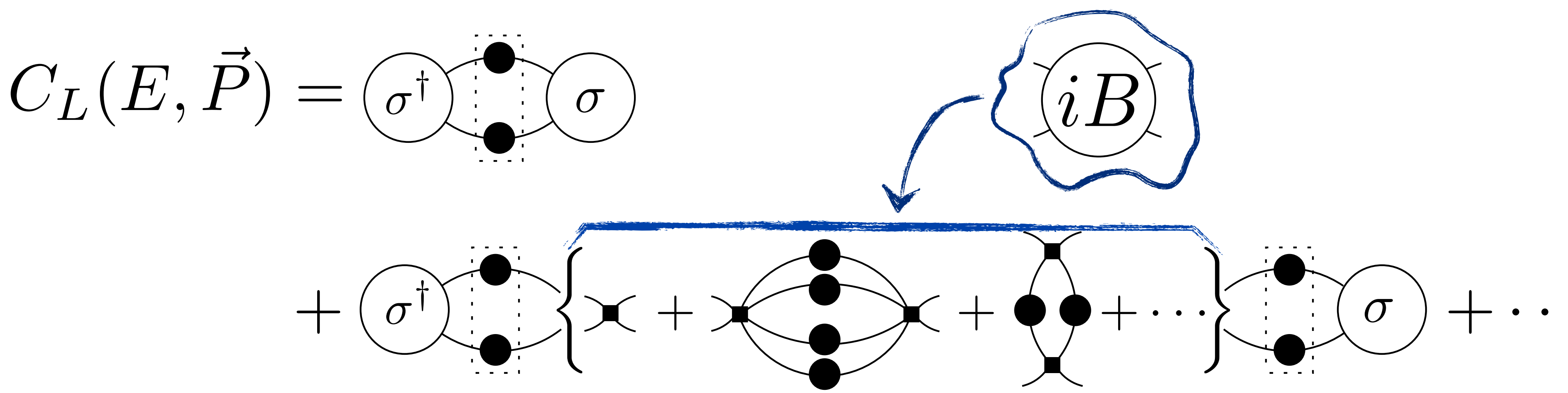
- Collect terms into infinite-volume Bethe-Salpeter kernels

$C_L(E,\vec{P}) =$

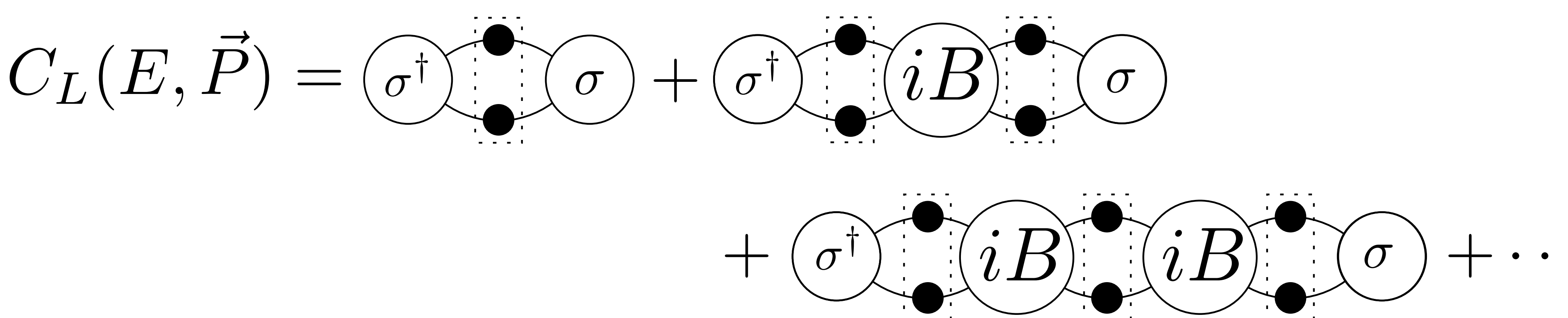


B-S kernel: 2-particle irreducible in the s-channel, i.e. no 2-particle cuts

- Apply previous analysis to 2-particle correlator
- Collect terms into infinite-volume Bethe-Salpeter kernels

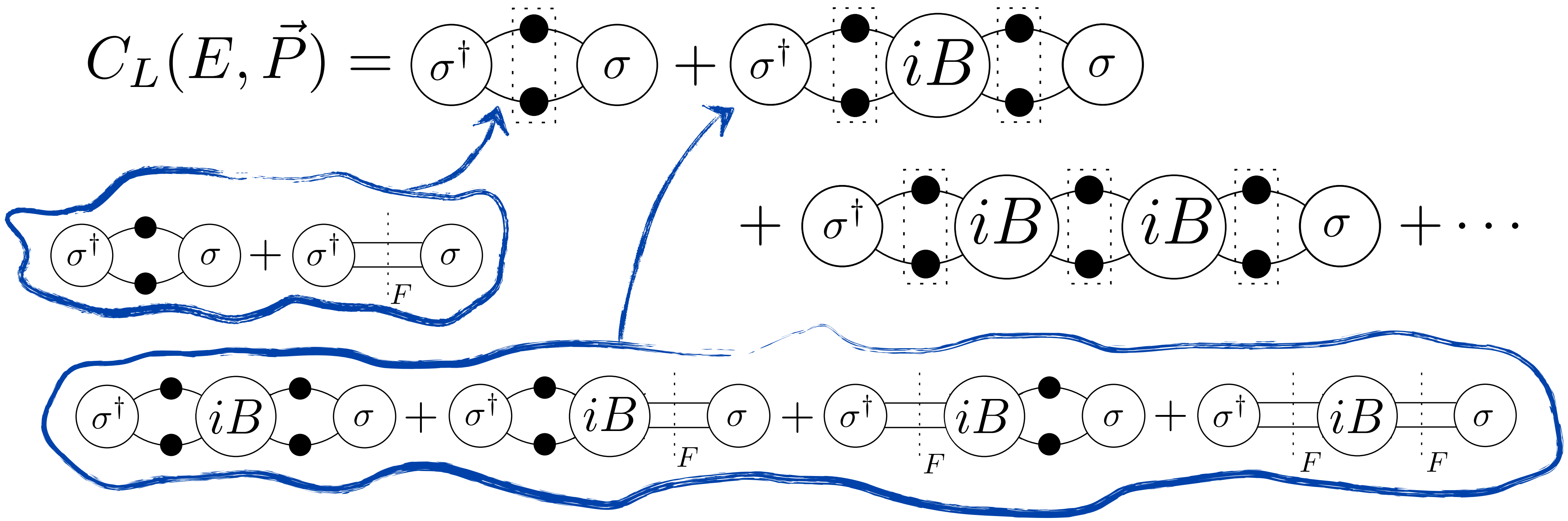


- Leading to

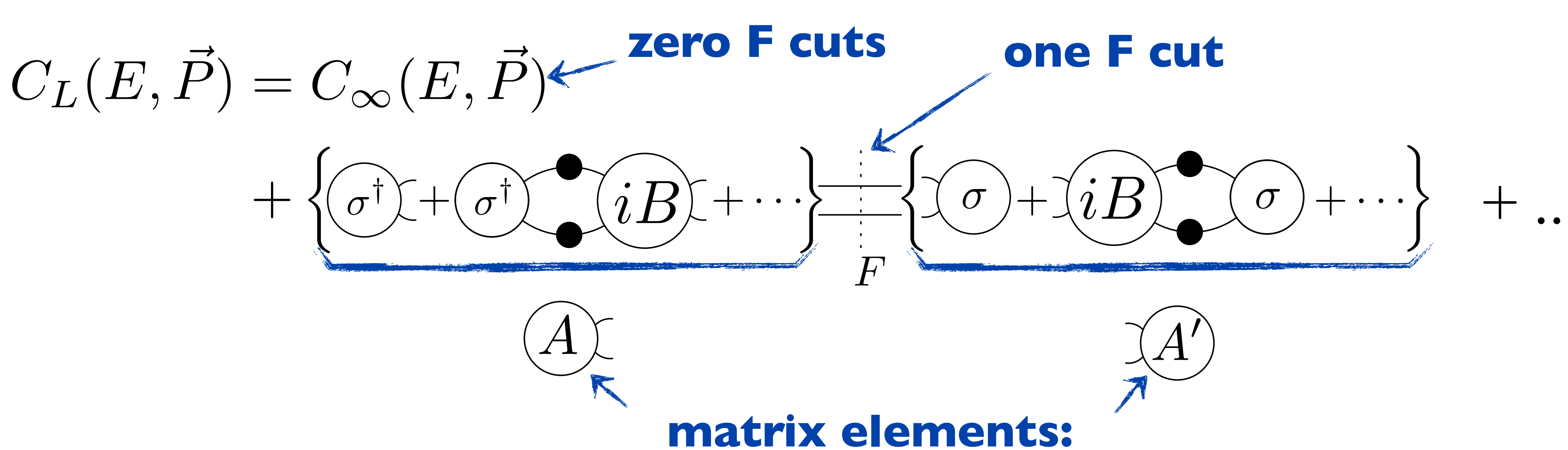


Similar structure to NREFT bubble-chain (e.g. in two nucleon system)

- Next use sum identity



- And regroup according to number of "F cuts"



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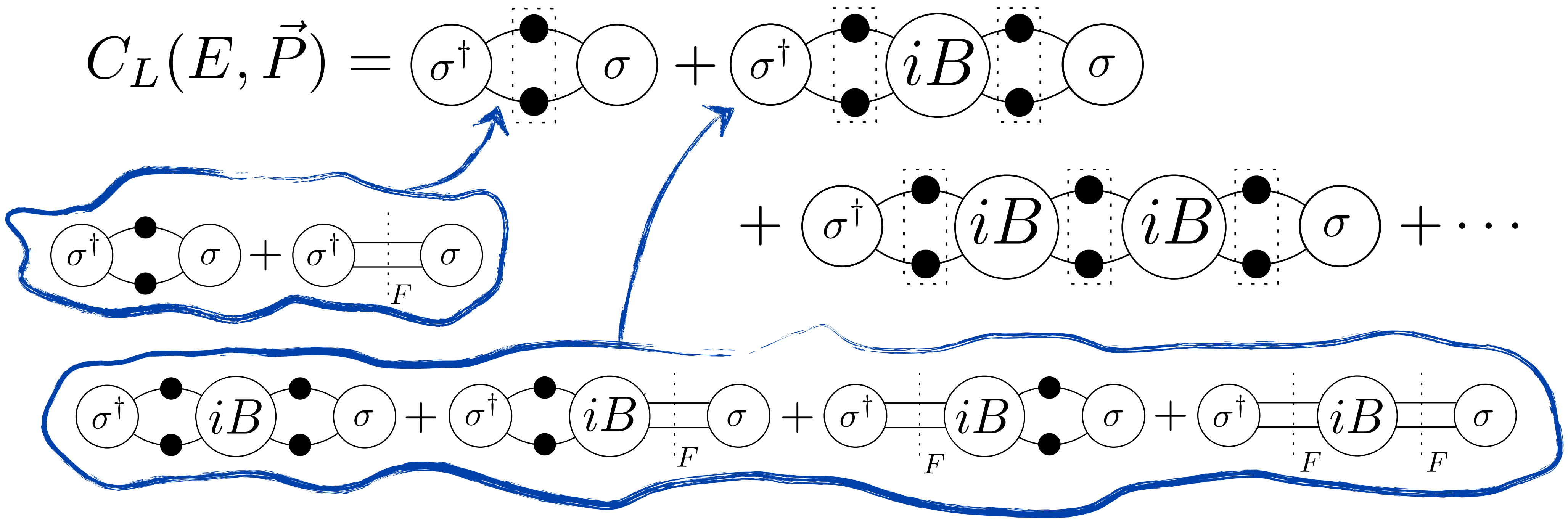
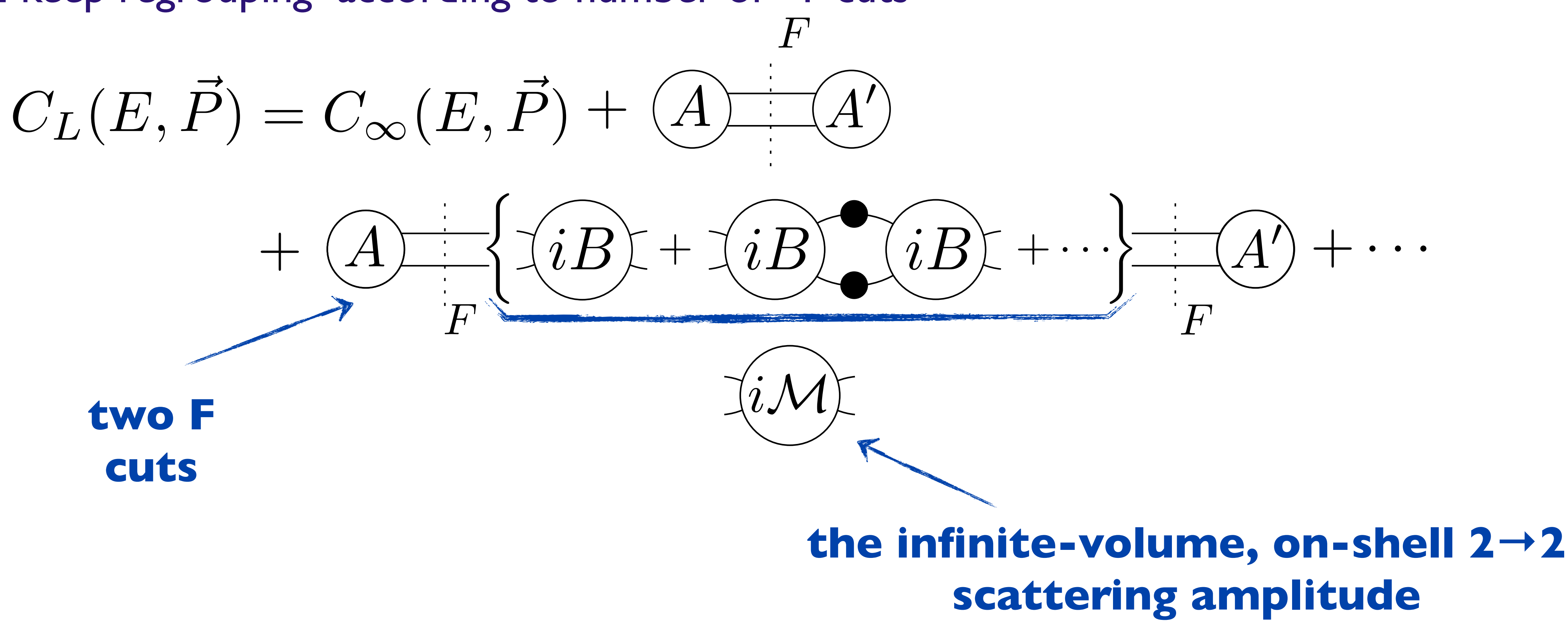
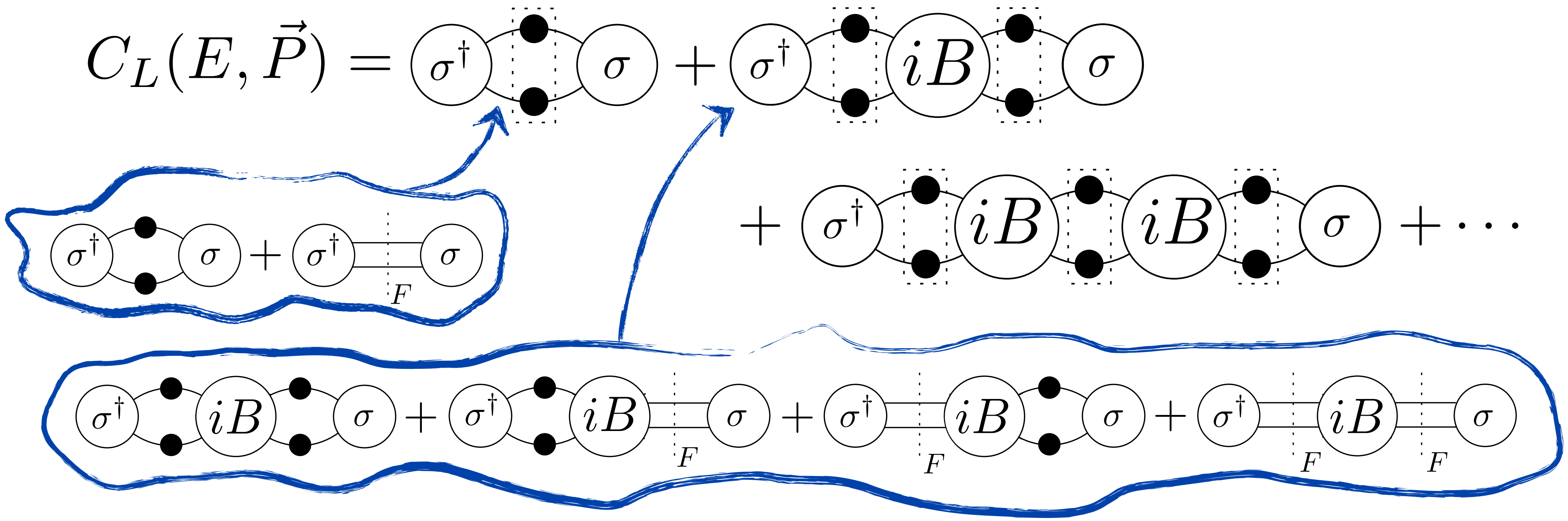


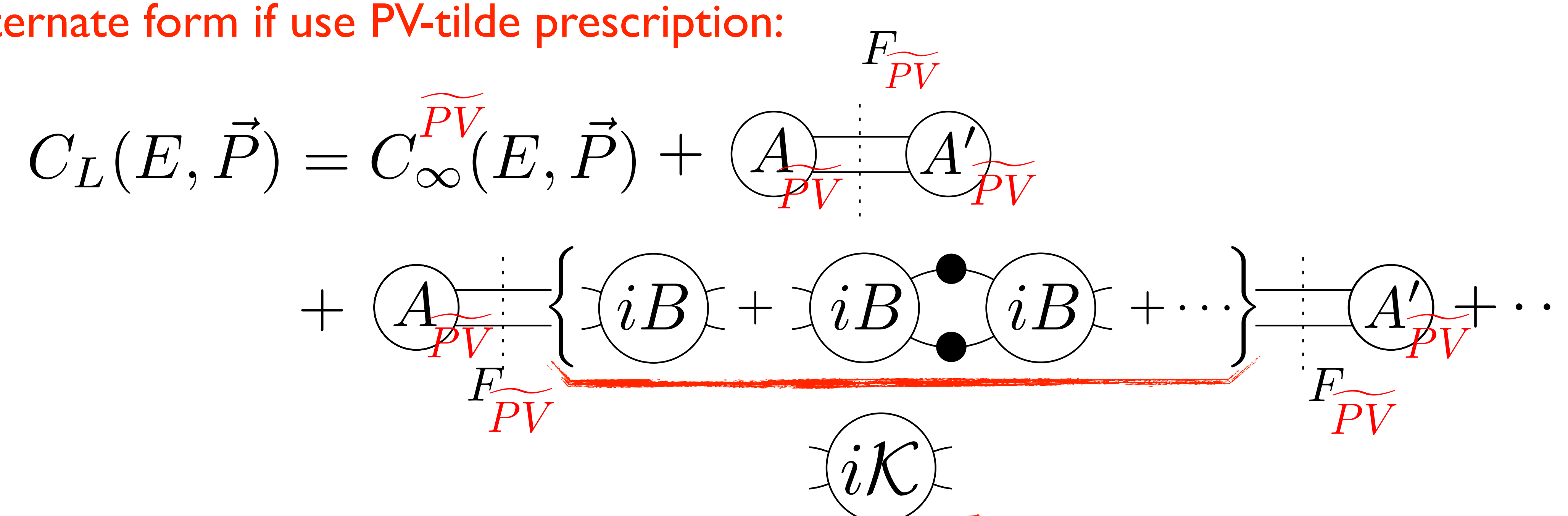
- And keep regrouping according to number of "F cuts"



- Next use sum identity



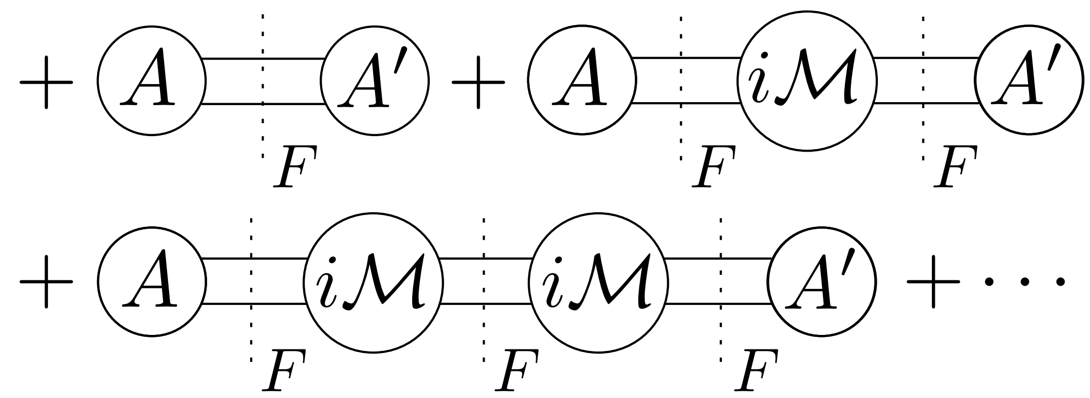
- Alternate form if use PV-tilde prescription:



the infinite-volume, on-shell 2→2 K-matrix

- Final result:

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$

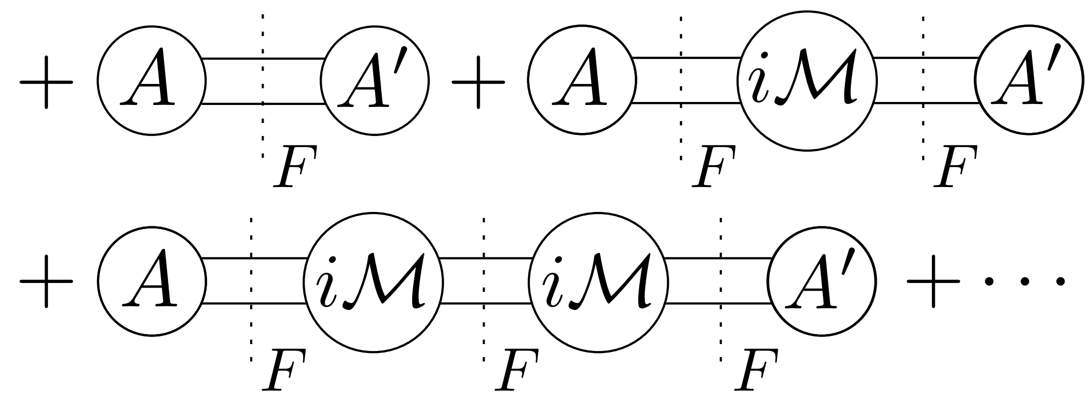


- $$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2\to2} iF]^n A$$

- Correlator is expressed in terms of infinite-volume, physical quantities and kinematic functions encoding the finite-volume effects

- Final result:

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$



- $$C_L(E,\overrightarrow{P}) = C_\infty(E,\overrightarrow{P}) + \sum_{n=0}^{\infty} A' iF\,[i\mathscr{M}_2\, iF]^n A$$

- $$C_L(E,\overrightarrow{P}) = C_\infty(E,\overrightarrow{P}) + \sum_{n=0}^{\infty} A' iF \frac{1}{1+\mathscr{M}_2 F} A$$

no poles, only cuts

no poles, only cuts

matrices in l,m space

no poles, only cuts

⇒ Poles in C_L occur when

$$\boxed{\det\left[F(E,\overrightarrow{P},L)^{-1} + \mathscr{M}_2(E^*)\right] = 0}$$

2-particle quantization condition

- At fixed L & **P**, the finite-volume spectrum E_1, E_2, ... is given by solutions of

$$\boxed{\det\left[F(E,\overrightarrow{P},L)^{-1}+\mathscr{M}_2(E^*)\right]=0}$$

For **P**=0 this equivalent to original result by [Lüscher]

Generalization to moving frame first obtained using RQM by [Rummukainen & Gottlieb]

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Generalization to moving frame first obtained using RQM by [Rummukainen & Gottlieb]

- F and $\mathcal{M}_2$ are matrices in l,m space:
  - $\mathcal{M}_2$ is diagonal; while F is off-diagonal, since the box violates rotation symmetry
- QC separates finite-volume (F) and infinite-volume quantities ($\mathcal{M}_2$)
- If $\mathcal{M}_2$ vanishes, solutions are free two-particle energies due to poles in F
- Each spectral energy gives information about all partial waves of $\mathcal{M}_2(E^*)$

2-particle quantization condition

- Equivalent form, obtained by using PV prescription throughout derivation, is

$$\boxed{\det\left[F_{PV}(E,\overrightarrow{P},L)^{-1}+\mathcal{K}_2(E^*)\right]=0}$$

- I prefer this as both $\mathcal{K}_2$, F_{PV} are real
- $\mathcal{K}_2$ contains the same information as $\mathcal{M}_2$, but is real and smooth (no threshold branch points)
- These differences are irrelevant for the two-particle QC—the two QCs are identical—but turn out to be important for the three-particle QC
- Beware when reading the literature, as each collaboration uses different notation for what I call F: sometimes B (box function), sometimes M

Applications of QC2

Truncation

$$\det\left[F_{PV}(E,\vec{P},L)^{-1} + \mathcal{K}_2(E^*)\right] = 0$$

- Near threshold $\mathcal{K}_2 \sim (q^*)^{2l}$ [familiar from QM due to the angular-momentum barrier]

- In practice, for $E^* \lesssim 1\mathrm{GeV}$, it is a good approximation to keep only the lowest one or two partial waves, i.e to set $\mathcal{K}_2^{(l)} = 0$ for $l > l_{\mathrm{max}}$

- If $\mathcal{K}_2$ (which is diagonal in l,m) vanishes for $l > l_{\mathrm{max}}$ then can show that need only keep $l \leq l_{\mathrm{max}}$ in F_{PV} (which is not diagonal)

- This leads to a finite-dimensional matrix condition that can be implemented numerically

- Can further reduce the dimensionality by projecting onto irreps of the cubic group [A_1^+, A_2^+, E^+, …—no time to discuss here]

Simplest case: single value of l

- If $l_{\max}=0$, or if $l_{\max}=1$ and one uses a cubic-group irrep that does not couple to $l=0$ (e.g. E^+ if $\boldsymbol{P}=0$), then only a single value of l contributes, and QC2 becomes algebraic, e.g.

$$\boxed{\det\left[F_{PV}(E,\vec{P},L)^{-1}+\mathcal{K}_2(E^*)\right]=0}$$

$$\Big\downarrow\ l_{\max}=0$$

$$\boxed{\mathcal{K}_2^{(\ell=0)}(E_n^*)=-\frac{1}{F_{PV;00;00}(E_n,\vec{P},L)}}$$

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$$\det\left[F_{PV}(E,\overrightarrow{P},L)^{-1}+\mathscr{K}_2(E^*)\right]=0$$

$\Downarrow$ $l_{\max}=0$

$E_n^*=\sqrt{E_n^2-\vec{P}^2}$

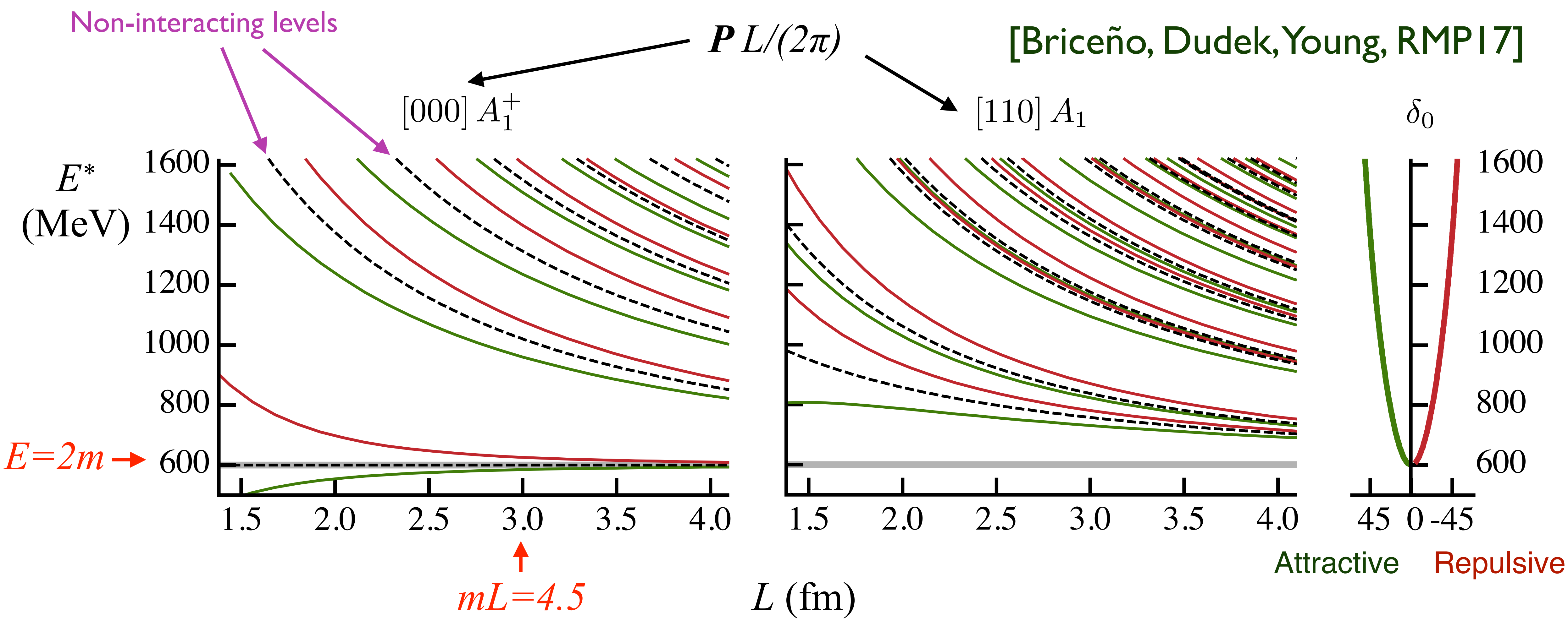
CM energy

$$\mathscr{K}_2^{(\ell=0)}(E_n^*)=-\frac{1}{F_{PV;00;00}(E_n,\overrightarrow{P},L)}$$

"measured" energy-level

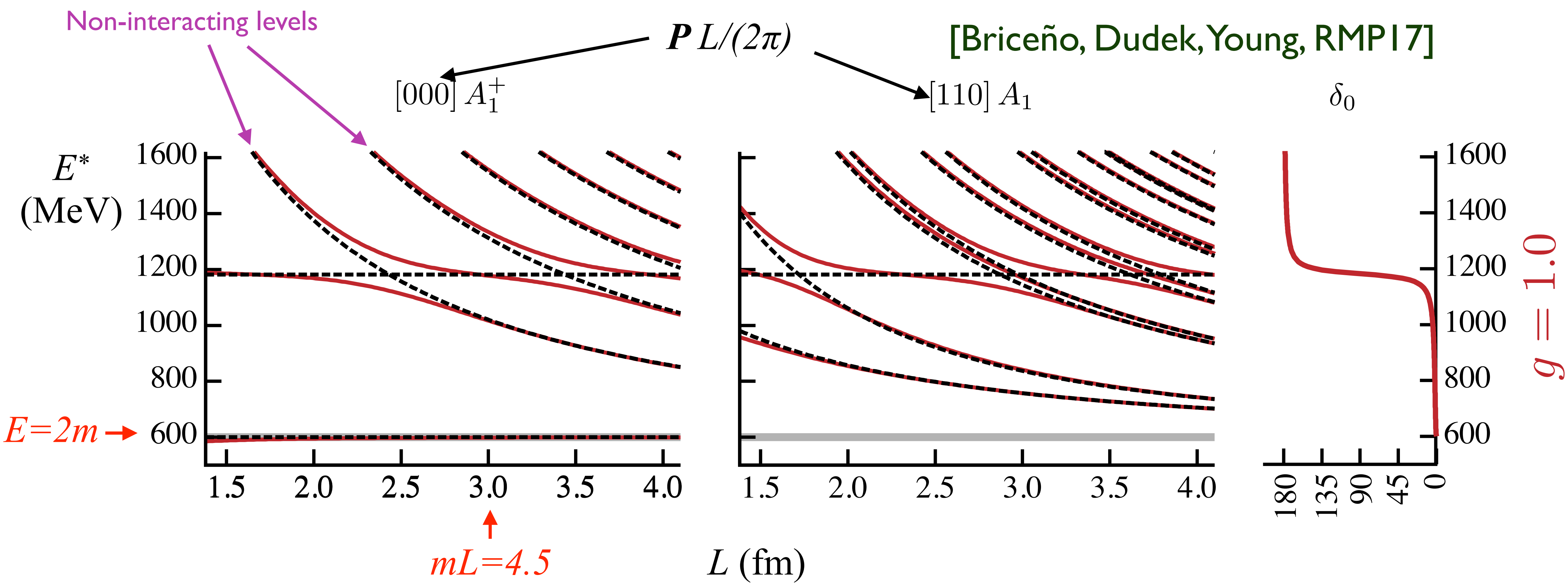
- One-to-one relation between energy levels and $\mathcal{K}_2\sim 1/(q^*\cot\delta)$

Overview of effects on spectrum



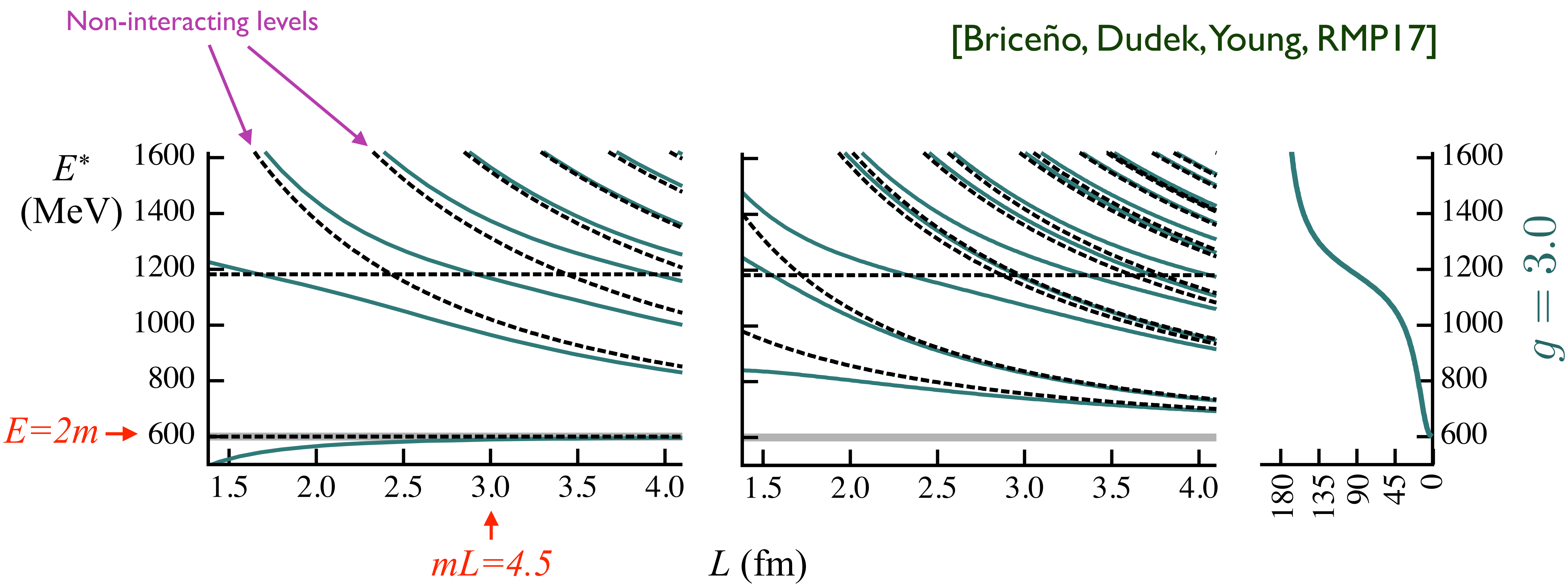
- Unphysical example for sake of illustration
- $l_{max}=0$, $m=300$ MeV, $a_0=\pm 0.32$fm ($m\,a_0=0.48$)
- Illustrates the power of using moving frames (**P**≠0) and multiple levels

Overview of effects on spectrum



- Narrow Brett-Wigner resonance at 1182 MeV
- Spectrum contains an additional level, and displays avoided level crossings

Overview of effects on spectrum



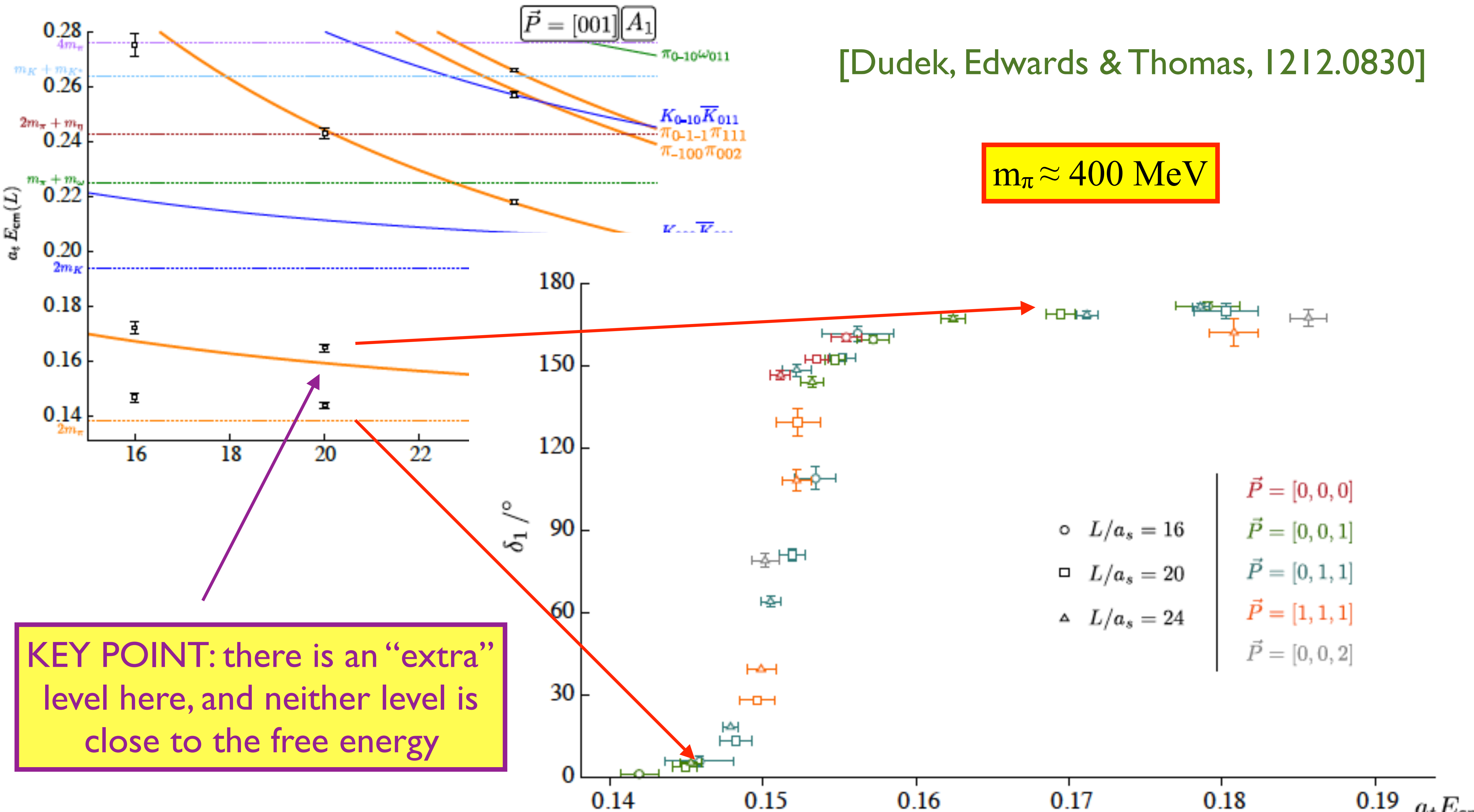
- Broad Brett-Wigner resonance at 1182 MeV
- Association of levels with "resonance" or "almost-free particles" no longer holds

ρ resonance from LQCD

- Most results to date assume $l_{max}=1$ and work with unphysical quark masses

[Dudek, Edwards & Thomas, 1212.0830]

$m_\pi \approx 400$ MeV

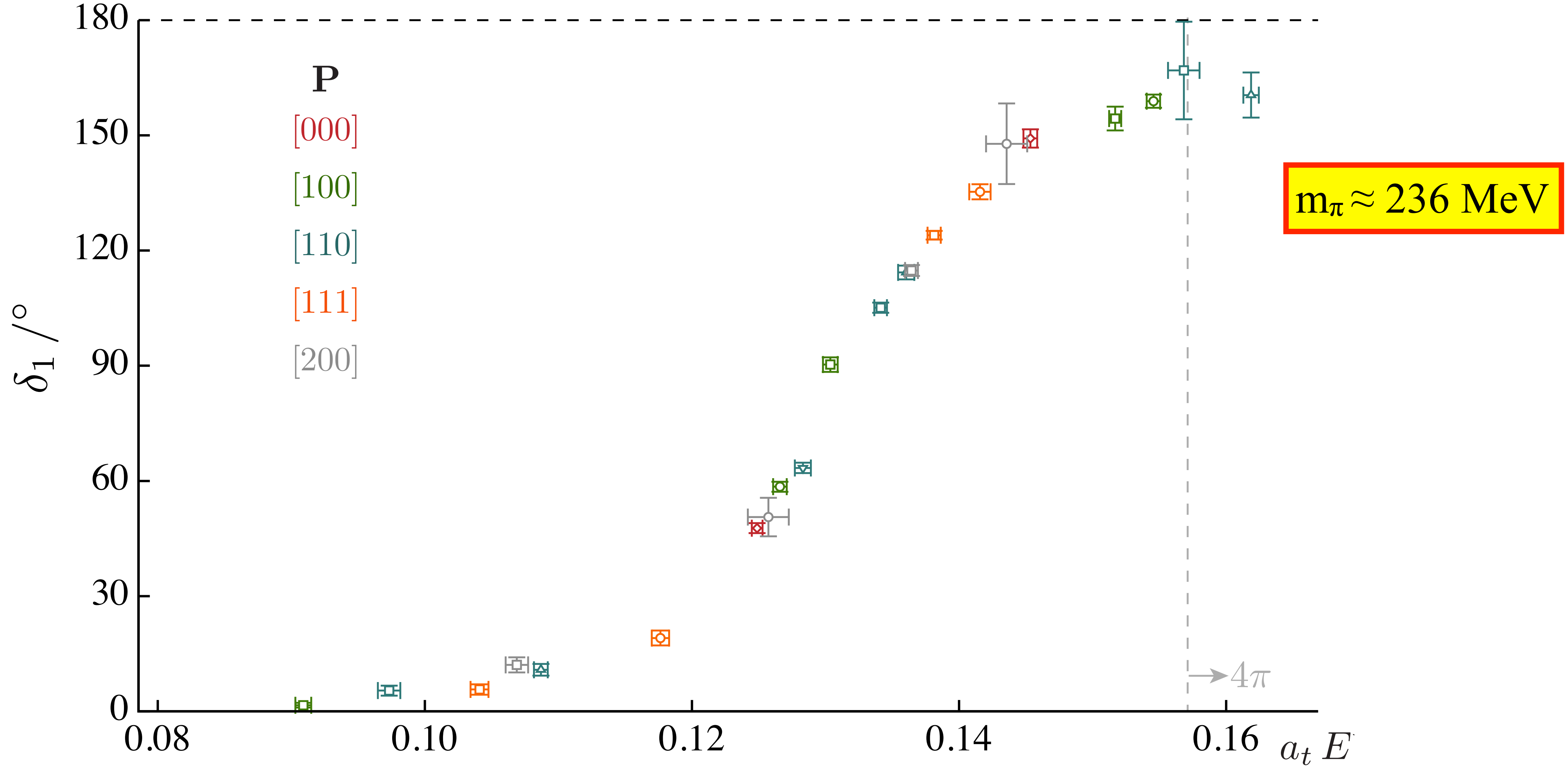


KEY POINT: there is an "extra" level here, and neither level is close to the free energy

ρ resonance from LQCD

- Most results to date assume $l_{max}=1$ and work with unphysical quark masses

[Wilson, Briceño, Dudek, Edwards & Thomas, 1507.02599]



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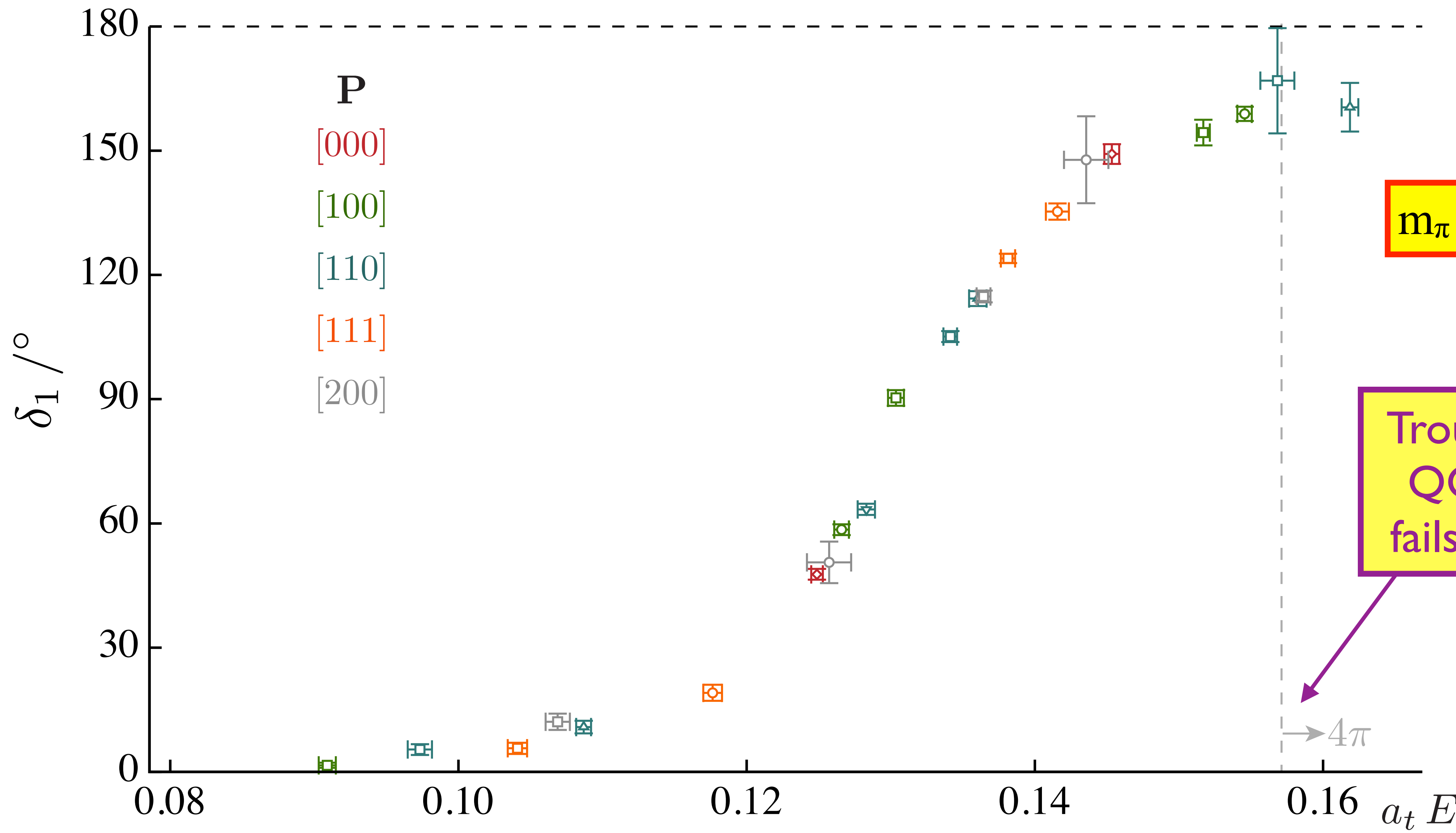
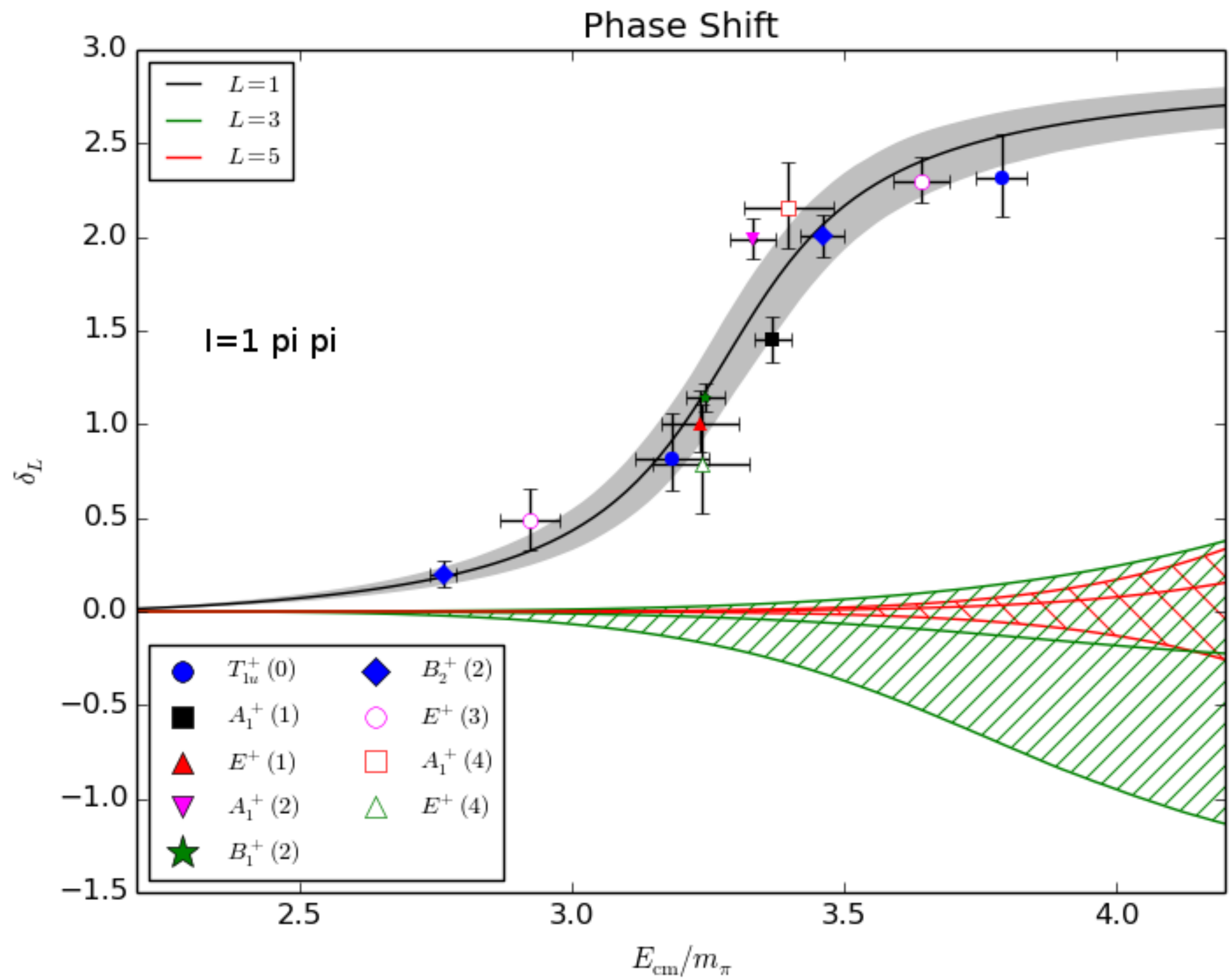


mπ ≈ 236 MeV

Trouble lurking? QC2 formally fails above here

ρ resonance from LQCD

- Some work includes higher partial waves, allowing better estimate of systematic errors



$m_\pi \approx 200$ MeV

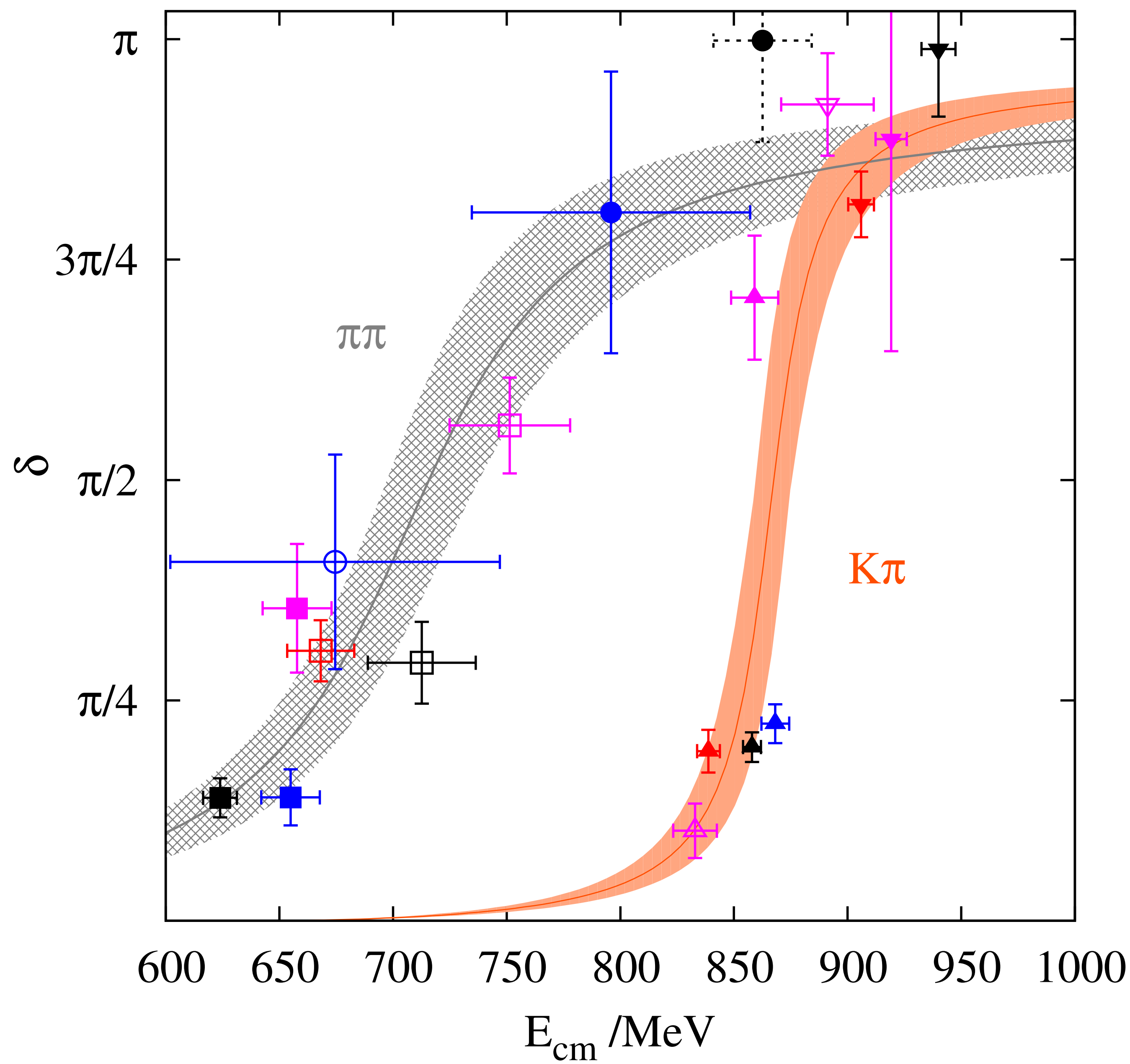
C. Morningstar Multihadron challenges

[Talk at MIAPP, October 2018]

ρ resonance from LQCD

- Pushing to physical quark masses (still with only a single lattice spacing)

[Bali *et al.* RQCD collab, 1512.08678]



$m_\pi \approx 150$ MeV

Generalizations

- Multiple two-particle channels [Hu, Feng & Liu, hep-lat/0504019; Lage, Meissner & Rusetsky, 0905.0069; Hansen & SS, 1204.0826; Briceño & Davoudi, 1204.1110]
  - e.g. $J^{PC} = 0^{++}$ $\pi\pi + K\bar{K}\,(+\eta\eta)$

$$\boxed{\det\left[\begin{pmatrix} F_{PV}^{\pi\pi}(E,\vec{P},L)^{-1} & 0 \\ 0 & F_{PV}^{K\bar{K}}(E,\vec{P},L)^{-1} \end{pmatrix} + \begin{pmatrix} \mathcal{K}_2^{\pi\pi}(E^*) & \mathcal{K}_2^{\pi K}(E^*) \\ \mathcal{K}_2^{\pi K}(E^*) & \mathcal{K}_2^{KK}(E^*) \end{pmatrix}\right] = 0}$$

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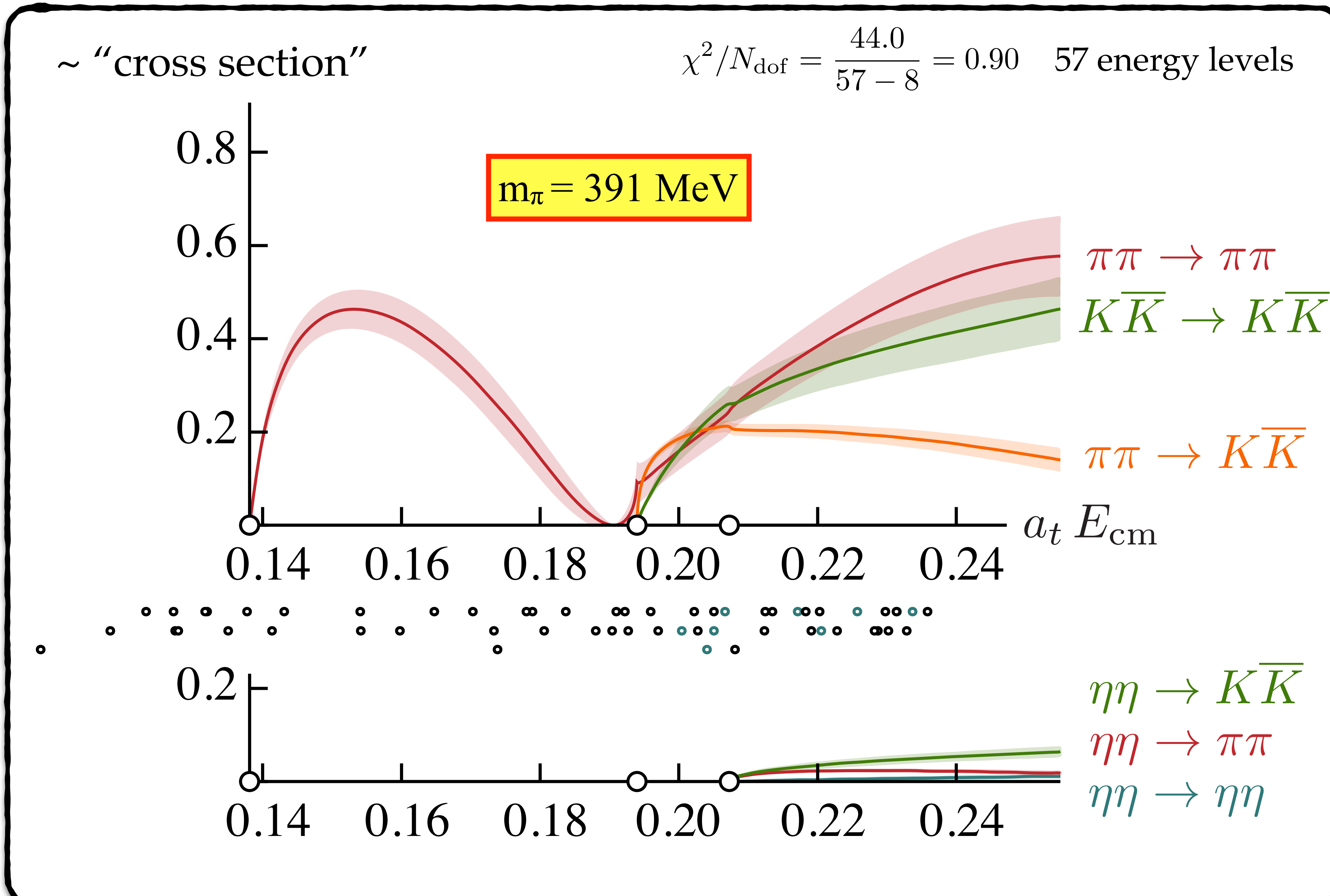
- Even if truncate to $l_{\max}=0$, there is no longer a one-to-one relation between energy levels and K-matrix elements
- Must parametrize the (enlarged) K matrix in some way and fit parameters to multiple spectral levels
- Using these parametrizations can study pole structure of scattering amplitude
- Approach is very similar to that used analyzing scattering data

Multiple-channel results

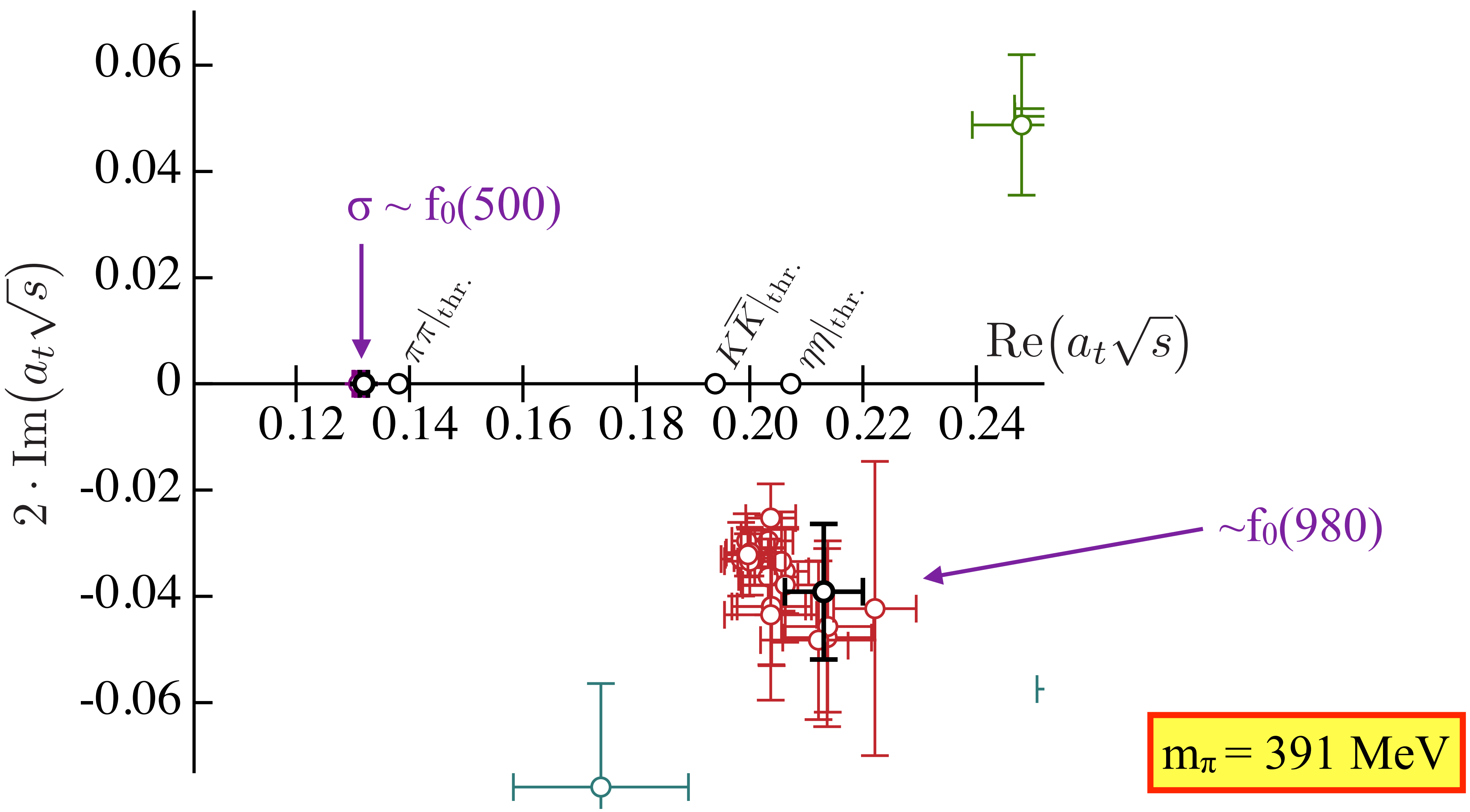
- S-wave above $2m_\pi$, $2m_K$, and $2m_\eta$
- Ansatz $\mathbf{K}^{-1}(s) = \begin{pmatrix} a+bs & c+ds & e \\ c+ds & f & g \\ e & g & h \end{pmatrix}$

[Briceño, Dudek, Edwards, & Wilson arXiv:1708.06667]



Multiple-channel results

- Parametrization-dependence of pole positions



[Briceño, Dudek, Edwards & Wilson
arXiv:1708.06667]

Multiple-channel results

- Very hot off the press!

arXiv:1907.03371v1 [hep-lat] 8 Jul 2019

A coupled-channel lattice study on the resonance-like structure $Z_c(3900)$

Ting Chen,[1] Ying Chen,[2] Ming Gong,[2] Chuan Liu,[3,*] Liuming Liu,[4] Yu-Bin Liu,[5] Zhaofeng Liu,[2] Jian-Ping Ma,[6] Markus Werner,[7] and Jian-Bo Zhang[8]
(CLQCD Collaboration)

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[3] *School of Physics and Center for High Energy Physics, Peking University, Beijing 100871, China*
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[4] *Institute of Modern Physics,Chinese Academy of Sciences, Lanzhou 730000, China*
University of Chinese Academy of Sciences, Beijing 100049, China
[5] *School of Physics, Nankai University, Tianjin 300071, China*
[6] *Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*
[7] *Helmholtz–Institut für Strahlen– und Kernphysik and Bethe Center for Theoretical Physics, Universität Bonn, D–53115 Bonn, Germany*
[8] *Department of Physics, Zhejiang University, Hangzhou 311027, China*

$\boxed{m_\pi = 320 \text{ MeV}}$

Two-particle decays & matrix elements

- Can extend analysis to obtain relation between $|\pi\pi\rangle_L$ and $|\pi\pi,\mathrm{in}\rangle$ (at same E^*)
  - In simplest case, involves phase shift and its derivative w.r.t. energy
  - Usually referred to as the Lellouch-Lüscher relation, after original derivation in non-moving frame [Lellouch & Lüscher, 2001]
  - Extended to moving frames in [KSS05; Kim, Christ & Yamazaki 2005]
  - General derivation and improved understanding given in [Briceño, Hansen & Walker-Loud, 2015]

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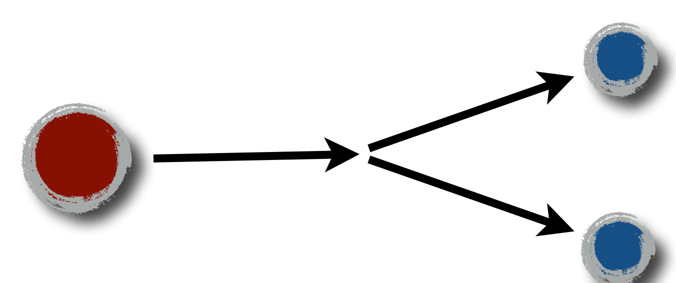
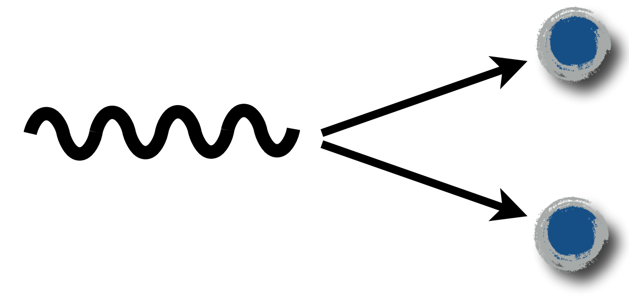
  - General derivation and improved understanding given in [Briceño, Hansen & Walker-Loud, 2015]
- First application: K→ππ decay amplitudes from ${}_L\langle\pi\pi|H_W|K\rangle_L$
  - Does QCD reproduce ΔI=½ rule? What is the prediction for ε'/ε?
  - Extensive work by [RBC-UKQCD collaboration]

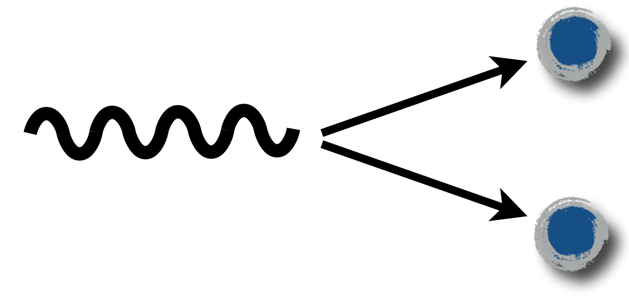
$$\langle\pi\pi,\mathrm{out}|\mathcal{H}|K\rangle \equiv$$



Time-like form factors $\langle \pi\pi,\ \mathrm{out}|\mathcal{J}_\mu|0\rangle \equiv$ 

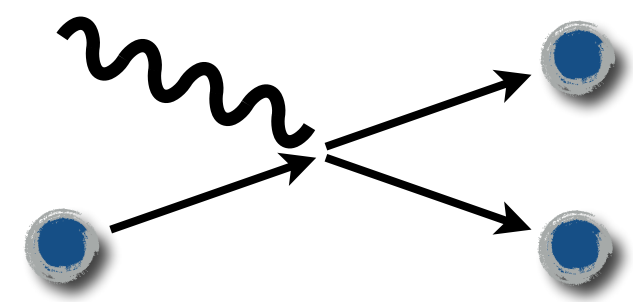
[Meyer, 2011]

Relevant for HVP contribution to muon g-2

Time-like form factors $\langle \pi\pi,\ \mathrm{out}|\mathcal{J}_\mu|0\rangle \equiv$ 

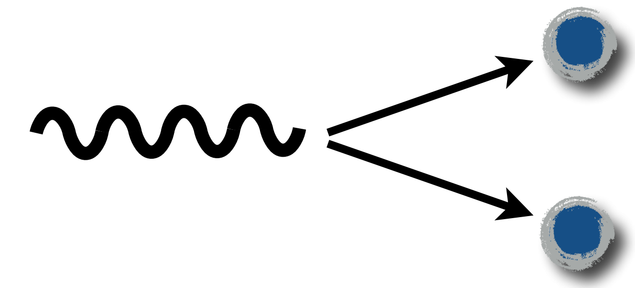
[Meyer, 2011]

Relevant for HVP contribution to muon g-2

Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

[Briceño, Hansen & Walker-Loud, 2015]

Time-like form factors $\langle \pi\pi,\ \mathrm{out}|\mathcal{J}_\mu|0\rangle \equiv$



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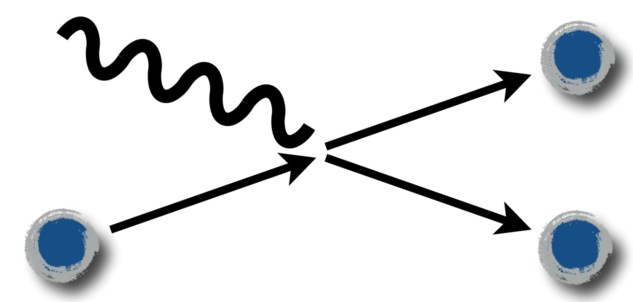
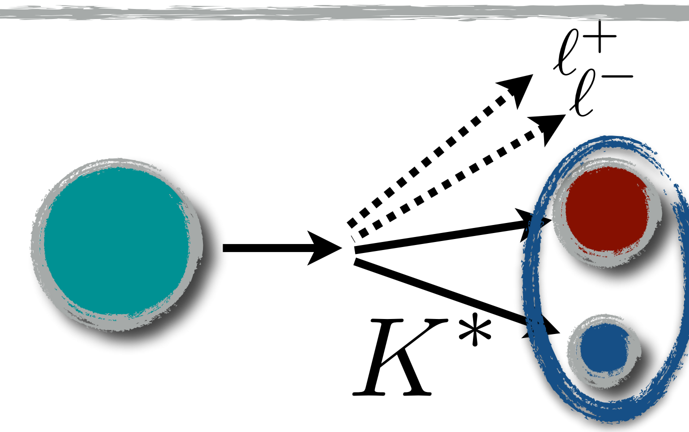
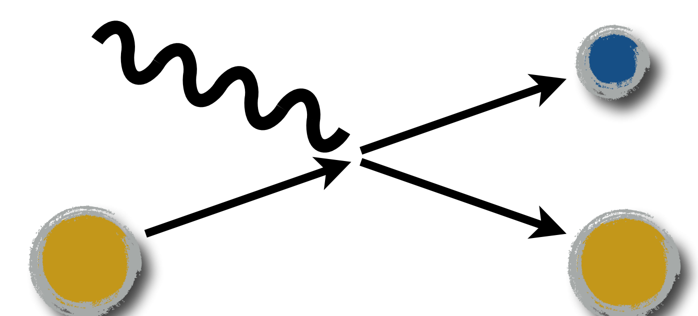


[Briceño, Hansen & Walker-Loud, 2015]

Resonance transition amplitudes $\langle K\pi, \mathrm{out}|\mathcal{J}_{\alpha\beta}|B\rangle \equiv$



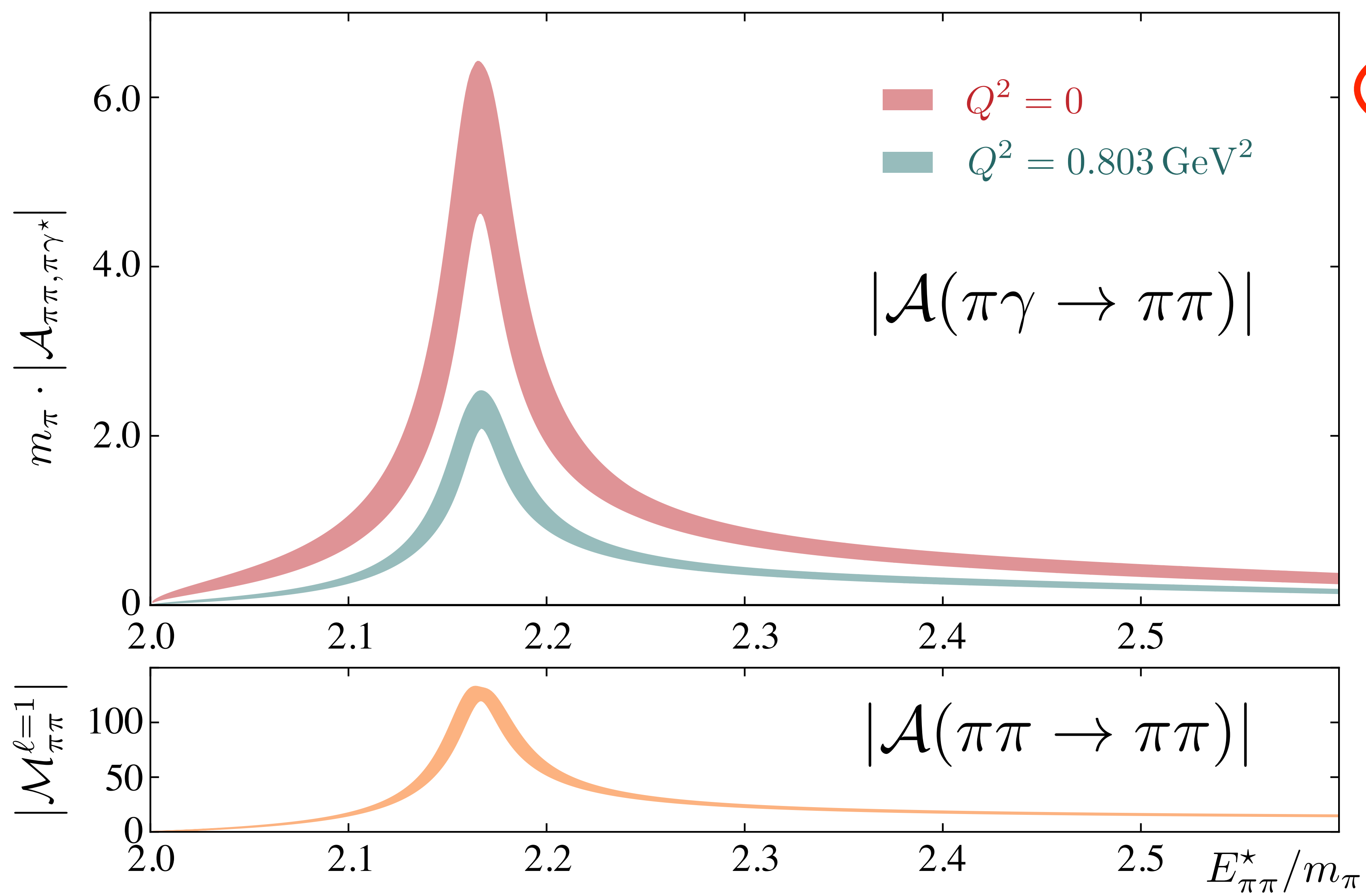
Particles with spin $\langle N\pi, \mathrm{out}|\mathcal{J}_\mu|N\rangle \equiv$



[Agadjanov *et al.*, 2014; Briceño, Hansen & Walker-Loud, 2015; Briceño & Hansen, 2016]

Numerical implementation

$$\pi\gamma \to \rho$$

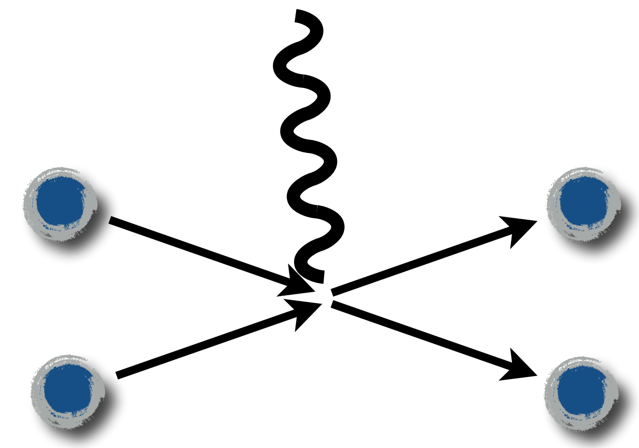


$m_\pi \approx 390$ MeV

Briceño, Dudek, Edwards, Shultz, Thomas, Wilson [HadSpec collab.]
arXiv:1604.03530

- Results also from [Leskovic, …, Meinel, …., arXiv:1611:00282]

Two-particle decays & matrix elements

ρ EM form factor $\langle \pi\pi, \mathrm{out} | \mathscr{J}_\mu | \pi\pi, \mathrm{in} \rangle =$ 

Under development: [Briceño, Hansen, 2015; Baroni, Briceño, Hansen & Ortega-Gama, 2018]

Summary of lecture 2

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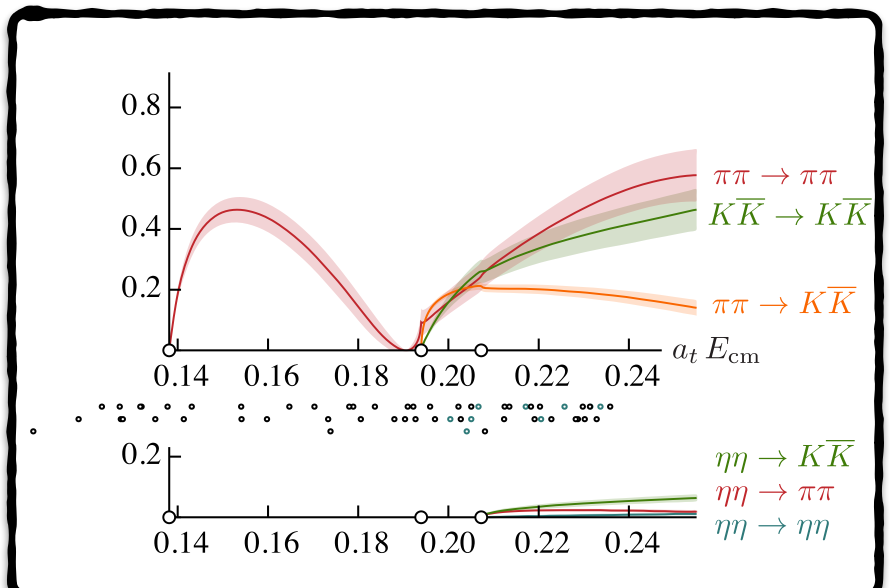
- Formalism for QC2 is developed, and widely implemented

$$\det\left[F_{PV}(E,\vec{P},L)^{-1}+\mathcal{K}_2(E^*)\right]=0$$



- Extensions to 2-particle matrix elements in various stages of development; expect all to reach maturity over next few years