Multiparticle scattering

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Outline

☑ Lecture 1

- Motivation/Background/Overview
- Deriving the two-particle quantization condition (QC2)

☐ Lecture 2

- Applying the QC2, in brief
- Deriving the three-particle quantization condition for identical scalars (QC3)

☐ Lecture 3

- Status of three-particle formalism
- Applications of QC3
- Outlook

Main references for this lecture

[Full list of references at end of lecture 3]

- Briceño, Dudek & Young, "Scattering processes & resonances from LQCD," 1706.06223, RMP 18
- Hansen & SS, "LQCD & three-particle decays of resonances," 1901.00483, to appear in ARNPS
- Hansen & SS 1408.5933, PRD14 & 1504.04248, PRD15 (derivation of QC3 in QFT)
- Blanton & SS 2007.16188, PRD20 (Alternative derivation of QC3 using time-ordered PT)

Outline for Lecture 2

- Applying the QC2, in brief
- Deriving of the three-particle quantization condition for identical scalars (QC3)
  - Presenting the QC3
  - Sketch of derivation using time-ordered PT (TOPT)
  - Integral equations related three-particle K matrix to $\mathscr{M}_3$

Applying the QC2, in brief

2-particle quantization condition

- At fixed $L, \vec{P}$ the spectrum is given (up to corrections $\propto e^{-M_{\min}L}$) by solutions of

$$\det\left[F_{PV}(E,\vec{P},L)^{-1} + \mathcal{K}_2(E^*)\right] = 0$$

- $F_{\rm PV}$ and $\mathcal{K}_2$ are matrices in ℓ, m space, with $[\mathcal{K}_2]_{\ell'm',\ell m} = \delta_{\ell'\ell}\delta_{m'm}\mathcal{K}_2^{(\ell)}$
- $\mathcal{K}_2^{(\ell)} = (16\pi E_2^*)/(q^*\cot\delta_\ell(q^*))$ is real and smooth (no threshold branch points)
- $F_{\rm PV}$ is a kinematic function

$$F_{{\rm PV};\ell'm';\ell m}(E,\vec{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - {\rm PV}\int\frac{d^3k}{(2\pi)^3}\right)\frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k})}$$

Truncation

$$\det\left[F_{PV}(E,\vec{P},L)^{-1}+\mathcal{K}_2(E^*)\right]=0$$

- Near threshold $\mathcal{K}_2^{(\ell)} \propto (q^*)^{2\ell}$ [familiar from QM]
- In practice, for $E^* \lesssim 1\text{GeV}$ it is a good approximation to keep only the lowest one or two partial waves, i.e to set $\mathcal{K}_2^{(\ell)}=0$ for $\ell > \ell_{\text{max}}$
  - Then can show that only need to keep $\ell \le \ell_{\text{max}}$ in F_{PV}
  - Leads to a finite-dimensional matrix that can be implemented numerically
- Can further reduce the dimensionality by projecting onto irreps of the cubic group [$A_1^+, A_2^+, E^+, \ldots$—no time to discuss here]

Simplest case: single value of ℓ

- If $\ell_{\max} = 0$, only a single value of ℓ contributes, and QC2 becomes algebraic

$$\det\left[F_{PV}(E, \vec{P}, L)^{-1} + \mathcal{K}_2(E^*)\right] = 0$$

$\downarrow$ $l_{\max}=0$

$E_n^* = \sqrt{E_n^2 - \vec{P}^2}$

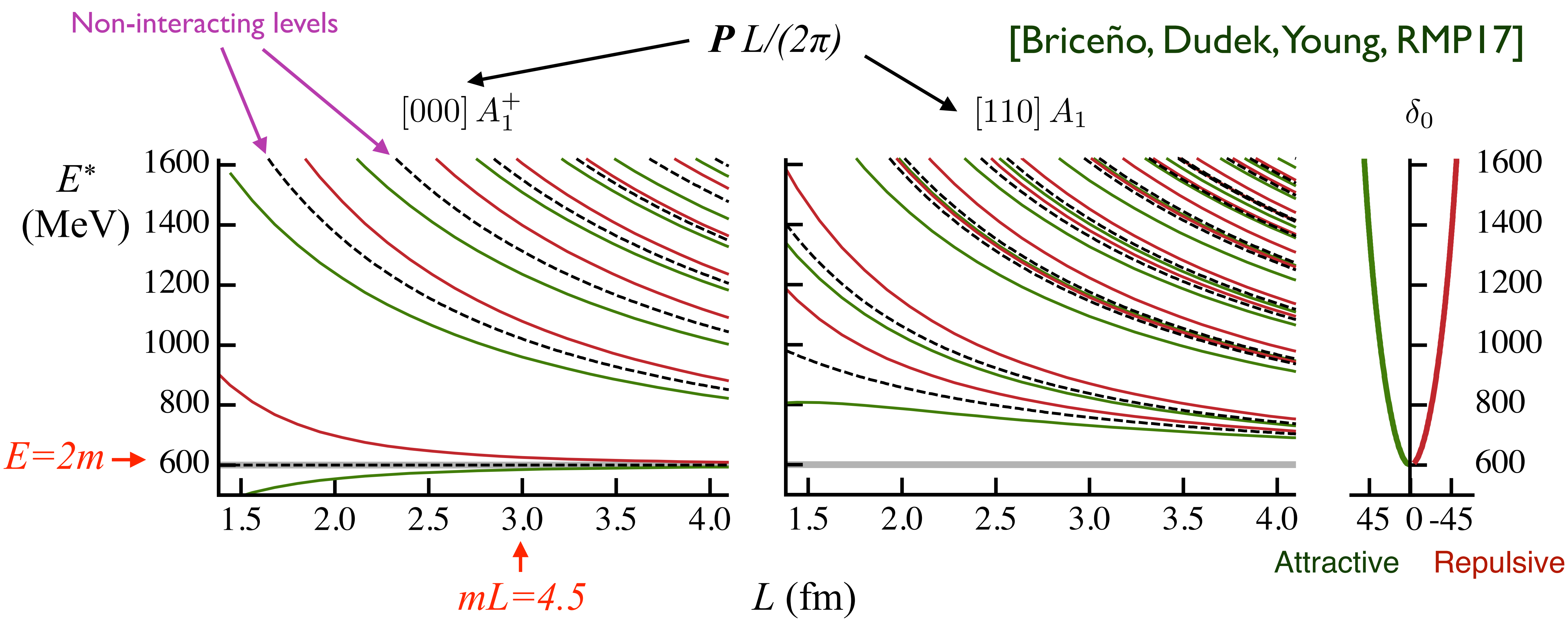
CM energy

"measured" energy-level

$$\mathcal{K}_2^{(\ell=0)}(E_n^*) = -\frac{1}{F_{PV;00;00}(E_n, \vec{P}, L)}$$

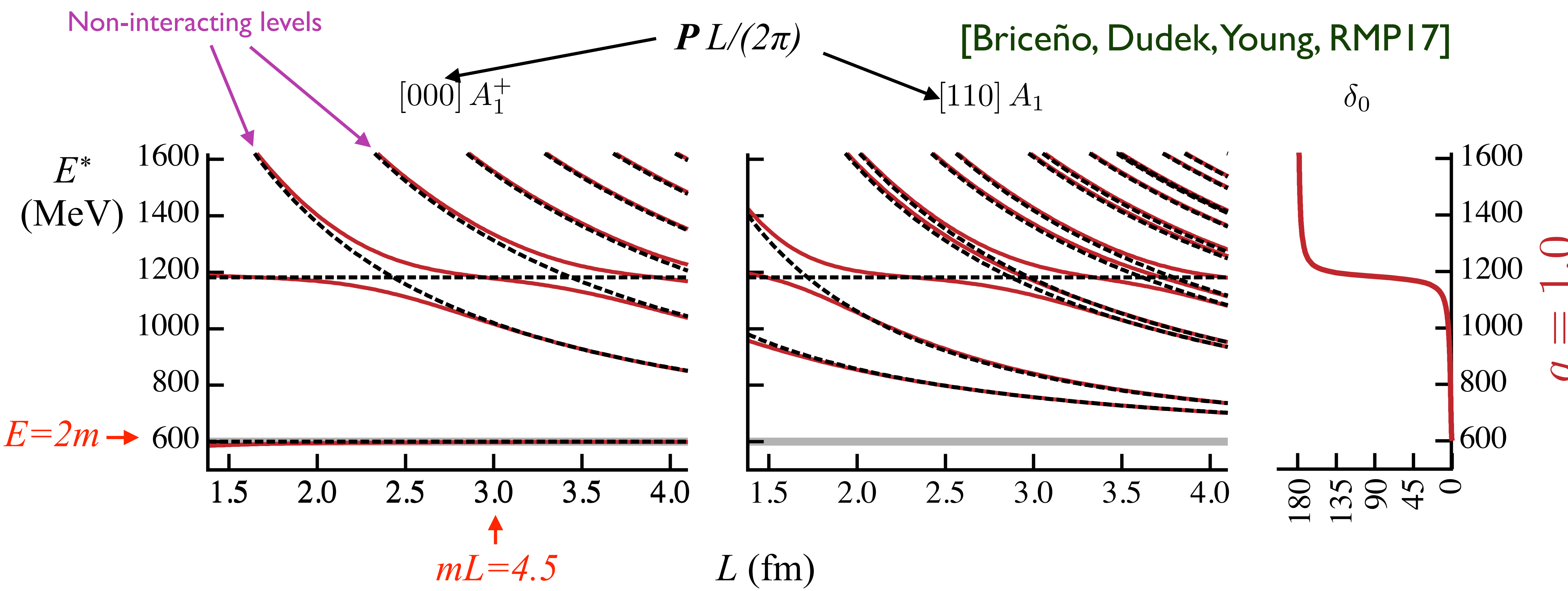
- One-to-one relation between energy levels and $K_2 \sim 1/(q^* \cot \delta)$
- Holds also if $\ell_{\max} = 1$ and one uses a cubic-group irrep that does not couple to $\ell = 0$
- Most state-of-the-art applications (e.g. $f_0 = \sigma$) involve multiple ℓ and multiple two-particle channels

Overview of effects on spectrum



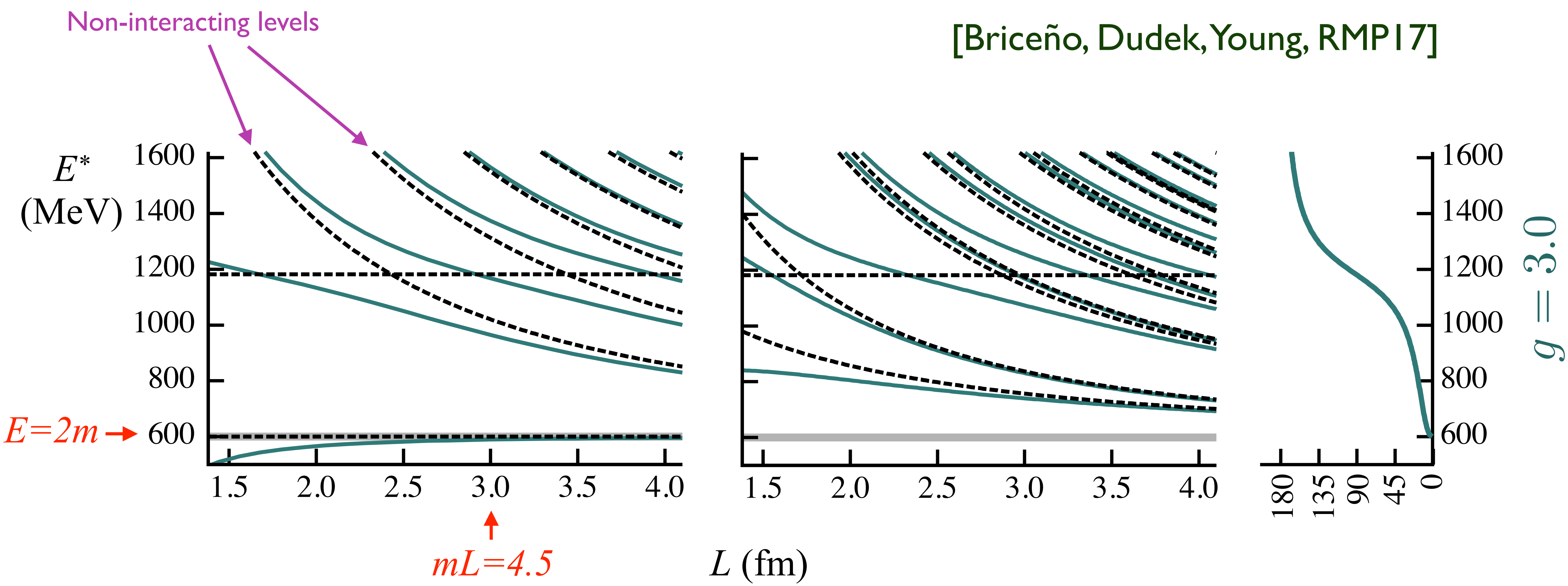
- Unphysical example for sake of illustration
- $\ell_{\max} = 0,\ m = 300\,\mathrm{MeV},\ a_0 = \pm\,0.32\,\mathrm{fm}\ (ma_0 = 0.48)$
- Illustrates the power of using moving frames ($\boldsymbol{P} \neq 0$) and multiple levels

Overview of effects on spectrum



- Narrow Brett-Wigner resonance at 1182 MeV
- Spectrum contains an additional level, and displays avoided level crossings

Overview of effects on spectrum



- Broad Breit-Wigner resonance at 1182 MeV
- Association of levels with "resonance" or "almost-free particles" no longer holds

QC3: Presenting the result

Sketch of history for three particles

- [Beane, Detmold, Savage et al. 07-11] studied ground state energies of $N\pi^+, MK^+, N\pi^+ + MK^+$ systems, and determined 3-particle interactions for particles at rest
- [Polejaeva & Rusetsky 12] showed in NREFT that 3 body spectrum determined by $2 \to 2$ & $3 \to 3$ infinite-volume scattering amplitudes
- [Hansen & SRS 14, 15] derived quantization condition (QC3) for 3 identical scalars in generic, relativistic EFT, working to all orders in Feynman-diagram expansion, keeping all angular momenta—"RFT approach"
- [Hammer & Rusetsky 17] derived QC3 using NREFT—greatly simplified derivation
- [Mai & Döring 17] obtained QC3 using unitary, relativistic representation of $3 \to 3$ amplitude—"FVU approach"
- [Blanton & SRS 20] showed equivalence of RFT & FVU approaches
- [Müller, Pang, Rusetsky & Wu 20] relativized NREFT approach
- [Müller & Rusetsky 20; Hansen, Romero-López & SRS 21] derived formalism for determining $K \to 3\pi$ amplitude

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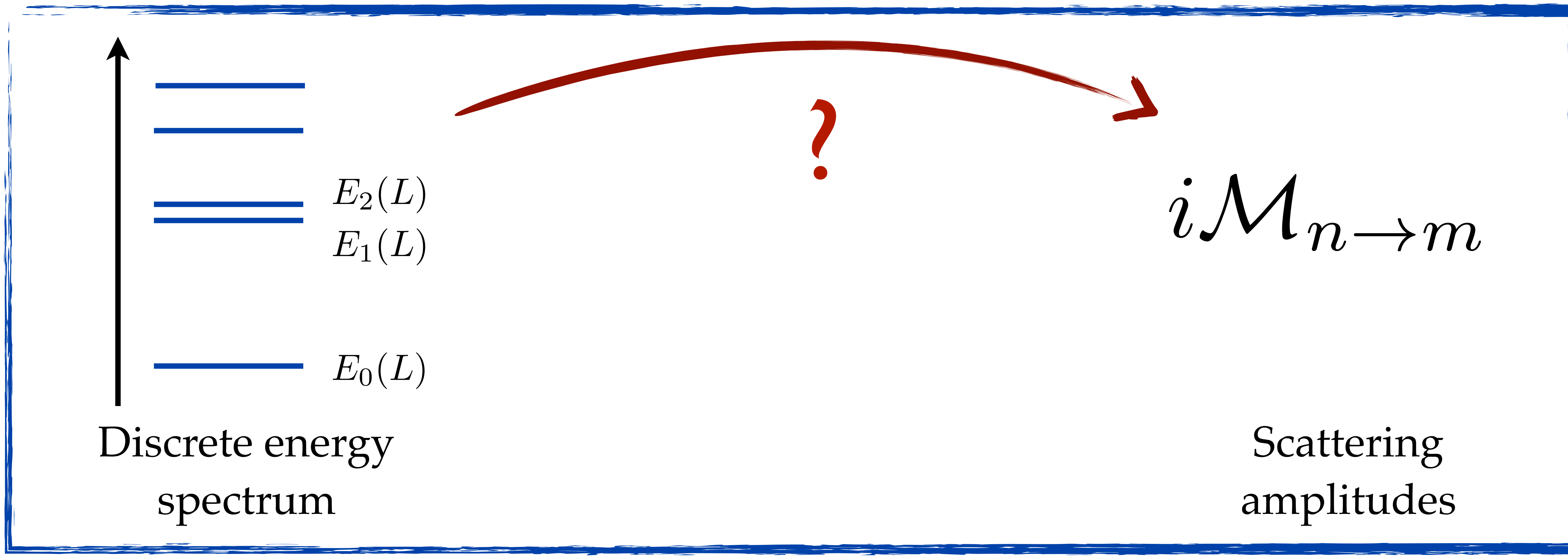
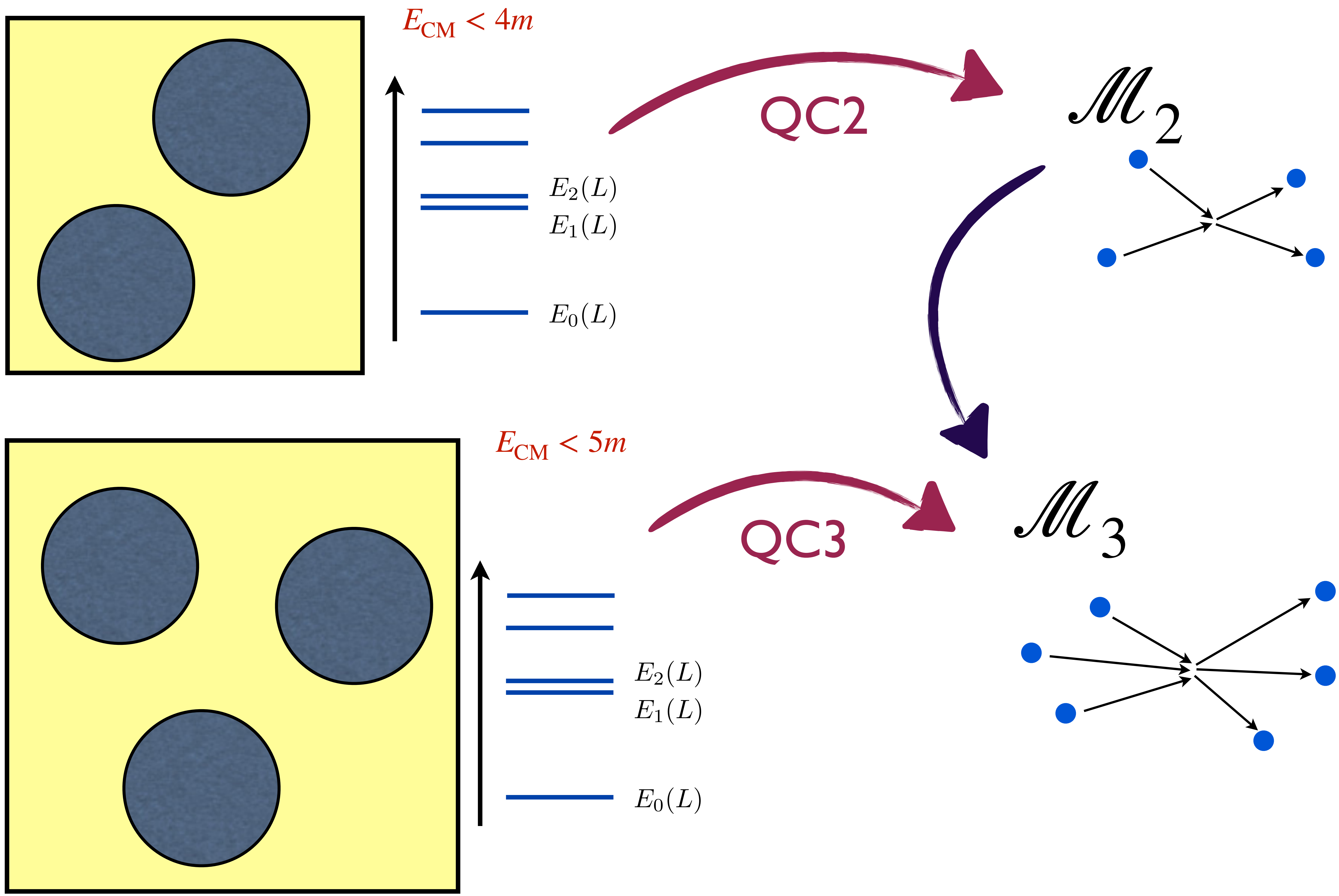
I will discuss the RFT approach

Problem in finite-volume QFT

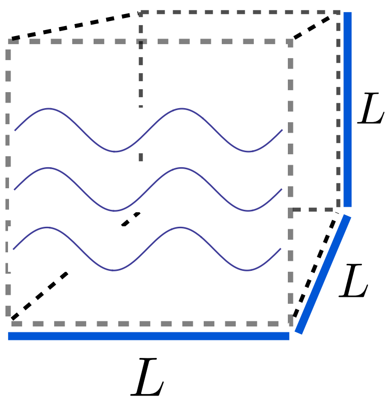


Structure of the result (Z_2 symmetry)



Two-step method

2 & 3 particle
Spectra from LQCD



Quantization conditions

QC2: $\det\left[F^{-1} + \mathcal{K}_2\right] = 0$

QC3: $\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$

[These are the RFT forms, and assume $\mathbb{Z}_2$ symmetry]

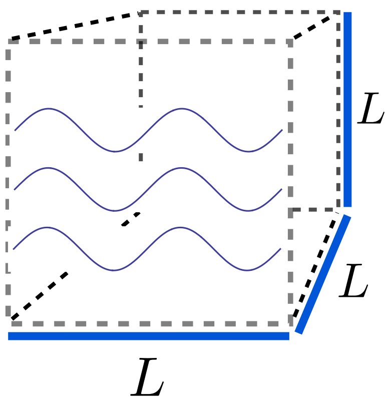
Integral equations in infinite volume

Incorporates initial- and final-state interactions

Scattering amplitude $\mathcal{M}_3$

Two-step method

2 & 3 particle
Spectra from LQCD



Infinite-volume K matrix:
Obtained from Feynman diagrams
using PV prescription for poles;
Real, free of unitary cuts

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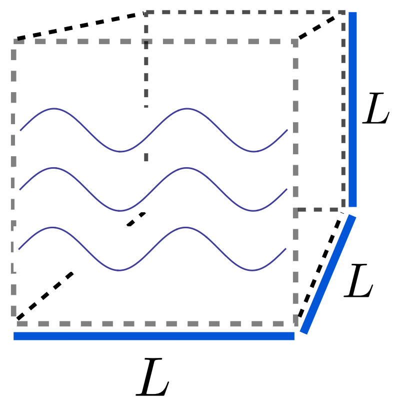

Infinite-volume K matrix: Obtained from Feynman diagrams using PV prescription for poles; Real, free of unitary cuts

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[These are the RFT forms, and assume $\mathbb{Z}_2$ symmetry]

Intermediate infinite-volume K matrix: A short-distance, real, three-particle interaction free of unitary cuts, and with physical divergences subtracted; unphysical since depends on cutoff

Integral equations in infinite volume

Incorporates initial- and final-state interactions

Scattering amplitude $\mathscr{M}_3$

QC2 $\longrightarrow$ QC3 [HS14]

$$\det\left[F_{\rm PV}(E,\vec{P},L)^{-1}+\mathscr{K}_2(E^*)\right]=0$$

- Total momentum (E, **P**)
- Matrix indices are l, m
- $F_{\rm PV}$ is a finite-volume geometric function
- $\mathscr{K}_2$ is an infinite-volume amplitude, which is real and has no unitary cusps
- $\mathscr{K}_2$ is algebraically related to $\mathscr{M}_2$

$$\frac{1}{\mathscr{M}_2^{(\ell)}}\equiv\frac{1}{\mathscr{K}_2^{(\ell)}}-i\rho$$

QC2 ⟶ QC3 [HS14]

$$\det\left[F_{\rm PV}(E,\vec{P},L)^{-1} + \mathcal{K}_2(E^*)\right] = 0$$

$$\det\left[F_3(E,\vec{P},L)^{-1} + \mathcal{K}_{\rm df,3}(E^*)\right] = 0$$

- Total momentum (E, **P**)
- Matrix indices are $l,\ m$
- $F_{\rm PV}$ is a finite-volume geometric function
- $\mathcal{K}_2$ is an infinite-volume amplitude, which is real and has no unitary cusps
- $\mathcal{K}_2$ is algebraically related to $\mathcal{M}_2$

$$\frac{1}{\mathcal{M}_2^{(\ell)}} \equiv \frac{1}{\mathcal{K}_2^{(\ell)}} - i\rho$$

- Total momentum (E, **P**)
- Matrix indices are $\boldsymbol{k},\ l,\ m$
- F_3 depends on geometric functions (F_{PV} and G) and also on $\mathcal{K}_2$
  - F_3 is known if first solve QC2
- $\mathcal{K}_{\rm df,3}$ is an infinite-volume 3-particle amplitude, which is real and has no unitary cusps
  - It is cutoff dependent ⇒ unphysical
- $\mathcal{K}_{\rm df,3}$ is related to $\mathcal{M}_3$ via integral equations [HS15]

Matrix indices in QC3

- All quantities are (infinite-dimensional) matrices, e.g. $[F_3]_{\boldsymbol{p}\ell' m', \boldsymbol{k}\ell m}$, with indices

[finite volume "spectator" momentum: $\boldsymbol{k} = 2\pi \boldsymbol{n}/L$ x [2-particle CM angular momentum: ℓ, m]

Describes three on-shell particles with total energy-momentum $(E, \boldsymbol{P})$

- For large $\boldsymbol{k}$ (at fixed E, L), the other two particles are below threshold

- Must include such configurations, by analytic continuation, up to a cut-off at $k \approx m$ [Polejaeva & Rusetsky, `12]

F_3 collects 2-particle interactions

$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\frac{1}{\mathcal{K}_2^{-1} + F + G}F\right]$$

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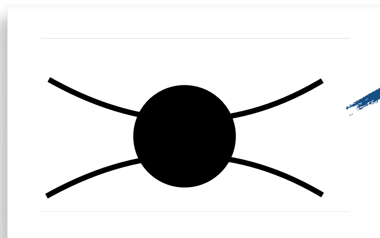

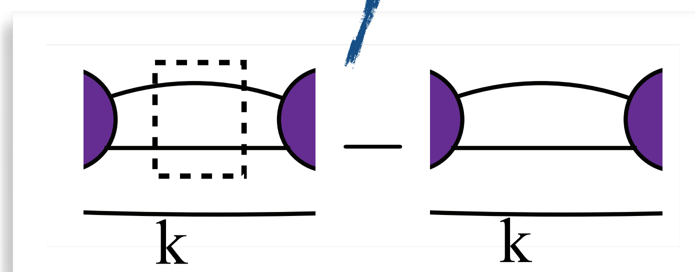

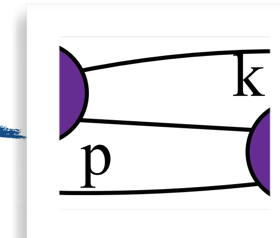

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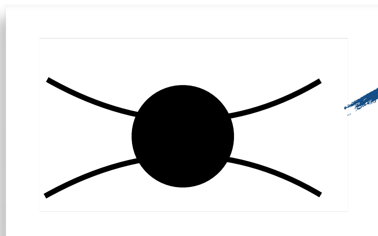
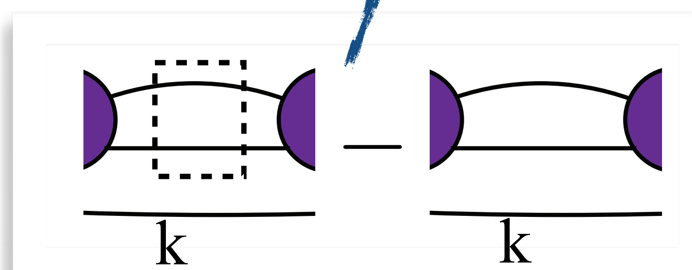
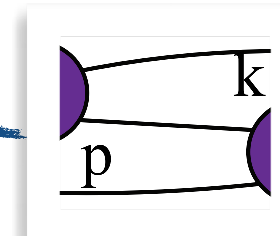
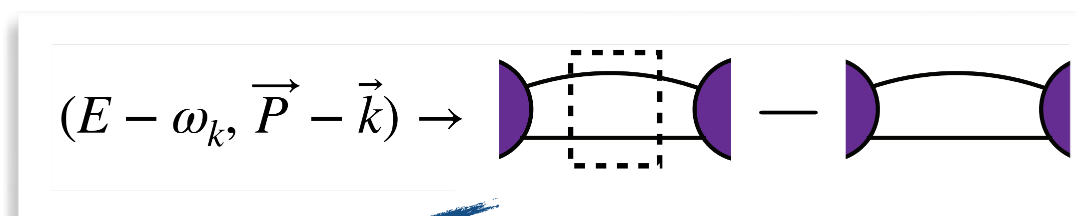






- F & G are known geometrical functions, containing cutoff function H



$$F_{p\ell'm';k\ell m} = \delta_{pk}\ H(\vec{k})\ F_{\mathrm{PV},\ell'm';\ell m}(E - \omega_k, \vec{P} - \vec{k}, L)$$

$$G_{p\ell'm';k\ell m} = \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell'm'}(\hat{k}^*)H(\vec{p})H(\vec{k})Y^*_{\ell m}(\hat{p}^*)}{(P-k-p)^2 - m^2}\left(\frac{p^*}{q_k^*}\right)^{\ell}\frac{1}{2\omega_k L^3}$$

Relativistic form introduced in [BHS17]

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$$F_{\mathrm{PV};\ell'm';\ell m}(E,\vec{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \mathrm{PV}\int\frac{d^3k}{(2\pi)^3}\right)\frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k})}$$

Relativistic form equivalent up to exponentially-suppressed terms

$$\mathcal{Y}_{\ell m}(\vec{k}^*) = \sqrt{4\pi}\left(\frac{k^*}{q^*}\right)^{\ell} Y_{\ell m}(\hat{k}^*)$$

Divergence-free K matrix

$$\det\left[F_3(E, \overrightarrow{P}, L)^{-1} + \mathscr{K}_{\mathrm{df},3}(E^*)\right] = 0$$

What is this? A quasi-local divergence-free 3-particle interaction

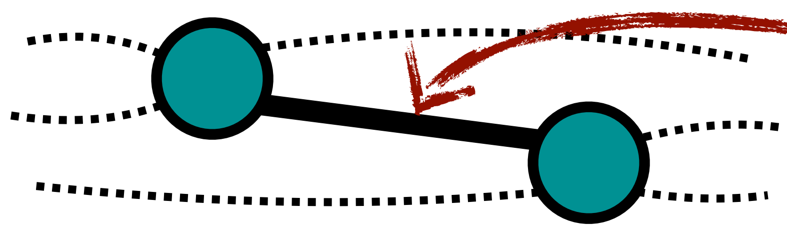
Divergence-free K matrix

$$\det\left[F_3(E,\vec{P},L)^{-1} + \mathcal{K}_{\mathrm{df},3}(E^*)\right] = 0$$

What is this? A quasi-local divergence-free 3-particle interaction

Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\to3} \equiv$ **fully connected correlator with six external legs amputated and projected on shell**

$=$  $+\cdots$

Certain external momenta put this on-shell!

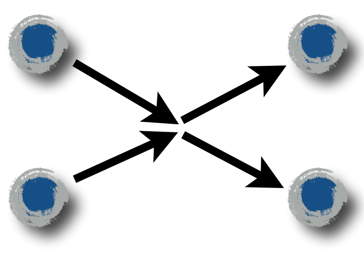
[Artwork from Hansen, HMI lectures]

- To have a nonsingular (divergence-free) quantity, need to subtract pole

Divergence-free K matrix

- $K_{df,3}$ has the same symmetries as M_3: relativistic invariance, particle interchange, T-reversal

M_2, K_2



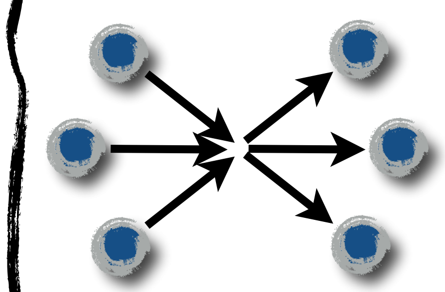
12 momentum components

-10 Poincaré generators

2 degrees of freedom

$s=E^{*2}$ + θ

M_3, $K_{df,3}$



18 momentum components

-10 Poincaré generators

8 degrees of freedom

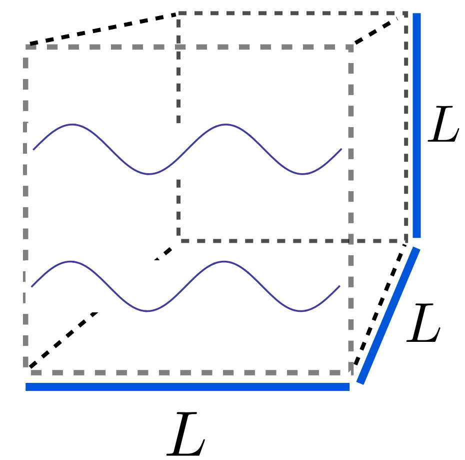
$s=E^{*2}$ + 7 "angles"

- Need more parameters to describe $\mathscr{K}_{\rm df,3}$ than $\mathscr{K}_2$ (will be discussed in lecture 3)
- Why $\mathscr{K}_2$ and $\mathscr{K}_{\rm df,3}$ appear in QC3, rather than $\mathscr{M}_2$ and $\mathscr{M}_{\rm df,3}$, will be explained shortly

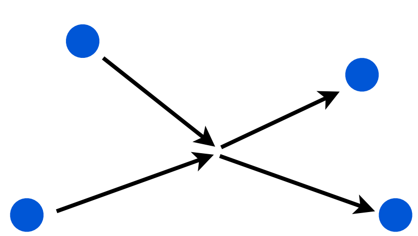
QC3: Sketch of derivation

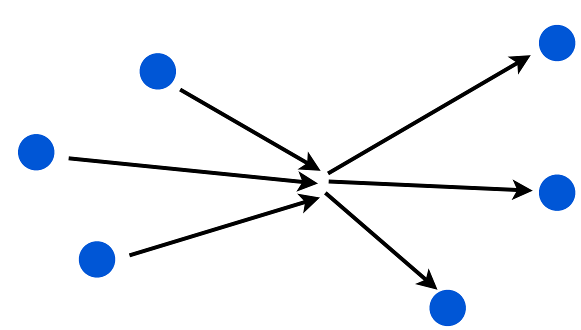
Set-up

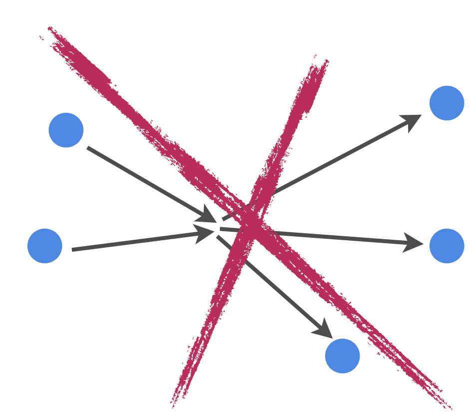
- Work in continuum (assume that LQCD can control discretization errors)



- Cubic box of size L with periodic BC, and infinite (Minkowski) time
    - Spatial loops are sums: $\frac{1}{L^3}\sum_{\vec{k}}$ $\qquad \vec{k} = \frac{2\pi}{L}\vec{n}$
- Consider identical, spinless particles with physical mass m, interacting *arbitrarily* except for a Z_2 (G-parity-like) symmetry
    - Only vertices are 2→2, 2→4, 3→3, 3→1, 3→5, 5→7, etc.
    - Even & odd particle-number sectors decouple







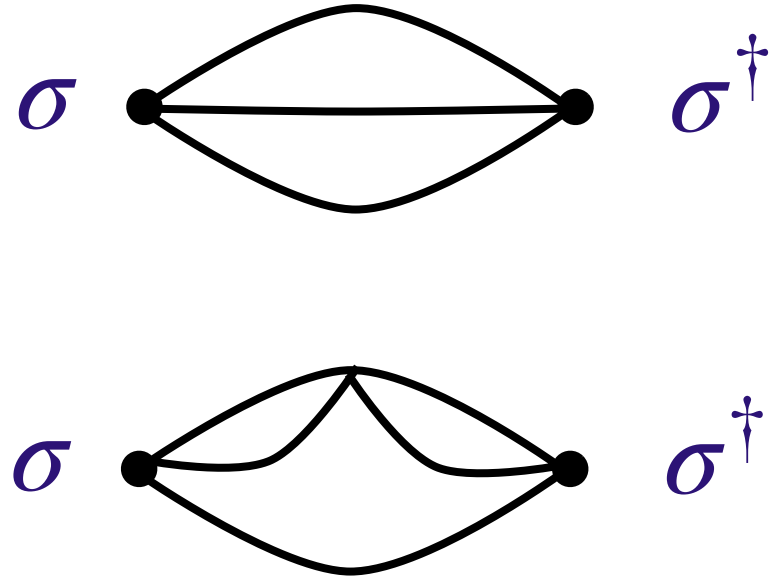
Use TOPT approach of [Blanton & SS,20]

- Calculate (for some $\boldsymbol{P} = 2\pi \boldsymbol{n}_P / L$)

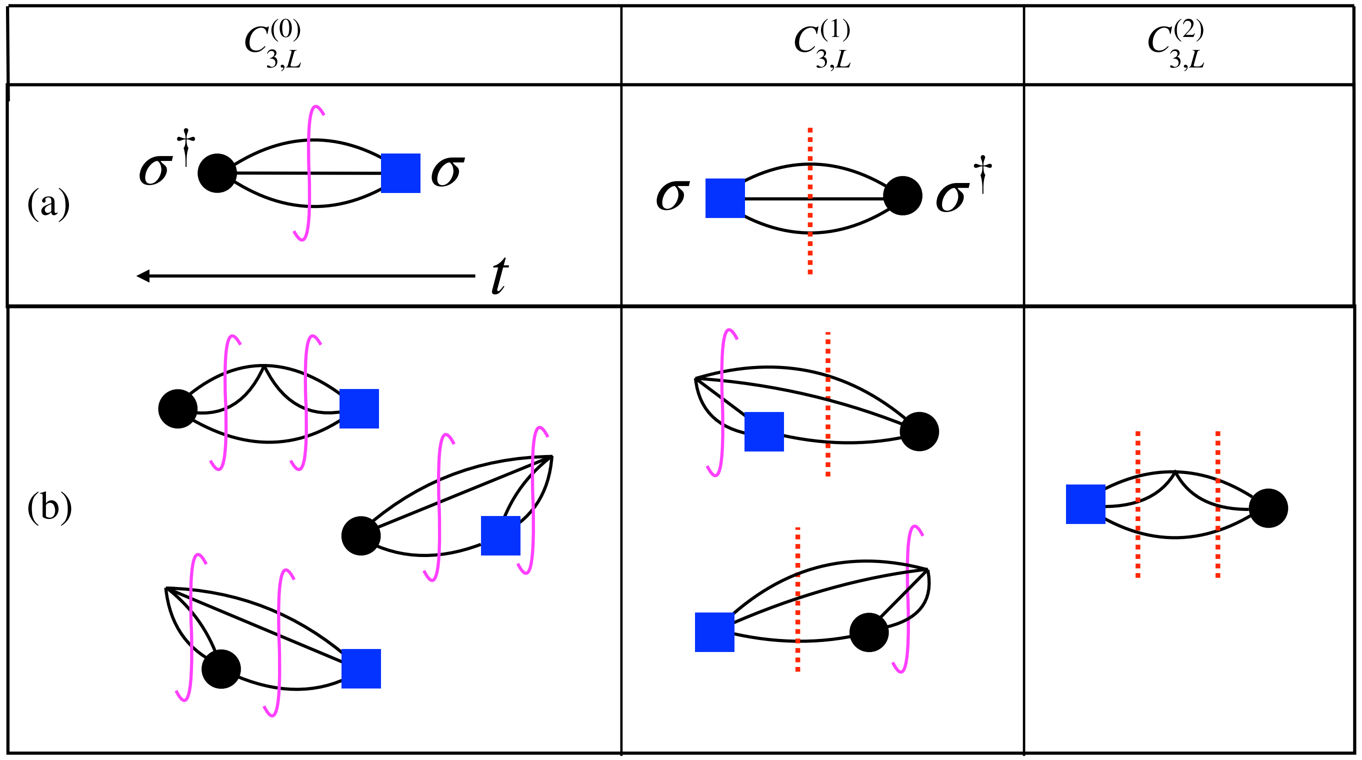
$$C_L(E, \boldsymbol{P}) = \int_x d^4x e^{iEt - i\boldsymbol{P}\cdot\boldsymbol{x}} \langle 0 | T\left\{\sigma(x)\sigma^\dagger(0)\right\} | 0 \rangle_L$$

$\sigma \sim \pi^3$

- Evaluate using time-ordered PT (obtained doing energy integrals)



Feynman



TOPT

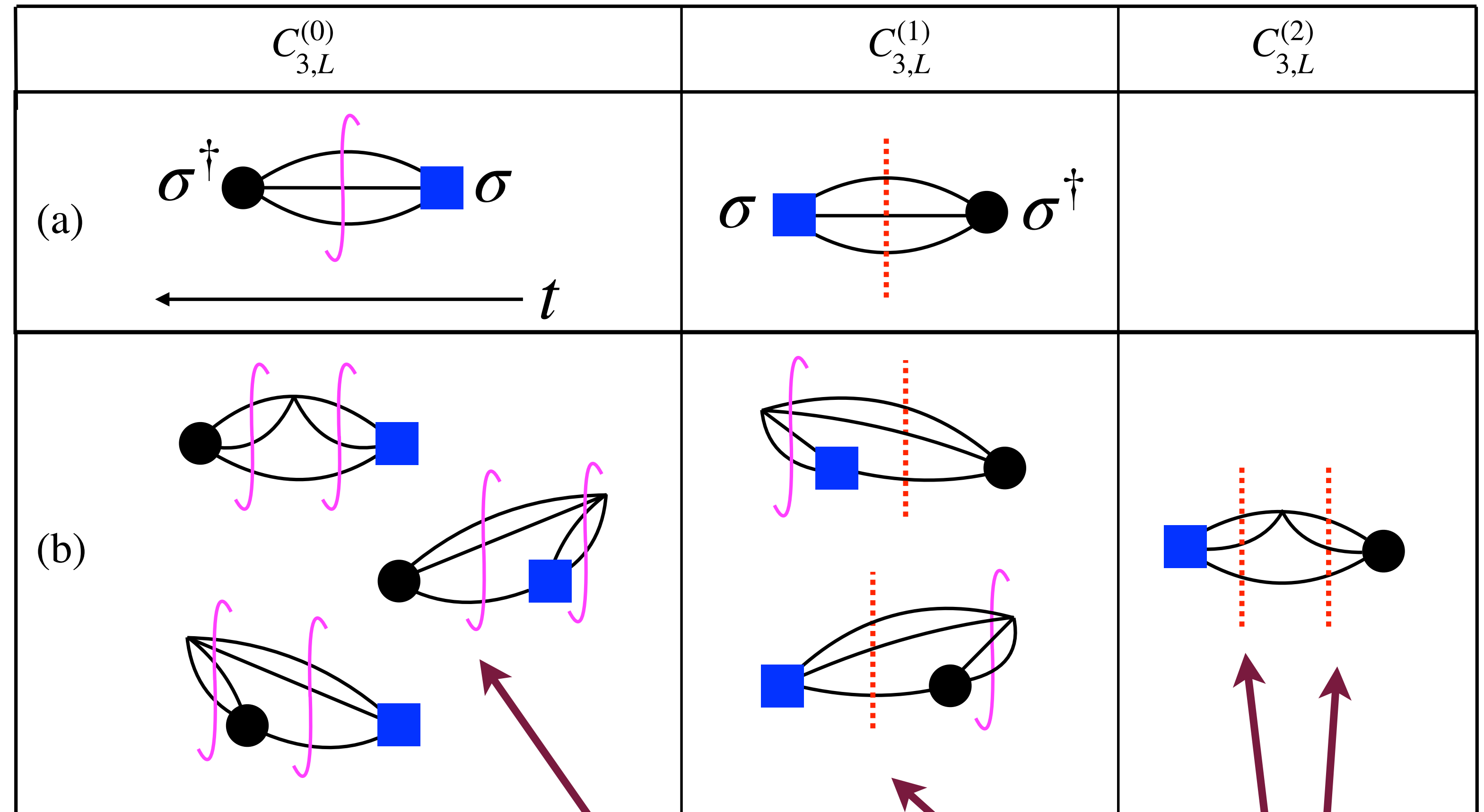
Energy denominators

- Associated with each cut:

$$\frac{1}{E_t - \sum_i \omega_i}$$

$$\omega_i = \sqrt{p_i^2 + M^2}$$

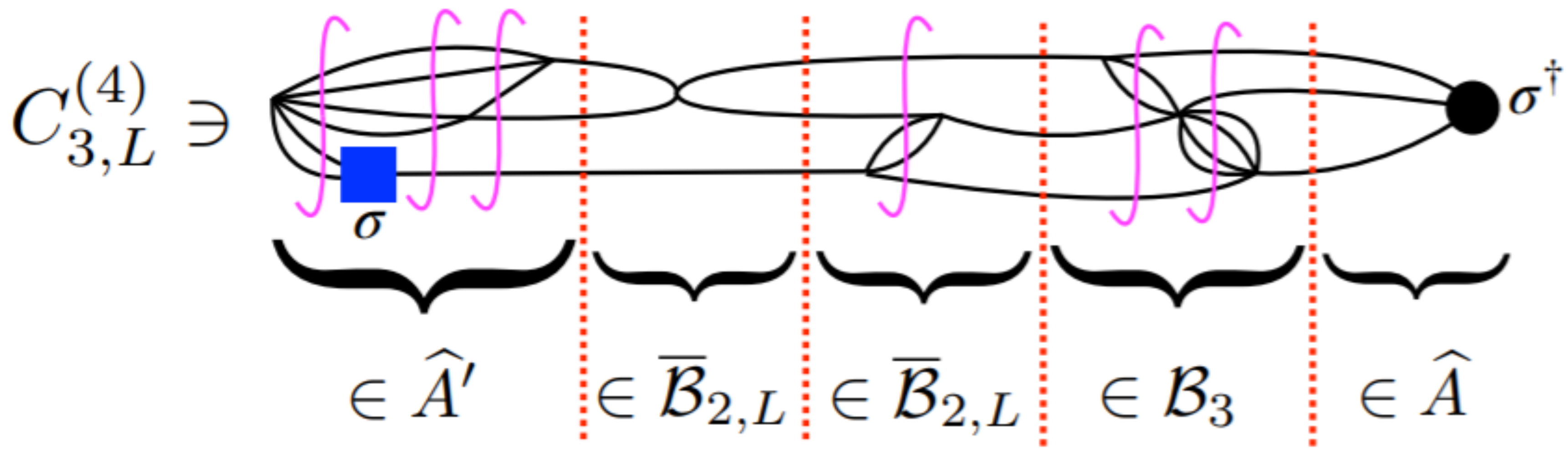
$$E_t \equiv \begin{cases} +E & \text{if } t_\sigma > t > t_{\sigma^\dagger} \\ -E & \text{if } t_{\sigma^\dagger} > t > t_\sigma \\ 0 & \text{otherwise,} \end{cases}$$



- If $E^* < 5M^2$ then only 3-particle cuts can be singular
  - Cuts are either relevant (have pole)
  - Or irrelevant (nonsingular) — can replace sums with integrals

Skeleton expansion

- Use skeleton expansion in terms of TOPT diagrams, ordered by the number of ``relevant cuts” e.g.



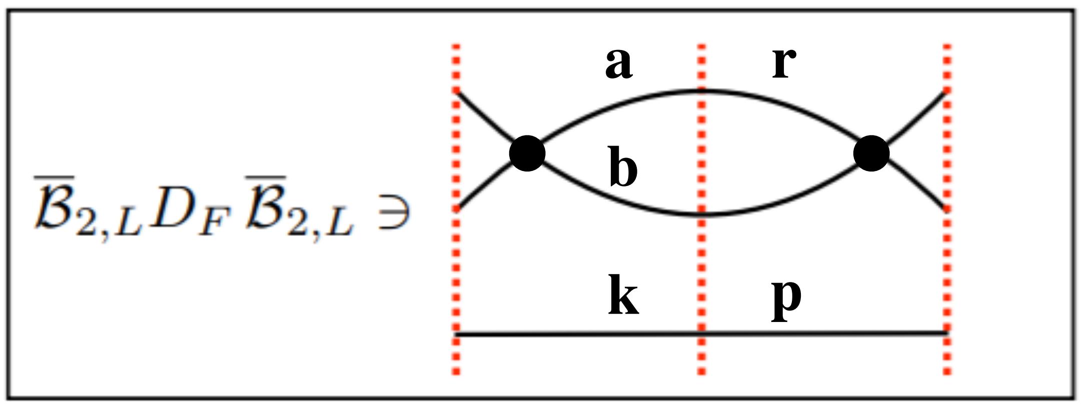
- Loops with only irrelevant cuts can be integrated; build up TOPT kernels $\widehat{A}'$, $\overline{\mathscr{B}}_{2,L} = 2\omega L^3 \mathscr{B}_2$, $\mathscr{B}_3$, $\widehat{A}$
  - Kernels are off shell ($E \neq \sum \omega_i$)
  - Kernels are matrices with indices associated with cuts: two independent three-momenta

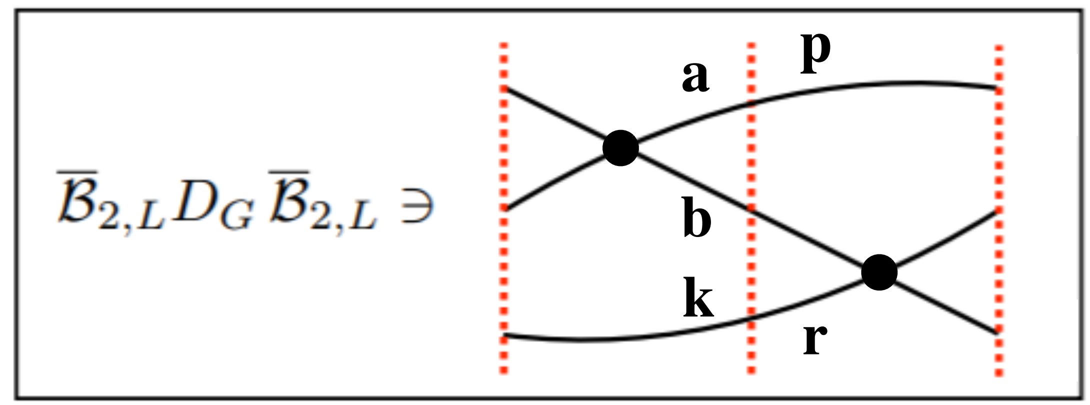
Two types of relevant cuts

- Two types of relevant cuts: F- and G-like

$$[iD_F]_{ka;pr} \equiv \delta_{kp}\delta_{ar}\frac{iD_{ka}}{2!}\,, \qquad [iD_G]_{ka;pr} \equiv \delta_{kr}\delta_{ap} iD_{kp}\,,$$

$$iD_{ka} \equiv \frac{1}{2\omega_k L^3}\frac{i}{2\omega_b(E-\omega_k-\omega_a-\omega_b)}\frac{1}{2\omega_a L^3}$$





"no switch"

"switch"

All orders summation

- Group diagrams according to the number of relevant cuts:
  - Use symmetry of $\hat{A}', \hat{A}, \mathscr{B}_3$ to write all cuts as $D_F + D_G$

$$C^{(1)}_{3,L}(E,\vec{P}) = \widehat{A}' i(D_F + D_G)\widehat{A}$$
$$C^{(2)}_{3,L}(E,\vec{P}) = \widehat{A}' i(D_F + D_G) i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)\widehat{A}$$
$$C^{(3)}_{3,L}(E,\vec{P}) = \widehat{A}' i(D_F + D_G) i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G) i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)\widehat{A}$$
$$\vdots$$
$$C^{(n)}_{3,L}(E,\vec{P}) = \widehat{A}' i(D_F + D_G)\left[i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)\right]^{n-1}\widehat{A}$$

$$\Rightarrow \boxed{C_{3,L}(E,\vec{P}) = C^{(0)}_{3,\infty}(E,\vec{P}) + \widehat{A}' i(D_F + D_G)\frac{1}{1 - i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)}\widehat{A}}$$

- Simple, explicit expression!
  - Clean separation of finite-vol. momentum sums and infinite-vol. integrals
  - (Relatively) straightforward to generalize to other systems (nondegenerate, multiple channels, …)

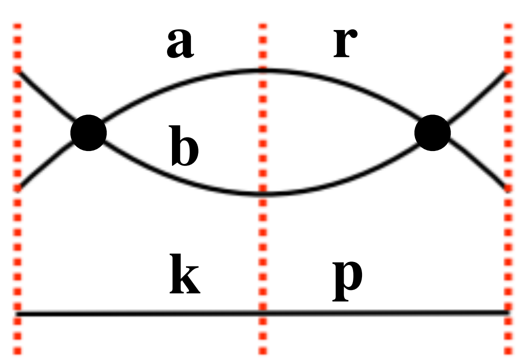
On-shell projection

- On-shell projection necessary to relate FV energies to physical infinite-vol amplitudes

$$[iD_F]_{ka;pr} \equiv \delta_{kp}\delta_{ar}\frac{iD_{ka}}{2!}\,, \qquad [iD_G]_{ka;pr} \equiv \delta_{kr}\delta_{ap} iD_{kp}\,,$$

$$iD_{ka} \equiv \frac{1}{2\omega_k L^3}\frac{i}{2\omega_b(E-\omega_k-\omega_a-\omega_b)}\frac{1}{2\omega_a L^3}$$

$\overline{\mathcal{B}}_{2,L} D_F \overline{\mathcal{B}}_{2,L} \ni$

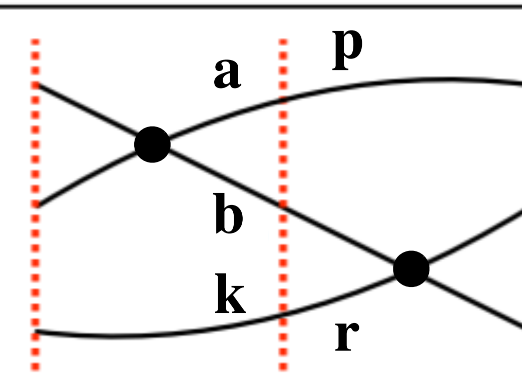


$$\frac{1}{L^6}\sum_k \sum_a = \int_k \mathrm{PV}\int_a + \frac{1}{L^3}\sum_k \underbrace{\left[\frac{1}{L^3}\sum_a - \mathrm{PV}\int_a\right]}_{\sim \widetilde{F}}$$

$$\Rightarrow D_F = \widetilde{I}_F + \widetilde{F}$$

Integral operator that sews kernels together

$\overline{\mathcal{B}}_{2,L} D_G \overline{\mathcal{B}}_{2,L} \ni$



- Decompose adjacent kernels into on-shell part and residue
- Residue cancels pole & leads to integral operator

$$\Rightarrow D_G = \widetilde{G} + \delta\widetilde{G}$$

Integral operator that sews kernels together

Project adjacent kernels on shell: $\{\mathbf{k}, \mathbf{a}\} \to \{\mathbf{k}\ell m\}$

Re-summing

- Simple rearrangement of geometric series leads to on-shell form

$$C_{3,L}(E,\vec{P}) = C^{(0)}_{3,\infty}(E,\vec{P}) + \widehat{A}' i(\widetilde{F}+\widetilde{G}+\widetilde{\mathcal{I}}_F+\delta\widetilde{G})\frac{1}{1-i(\overline{\mathcal{B}}_{2,L}+\mathcal{B}_3)i(\widetilde{F}+\widetilde{G}+\widetilde{\mathcal{I}}_F+\delta\widetilde{G})}\widehat{A}$$

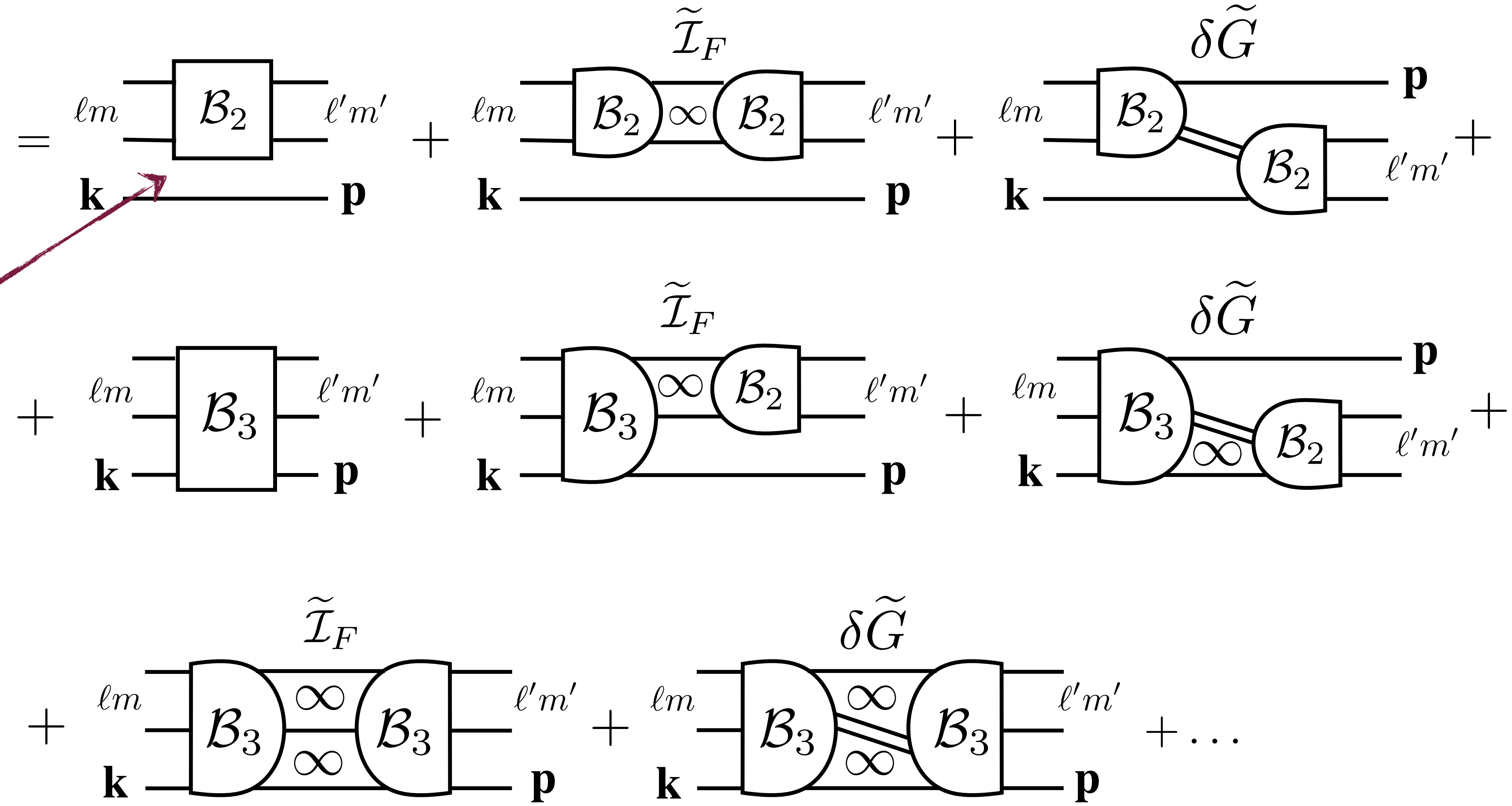
$$= \widetilde{C}_{3,\infty}(E,\vec{P}) + \widetilde{A}'^{(u)} i(\widetilde{F}+\widetilde{G})\frac{1}{1-i\left(2\omega L^3\mathcal{K}_2+\widetilde{\mathcal{K}}^{(u,u)}_{\mathrm{df},3}\right)i(\widetilde{F}+\widetilde{G})}\widetilde{A}^{(u)}$$

where $$i\left(2\omega L^3\mathcal{K}_2+\widetilde{\mathcal{K}}^{(u,u)}_{\mathrm{df},3}\right) = \frac{1}{1-i(\overline{\mathcal{B}}_{2,L}+\mathcal{B}_3)i(\widetilde{\mathcal{I}}_F+\delta\widetilde{G})}i(\overline{\mathcal{B}}_{2,L}+\mathcal{B}_3)$$

- Simplicity of expression is due to combining 2- and 3-particle K matrices
- $\widetilde{A}'^{(u)}$, $\mathscr{K}_2$, $\widetilde{\mathscr{K}}^{(u,u)}_{\mathrm{df},3}$, & $\widetilde{A}^{(u)}$ are on-shell, infinite-volume quantities, with $\mathscr{K}_2$ the same two-particle K matrix as in the Feynman-diagram approach
- "(u)" & "(u,u)" indicate asymmetry due to factors of $\overline{\mathscr{B}}_{2,L}$
- Have assumed that $\mathscr{B}_{2,L}$ has no poles as a function of $\boldsymbol{k}$; if it does, can use a modified PV prescription to avoid the issue

Meaning of asymmetry

$$2\omega L^3 \mathscr{K}_2 + \widetilde{\mathscr{K}}^{(u,u)}_{\mathrm{df},3} =$$



On-shell kernels shown by flat ends

- Momenta **k**, **p** spectate if external interaction involves two particles

Obtaining QC3 (asymmetric form)

$$C_{3,L} - \widetilde{C}_{3,\infty} = \widetilde{A}'^{(u)} i(\widetilde{F} + \widetilde{G}) \frac{1}{1 - i\left(2\omega L^3 \mathcal{K}_2 + \widetilde{\mathcal{K}}_{\text{df},3}^{(u,u)}\right) i(\widetilde{F} + \widetilde{G})} \widetilde{A}^{(u)}$$

- Spectrum determined by poles in $C_{3,L}(E, \mathbf{P})$

$$\Rightarrow \quad \boxed{\det\left[1 + \left(2\omega L^3 \mathcal{K}_2 + \widetilde{\mathcal{K}}_{\text{df},3}^{(u,u)}\right)\left(\widetilde{F} + \widetilde{G}\right)\right] = 0}$$

- $\widetilde{\mathscr{K}}_{df,3}^{(u,u)}$ related to $\mathscr{M}_3$ by known integral equations
- "df"="divergence-free"; absence of divergences is manifest given explicit form
- This is the form of the QC3 that can be shown to be equivalent to that in the FVU approach [BS20b]

Comparing QC3s

[HS14, HS15]

$$\det[1 + F_3 \mathcal{K}_{\mathrm{df},3}] = 0$$

$$F_3 = \widetilde{F}\left[\frac{1}{3} - \frac{1}{1/(2\omega L^3 \mathcal{K}_2) + \widetilde{F} + \widetilde{G}}\widetilde{F}\right]$$

- ☐ Complicated derivation, hard to generalize
- ☐ Implicit, constructive definitions
- ☑ $\mathcal{K}_{\mathrm{df},3}$ is Lorentz invariant
- ☑ $\mathcal{K}_{\mathrm{df},3}$ is symmetric under particle exchange, so easier to parametrize

[BS20a]

$$\det[1 + (2\omega L^3 \mathcal{K}_2 + \widetilde{\mathcal{K}}^{(u,u)}_{\mathrm{df},3})(\widetilde{F} + \widetilde{G})] = 0$$

- ☑ Greatly simplified derivation, easy to generalize
- ☑ Explicit expressions for all quantities
- ☑ Clean separation of infinite- and finite-volume quantities
- ☐ $\widetilde{\mathcal{K}}^{(u,u)}_{\mathrm{df},3}$ is not Lorentz invariant (because we used TOPT)
- ☐ Asymmetry of $\widetilde{\mathcal{K}}^{(u,u)}_{\mathrm{df},3}$ implies that description requires additional parameters

Best of both worlds

$$\det[1 + (2\omega L^3 \mathscr{K}_2 + \widetilde{\mathscr{K}}_{\text{df},3}^{(u,u)})(\widetilde{F} + \widetilde{G})] = 0$$

can symmetrize to original form

$$\det[1 + F_3 \widetilde{\mathscr{K}}'_{\text{df},3}] = 0$$

- $\widetilde{\mathscr{K}}'_{\text{df},3}$ obtained from $\widetilde{\mathscr{K}}_{\text{df},3}^{(u,u)}$ by solving an integral equation and symmetrizing
- Can show that $\widetilde{\mathscr{K}}'_{\text{df},3} = \mathscr{K}_{\text{df},3}$ (since both related to $\mathscr{M}_3$ by same integral equation) so obtain exactly the original [HS14] QC3
    - Thus symmetrization also restores Lorentz invariance!

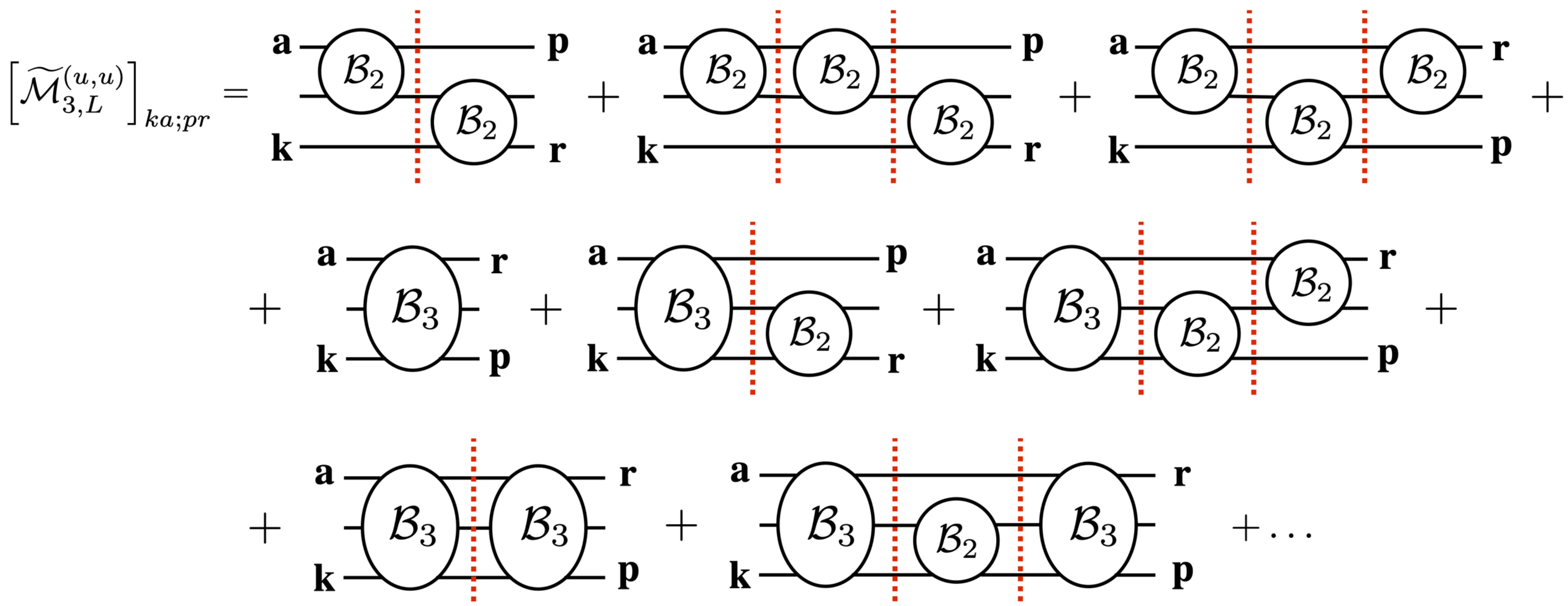
Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

Overview of method [HS15/BS20a]

- Introduce asymmetric finite-volume 3-to-3 amplitude, $\mathscr{M}_{3,L}^{(u,u)}$, such that, when take $L \to \infty$ limit appropriately, and symmetrize, obtain $\mathscr{M}_3$
- Derive expression for $\mathscr{M}_{3,L}^{(u,u)}$ in terms of $\mathscr{K}_{\text{df},3}$ using TOPT/symmetrization
- Take $L \to \infty$ limit and obtain integral equations

Obtaining TOPT result for $\mathscr{M}_{3,L}^{(u,u)}$

- TOPT form of $\mathscr{M}_{3,L}^{(u,u)}$ before on-shell projection
  - Initial/Final states are at $t = \pm\infty$
  - Kernels are the same as in expression for C_L



- Obtain geometric series if include disconnected part

$$i(\overline{\mathscr{M}}_{2,L} + \mathscr{M}_{3,L}^{(u,u)}) = i(\overline{\mathscr{B}}_{2,L} + \mathscr{B}_3)\frac{1}{1 - i(D_F + D_G)i(\overline{\mathscr{B}}_{2,L} + \mathscr{B}_3)}$$

On-shell projection

- Obtain geometric series if include disconnected part

$$i\left(2\omega L^3 \mathcal{M}_2 + \mathcal{M}_{3,L}^{(u,u)}\right)_{\text{off}} = i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3)\frac{1}{1 - i(D_F + D_G)i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3)}$$

- Project on shell as before, both "internally" and "externally"

$$i\left(2\omega L^3 \mathcal{M}_2 + \mathcal{M}_{3,L}^{(u,u)}\right)_{\text{on}} = i\left(2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{\text{df},3}^{(u,u)}\right)\frac{1}{1 - i(\tilde{F} + \tilde{G})i(2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{\text{df},3}^{(u,u)})}$$

  - Gives expression for $\mathcal{M}_{3,L}^{(u,u)}$ in terms of same quantities that appear in QC3

Symmetrization

- On-shell form in terms of asymmetric $\mathcal{M}_{\rm df,3}$

$$i\left(2\omega L^3\mathcal{M}_2+\mathcal{M}_{3,L}^{(u,u)}\right)_{\rm on}=i\left(2\omega L^3\mathcal{K}_2+\mathcal{K}_{\rm df,3}^{(u,u)}\right)\frac{1}{1-i(\tilde F+\tilde G)i(2\omega L^3\mathcal{K}_2+\mathcal{K}_{\rm df,3}^{(u,u)})}$$

- Use symmetrization identities, plus extensive algebraic gymnastics, to find

$$\mathcal{M}_{3,L}=\mathcal{S}\left\{\mathcal{D}_L^{(u,u)}+\mathcal{M}_{\rm df,3,L}^{(u,u)}\right\}$$

$$i\mathcal{D}_L^{(u,u)}=i\mathcal{M}_{2,L}i\tilde G i\mathcal{M}_{2,L}\frac{1}{1-i\tilde G i\mathcal{M}_{2,L}},\qquad \mathcal{M}_{2,L}=2\omega L^3\mathcal{M}_2$$

$$i\mathcal{M}_{\rm df,3,L}^{(u,u)}=\mathcal{L}_L^{(u)}i\mathcal{K}_{\rm df,3}\frac{1}{1-iF_3 i\mathcal{K}_{\rm df,3}}\mathcal{L}_L^{(u)\dagger}$$

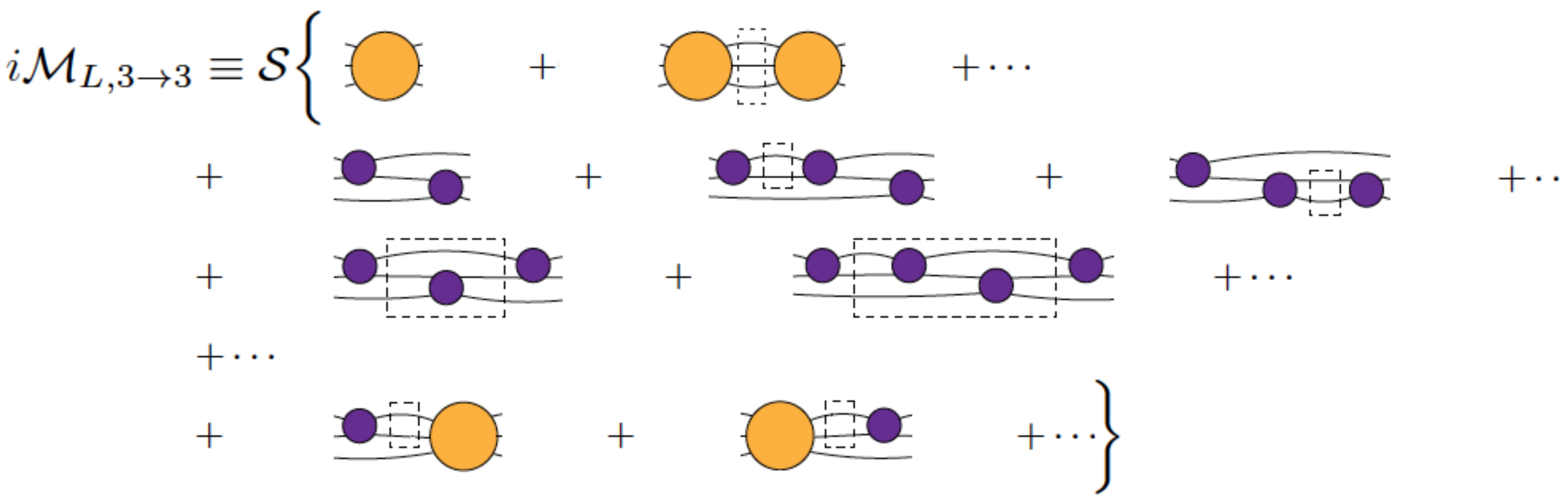
$$\mathcal{L}_L^{(u)}=\frac{1}{3}+\frac{1}{1-i\mathcal{M}_{2,L}i\tilde G}i\mathcal{M}_{2,L}i\tilde F$$

Composed of same quantities as symmetric QC3

Diagrammatic interpretation

$$\mathscr{M}_{3,L} = \mathcal{S}\left\{\mathscr{D}_L^{(u,u)} + \mathscr{M}_{\mathrm{df,3,L}}^{(u,u)}\right\}$$

$$i\mathscr{D}_L^{(u,u)} = i\mathscr{M}_{2,L} i\tilde{G} i\mathscr{M}_{2,L}\frac{1}{1 - i\tilde{G}\mathscr{M}_{2,L}}, \qquad i\mathscr{M}_{\mathrm{df,3,L}}^{(u,u)} = \mathscr{L}_L^{(u)} i\mathscr{K}_{\mathrm{df,3}}\frac{1}{1 - iF_3 i\mathscr{K}_{\mathrm{df,3}}}\mathscr{L}_L^{(u)\dagger}$$



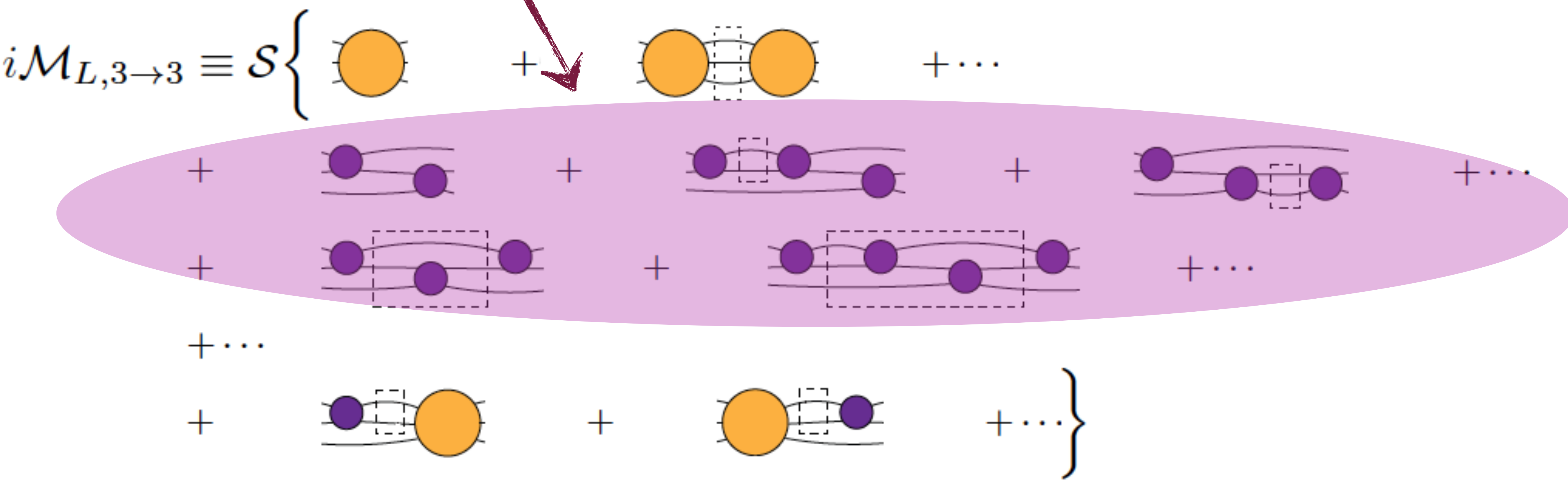
Diagrammatic interpretation

$$\mathscr{M}_{3,L} = \mathcal{S}\left\{\mathscr{D}_L^{(u,u)} + \mathscr{M}_{\text{df,3,L}}^{(u,u)}\right\}$$

$$i\mathscr{D}_L^{(u,u)} = i\mathscr{M}_{2,L}\, i\tilde{G}\, i\mathscr{M}_{2,L}\frac{1}{1 - i\tilde{G}\mathscr{M}_{2,L}},$$

$$i\mathscr{M}_{\text{df,3,L}}^{(u,u)} = \mathscr{L}_L^{(u)}\, i\mathscr{K}_{\text{df,3}}\frac{1}{1 - iF_3\, i\mathscr{K}_{\text{df,3}}}\mathscr{L}_L^{(u)\dagger}$$

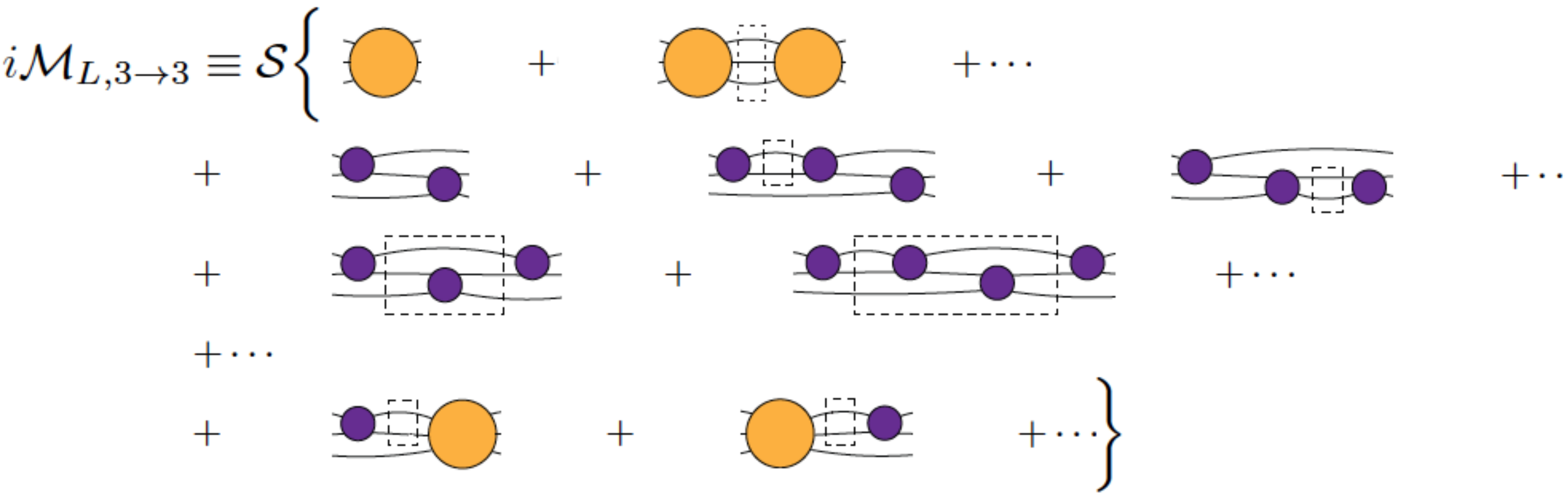
Part of $\mathscr{M}_{3,L}$ that leads to on-shell divergences in $\mathscr{M}_3$



Diagrammatic interpretation

$$\mathscr{M}_{3,L} = \mathcal{S}\left\{\mathscr{D}_L^{(u,u)} + \mathscr{M}_{\text{df,3,L}}^{(u,u)}\right\}$$

$$i\mathscr{D}_L^{(u,u)} = i\mathscr{M}_{2,L} i\tilde{G} i\mathscr{M}_{2,L} \frac{1}{1 - i\tilde{G}\mathscr{M}_{2,L}}, \qquad i\mathscr{M}_{\text{df,3,L}}^{(u,u)} = \mathscr{L}_L^{(u)} i\mathscr{K}_{\text{df,3}} \frac{1}{1 - iF_3 i\mathscr{K}_{\text{df,3}}} \mathscr{L}_L^{(u)\dagger}$$

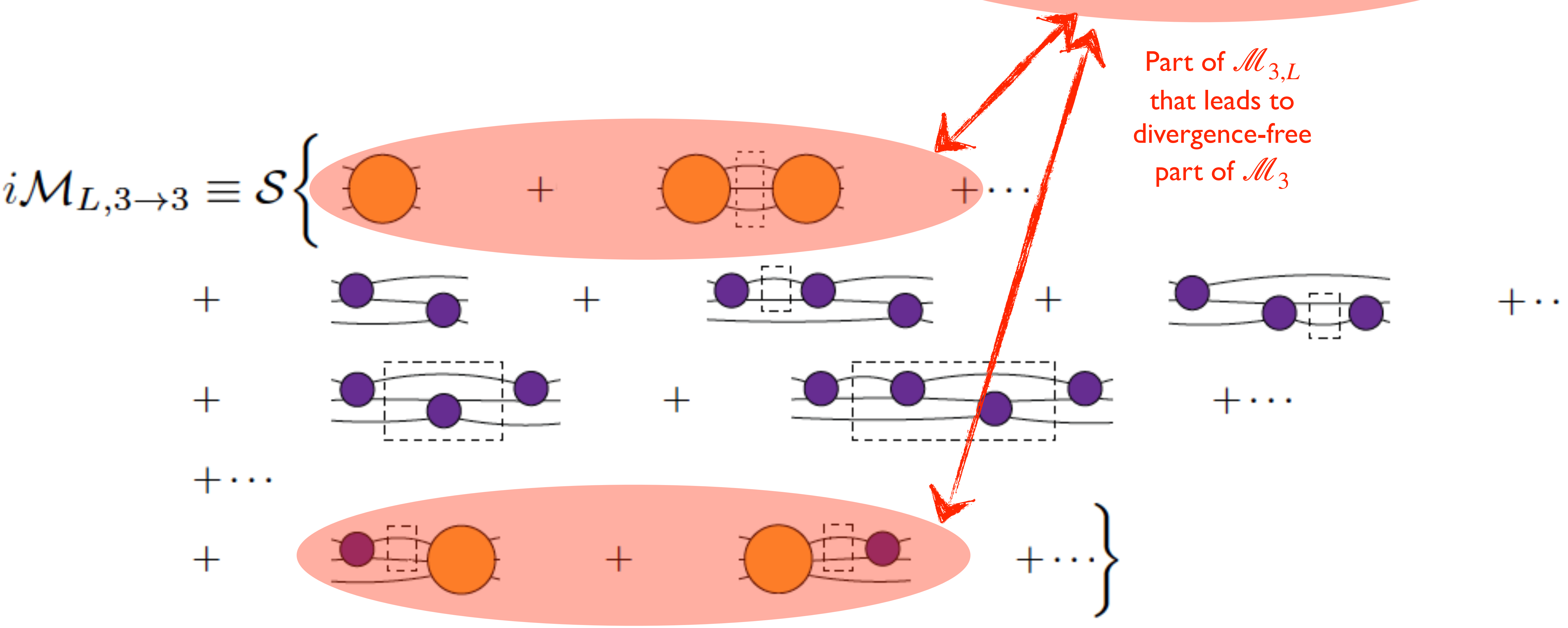


Diagrammatic interpretation

$$\mathscr{M}_{3,L} = \mathcal{S}\left\{\mathscr{D}_L^{(u,u)} + \mathscr{M}_{\mathrm{df,3,L}}^{(u,u)}\right\}$$

$$i\mathscr{D}_L^{(u,u)} = i\mathscr{M}_{2,L} i\tilde{G} i\mathscr{M}_{2,L} \frac{1}{1 - i\tilde{G}\mathscr{M}_{2,L}},$$

$$i\mathscr{M}_{\mathrm{df,3,L}}^{(u,u)} = \mathscr{L}_L^{(u)} i\mathscr{K}_{\mathrm{df},3} \frac{1}{1 - iF_3 i\mathscr{K}_{\mathrm{df},3}} \mathscr{L}_L^{(u)\dagger}$$



Infinite-volume limit

- Reintroduce $i\epsilon$ in energy denominators (irrelevant in finite volume)
- Take $L \to \infty$, and sum of finite-volume TOPT (or Feynman) diagrams becomes exactly the diagrams leading to $\mathcal{M}_3$
    - Matrix equations involving $\boldsymbol{k}$ become integrals
    - Sums over ℓ, m remain
- Obtain a set of nested integral equations and applications of integral operators
    - E.g. for "ladder series" that contains divergences

$$i\mathcal{D}_L^{(u,u)} = i\mathcal{M}_{2,L} i\tilde{G} i\mathcal{M}_{2,L} \frac{1}{1 - i\tilde{G} i\mathcal{M}_{2,L}}$$

$$i\mathcal{D}_L^{(u,u)} = i\mathcal{M}_{2,L} i\tilde{G} i\mathcal{M}_{2,L} + i\tilde{G} i\mathcal{M}_{2,L} i\mathcal{D}_L^{(u,u)}$$

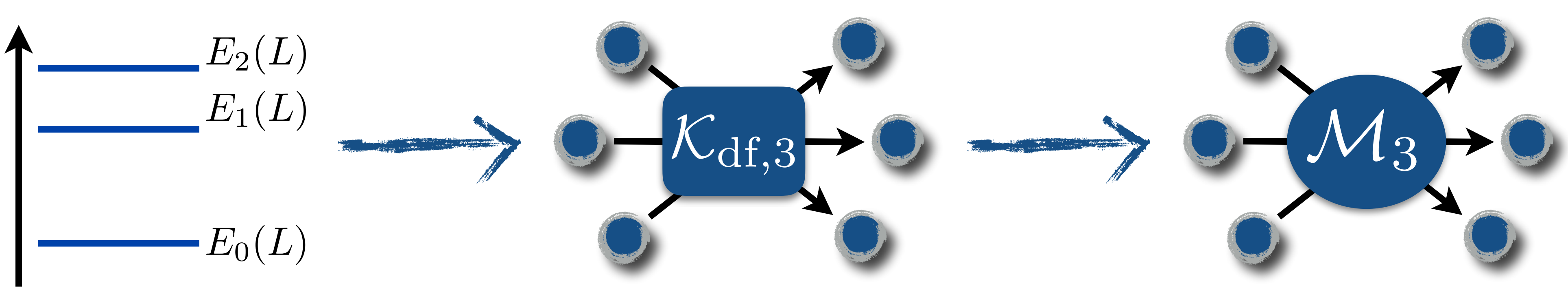
Summary of Lecture 2

Summary of Lecture 2

- Applications of QC2 are at a mature and sophisticated stage
- Derivation of QC3 is complicated, but final result is relatively simple
  - Intuitive interpretation of each part

$$\det[1 + F_3 \mathscr{K}_{\mathrm{df},3}] = 0$$

$$F_3 = \widetilde{F}\left[\frac{1}{3} - \frac{1}{1/(2\omega L^3 \mathcal{K}_2) + \widetilde{F} + \widetilde{G}}\widetilde{F}\right]$$



Thank you!
Questions?

Backup Slides

Generalizations of QC2

- **Multiple two-particle channels** [Hu, Feng & Liu, hep-lat/0504019; Lage, Meissner & Rusetsky, 0905.0069; Hansen & SS, 1204.0826; Briceño & Davoudi, 1204.1110]
  - e.g. $J^{PC} = 0^{++}$ $\pi\pi + K\bar{K}\,(+\eta\eta)$

$$\det\left[\begin{pmatrix} F^{\pi\pi}_{PV}(E,\vec{P},L)^{-1} & 0 \\ 0 & F^{K\bar{K}}_{PV}(E,\vec{P},L)^{-1}\end{pmatrix} + \begin{pmatrix}\mathscr{K}_2^{\pi\pi}(E^*) & \mathscr{K}_2^{\pi K}(E^*) \\ \mathscr{K}_2^{\pi K}(E^*) & \mathscr{K}_2^{KK}(E^*)\end{pmatrix}\right] = 0$$

Generalizations of QC2

- Multiple two-particle channels [Hu, Feng & Liu, hep-lat/0504019; Lage, Meissner & Rusetsky, 0905.0069; Hansen & SS, 1204.0826; Briceño & Davoudi, 1204.1110]
  - e.g. $J^{PC} = 0^{++} \quad \pi\pi + K\bar{K}\,(+\eta\eta)$

$$\det\left[\begin{pmatrix} F_{PV}^{\pi\pi}(E,\vec{P},L)^{-1} & 0 \\ 0 & F_{PV}^{K\bar{K}}(E,\vec{P},L)^{-1} \end{pmatrix} + \begin{pmatrix} \mathcal{K}_2^{\pi\pi}(E^*) & \mathcal{K}_2^{\pi K}(E^*) \\ \mathcal{K}_2^{\pi K}(E^*) & \mathcal{K}_2^{KK}(E^*) \end{pmatrix}\right] = 0$$

- Even if truncate to $l_{\max}=0$, there is no longer a one-to-one relation between energy levels and K-matrix elements
- Must parametrize the (enlarged) K matrix in some way and fit parameters to multiple spectral levels
- Using these parametrizations can study pole structure of scattering amplitude
- Approach is very similar to that used analyzing scattering data