Multiparticle scattering

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Outline

☑ Lecture 1

- Motivation/Background/Overview
- Deriving the two-particle quantization condition (QC2)

☑ Lecture 2

- Applying the QC2, in brief
- Deriving the three-particle quantization condition for identical scalars (QC3)

☐ Lecture 3

- Status of three-particle formalism
- Applications of QC3
- Outlook

Main references for this lecture

[Full list of references at end of lecture 3]

- Blanton & SS, 2105.12904, PRD21 (Formalism for 2+1 systems)
- Blanton, Romero-López & SS, 2111.12734, JHEP22, "Implementing the three-particle quantization condition for $\pi^+\pi^+K^+$ and related systems"
- Blanton, Hanlon, Hörz, Morningstar, Romero-López & SS, 2106.05590, JHEP 21, "$3\pi^+$ & $3K^+$ interactions beyond leading order from lattice QCD"
- Draper, Hanlon, Hörz, Morningstar, Romero-López & SS, 2302.13587, JHEP 23, "Interactions of πK, $\pi\pi K$ and $KK\pi$ systems at maximal isospin from lattice QCD"
- Blanton, Romero-López & SS, 1909.02973, PRL, "I=3 three-pion scattering amplitude from lattice QCD" (includes LO ChPT calculation of $\mathcal{K}_{\rm df,3}$)
- Baeza-Ballesteros, Bijnens, Husek, Romero-López, SRS & Sjö, 2303.13206, JHEP, "The isospin-3 three-particle K-matrix at NLO in ChPT"

Outline for Lecture 3

- Status of three-particle formalism
  - Focus on studies of resonances (rather than electroweak decays)
- Applications of the three-particle formalism
  - Overview
  - $\pi^+\pi^+K^+$ and $K^+K^+\pi^+$ amplitudes using LQCD
  - NLO ChPT results for $\mathscr{K}_{\rm df,3}$ for $3\pi^+ \to 3\pi^+$
- Summary and Outlook

Status of three-particle formalism (focusing on resonances, rather than electroweak decays)

Status: mid 2023

- 3 identical spinless particles [Hansen & SRS 14,15 (RFT); Hammer, Pang, Rusetsky 17 (NREFT); Mai, Döring 17 (FVU)]
  - Applications: $3\pi^+,\ 3K^+,\ 3D^+, \ldots$ as well as ϕ^4 theory
- Mixing of two- and three-particle channels for identical spinless particles [Briceño, Hansen, SRS 17]
  - Applications: Step on the way to $N(1440) \to N\pi,\ N\pi\pi$, etc.
- 3 degenerate but distinguishable spinless particles, e.g 3π with isospin 0, 1, 2, 3 [Hansen, Romero-López, SRS 20]; $I = 1$ case in FVU approach [Mai et al., 21]
  - Applications:

Resonance	$I_{\pi\pi\pi}$	J^P	Decays
$\omega(782)$	0	1^-	$\pi^+\pi^0\pi^-$
$h_1(1170)$	0	1^+	$\rho\pi \to 3\pi$
$\omega_3(1670)$	0	3^-	$3\pi,\ 5\pi$
$\pi(1300)$	1	0^-	$\rho\pi \to 3\pi$
$a_1(1260)$	1	1^+	$3\pi,\ K\bar{K}\pi$
$\pi_1(1400)$	1	1^-	$\eta\pi,\ 3\pi?$
$\pi_2(1670)$	1	2^-	$3\pi,\ K\bar{K}\pi$
$a_2(1320)$	1	2^+	$3\pi,\ K\bar{K},\ 5\pi,\ \eta\pi$
$a_4(1970)$	1	4^+	$3\pi,\ K\bar{K},\ 5\pi,\ \eta\pi$

Status: mid 2023 (continued)

- 3 nondegenerate spinless particles [Blanton, SRS 20]
  - Applications: $D_s^+ D^0 \pi^-$
- 2 identical +1 different spinless particles [Blanton, SRS 21]
  - Applications: $\pi^+\pi^+K^+, K^+K^+\pi^+$
- 3 identical spin-½ particles [Draper, Hansen, Romero-López, SRS 23]
  - Applications: $3n,\ 3p,\ 3\Lambda$

Form of generalized results

- Result for spinless particles presented in Lecture 2:

$$\det[1 + F_3 \mathscr{K}_{\mathrm{df},3}] = 0$$

$$F_3 = \widetilde{F}\left[\frac{1}{3} - \frac{1}{1/(2\omega L^3 \mathcal{K}_2) + \widetilde{F} + \widetilde{G}}\widetilde{F}\right]$$

Form of generalized results

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- Form for all the generalizations

$$\det[1 + \widehat{F}_3 \widehat{\mathscr{K}}_{\mathrm{df},3}] = 0$$

$$\widehat{F}_3 = \hat{F}\left[\frac{1}{3} - \frac{1}{1/(2\omega L^3 \hat{\mathscr{K}}_2 + \hat{F} + \hat{G}}\hat{F}\right]$$

What is different?

What is different?

$$\det[1 + \widehat{F}_3 \widehat{\mathcal{K}}_{\mathrm{df},3}] = 0$$

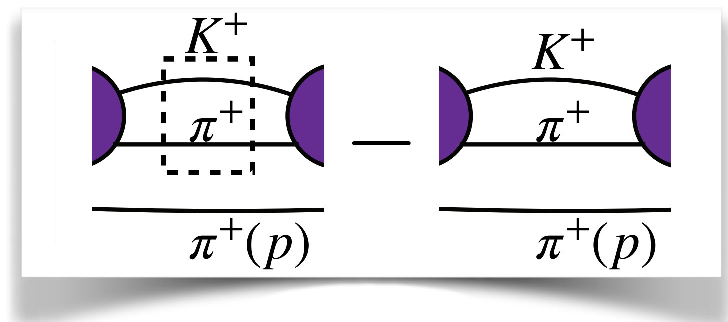
$$\widehat{F}_3 = \hat{F}\left[\frac{1}{3} - \frac{1}{1/(2\omega L^3 \hat{\mathcal{K}}_2 + \hat{F} + \hat{G}}\hat{F}\right]$$

- "Hats" indicate the presence of additional matrix indices (beyond $k\ell m$)
- Additional kinematic differences (e.g. for nondegenerate particles) are buried in the definitions of the symbols
- Illustrate for example of "2+1" system: $\pi^+\pi^+K^+$
  - Indices enlarge to $k\ell mi$, with i labeling the spectator flavor
  - Spectator is π^+ ($i = 1 \Rightarrow \pi^+K^+$ scattering) or K^+ ($i = 2 \Rightarrow \pi^+\pi^+$ scattering)
  - All partial waves contribute to π^+K^+ scattering, while only even waves contribute to $\pi^+\pi^+$ scattering

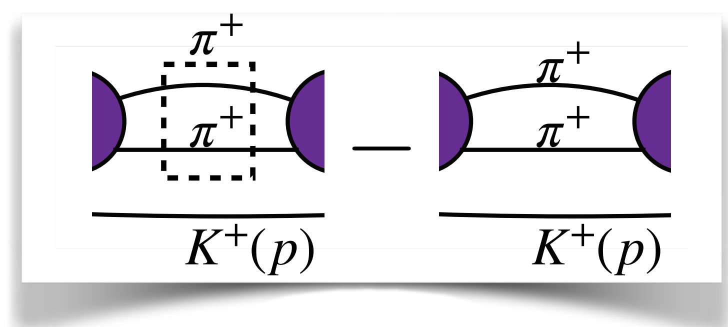
Details on matrices: $\widehat{F}$

$$\det\left[\widehat{F}_3^{-1}(E,\boldsymbol{P},L)+\widehat{\mathcal{K}}_{\text{df},3}(E^*)\right]=0$$

$$\widehat{F}_3=\frac{\widehat{F}}{3}-\widehat{F}\frac{1}{\widehat{\mathcal{K}}_{2,L}^{-1}+\widehat{F}+\widehat{G}}\widehat{F}$$



$$\widehat{F}=\begin{pmatrix}\tilde{F}^{(1)} & 0\\ 0 & \tilde{F}^{(2)}\end{pmatrix}$$



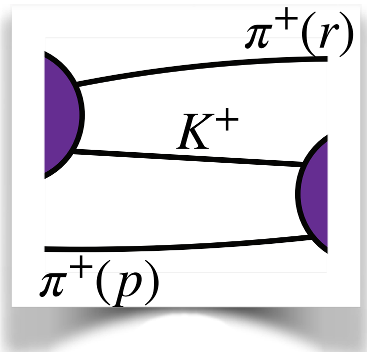
$$\left[\widetilde{F}^{(i)}\right]_{p'\ell'm';p\ell m}=\delta_{\boldsymbol{p}'\boldsymbol{p}}\frac{H^{(i)}(\boldsymbol{p})}{2\omega_p^{(i)}L^3}\left[\frac{1}{L^3}\sum_{\boldsymbol{a}}^{\text{UV}}-\text{PV}\int^{\text{UV}}\frac{d^3a}{(2\pi)^3}\right]$$
$$\times\left[\frac{\mathcal{Y}_{\ell'm'}(\boldsymbol{a}^{*(i,j,p)})}{(q_{2,p}^{*(i)})^{\ell'}}\frac{1}{4\omega_a^{(j)}\omega_b^{(k)}(E-\omega_p^{(i)}-\omega_a^{(j)}-\omega_b^{(k)})}\frac{\mathcal{Y}_{\ell m}(\boldsymbol{a}^{*(i,j,p)})}{(q_{2,p}^{*(i)})^{\ell}}\right]$$

- $H^{(i)}(p)$ is transition/cutoff function
- Only even ℓ contribute if $i=2$

Details on matrices: $\widehat{G}$

$$\det\left[\widehat{F}_3^{-1}(E,\boldsymbol{P},L)+\widehat{\mathcal{K}}_{\mathrm{df},3}(E^*)\right]=0$$

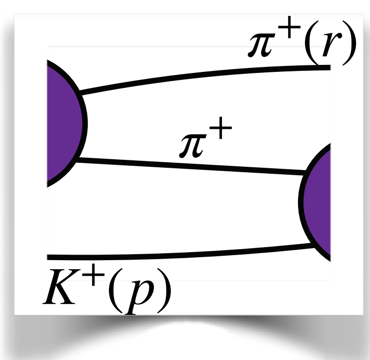
$$\widehat{F}_3=\frac{\widehat{F}}{3}-\widehat{F}\frac{1}{\widehat{\mathcal{K}}_{2,L}^{-1}+\widehat{F}+\widehat{G}}\widehat{F}$$



Symmetry factor

Basis-change factor of $(-1)^\ell$

$$\widehat{G}=\begin{pmatrix}\widetilde{G}^{(11)} & \sqrt{2}P_L\widetilde{G}^{(12)}\\ \sqrt{2}\widetilde{G}^{(21)}P_L & 0\end{pmatrix}$$



No "switch" between K^+ spectators, since only one of them

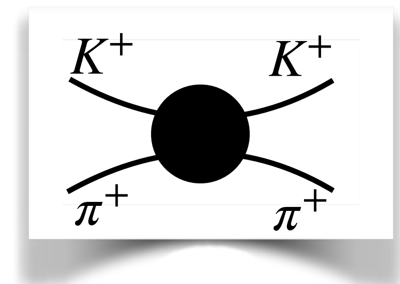
$$\left[\widetilde{G}^{(ij)}\right]_{p\ell'm';r\ell m}=\frac{1}{2\omega_p^{(i)}L^3}\frac{\mathcal{Y}_{\ell'm'}(\boldsymbol{r}^{*(i,j,p)})}{(q_{2,p}^{*(i)})^{\ell'}}\frac{H^{(i)}(\boldsymbol{p})H^{(j)}(\boldsymbol{r})}{b_{ij}^2-m_k^2}\frac{\mathcal{Y}_{\ell m}(\boldsymbol{p}^{*(j,i,r)})}{(q_{2,r}^{*(j)})^{\ell}}\frac{1}{2\omega_r^{(j)}L^3}$$

- Same $H^{(i)}(p)$ as in $\tilde{F}^{(i)}$

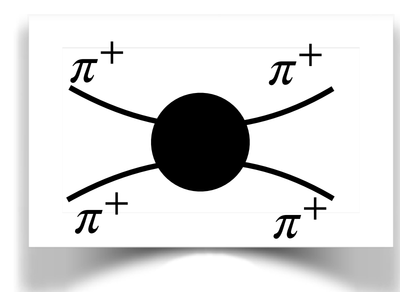
Details on matrices: $\widehat{\overline{\mathcal{K}}}_{2,L}$

$$\det\left[\widehat{F}_3^{-1}(E,\boldsymbol{P},L)+\widehat{\mathcal{K}}_{\mathrm{df},3}(E^*)\right]=0$$

$$\widehat{F}_3=\frac{\widehat{F}}{3}-\widehat{F}\frac{1}{\widehat{\overline{\mathcal{K}}}_{2,L}^{-1}+\widehat{F}+\widehat{G}}\widehat{F}$$



$$\widehat{\overline{\mathcal{K}}}_{2,L}=\begin{pmatrix}\overline{\mathcal{K}}_{2,L}^{(1)} & 0\\ 0 & \frac{1}{2}\overline{\mathcal{K}}_{2,L}^{(2)}\end{pmatrix}$$



Symmetry factor

$$\left[\overline{\mathcal{K}}_{2,L}^{(i)}\right]_{p\ell'm';r\ell m}=\delta_{\boldsymbol{pr}}2\omega_r^{(i)}L^3\left[\mathcal{K}_2^{(i)}(\boldsymbol{r})\right]_{\ell'm';\ell m},$$

$$\left[\mathcal{K}_2^{(i)}(\boldsymbol{r})^{-1}\right]_{\ell'm';\ell m}=\delta_{\ell'\ell}\delta_{m'm}\frac{\eta_i}{8\pi\sqrt{\sigma_i}}\left\{q_{2,r}^{*(i)}\cot\delta_\ell^{(i)}(q_{2,r}^{*(i)})+|q_{2,r}^{*(i)}|[1-H^{(i)}(\boldsymbol{r})]\right\}$$

Details on matrices: $\widehat{\mathscr{K}}_{\rm df,3}$

$$\det\left[\widehat{F}_3^{-1}(E,\boldsymbol{P},L)+\widehat{\mathcal{K}}_{\rm df,3}(E^*)\right]=0$$

$$\widehat{F}_3=\frac{\widehat{F}}{3}-\widehat{F}\frac{1}{\widehat{\mathcal{K}}_{2,L}^{-1}+\widehat{F}+\widehat{G}}\widehat{F}$$

$$\widehat{\mathcal{K}}_{\rm df,3}=\begin{pmatrix}[\mathcal{K}_{\rm df,3}]_{p\ell'm'1;k\ell m1} & [\mathcal{K}_{\rm df,3}]_{p\ell'm'1;k\ell m2}/\sqrt{2}\\ [\mathcal{K}_{\rm df,3}]_{p\ell'm'2;k\ell m1}/\sqrt{2} & [\mathcal{K}_{\rm df,3}]_{p\ell'm'2;k\ell m2}/2\end{pmatrix}$$

Symmetry factors

- Each entry involves the ***same*** infinite-volume amplitude, $\mathscr{K}_{\rm df,3}(p_1,p_{1'},p_2;k_1,k_{1'},k_2)$, decomposed in different coords
- $\mathscr{K}_{\rm df,3}$ is smooth (no cuts or two-particle poles) aside from possible poles associated with three-particle resonances
- $\mathscr{K}_{\rm df,3}$ is invariant under Lorentz transformations, T, P, and interchange of identical particles in initial and/or final states
- $\mathscr{K}_{\rm df,3}$ depends on cutoff function H, and is thus not physical
- Related to $\mathscr{M}_3$ by integral equations of similar form to those discussed in Lecture 2

Applications of the three-particle formalism: Overview

Status: applications

[Detailed references at end of slides]

- $3\pi^+$: determined parameters in threshold expansion of $\mathcal{K}_{\rm df,3}$, including pair interactions in s- and d-waves; integral equations solved for s-wave interactions only [Blanton et al., 19, 21; Mai et al. 19; Culver et al. 19, Fisher et al. 20, Hansen et al. 20, Brett et al. 21]
- $3K^+$: determined s- and d-wave parameters in $\mathcal{K}_{\rm df,3}$ [Alexandru et al. 20; Blanton et al. 21]
- ϕ^4: extracted $\mathcal{K}_{\rm df,3}$ in single-scalar theory; extracted 3-particle resonance parameters in two-scalar theory with both RFT and FVU approaches [Romero-López et al. 18, Garofalo et al. 22]
- 3π with $I = 1$: first study of $a_1(1260)$ with formalism based on 2 levels; solved integral equations in FVU approach [Mai et al., 21]
- $\pi^+\pi^+K^+$ & $K^+K^+\pi^+$: determined s- and p-wave parameters in $\mathcal{K}_{\rm df,3}$; found evidence for small discretization effects [Draper et al, 23]
- Integral equations solved for complex energies for simple system with near-unitary two-particle interactions and Efimov states (bound or resonant) [Jakura et al. 20, Dawid et al., 23]
- ChPT: LO results for $3\pi^+$, $\pi^+\pi^+K^+$, $K^+K^+\pi^+$, $3K^+$, including a^2 effects: agree in rough magnitude but not in detail with results from LQCD calculations [Blanton et al., 19, 21]
- ChPT: NLO result for $3\pi^+$; greatly improves agreement with LQCD results [Baeza-Ballesteros et al., 23]

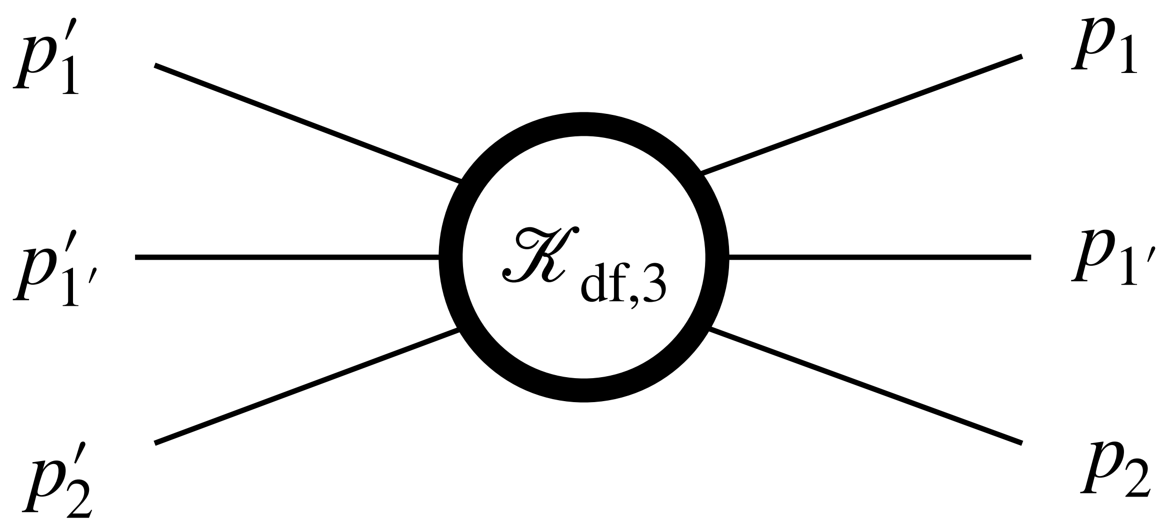
Truncation

$$\det\left[\widehat{F}_3^{-1} + \widehat{\mathcal{K}}_{\mathrm{df},3}\right] = 0$$

matrices with indices $\boldsymbol{k}\ell m i$

- To use quantization condition, one must truncate matrix space, as for the two-particle case
- Spectator-momentum space is truncated by cut-off function H(**k**)
- Need to truncate sums over ℓ in $\mathcal{K}_2$, $\widehat{\mathcal{K}}_{\mathrm{df},3}$
  - Automatically truncates $\widehat{F}$, $\widehat{G}$
- Illustrate with example of 2+1 system

Threshold expansion for $\mathcal{K}_{\mathrm{df},3}$



Useful invariants:

$$\Delta = \frac{s-M}{M^2}\,, \qquad s = (p_1 + p_{1'} + p_2)^2 = P^2\,,$$

$$\Delta_2^S = \Delta_2 + \Delta_2'\,, \quad \Delta_2 = \frac{(p_1+p_{1'})^2 - 4m_1^2}{M^2}\,, \quad \Delta_2' = \frac{(p_1'+p_{1'}')^2 - 4m_1^2}{M^2}\,,$$

$$\tilde{t}_{22} = \frac{t_{22}}{M^2} = \frac{(p_2 - p_2')^2}{M^2}\,, \qquad M = 2m_1 + m_2\,.$$

- Expand in powers of $\Delta \sim \Delta_2^S \sim \tilde{t}_{22}$
- 1 term of $\mathcal{O}(\Delta^0)$, 3 terms of $\mathcal{O}(\Delta)$, 11 terms of $\mathcal{O}(\Delta^2)$
- In practice, work to linear order, so that there are 4 undetermined constants:

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\Delta + \mathcal{K}_{\mathrm{df},3}^{B,1}\Delta_2^S + \mathcal{K}_{\mathrm{df},3}^{E,1}\tilde{t}_{22}$$

Properties of the terms

$$\mathcal{K}_{\rm df,3} = \mathcal{K}_{\rm df,3}^{\rm iso,0} + \mathcal{K}_{\rm df,3}^{\rm iso,1}\Delta + \mathcal{K}_{\rm df,3}^{B,1}\Delta_2^S + \mathcal{K}_{\rm df,3}^{E,1}\widetilde{t}_{22}$$

Independent of relative momenta & angles

Depends on relative momenta & angles, but only contains $J_{\rm tot} = 0$, and only appears in trivial FV irreps

Depends on relative momenta & angles, contains $J_{\rm tot} = 0$ & 1, and appears in some nontrivial FV irreps

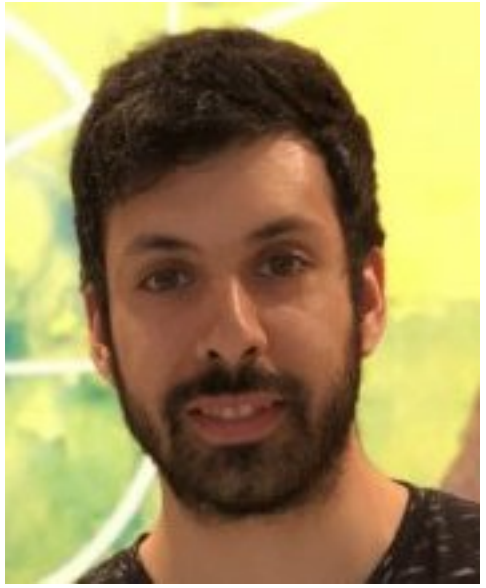
- Even though $\mathscr{K}_{\rm df,3}^{E,1}$ term is of higher order than $\mathscr{K}_{\rm df,3}^{\rm iso,0}$ term, it can be easier to determine as it appears in more FV irreps
- When decompose into p, ℓ, m, i basis (a straightforward but very tedious exercise)
  - Isotropic terms lead only to terms with $\ell' = \ell = 0$
  - $\mathscr{K}_{\rm df,3}^{B,1}$ & $\mathscr{K}_{\rm df,3}^{E,1}$ contain $\ell', \ell = 0{,}1$ terms
  - Only $\mathscr{K}_{\rm df,3}^{E,1}$ contains $\ell' = \ell = 1$ terms
- For consistency, truncate effective-range expansion of $\mathscr{K}_2$ at linear order in q^2
- Decompose all terms in QC3 into irreps of appropriate little group (subgroup of cubic group that leaves total momentum $\boldsymbol{P}$ unchanged)

Applications of the three-particle formalism: $\pi^+\pi^+K^+$ and $K^+K^+\pi^+$ amplitudes using LQCD

[Draper, Hanlon, Hörz, Morningstar, Romero-López & SRS, 2302.13587 (JHEP)]







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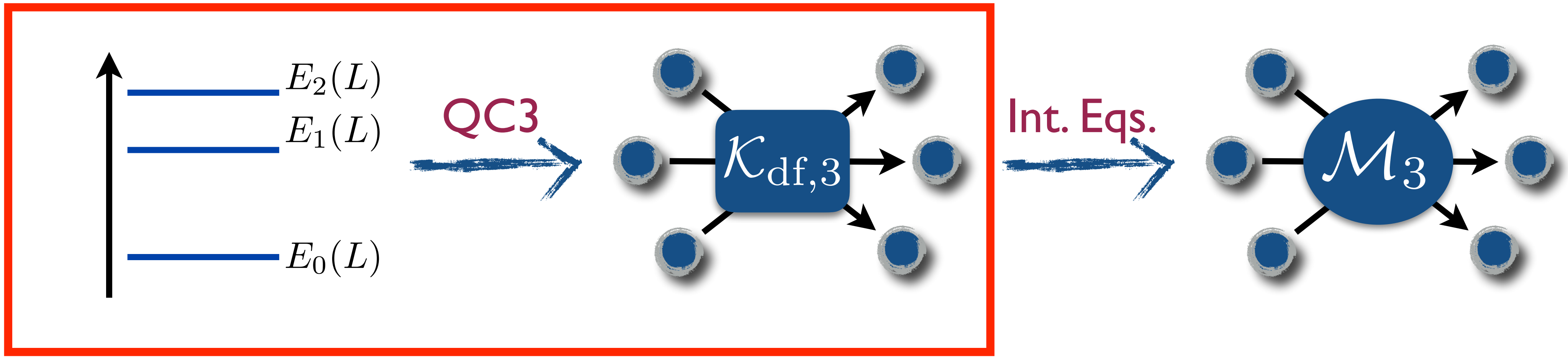
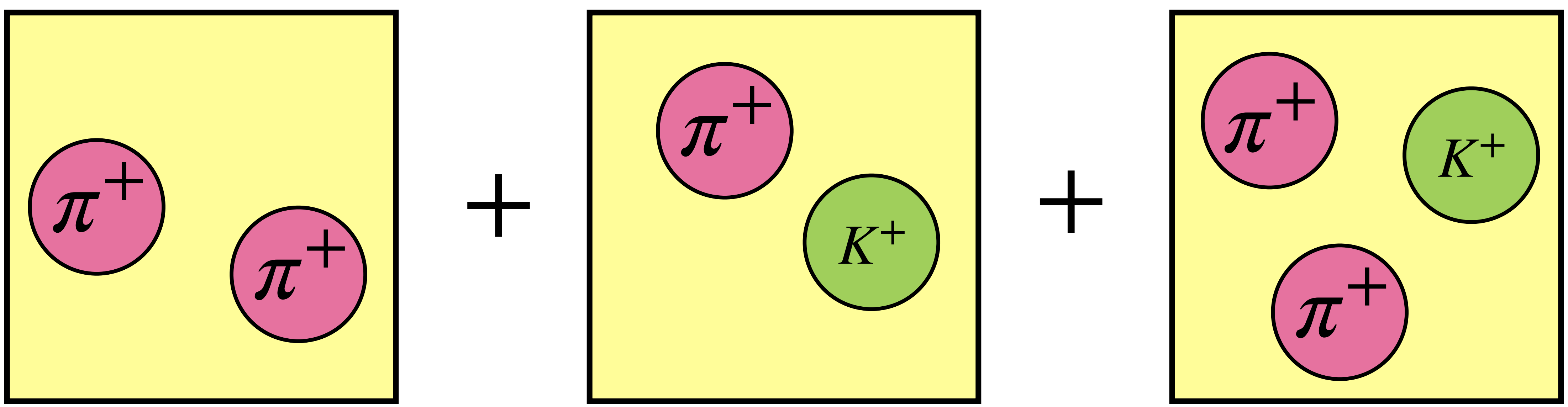
Strategy



- Consider multiparticle system with weakly repulsive interactions—pions and kaons at maximal isospin ($2\pi^+/3\pi^+, 2K^+/3K^+, 2\pi^+/\pi^+K^+/3K^+, 2K^+/\pi^+K^+/3K^+$)
  - No resonances in two-particle subchannels or in three-particle system
  - Simultaneously fit to several spectra using threshold expansions for K matrices
    - For example, to obtain the $\pi^+\pi^+K^+$ interaction need:

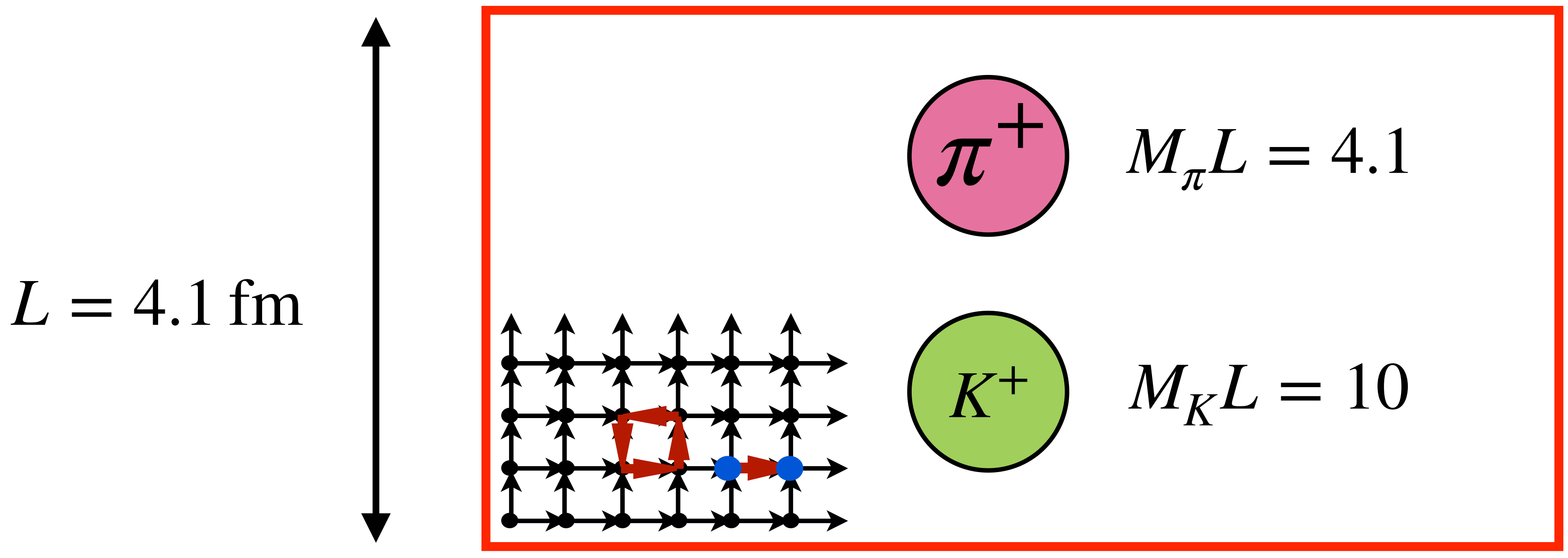


Lattices used in pilot calculation

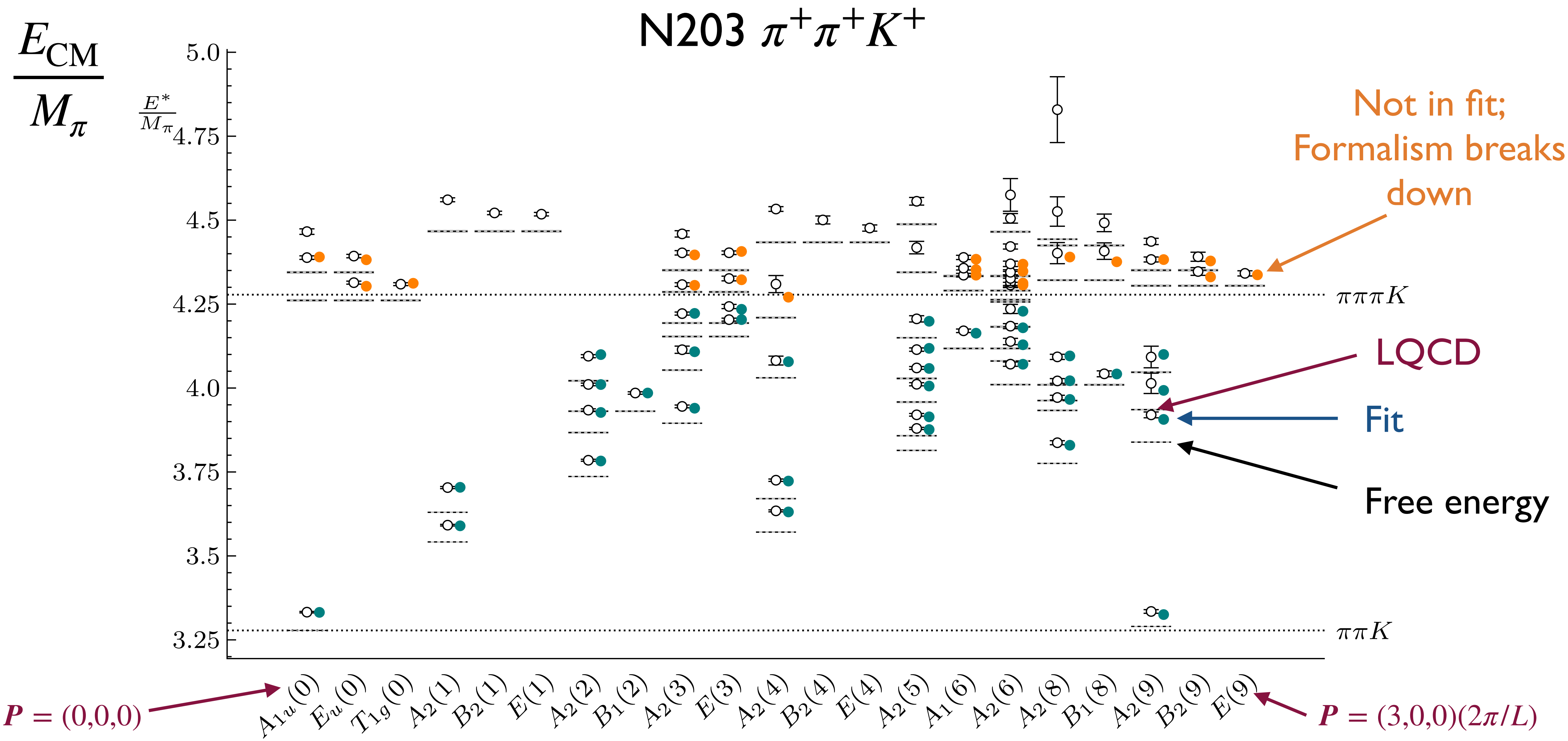
- Improved Wilson fermions at $a = 0.064\,\mathrm{fm}$ (CLS lattices)

	$(L/a)^3 \times (T/a)$	M_π [MeV]	M_K [MeV]	N_{cfg}	t_{src}/a	N_{ev}	dilution	$N_r(\ell/s)$
N203	$48^3 \times 128$	340	440	771	32, 52	192	(LI12,SF)	6/3
D200	$64^3 \times 128$	200	480	2000	35, 92	448	(LI16,SF)	6/3

D200 configurations



Example of fit

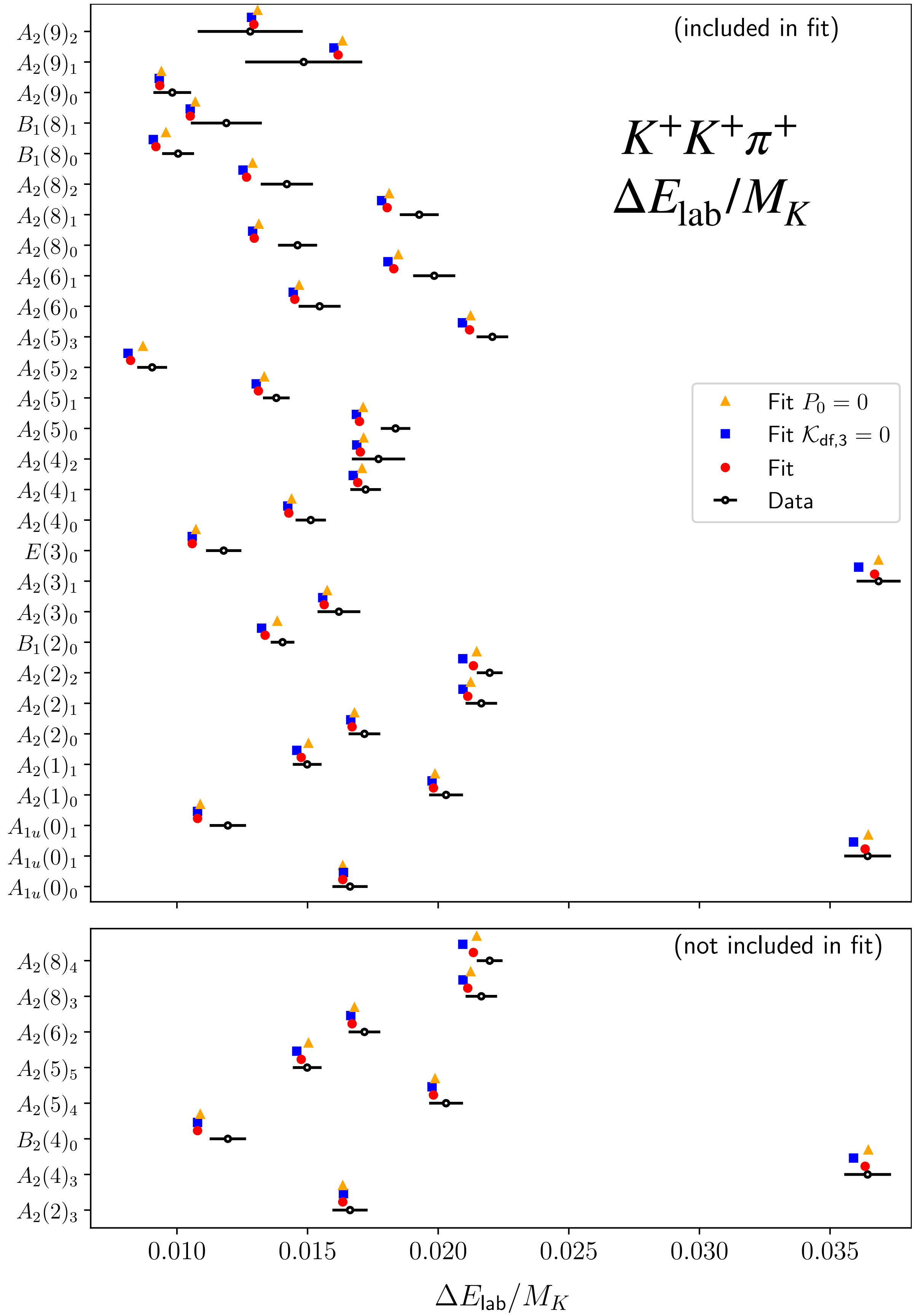


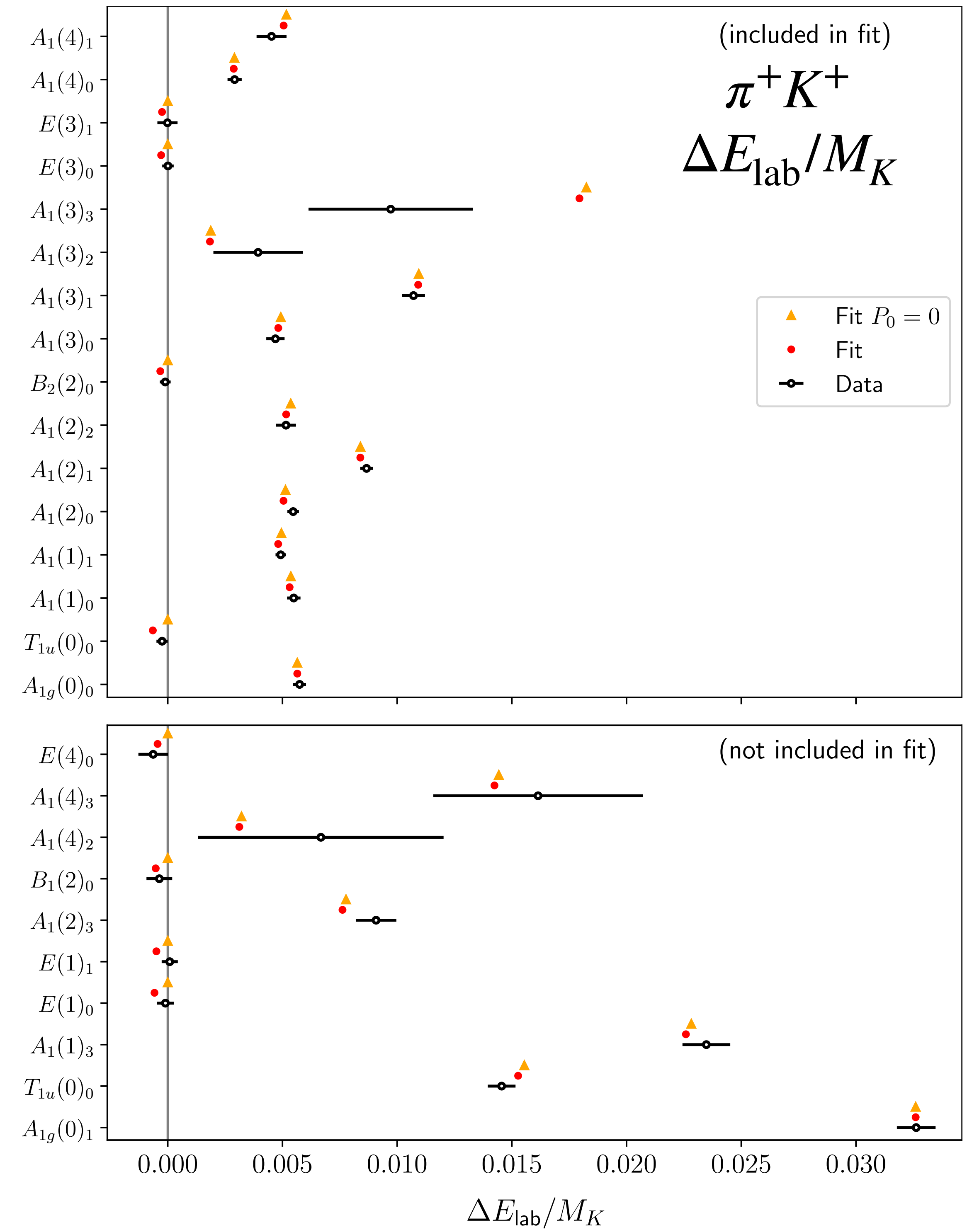
Simultaneous fit to 27 $\pi^+\pi^+$, 19 π^+K^+, & 36 $\pi^+\pi^+K^+$ levels with 9 parameters

$\chi^2/\mathrm{DOF} = 119/(82-9)$

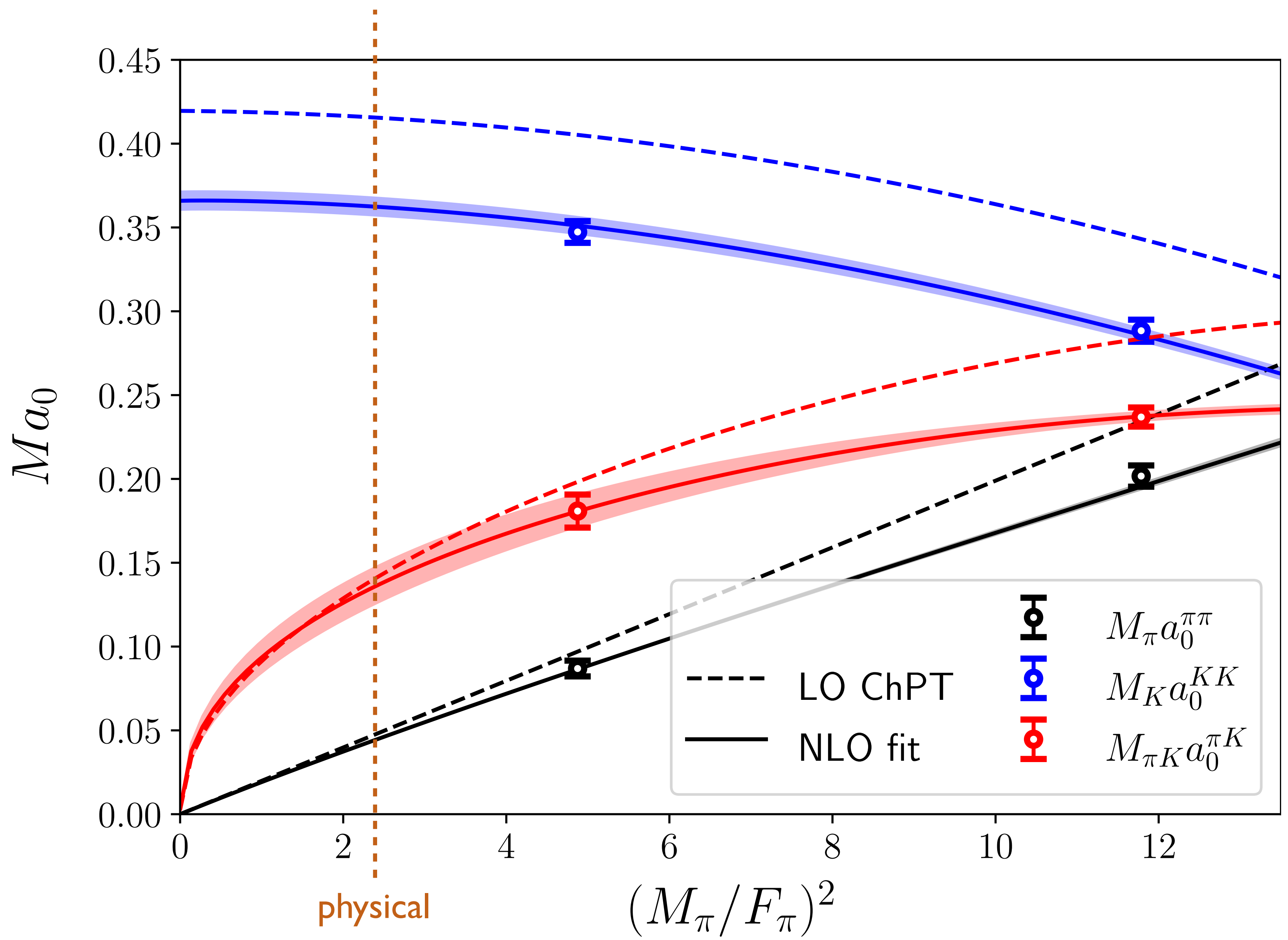
Fit is to lab-frame shifts

Simultaneous fit to 28 K^+K^+, 16 π^+K^+, & 29 $K^+K^+\pi^+$ levels with 10 parameters on D200: $\chi^2/\mathrm{DOF} = 162/(73-10)$



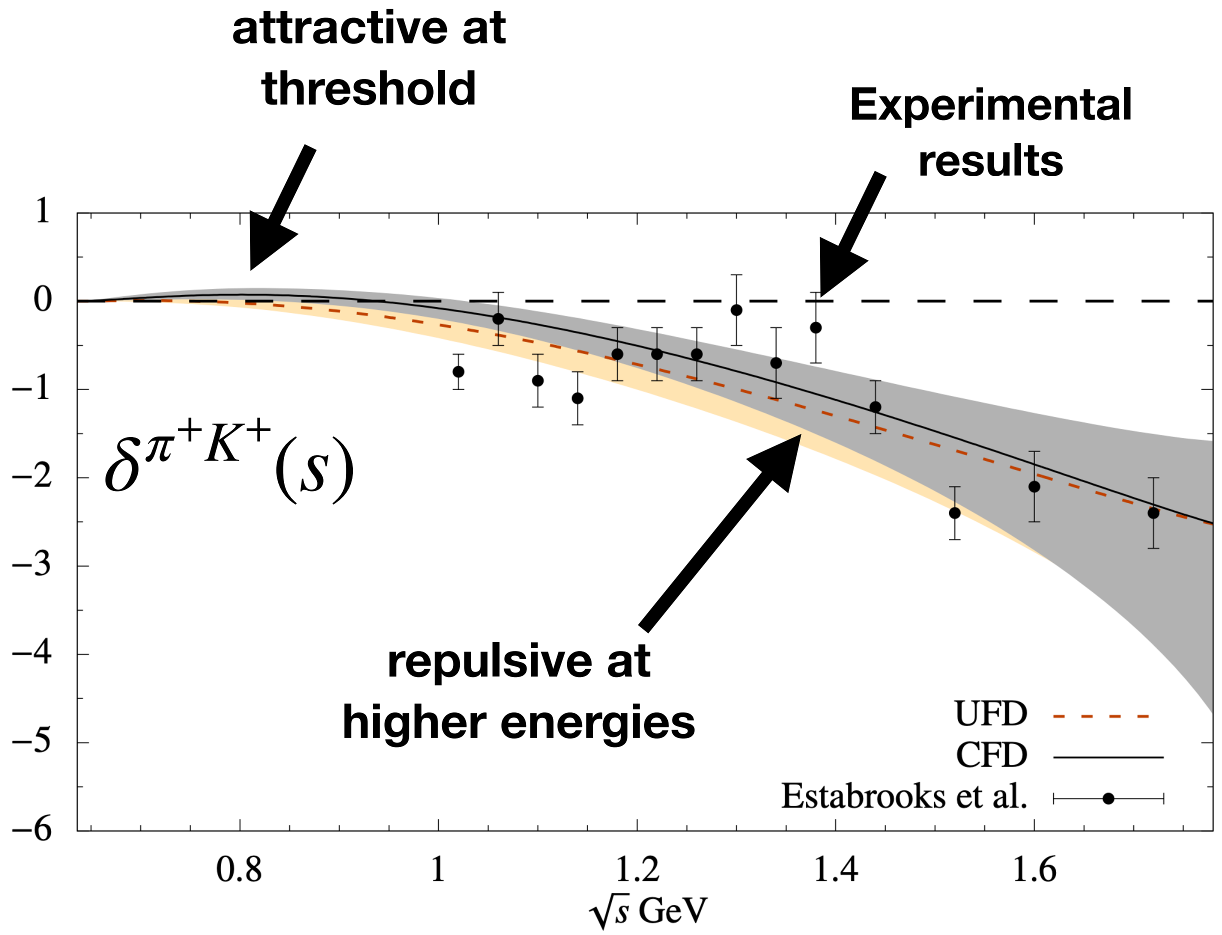


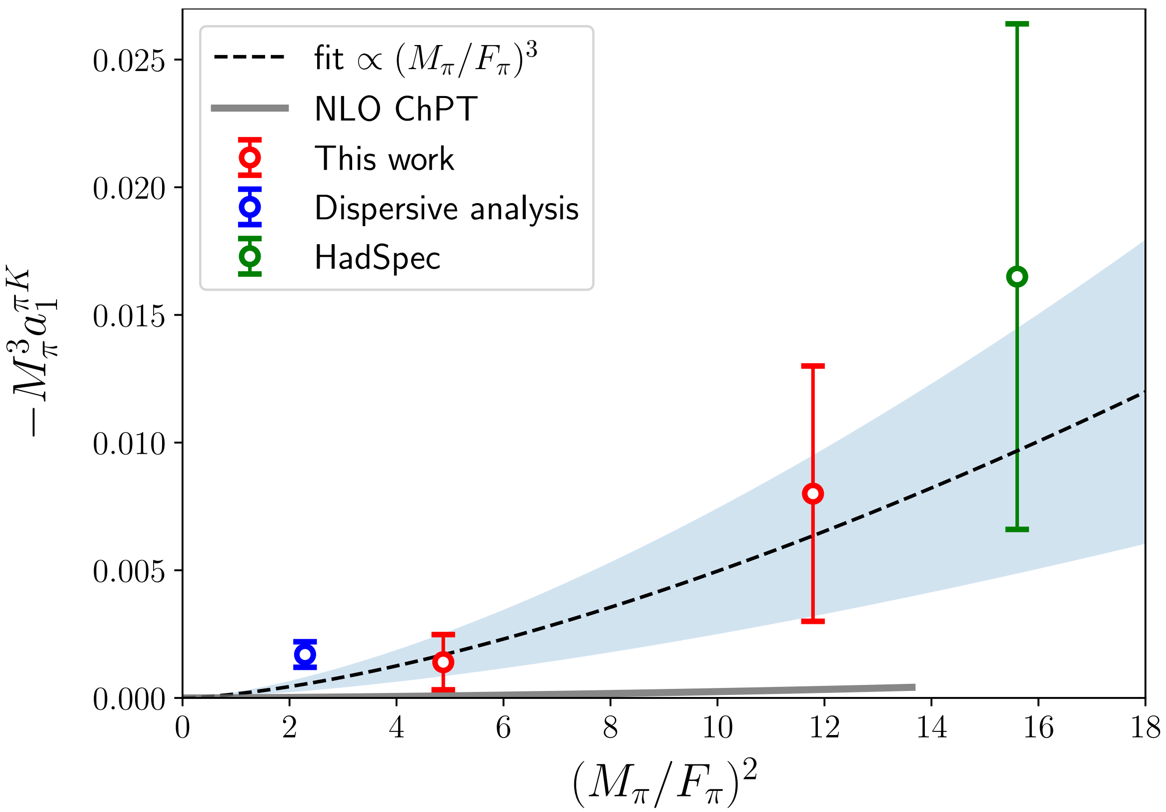
Results: scattering lengths



- 2-particle s-wave scattering lengths are well determined
- All are repulsive and consistent with ChPT
  - Evidence for small discretization errors from fits to "Wilson ChPT"

P-wave π^+K^+ scatt. Length

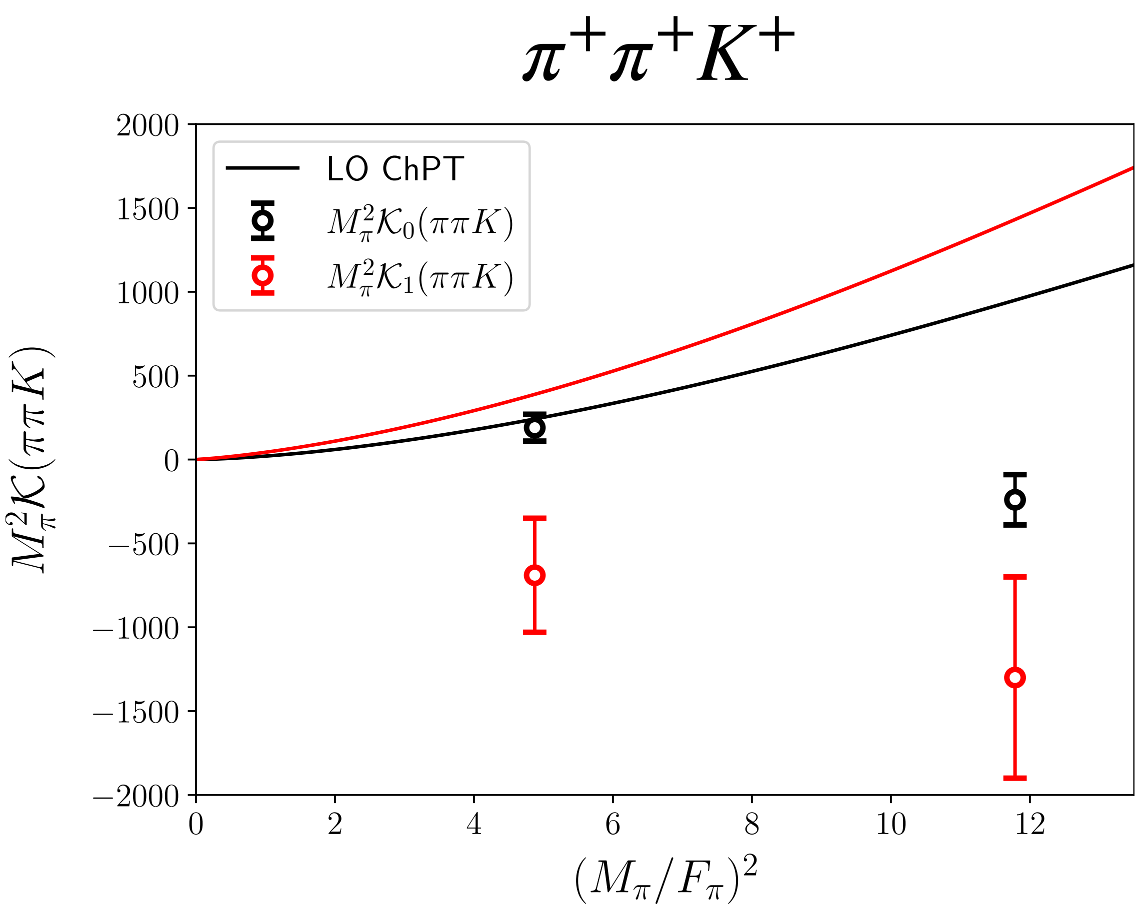


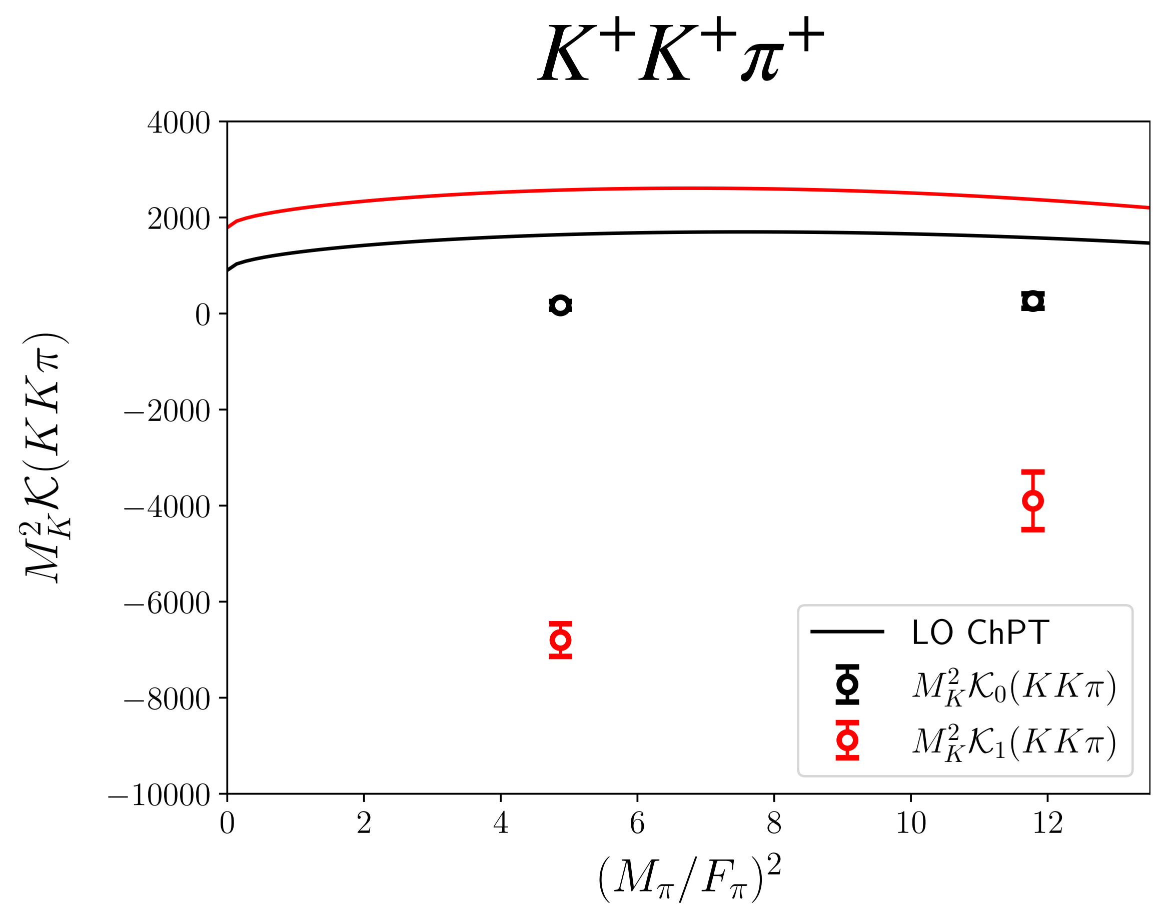


[Pelaez, Rodas, 2010.11222]

- Find evidence for **attractive** p-wave scattering length
  - Consistent with dispersive analysis

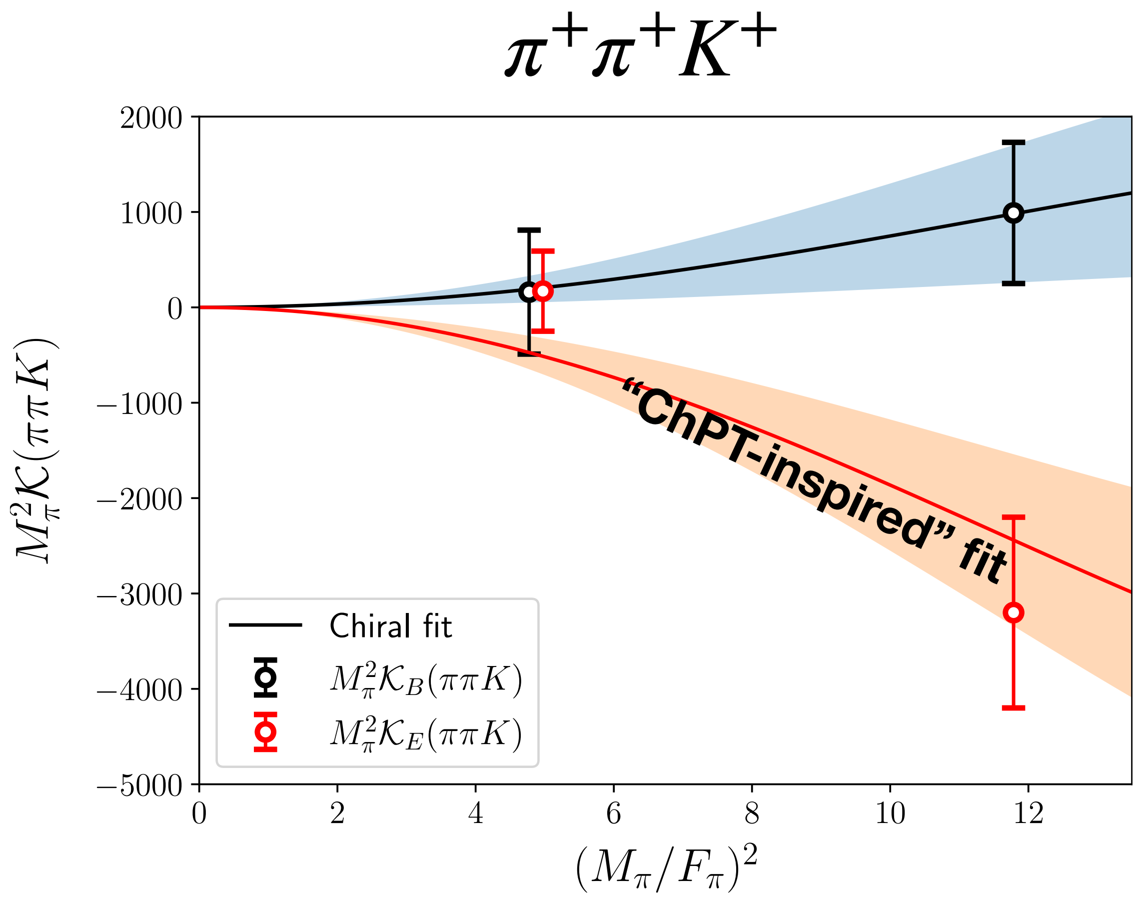
s-wave contributions to $\mathscr{K}_{\rm df,3}$

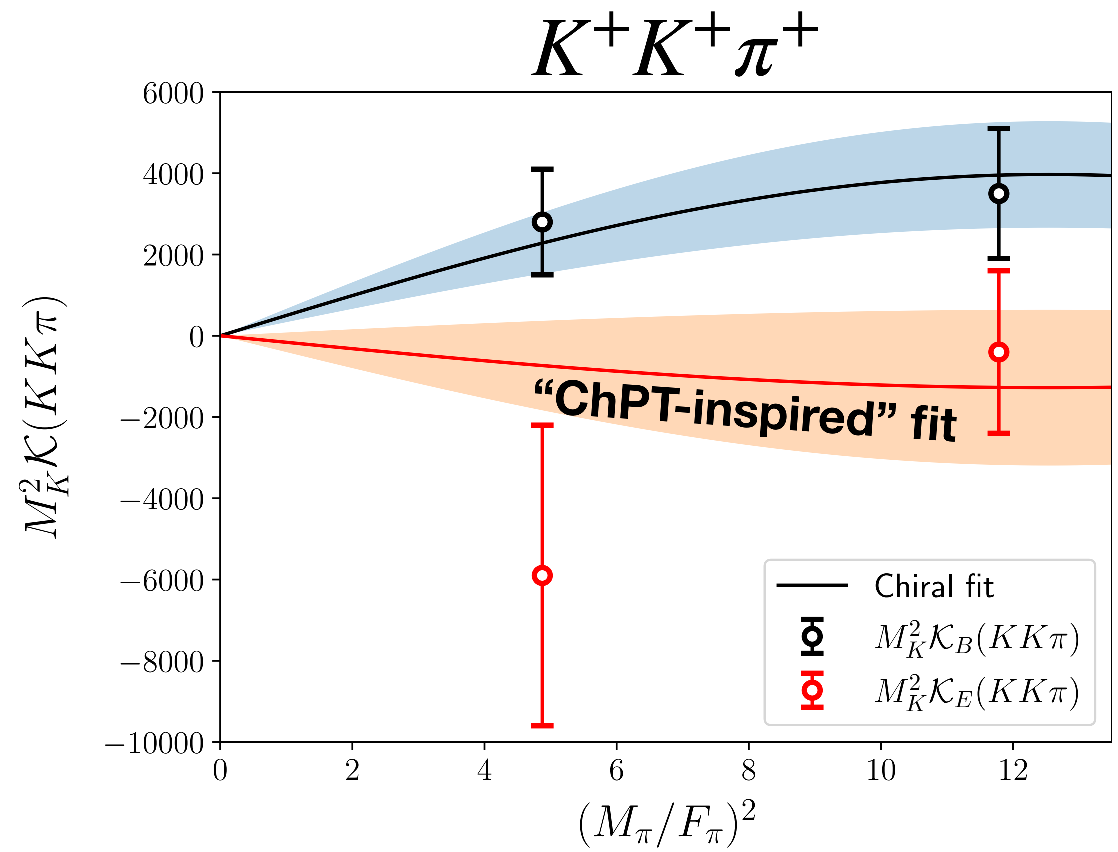




- Evidence for nonzero values ($2-5\sigma$)
- Overall effect of $\mathscr{K}_{\rm df,3}$ is repulsive
- LO ChPT predicts opposite sign (but see later)

p-wave contributions to $\mathcal{K}_{\text{df},3}$

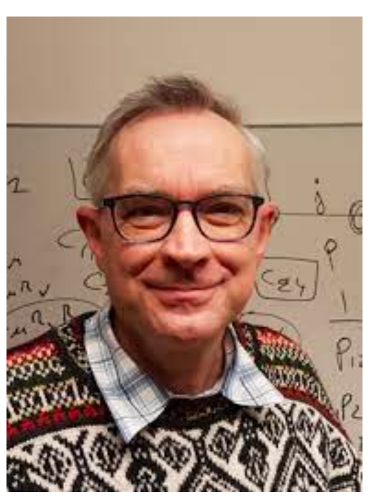


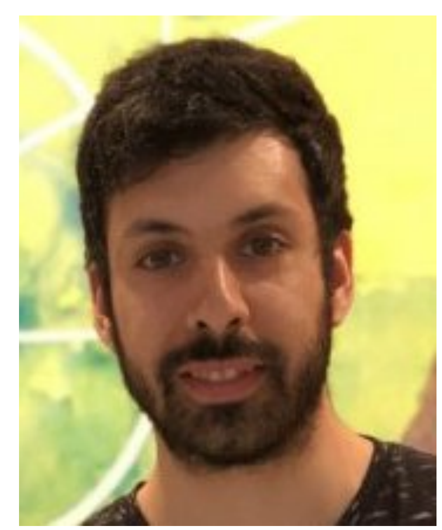
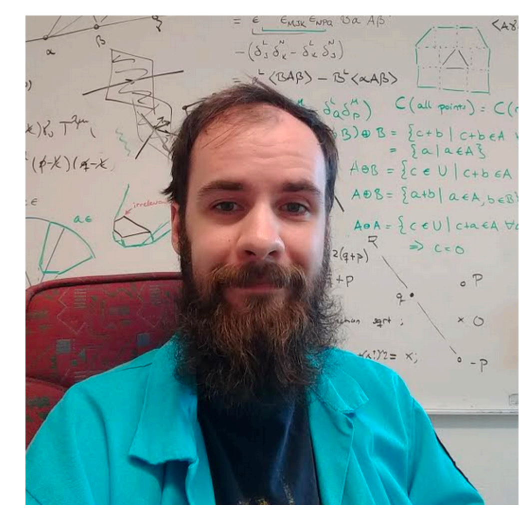


- Evidence for nonzero values in some cases
  - $\mathcal{K}_E$ is only contribution of $\mathcal{K}_{\text{df},3}$ to nontrivial irreps
- Appear at NLO in ChPT—prediction not yet available

Applications of the three-particle formalism: NLO ChPT results for $\mathcal{K}_{\rm df,3}$ for $3\pi^+ \to 3\pi^+$

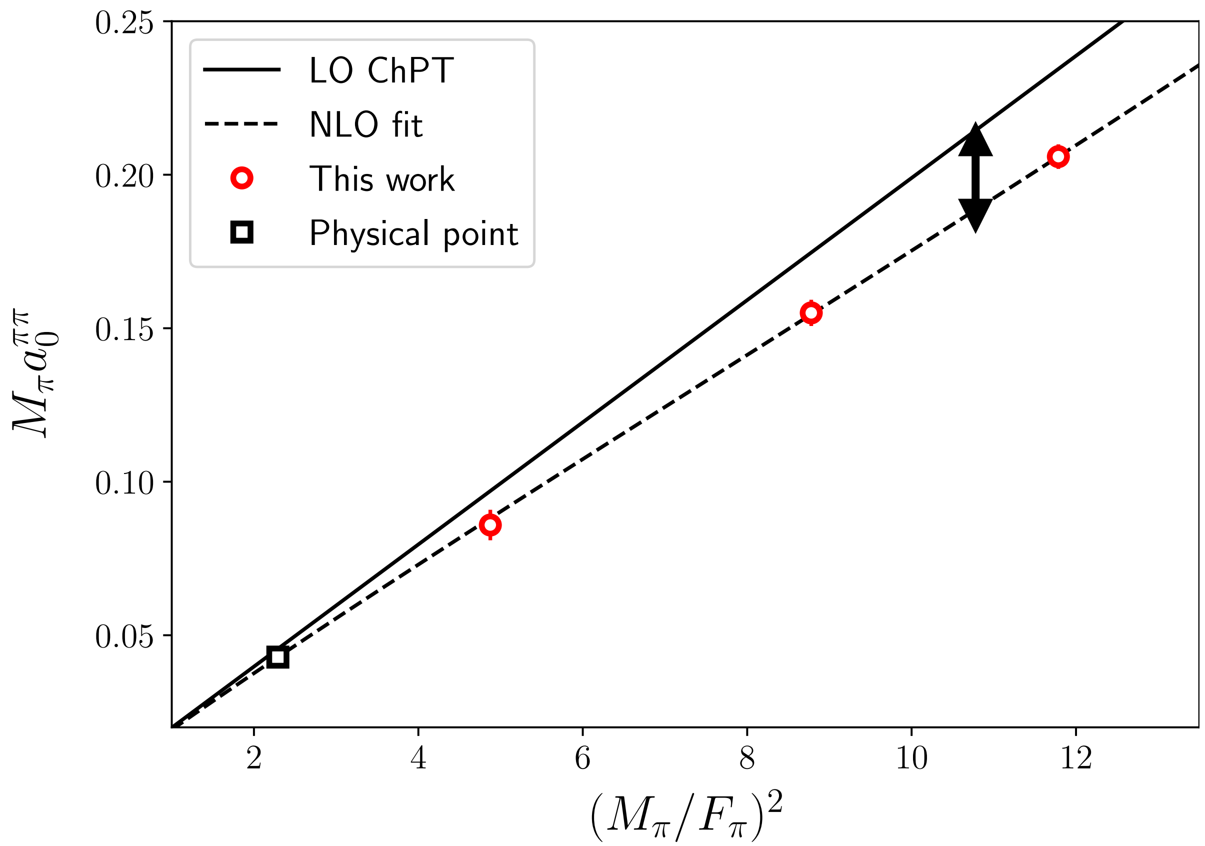
[Baeza-Ballesteros, Bijnens, Husek, Romero-López, SRS, Sjö, 2303.13206]

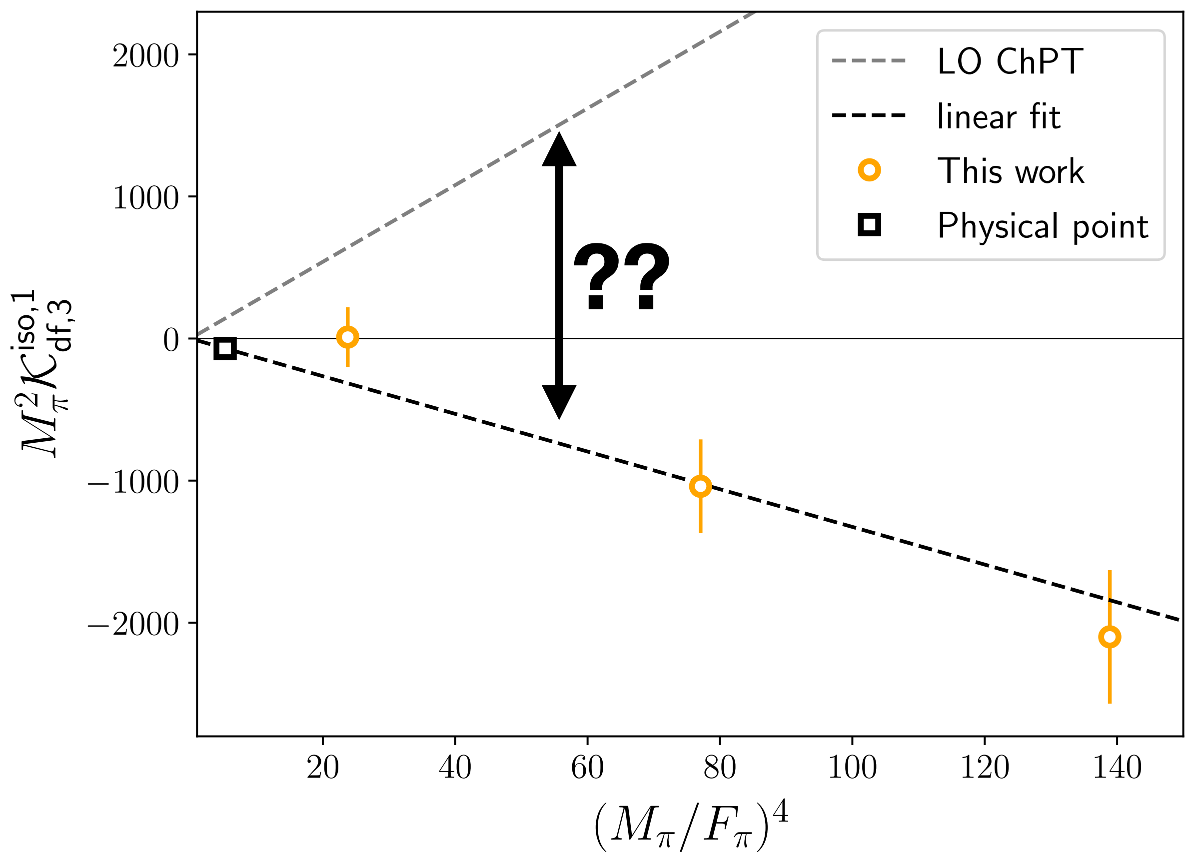



$2\pi/3\pi$ K matrices vs ChPT

$2\pi^+$ scattering length



$3\pi^+$ K matrix

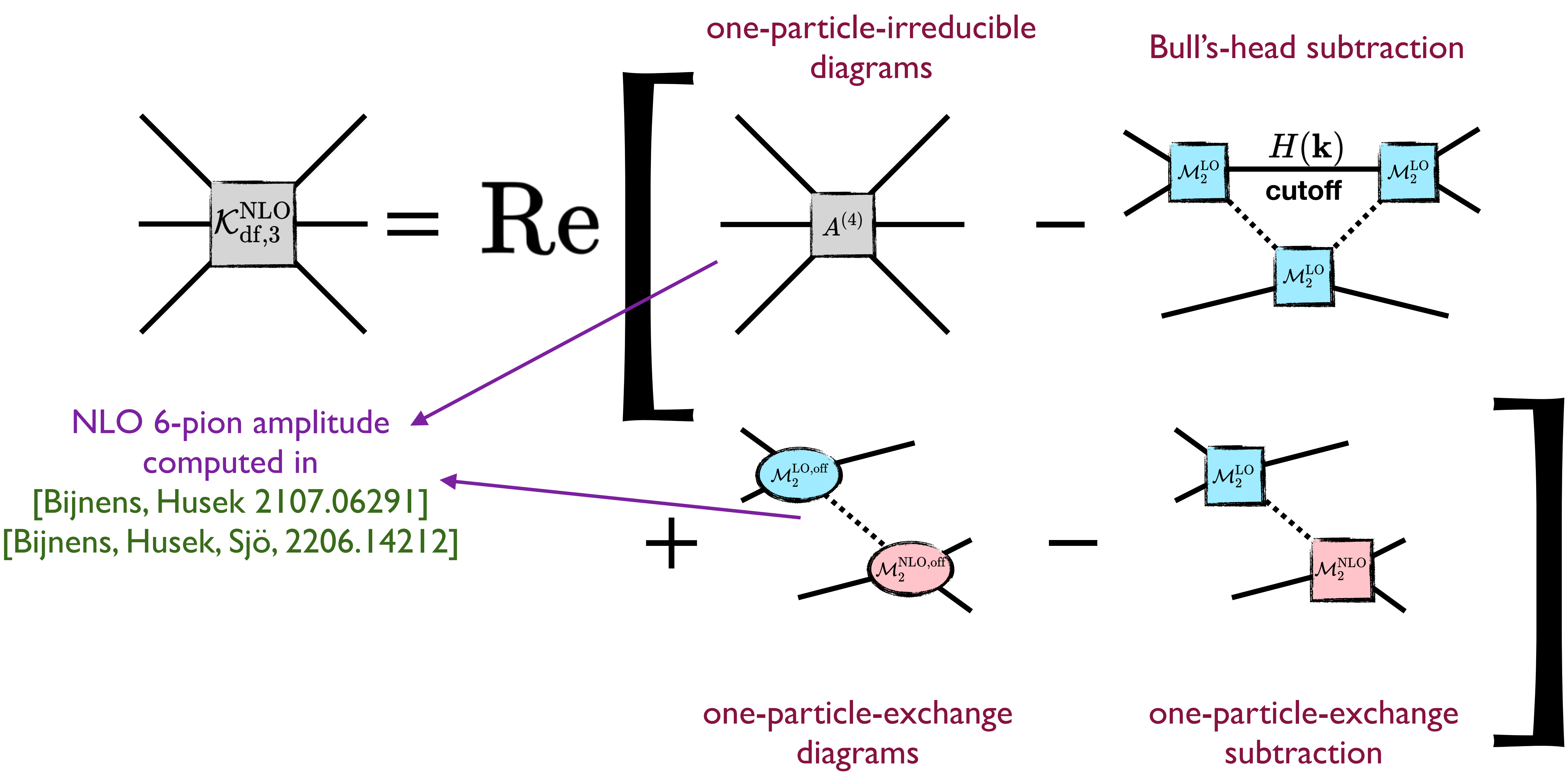


[Results from Blanton, Hanlon, Hörz, Morningstar, Romero-López, SRS, 2106.05590 (JHEP)]

- LO ChPT describes 2-pion sector well
- Large discrepancy in 3-pion sector!

NLO ChPT for $\mathcal{K}_{\mathrm{df},3}$

- Integral equations simplify to: $\mathcal{K}_{\mathrm{df},3}^{\mathrm{NLO}} = \mathrm{Re}\, \mathcal{M}_{\mathrm{df},3}^{\mathrm{NLO}}$



Threshold expansion for $\mathcal{K}_{\mathrm{df},3}$

- $\mathcal{K}_{\mathrm{df},3}$ is a real, smooth function which is Lorentz, P and T invariant
- Expand about threshold in powers of $\Delta = (s - 9M_\pi^2)/9M_\pi^2$, $\tilde{t}_{ij} = (p_i' - p_j)^2/9M_\pi^2$, …

Depend on CM energy

Angular dependence

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},\,1}\Delta + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},\,2}\Delta^2 + \mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B + \mathcal{O}(\Delta^3)$$

$$\Delta_B = \sum_{i,j=1}^{3} \tilde{t}_{ij}^2 - \Delta^2,$$

- Can separate terms in fit based on dependence on energy and rotational properties
  - E.g. only $\mathcal{K}_B$ contributes to nontrivial irreps

NLO ChPT results for $\mathcal{K}_{\mathrm{df},3}$

$\kappa = 1/(16\pi^2)$

$$\mathcal{K}_0 = \left(\frac{M_\pi}{F_\pi}\right)^4 18 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-3\kappa(35 + 12\log 3) - \mathcal{D}_0 + 111L + \ell^{\mathrm{r}}_{(0)}\right],$$

$$\mathcal{K}_1 = \left(\frac{M_\pi}{F_\pi}\right)^4 27 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-\frac{\kappa}{20}\left(1999 + 1920\log 3\right) - \mathcal{D}_1 + 384L + \ell^{\mathrm{r}}_{(1)}\right],$$

$$\mathcal{K}_2 = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{207\kappa}{1400}(2923 - 420\log 3) - \mathcal{D}_2 + 360L + \ell^{\mathrm{r}}_{(2)}\right],$$

$$\mathcal{K}_\mathrm{A} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{9\kappa}{560}(21809 - 1050\log 3) - \mathcal{D}_\mathrm{A} - 9L + \ell^{\mathrm{r}}_{(\mathrm{A})}\right],$$

$$\mathcal{K}_\mathrm{B} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{27\kappa}{1400}(6698 - 245\log 3) - \mathcal{D}_\mathrm{B} + 54L + \ell^{\mathrm{r}}_{(\mathrm{B})}\right].$$

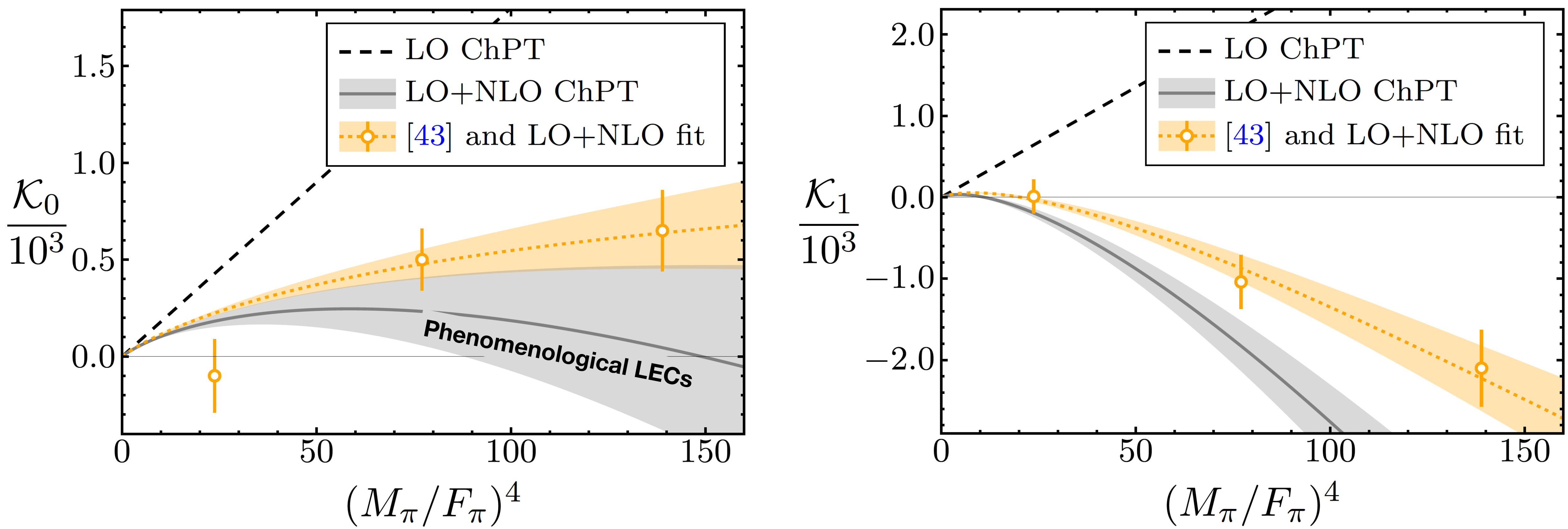
$L \equiv \kappa \log(M_\pi^2/\mu^2)$

LECs

Numerical coefficients
Depend on cutoff H(**k**)

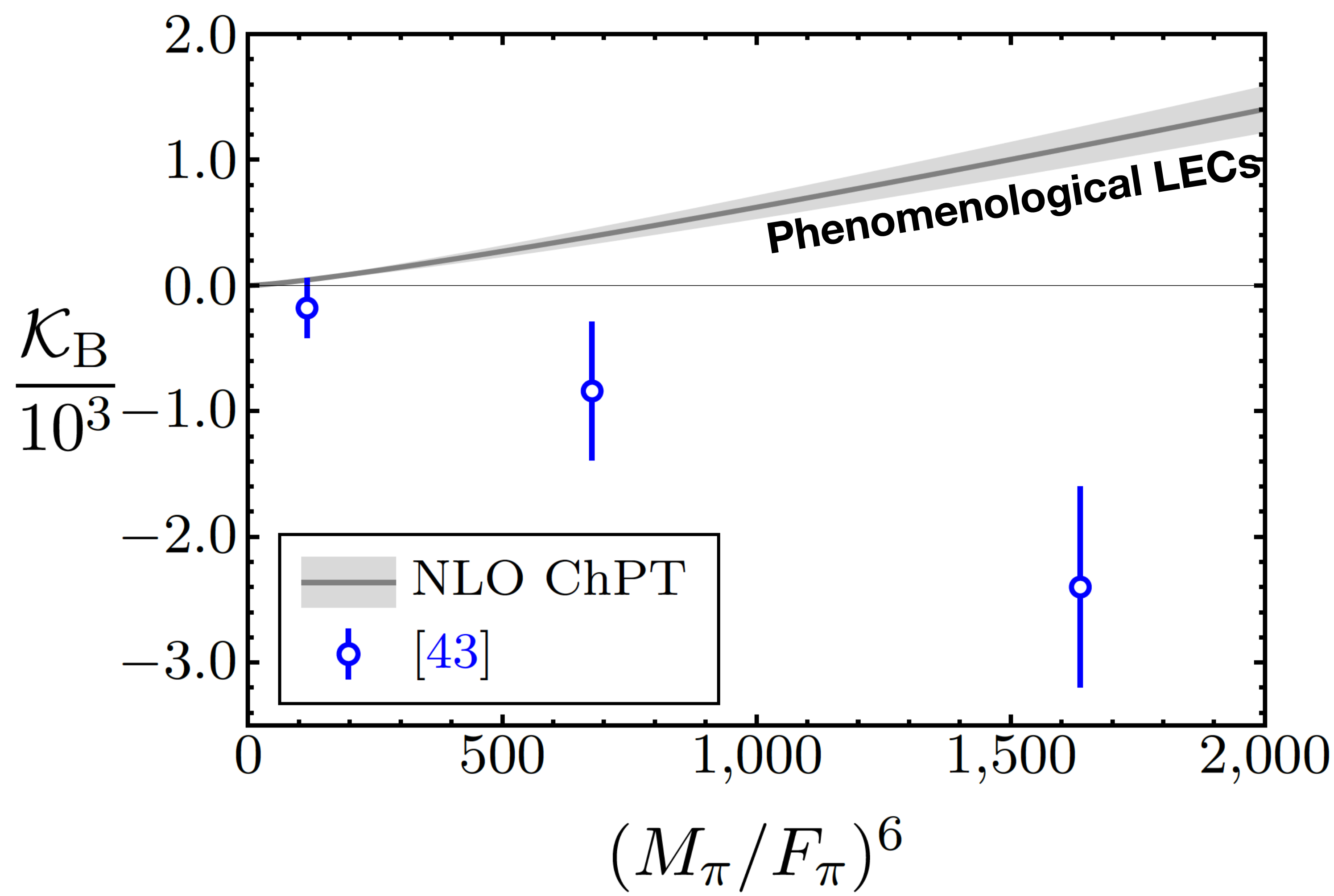
μ-dependence cancels

Comparison to LQCD



- (Very) large NLO corrections
  - Discrepancy with LO ChPT resolved!
  - ChPT not trustworthy for $\mathscr{K}_1$

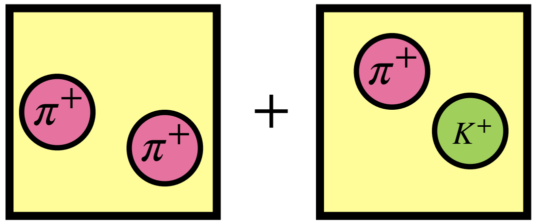
Comparison to LQCD



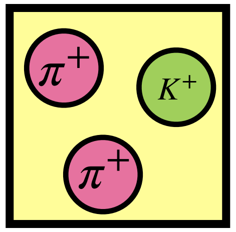
- $\mathcal{K}_B$ first appears at NLO in ChPT
- Discrepancy may be resolved by NNLO terms?

Summary and Outlook

Summary

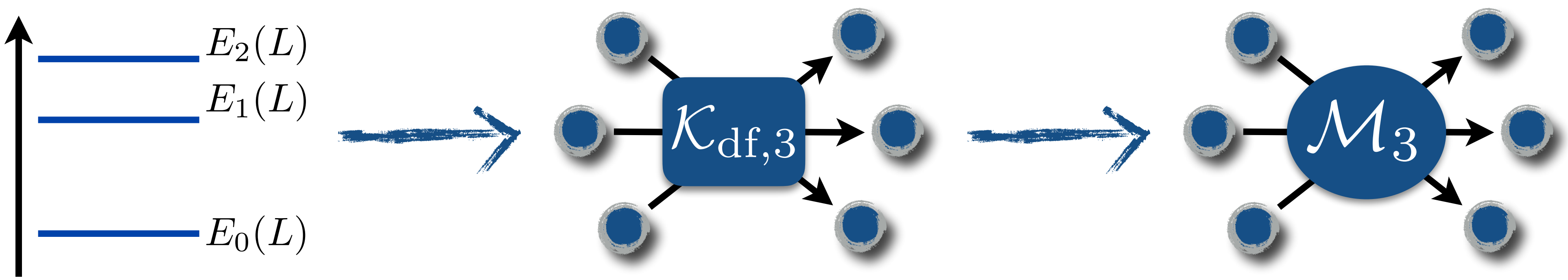


- Two-particle sector is entering precision phase
  - Frontier is two nucleons, which are more challenging for LQCD
- Major steps have been taken in the three-particle sector



  - Formalism well established & cross checked, and almost complete
  - Several applications to three-particle spectra from LQCD
  - Initial discrepancy with LO ChPT explained by large NLO contributions
  - Integral equations solved in several cases
  - Path to a calculation of $K \to 3\pi$ decay amplitudes is now open

Example of complete application



[Hansen, Briceño, Dudek, Edwards, Wilson (HADSPEC collaboration) 2009.04931 PRL 21]

$M_\pi \approx 390$ MeV, $a \approx 0.12$ fm, $L \approx 2.5$ & 2.9 fm

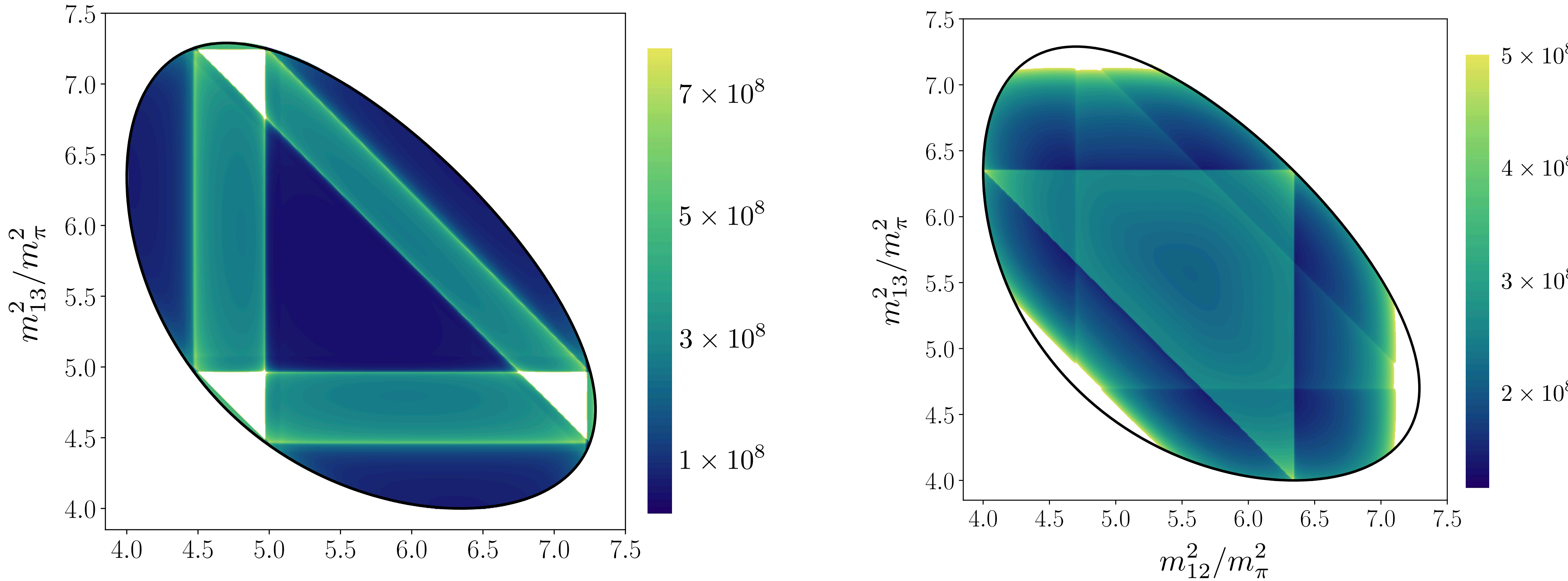


FIG. 3. *Top:* Dalitz-like plot of $m_\pi^4|\mathcal{M}_3|^2$ for $\sqrt{s_3} = 3.7m$ with final kinematics fixed to $\{\boldsymbol{p}_1'^2, \boldsymbol{p}_2'^2\} = \{0.01m_\pi^2, 0.7m_\pi^2\} \implies \{m_{12}', m_{13}'\} = \{2.1m_\pi, 2.25m_\pi\}$. *Bottom:* Same total energy, now with incoming and outgoing kinematics set equal, as discussed in the text.

Outlook

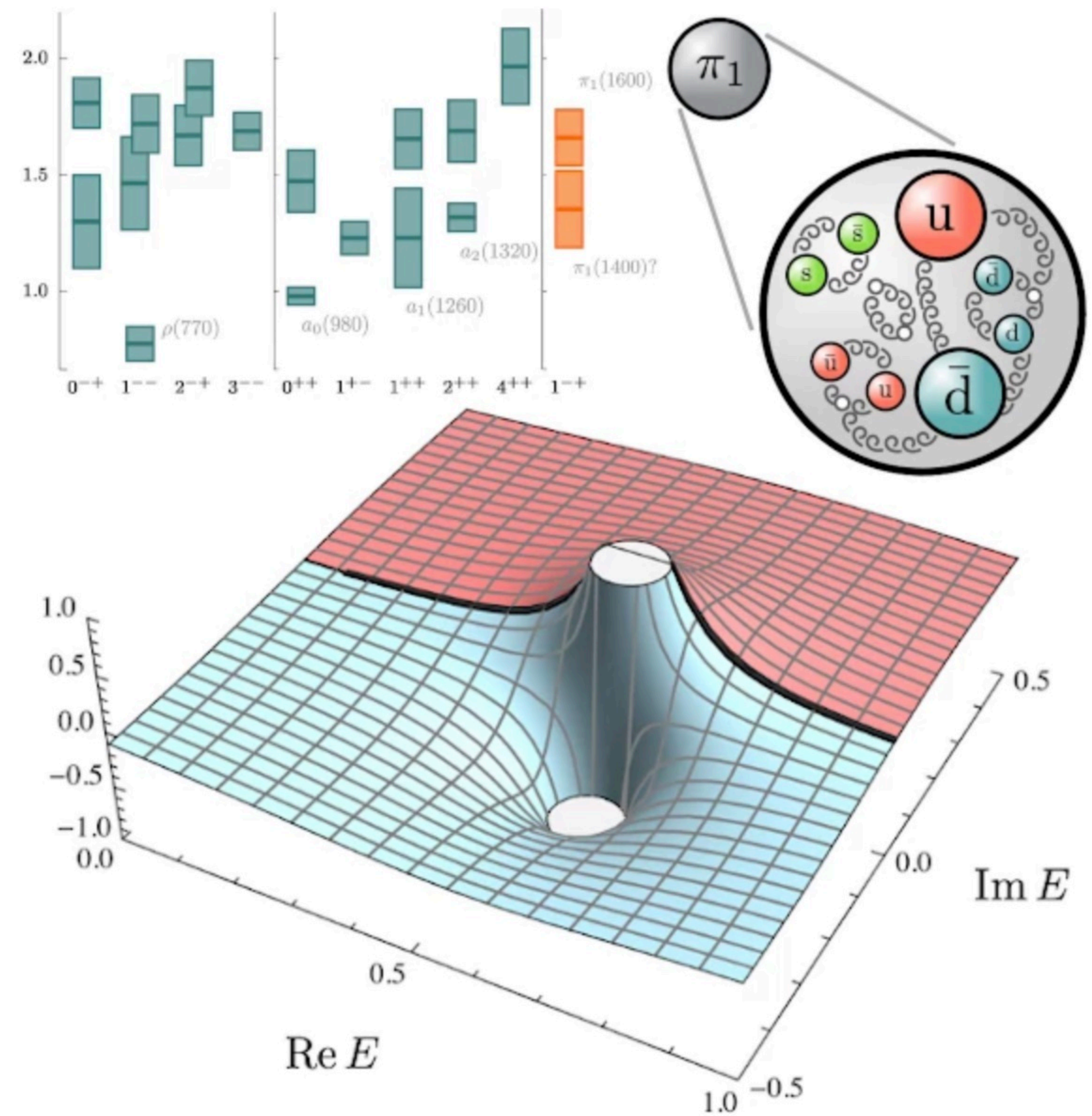
- Generalize formalism to broaden applications
  - 3 nucleons with $I = \frac{1}{2}$ (nnp & ppn)
  - $T_{cc}(3875,\, I = 0,\, J^P = 1^+?) \to D^0D^0\pi^+,\, D^+D^0\pi^0,\, D^+D^+\pi^-$
  - Accessing the WZW term: $K\bar{K} \leftrightarrow \pi^+\pi^0\pi^-(I = 0)$
  - $N(1440,\, J^P = \frac{1}{2}^+) \to N\pi,\, N\pi\pi$
  - $J^{PC}, I^G = 1^{-+}, 1^- :\;\; \pi_1(1600) \to \eta\pi,\, 3\pi,\, KK\pi\pi,\, \eta\pi\pi\pi,\, 5\pi$
- Extend ChPT calculations to provide cross/sanity checks for $\mathscr{K}_{\rm df,3}$ results
  - NLO calculation for $I = 0,1,2$ underway
- Extend implementations using LQCD simulations
  - $3\pi^+,\, 3K^+,\, \pi^+\pi^+K^+,\, K^+K^+\pi^+$ at physical quark masses
  - $I = 0,1$ three-particle resonances $(\omega,\, a_1,\, \ldots)$
- Extend applications of integral equations in the presence of three-particle resonances, e.g. T_{cc}
- Move on to 4 particles!

ExoHad collaboration

ExoHad Collaboration exohad.org People Events Talks Publications

EXOHAD
EXOTIC HADRONS TOPICAL COLLABORATION

The Exo(tic) Had(ron) Collaboration started in 2023 to explore all aspects of exotic hadron physics, from predictions within lattice QCD, through reliable extraction of their existence and properties from experimental data, to descriptions of their structure within phenomenological models.



Thank you!
Questions?

References

RFT 3-particle papers

Max Hansen & SRS:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]

Raúl Briceño, Max Hansen & SRS:

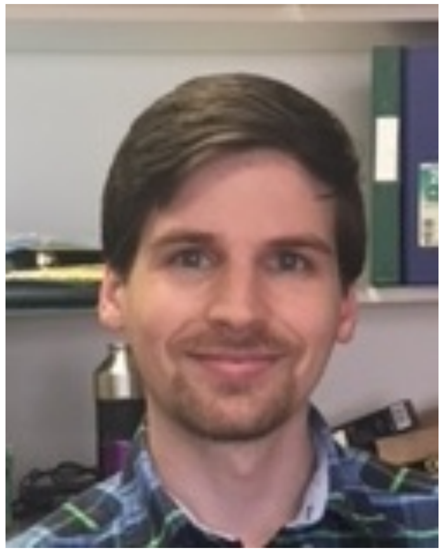
"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]

"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

SRS

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory," arXiv:1707.04279 (PRD) [SPT17]



Tyler Blanton, Fernando Romero-López & SRS:

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]

"I=3 three-pion scattering amplitude from lattice QCD," arXiv:1909.02973 (PRL) [BRS-PRL19]

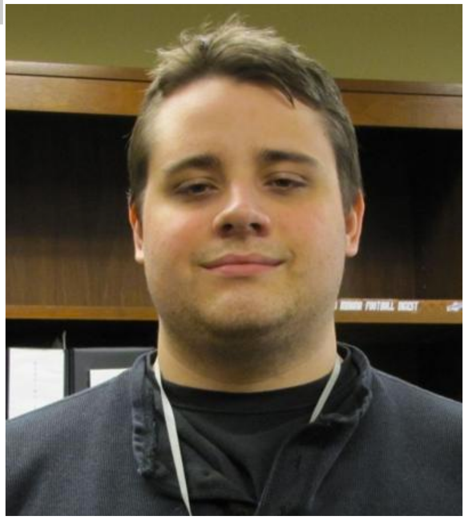
"Implementing the three-particle quantization condition for $\pi^+\pi^+K^+$ and related systems" 2111.12734 (JHEP)

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP) [BBHRS19]

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)



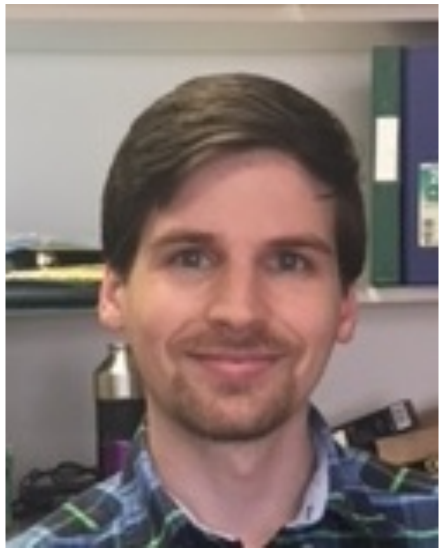
Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

"Generalizing the relativistic quantization condition to include all three-pion isospin channels", arXiv:2003.10974 (JHEP) [HRS20]

"Decay amplitudes to three particles from finite-volume matrix elements," arXiv: 2101.10246 (JHEP)



<u>Tyler Blanton & SRS:</u>

"Alternative derivation of the relativistic three-particle quantization condition," arXiv:2007.16188 (PRD) [BS20a]

"Equivalence of relativistic three-particle quantization conditions," arXiv:2007.16190 (PRD) [BS20b]

"Relativistic three-particle quantization condition for nondegenerate scalars," arXiv:2011.05520 (PRD)

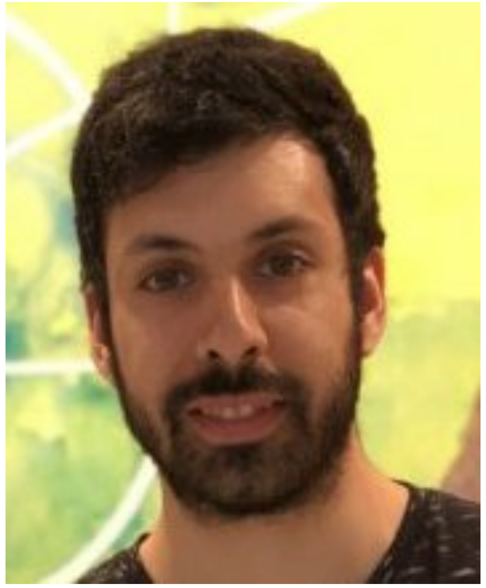
"Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ & related systems," arXiv:2105.12904 (PRD)

<u>Tyler Blanton, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS</u>

"$3\pi^+$ & $3K^+$ interactions beyond leading order from lattice QCD," arXiv:2106.05590 (JHEP)

<u>Zack Draper, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS</u>

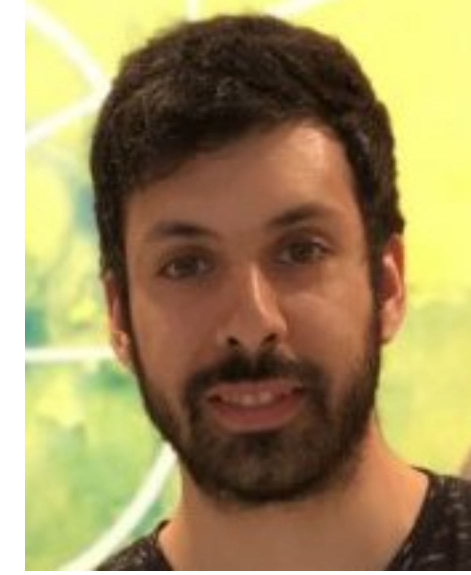
"Interactions of πK, $\pi\pi K$ and $KK\pi$ systems at maximal isospin from lattice QCD," arXiv:2302.13587



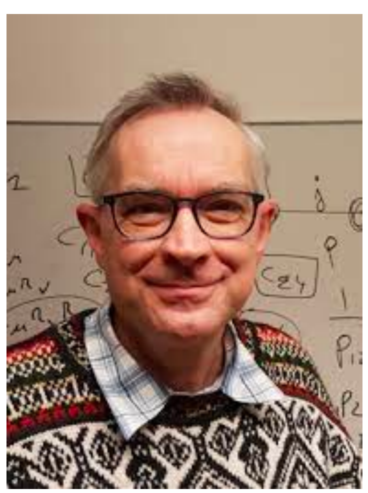
Zach Draper, Max Hansen, Fernando Romero-López & SRS:

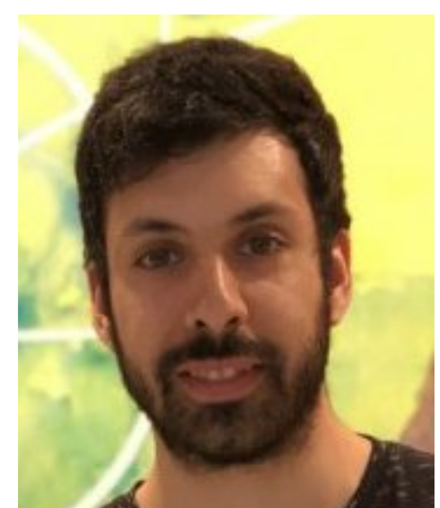
“Three relativistic neutrons in a finite volume,”

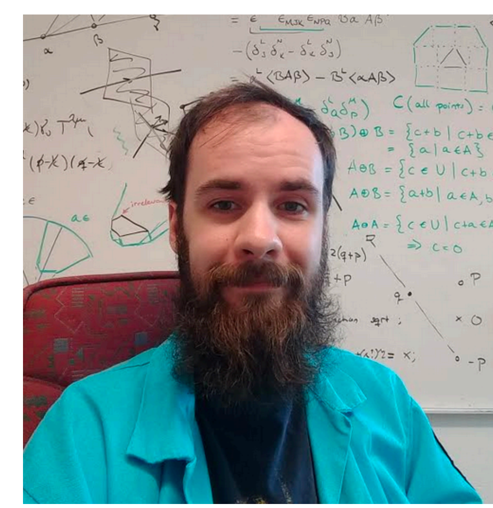
arXiv:2303.10219



Jorge Baeza-Ballesteros, Johan Bijnens, Tomas Husek, Fernando Romero-López, SRS & Mattias Sjö: **“The isospin-3 three-particle K-matrix at NLO in ChPT,” arXiv:2303.13206**







Other work

★ **Implementing RFT integral equations**

- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating $3\pi^+$ spectrum and using to determine three-particle scattering amplitude]
- A. Jackura et al., 2010.09820, PRD [Solving s-wave RFT integral equations in presence of bound states]
- S. Dawid, Md. Islam and R. Briceño, 2303.04394 [Analytic continuation of 3-particle amplitues]

★ **Reviews**

- A. Rusetsky, 1911.01253 [LATTICE 2019 plenary]
- M. Mai, M. Döring and A. Rusetsky, 2103.00577 [Review of formalisms and chiral extrapolations]
- F. Romero-López, 2112.05170, [Three-particle scattering amplitudes from lattice QCD]

★ **Other numerical simulations**

- F. Romero-López, A. Rusetsky, C. Urbach, 1806.02367, JHEP [2- & 3-body interactions in φ^4 theory]
- M. Fischer et al., 2008.03035, Eur.Phys.J.C [$2\pi^+$ & $3\pi^+$ at physical masses]
- M. Garofolo et al., 2211.05605, JHEP [3-body resonances in φ^4 theory]

Other work

★ **NREFT approach**

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, 1706.07700, JHEP & 1707.02176 , JHEP [Formalism & examples]
- M. Döring et al., 1802.03362 , PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111 , PRD [large volume expansion for excited levels]
- F. Müller, T. Yu & A. Rusetsky, 2011.14178, PRD [large volume expansion for I=1 three pion ground state]
- F. Romero-López, A. Rusetsky, N. Schlage & C. Urbach, 2010.11715, JHEP [generalized large-volume exps]
- F. Müller & A. Rusetsky, 2012.13957, JHEP [Three-particle analog of Lellouch-Lüscher formula]
- J-Y. Pang, M. Ebert, H-W. Hammer, F. Müller, A. Rusetsky, 2204.04807 , JHEP, [Spurious poles in a finite volume]
- F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2110.09351, JHEP [Relativistic-invariant formulation of the NREFT three-particle quantization condition]
- J. Lozano, U. Meißner, F. Romero-López, A. Rusetsky & G. Schierholz, 2205.11316 , JHEP [Resonance form factors from finite-volume correlation functions with the external field method]
- F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2211.10126, JHEP [3-particle Lellouch-Lüscher formalism in moving frames
- R. Bubna, F. Müller, A. Rusetsky, 2304.13635 [Finite-volume energy shift of the three-nucleon ground state]

Alternate 3-particle approaches

★ **Finite-volume unitarity (FVU) approach**

- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of M_3 involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749, PRD [applying FVU approach to $3\pi^+$ spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]
- A. Alexandru et al., 2009.12358, PRD [calculating $3K^-$ spectrum and comparing with FVU predictions]
- R. Brett et al., 2101.06144, PRD [determining $3\pi^+$ interaction from LQCD spectrum]
- M. Mai et al., 2107.03973, PRL [three-body dynamics of the $a_1(1260)$ from LQCD]
- D. Dasadivan et al., 2112.03355, PRD [pole position of $a_1(1260)$ in a unitary framework]

★ **HALQCD approach**

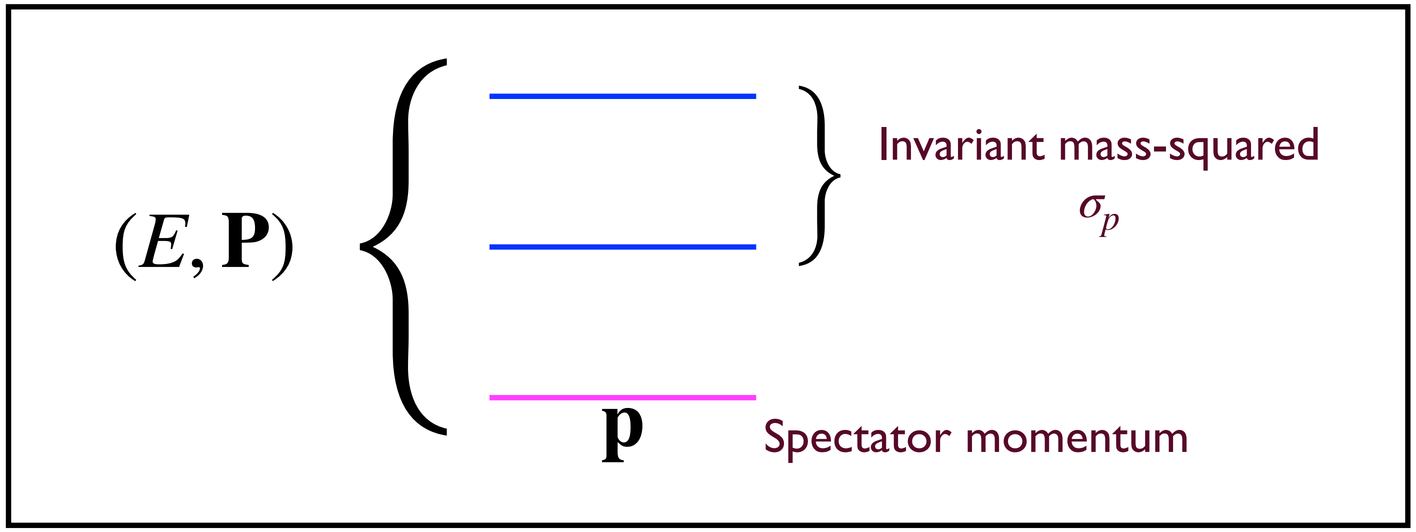
- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]

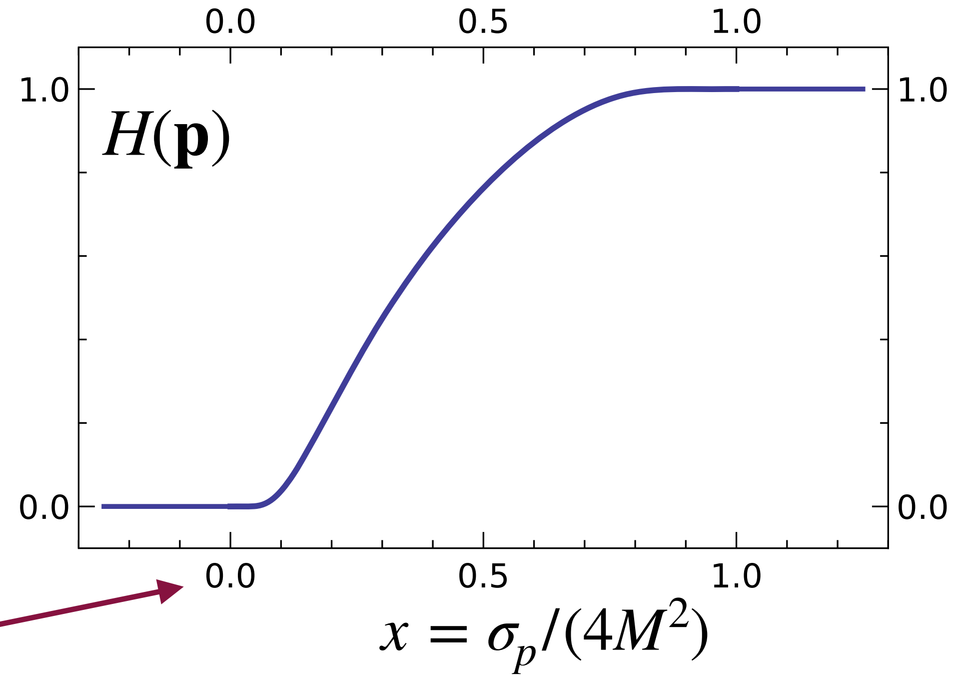
Backup Slides

Cutoff/transition function

- In RFT derivation, need cutoff function to truncate matrix indices and to avoid LH cut
  - Must be smooth to avoid power-law finite-volume (FV) effects

Form for degenerate particles





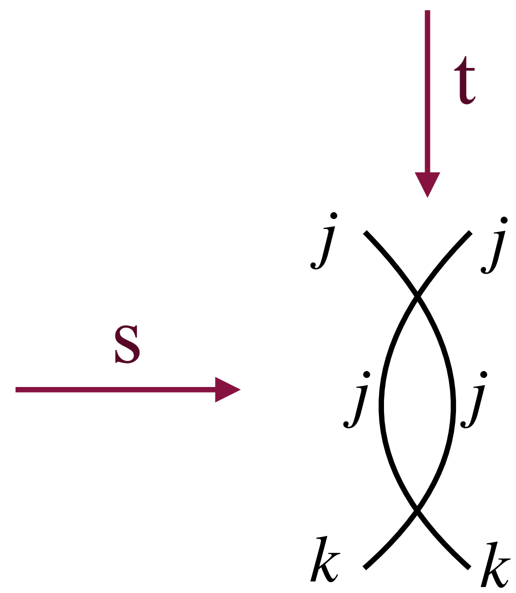
Position of left-hand cut in pair interaction ($\mathcal{K}_2$ or $\mathcal{M}_2$): $s = u = 0,\ t = 4M^2$

- May be possible to raise the cutoff, following the arguments used to relativize the NREFT approach [F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2110.09351, JHEP]

Cutoff/transition function

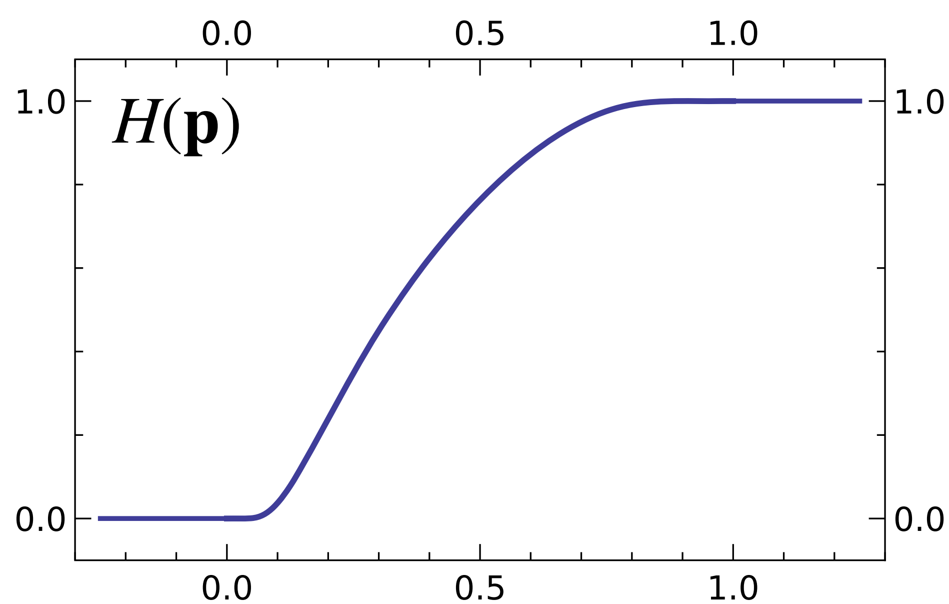
- For nondegenerate particles, LH cut moves, and must change cutoff function accordingly

Assume $M_j < M_k$:



$$t = 4M_j^2 = 4M_{\min,jk}^2$$

$$s = u = |M_k^2 - M_j^2| > 0$$



$$x = (1 + \epsilon_H) \frac{\sigma_p - |M_k^2 - M_j^2|}{(M_j + M_k)^2 - |M_k^2 - M_j^2|}$$

- Same functional form, but argument adjusted so $H(\mathbf{p})$ vanishes at position of left-hand cut
- Strictly speaking, to avoid power-law FV effects, need $\epsilon_H > 0$ (though in practice set to zero)