Multiparticle scattering

Steve Sharpe
University of Washington

Outline

❑ Lecture 1

- Motivation/Background/Overview
- Deriving the two-particle quantization condition (QC2)

❑ Lecture 2

- Applying the QC2, in brief
- Deriving the three-particle quantization condition for identical scalars (QC3)

❑ Lecture 3

- Status of three-particle formalism
- Applications of QC3
- Outlook

Main references for this lecture

[Full list of references at end of lecture 3]

- Briceño, Dudek & Young, "Scattering processes & resonances from LQCD," 1706.06223, RMP 18
- Hansen & SS, "LQCD & three-particle decays of resonances," 1901.00483, ARNPS 20
- Lectures by Dudek, Hansen & Meyer at HMI Institute on "Scattering from the lattice: applications to phenomenology and beyond," May 2018, https://indico.cern.ch/event/690702/
- Lüscher, Commun.Math.Phys. 105 (1986) 153-188; Nucl.Phys. B354 (1991) 531-578 & B364 (1991) 237-251 (foundational papers)
- Kim, Sachrajda & SS, hep-lat/0507006 , NPB 2015 (direct derivation in QFT of QC2)

Outline for Lecture 1

- Background: hadronic resonances
- Further motivation for studying multiparticle states
- Some scattering basics
- Derivation of QC2 = "Lüscher quantization condition"

Background:
hadronic resonances

Stable hadrons in isosymmetric QCD

- QCD with $m_u=m_d$, and no EM (or weak) interactions
  - Theory studied in majority of LQCD simulations
  - Differs from real world at ~1% level

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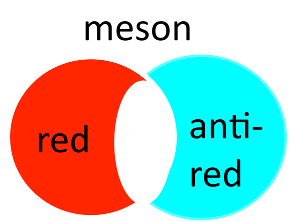

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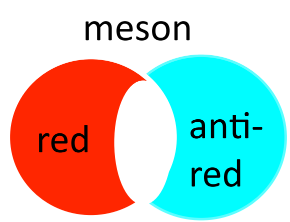

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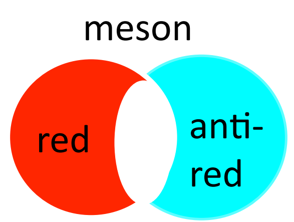

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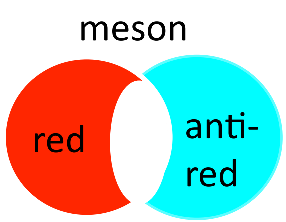
- Mesons 

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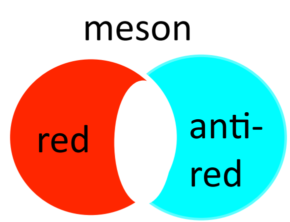

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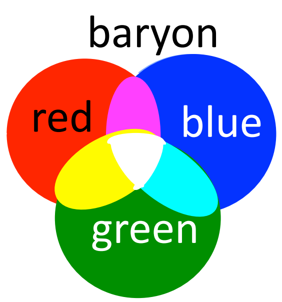
- Baryons



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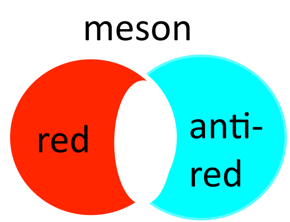

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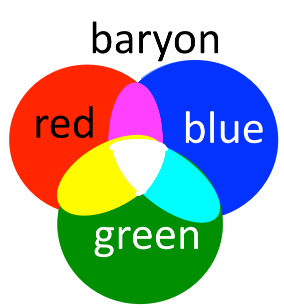

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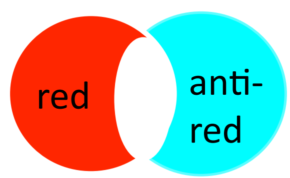

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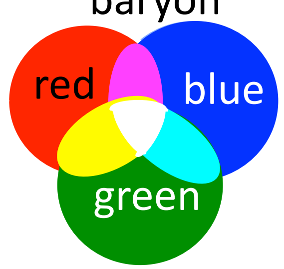

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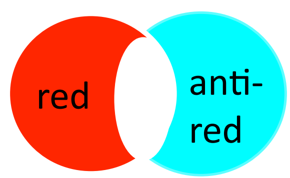

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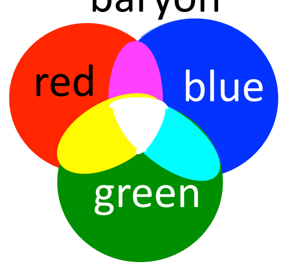

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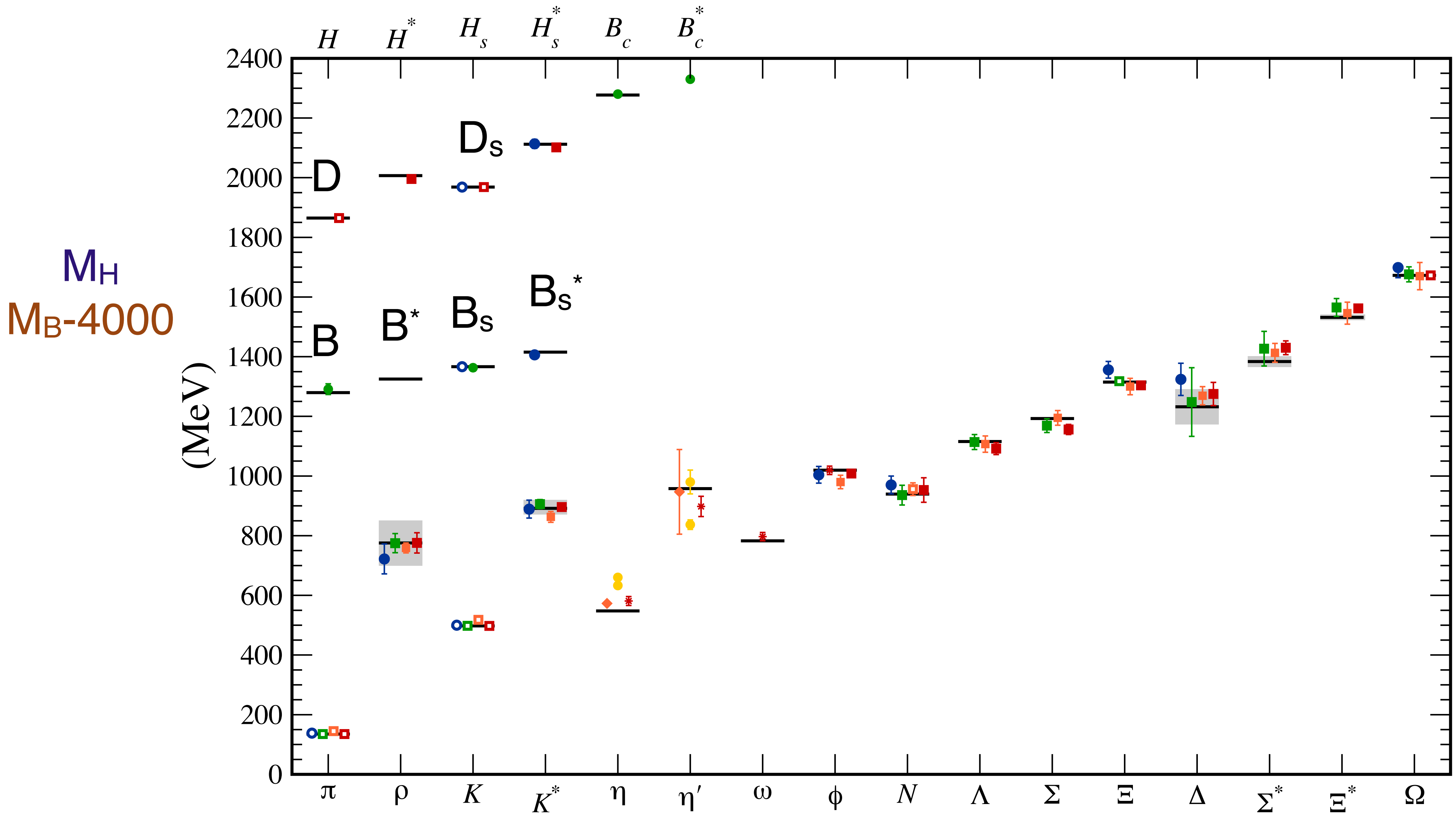

[Kronfeld, 1203.1204]

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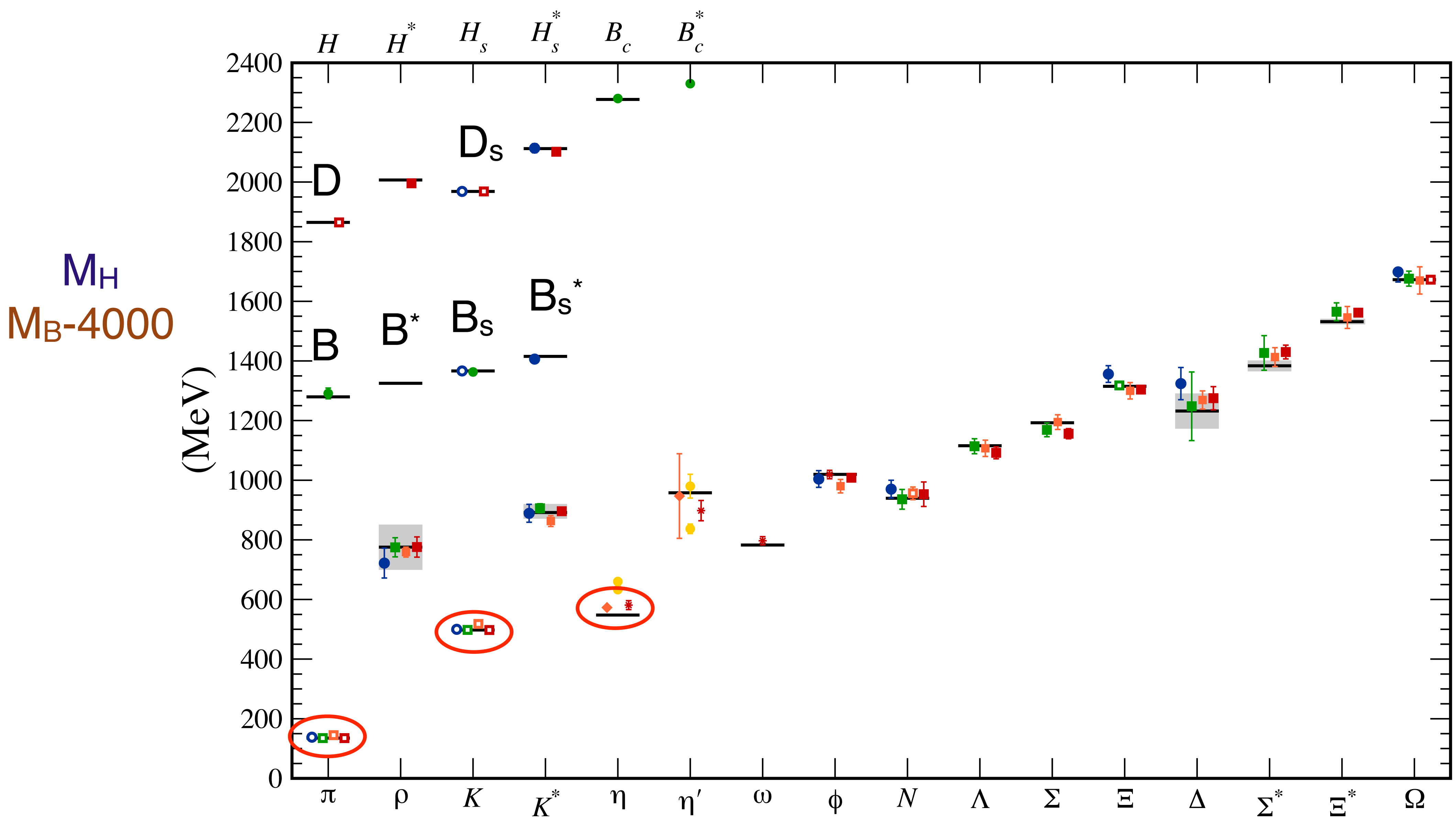

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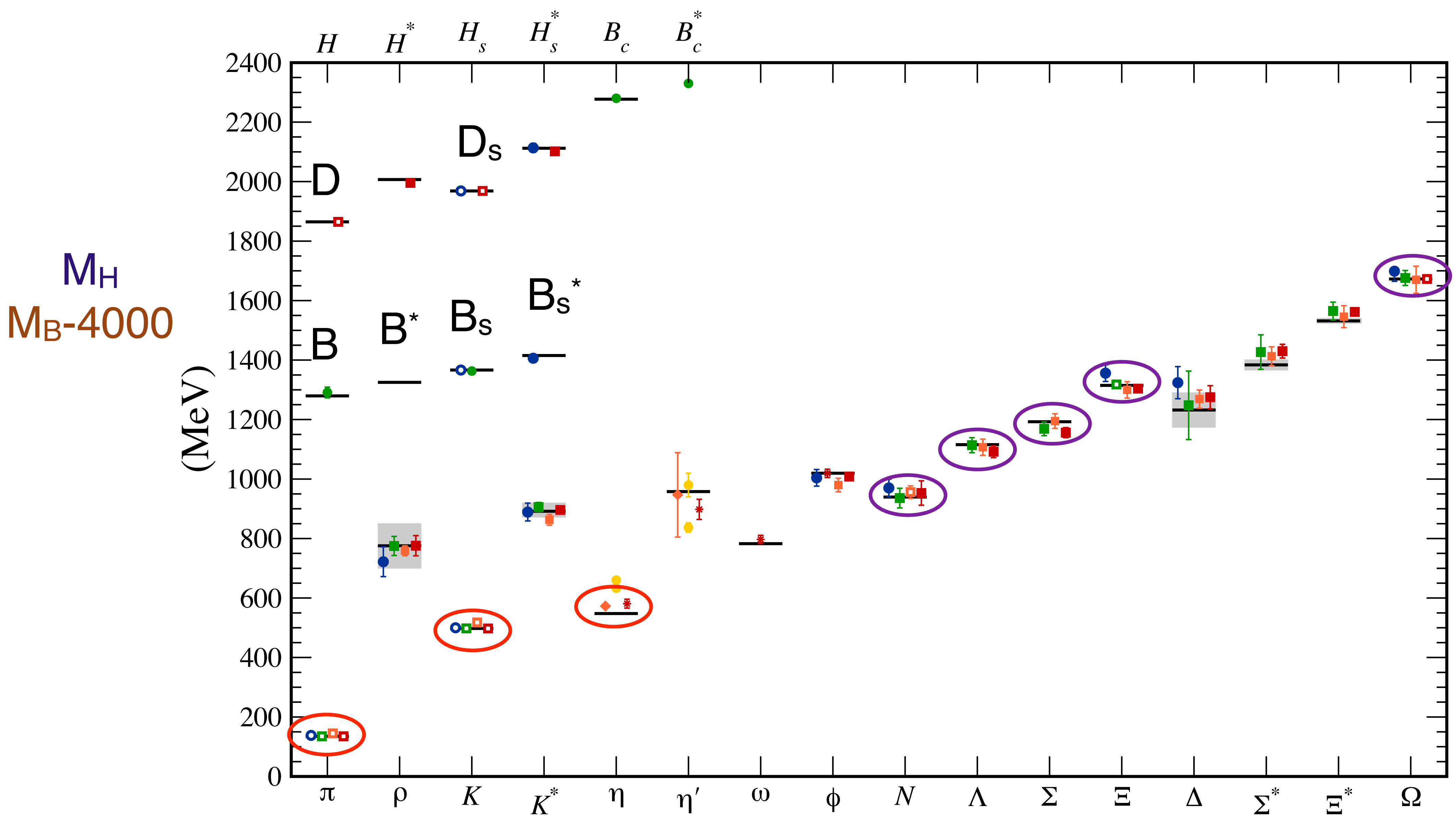

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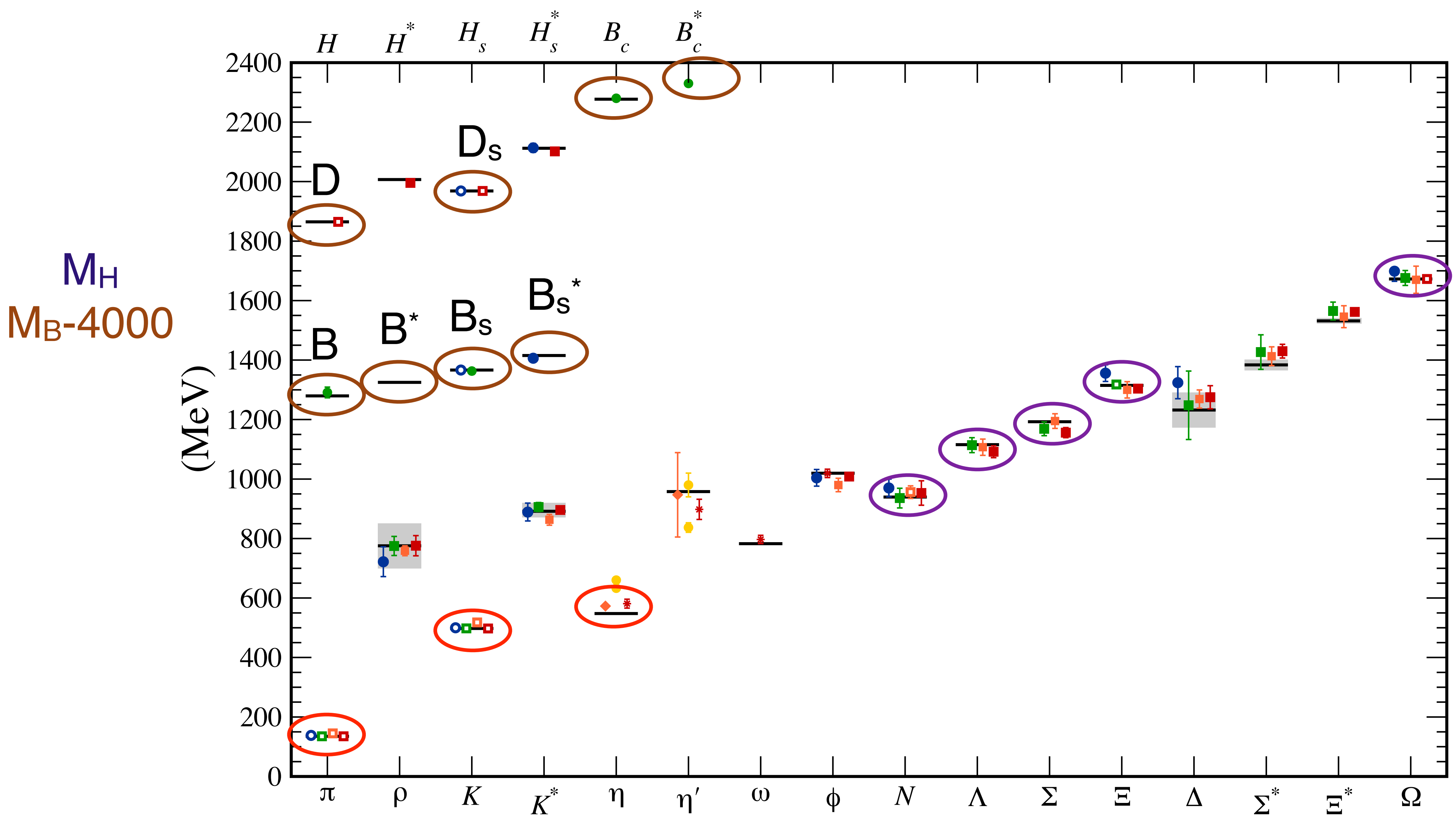

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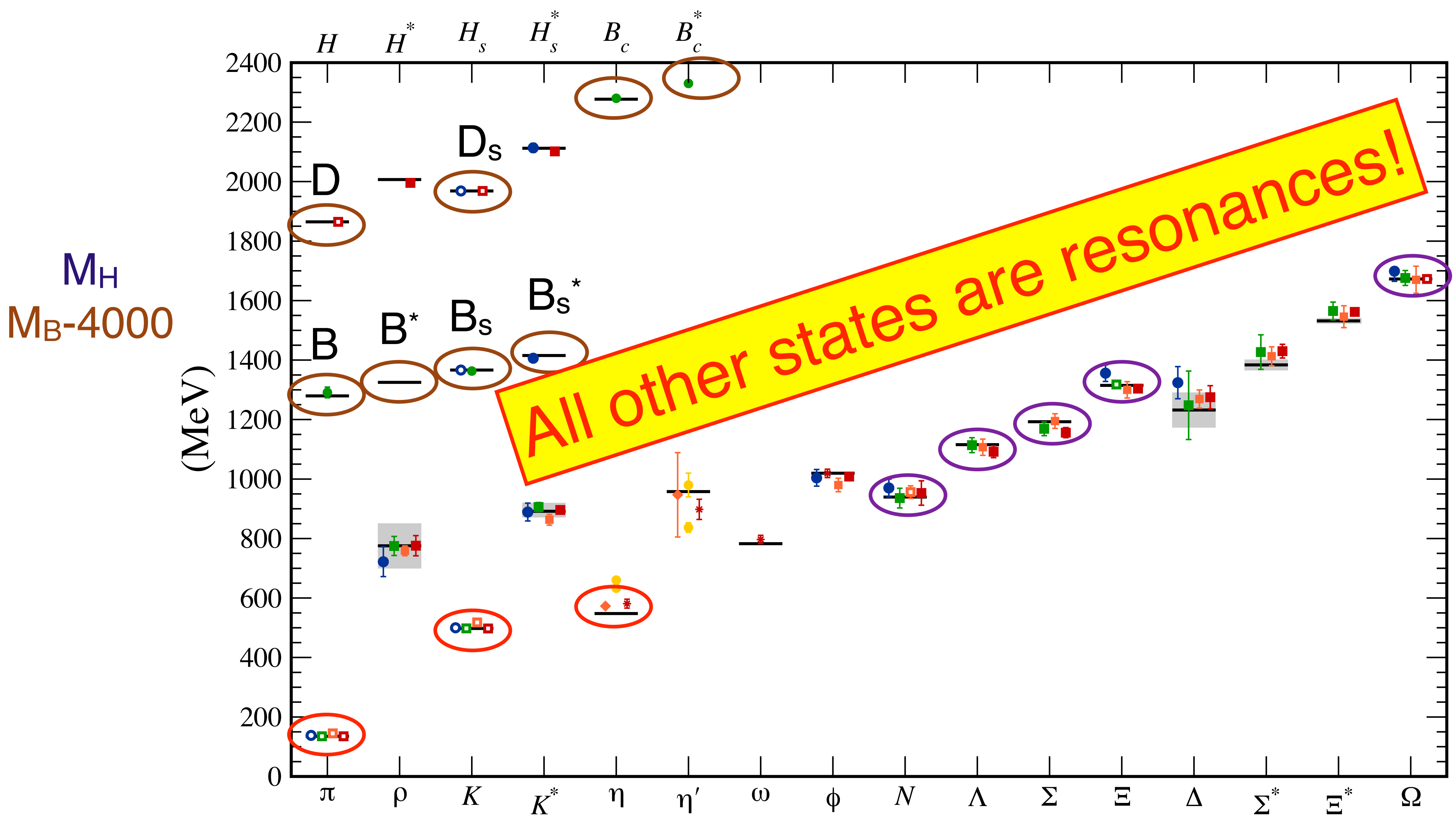


[Kronfeld, 1203.1204]

Plethora of resonances

- Most hadrons are resonances!

pdg meson listings

LIGHT UNFLAVORED ($S = C = B = 0$)

	$I^G(J^{PC})$		$I^G(J^{PC})$
• $\pi^\pm$	$1^-(0^-)$	• $\rho_3(1690)$	$1^+(3^{--})$
• π^0	$1^-(0^{-+})$	• $\rho(1700)$	$1^+(1^{--})$
• η	$0^+(0^{-+})$	$a_2(1700)$	$1^-(2^{++})$
• $f_0(500)$	$0^+(0^{++})$	• $f_0(1710)$	$0^+(0^{++})$
• $\rho(770)$	$1^+(1^{--})$	$\eta(1760)$	$0^+(0^{-+})$
• $\omega(782)$	$0^-(1^{--})$	• $\pi(1800)$	$1^-(0^{-+})$
• $\eta'(958)$	$0^+(0^{-+})$	$f_2(1810)$	$0^+(2^{++})$
• $f_0(980)$	$0^+(0^{++})$	$X(1835)$	$?^?(0^{-+})$
• $a_0(980)$	$1^-(0^{++})$	$X(1840)$	$?^?(?^{??})$
• $\phi(1020)$	$0^-(1^{--})$	$a_1(1420)$	$1^-(1^{++})$
• $h_1(1170)$	$0^-(1^{+-})$	• $\phi_3(1850)$	$0^-(3^{--})$
• $b_1(1235)$	$1^+(1^{+-})$	$\eta_2(1870)$	$0^+(2^{-+})$
• $a_1(1260)$	$1^-(1^{++})$	• $\pi_2(1880)$	$1^-(2^{-+})$
• $f_2(1270)$	$0^+(2^{++})$	$\rho(1900)$	$1^+(1^{--})$
• $f_1(1285)$	$0^+(1^{++})$	$f_2(1910)$	$0^+(2^{++})$
• $\eta(1295)$	$0^+(0^{-+})$	$a_0(1950)$	$1^-(0^{++})$
• $\pi(1300)$	$1^-(0^{-+})$	• $f_2(1950)$	$0^+(2^{++})$
• $a_2(1320)$	$1^-(2^{++})$	$\rho_3(1990)$	$1^+(3^{--})$
• $f_0(1370)$	$0^+(0^{++})$	• $f_2(2010)$	$0^+(2^{++})$
$h_1(1380)$	$?^-(1^{+-})$	$f_0(2020)$	$0^+(0^{++})$
• $\pi_1(1400)$	$1^-(1^{-+})$	• $a_4(2040)$	$1^-(4^{++})$
• $\eta(1405)$	$0^+(0^{-+})$	• $f_4(2050)$	$0^+(4^{++})$
• $f_1(1420)$	$0^+(1^{++})$	$\pi_2(2100)$	$1^-(2^{-+})$
• $\omega(1420)$	$0^-(1^{--})$	$f_0(2100)$	$0^+(0^{++})$
$f_2(1430)$	$0^+(2^{++})$	$f_2(2150)$	$0^+(2^{++})$

STRANGE ($S = \pm1$, $C = B = 0$)

	$I(J^P)$
• $K^\pm$	$1/2(0^-)$
• K^0	$1/2(0^-)$
• K^0_S	$1/2(0^-)$
• K^0_L	$1/2(0^-)$
$K^*_0(800)$	$1/2(0^+)$
• $K^*(892)$	$1/2(1^-)$
• $K_1(1270)$	$1/2(1^+)$
• $K_1(1400)$	$1/2(1^+)$
• $K^*(1410)$	$1/2(1^-)$
• $K^*_0(1430)$	$1/2(0^+)$
• $K^*_2(1430)$	$1/2(2^+)$
$K(1460)$	$1/2(0^-)$
$K_2(1580)$	$1/2(2^-)$
$K(1630)$	$1/2(?^?)$
$K_1(1650)$	$1/2(1^+)$
• $K^*(1680)$	$1/2(1^-)$
• $K_2(1770)$	$1/2(2^-)$
• $K^*_3(1780)$	$1/2(3^-)$
• $K_2(1820)$	$1/2(2^-)$
$K(1830)$	$1/2(0^-)$
$K^*_0(1950)$	$1/2(0^+)$
$K^*_2(1980)$	$1/2(2^+)$
• $K^*_4(2045)$	$1/2(4^+)$
$K_2(2250)$	$1/2(2^-)$
$K_3(2320)$	$1/2(3^+)$

CHARMED, STRANGE ($C = S = \pm1$)

	$I(J^P)$
• $D^\pm_s$	$0(0^-)$
• $D^{*\pm}_s$	$0(?^?)$
• $D^*_{s0}(2317)^\pm$	$0(0^+)$
• $D_{s1}(2460)^\pm$	$0(1^+)$
• $D_{s1}(2536)^\pm$	$0(1^+)$
• $D_{s2}(2573)$	$0(2^+)$
• $D^*_{s1}(2700)^\pm$	$0(1^-)$
$D^*_{s1}(2860)^\pm$	$0(1^-)$
$D^*_{s3}(2860)^\pm$	$0(3^-)$
$D_{sJ}(3040)^\pm$	$0(?^?)$

BOTTOM ($B = \pm1$)

• $B^\pm$	$1/2(0^-)$
• B^0	$1/2(0^-)$
• $B^\pm/B^0$ ADMIXTURE	
• $B^\pm/B^0/B^0_s/b$-baryon ADMIXTURE	
V_{cb} and V_{ub} CKM Matrix Elements	
• B^*	$1/2(1^-)$
• $B_1(5721)^+$	$1/2(1^+)$
• $B_1(5721)^0$	$1/2(1^+)$
$B^*_J(5732)$	$?(?^?)$
• $B^*_2(5747)^+$	$1/2(2^+)$
• $B^*_2(5747)^0$	$1/2(2^+)$

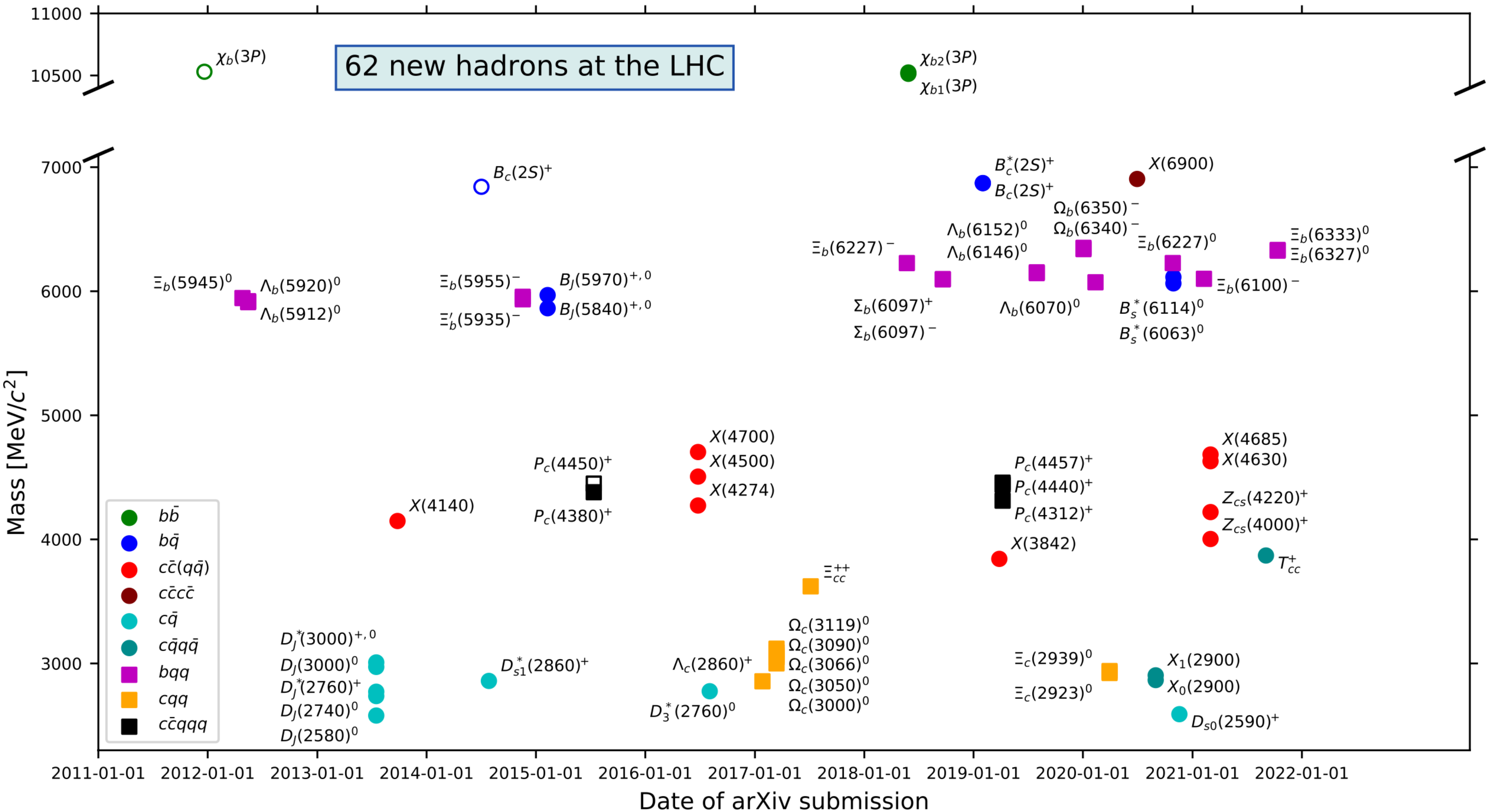
$c\bar{c}$

	$I^G(J^{PC})$
• $\eta_c(1S)$	$0^+(0^{-+})$
• $J/\psi(1S)$	$0^-(1^{--})$
• $\chi_{c0}(1P)$	$0^+(0^{++})$
• $\chi_{c1}(1P)$	$0^+(1^{++})$
• $h_c(1P)$	$?^?(1^{+-})$
• $\chi_{c2}(1P)$	$0^+(2^{++})$
• $\eta_c(2S)$	$0^+(0^{-+})$
• $\psi(2S)$	$0^-(1^{--})$
• $\psi(3770)$	$0^-(1^{--})$
• $\psi(3823)$	$?^?(2^{--})$
• $X(3872)$	$0^+(1^{++})$
• $X(3900)$	$1^+(1^{+-})$
• $X(3915)$	$0^+(0/2^{++})$
• $\chi_{c2}(2P)$	$0^+(2^{++})$
$X(3940)$	$?^?(?^{??})$
• $X(4020)$	$1(?^?)$
• $\psi(4040)$	$0^-(1^{--})$
$X(4050)^\pm$	$?(?^?)$
$X(4055)^\pm$	$?(?^?)$
• $X(4140)$	$0^+(1^{++})$
• $\psi(4160)$	$0^-(1^{--})$
$X(4160)$	$?^?(?^{??})$
$X(4200)^\pm$	$?(1^+)$
$X(4230)$	$?^?(1^{--})$
$X(4240)^\pm$	$?^?(0^-)$
$X(4250)^\pm$	$?(?^?)$

Stable under strong ints

pdg.lbl.gov

Cornucopia of exotics



[I. Danilkin, talk at INT workshop, March 23]

+ data from Babar, Belle, COMPASS, …

Examples of resonances

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- Very short lived, with decays into 2, 3, … stable hadrons

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  - Example 1: single-channel decay of s-wave spin-triplet q q-bar state:

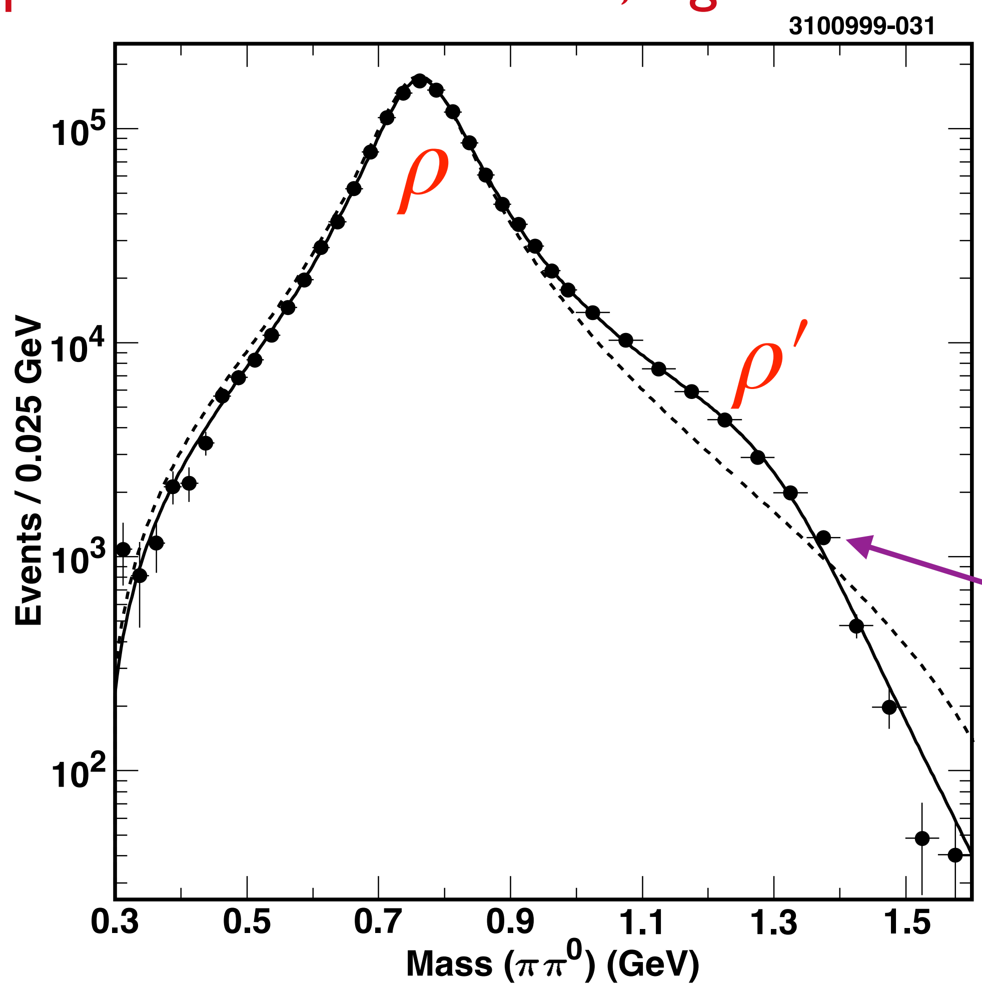
$$I^G J^{PC} = 1^+1^{--} : \rho \to \pi\pi\,,\ M_\rho \approx 775 MeV\,,\ \Gamma_\rho \approx 150 MeV\ (\tau = 4\times 10^{-23} s)$$

  - Many production mechanisms, e.g. $\tau^- \to \pi^-\pi^0\nu_\tau$

ρ is produced by the vector part of the weak current $\bar{u}\gamma^\mu d$

[CLEO collab., hep-ex/9910046]

$e^+e^- \to \tau^+\tau^- + X$



Fitting the spectrum involves models & uncertainties

Examples of resonances

- Example 2: multi-channel decay of p-wave $q\bar{q}$ state:

pdg summary entry

$a_2(1320)$ $I^G(J^{PC}) = 1^-(2^{++})$

Mass $m = 1318.3^{+0.5}_{-0.6}$ MeV

Full width $\Gamma = 107 \pm 5$ MeV

$a_2(1320)$ DECAY MODES	Fraction (Γ_i/Γ)
3π	(70.1 ± 2.7) %
$\eta\pi$	(14.5 ± 1.2) %
$\omega\pi\pi$	(10.6 ± 3.2) %
$K\overline{K}$	(4.9 ± 0.8) %
$\eta'(958)\pi$	$(5.5 \pm 0.9) \times 10^{-3}$
$\pi^\pm\gamma$	$(2.91 \pm 0.27) \times 10^{-3}$
$\gamma\gamma$	$(9.4 \pm 0.7) \times 10^{-6}$

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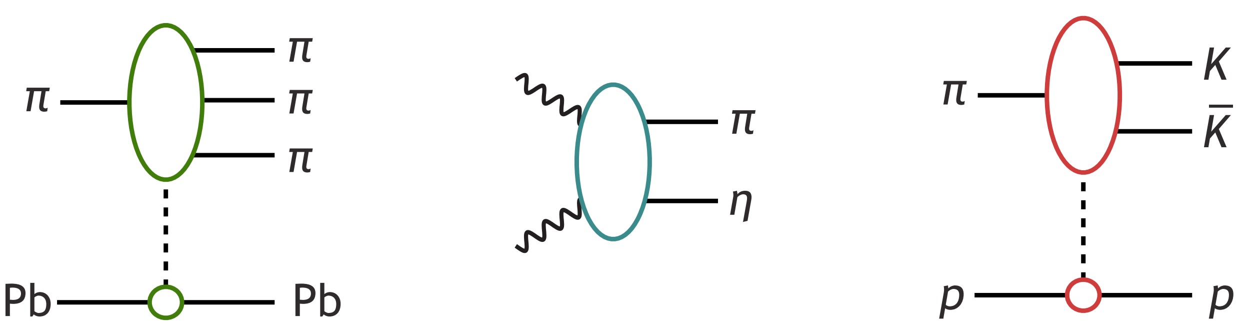
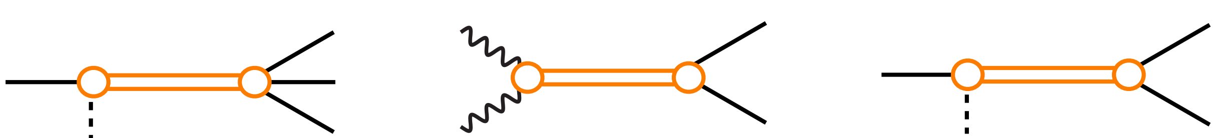
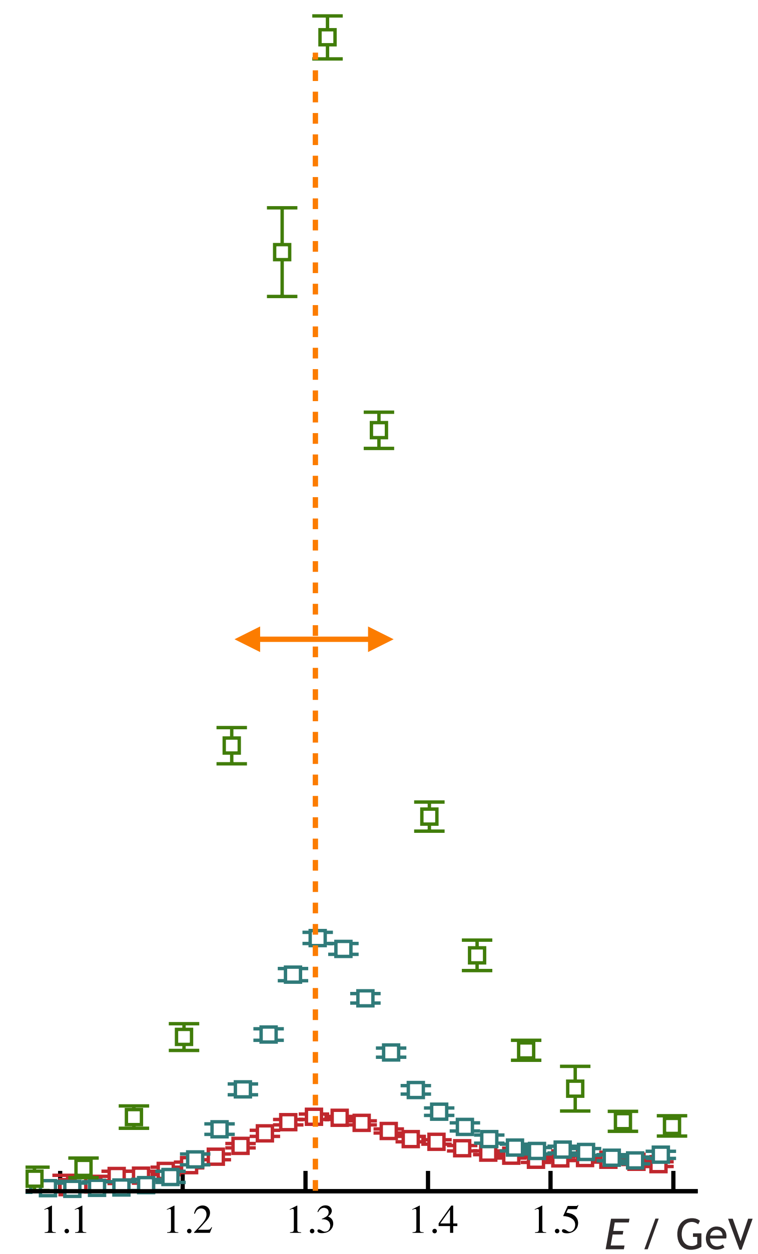
same 'bump' appears in multiple different processes ...



... due to same a_2 resonance



$a_2(1320)$



$\pi\,\mathrm{Pb} \to \pi\rho\,\mathrm{Pb}$ COMPASS

$\gamma\gamma \to \pi\eta$ Belle

$\pi p \to K\bar{K}p$ CERN SPS

[Figures from HMI slides of Jo Dudek]

Lessons

- Extracting resonance parameters from experiment is indirect & challenging
  - Resonance is defined as a pole in a scattering amplitude—not directly accessible
- Typical resonances have multiple decay channels, each involving 2 or 3 (or more) particles
- Quark model (or other models) fails to explain presence or properties of an increasing number of resonances
  - X, Y, Z resonances, glueballs, hybrids, tetraquarks, pentaquark, …
- Resonances are a largely unexplored frontier in our attempts to understand hadronic physics (i.e. the properties of a strongly-coupled QFT) from first principles

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- Quark model (or other mo[illegible]ain presence or properties of an increasing number [illegible]
  - X, Y, Z reso[illegible]s, glueballs, hybrids, tetraquarks, pentaquark, …
- Resonances are a largely unexplored frontier in our attempts to understand hadronic physics (i.e. the properties of a strongly-coupled QFT) from first principles

A major challenge for LQCD!

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  - X,Y, Z resonances, glueballs, hybrids, tetraquarks, pentaquark, …
  - LQCD calculations must use large bases of operators to allow understanding of structure of hadrons—any input is useful!
  - Varying the quark masses can provide additional useful information

Personal note

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HYBRIDS: MIXED STATES OF QUARKS AND GLUONS*

Nuclear Physics B222 (1983) 211–244

Michael CHANOWITZ and Stephen SHARPE

Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720, USA

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Submitted for publication

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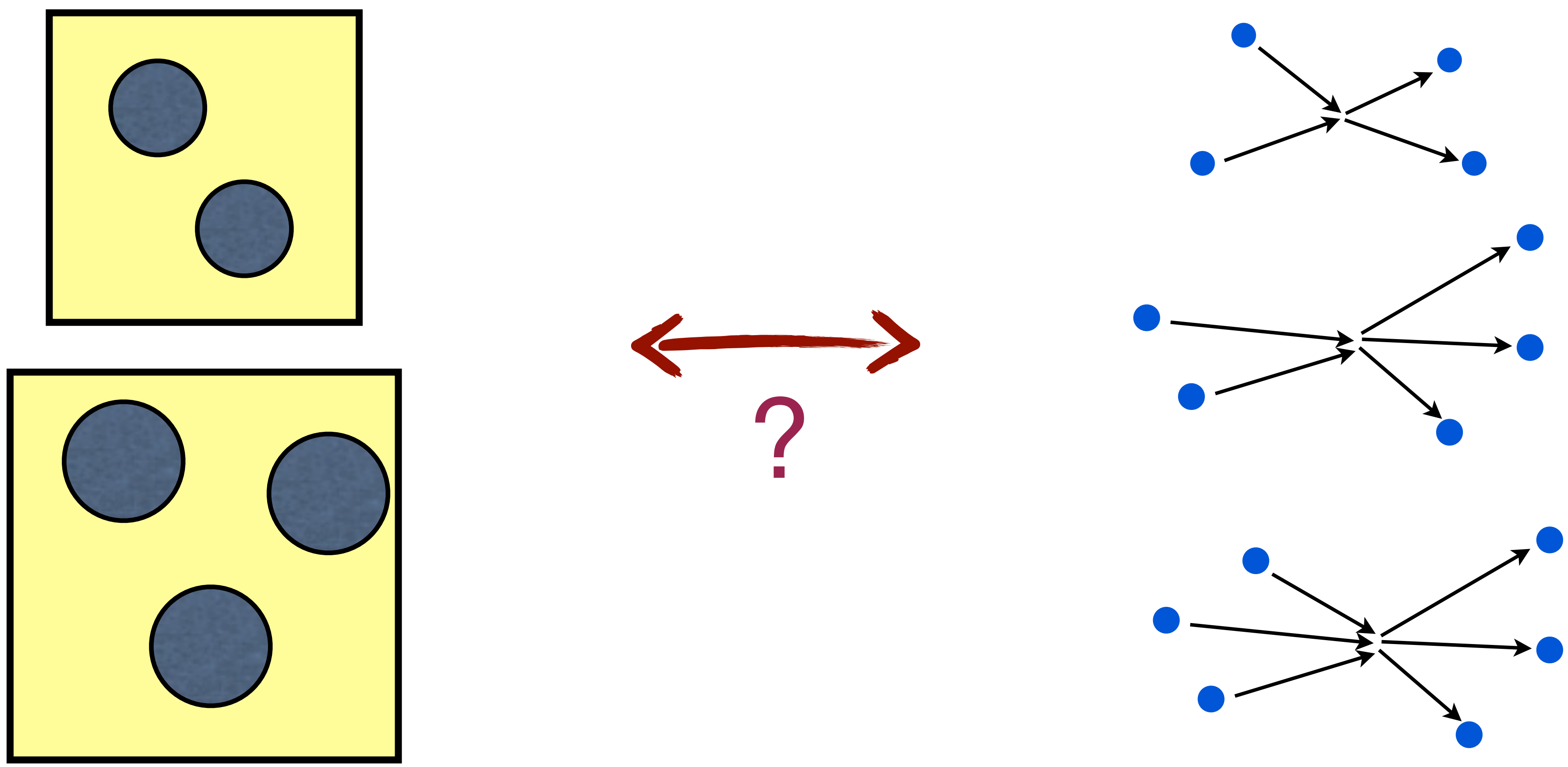
Noise!

- There are now increasingly sophisticated calculations of hybrid meson properties, and these will eventually be based on the formalism I will describe in these lectures

Preview

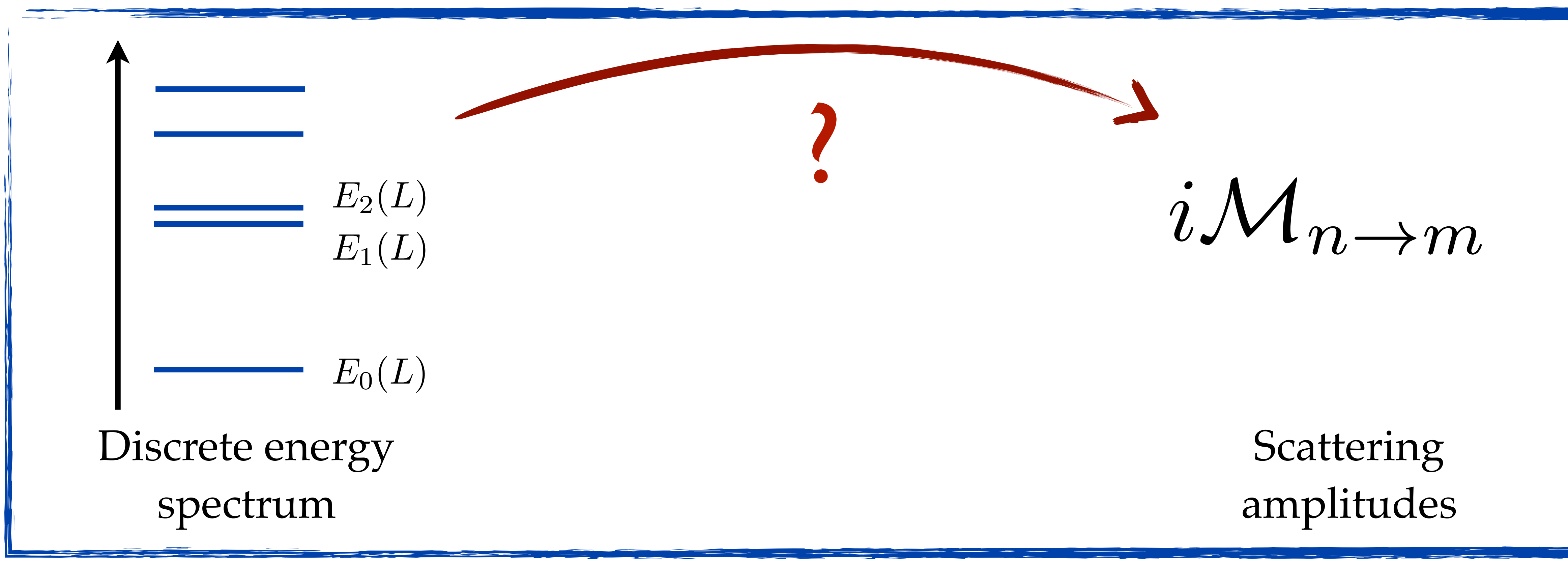
- Fundamental issue:
  - LQCD simulations are done in finite volumes, with imaginary time
  - Experiments are done in infinite volume in real time



How do we connect?

Fundamental Issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?



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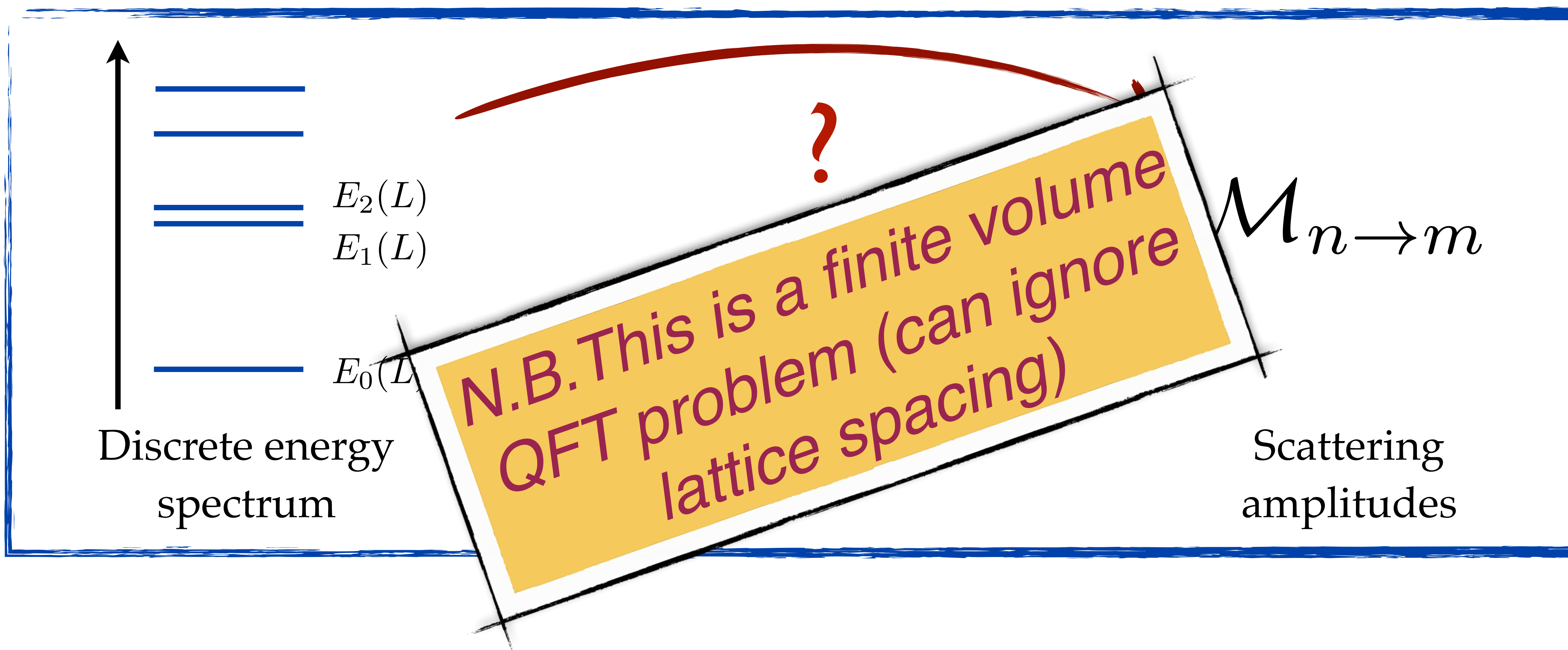


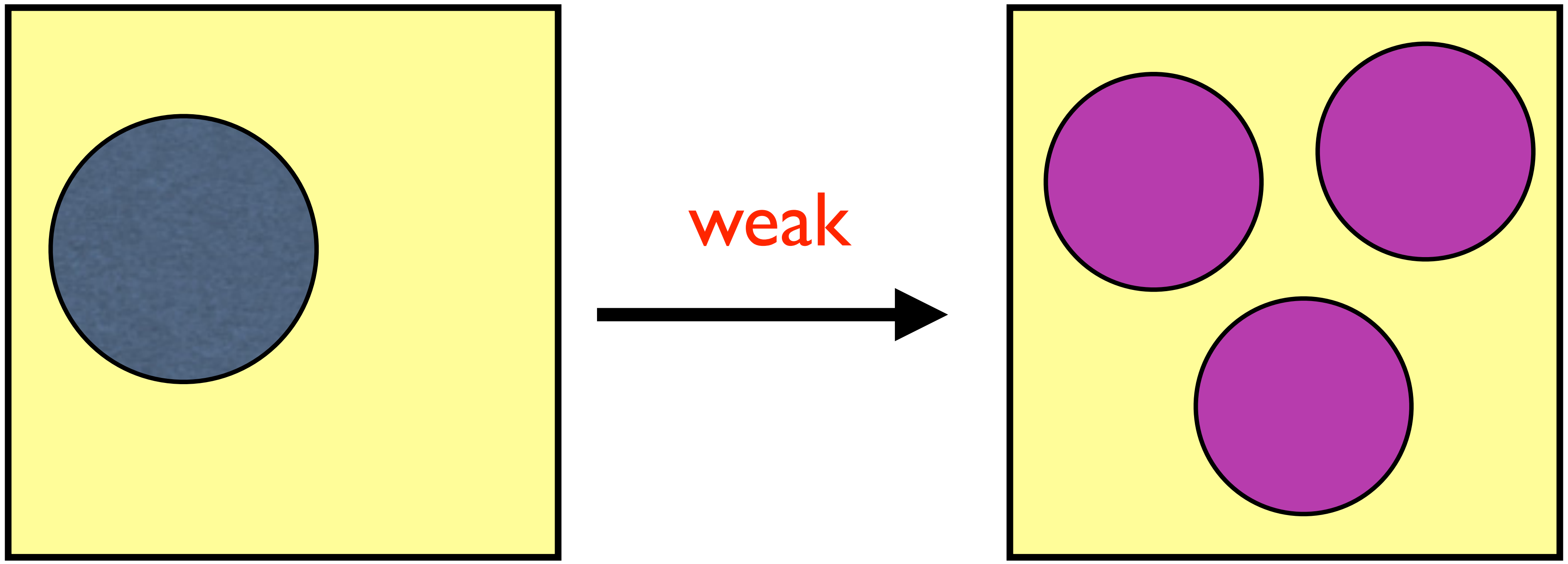
Further motivations for studying multiparticle states

Motivations

- Calculating electroweak decay and transition amplitudes for processes involving multiple particles
- Determining NN and NNN interactions as input for predicting properties of nuclei and nuclear matter
  - NNN interactions needed as input for EFT treatments of large nuclei, and for the neutron-star equation of state
- $\pi\pi\pi, \pi K\bar{K}, \ldots$ interactions needed as input to study pion & kaon condensation

Electroweak decays

e.g. $K\to\pi\pi\pi$ decay amplitudes



- Does the SM reproduce the observed CP violation in $K\to\pi\pi\pi$ decays?
- Formalism to study this now exists [Hansen, Romero-López, SRS, 2021]

A more distant motivation

Observation of CP violation in charm decays

CERN-EP-2019-042
13 March 2019

LHCb collaboration[†]

Abstract

A search for charge-parity (CP) violation in $D^0 \to K^- K^+$ and $D^0 \to \pi^- \pi^+$ decays is reported, using pp collision data corresponding to an integrated luminosity of $6\,\mathrm{fb}^{-1}$ collected at a center-of-mass energy of 13 TeV with the LHCb detector. The flavor of the charm meson is inferred from the charge of the pion in $D^*(2010)^+ \to D^0 \pi^+$ decays or from the charge of the muon in $\overline{B} \to D^0 \mu^- \bar{\nu}_\mu X$ decays. The difference between the CP asymmetries in $D^0 \to K^- K^+$ and $D^0 \to \pi^- \pi^+$ decays is measured to be $\Delta A_{CP} = [-18.2 \pm 3.2\,(\text{stat.}) \pm 0.9\,(\text{syst.})] \times 10^{-4}$ for π-tagged and $\Delta A_{CP} = [-9 \pm 8\,(\text{stat.}) \pm 5\,(\text{syst.})] \times 10^{-4}$ for μ-tagged D^0 mesons. Combining these with previous LHCb results leads to

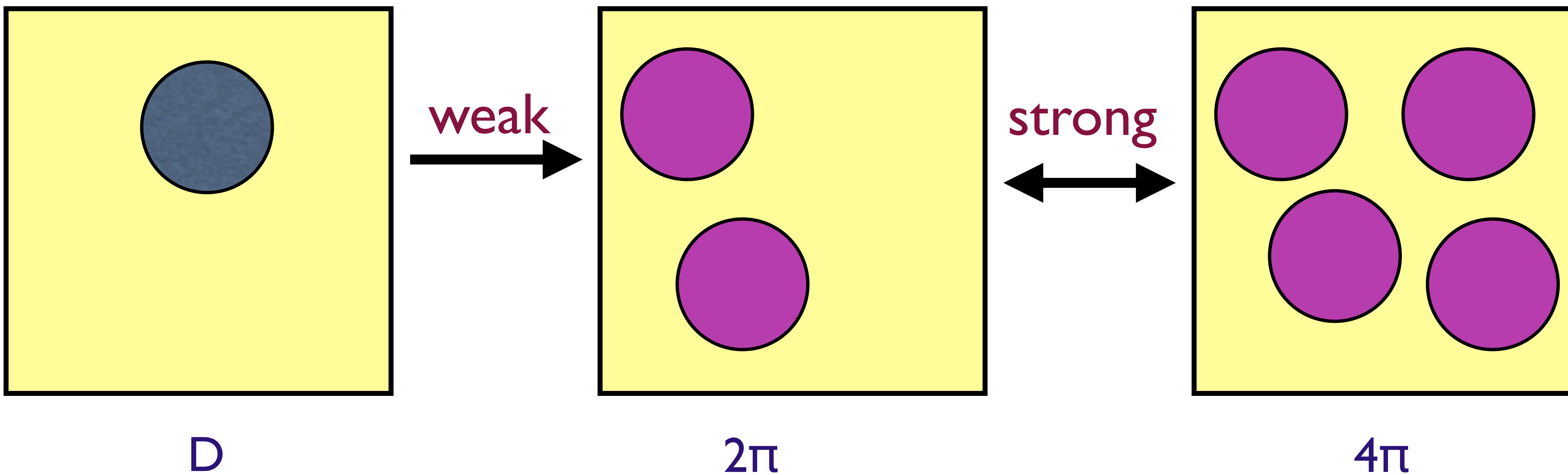
$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4},$$

where the uncertainty includes both statistical and systematic contributions. The measured value differs from zero by more than five standard deviations. This is the first observation of CP violation in the decay of charm hadrons.

5.3σ effect

A more distant motivation

- Calculating CP-violation in $D\to\pi\pi, K\bar{K}$ in the Standard Model
- Finite-volume state is a mix of $2\pi, K\bar{K}, \eta\eta, 4\pi, 6\pi, \ldots$
- Need 4 (or more) particles in the box!



Scattering basics (infinite-volume)

$\mathcal{M}_2$

- Recall some details of the simplest scattering process: 2 → 2
  - We will mainly discuss spinless particles in these lectures, e.g. pions
  - We will consider both identical particles, e.g. $\pi^+\pi^+$, and nonidentical, e.g. π^+K^+
- Scattering amplitude related to the S matrix

$$S = 1 + iT \qquad \langle f | T | i \rangle = (2\pi)^4 \delta^4(P_f - P_i) \mathcal{M}_{fi}$$

- In a given theory, can calculate in perturbation theory (PT), e.g. in ϕ^4 theory

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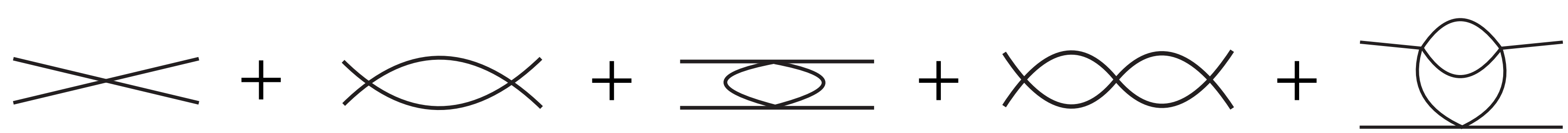
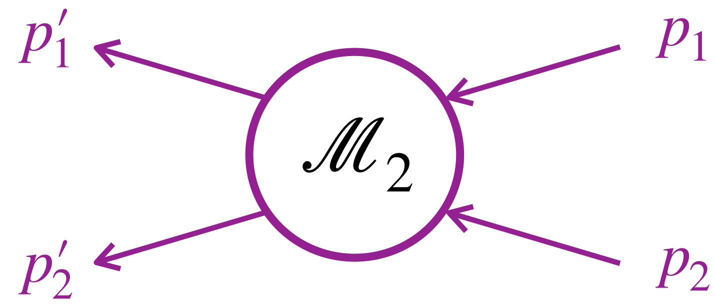
$i\mathcal{M}_2 =$  $+ \ldots$

- We will not assume a particular theory, e.g. ChPT or ϕ^4; instead we use a generic relativistic QFT, with all possible vertices, and work to all orders in PT

Properties of $\mathcal{M}_2$

- Poincaré invariance ⇒ $\mathcal{M}_2$ depends on the two independent Mandelstam variables



$$\mathcal{M}_2 = \mathcal{M}_2(s,t), \quad s = (p_1 + p_2)^2,\ t = (p_1 - p_1')^2,\ u = (p_1 - p_2')^2 = 4m^2 - s - t$$

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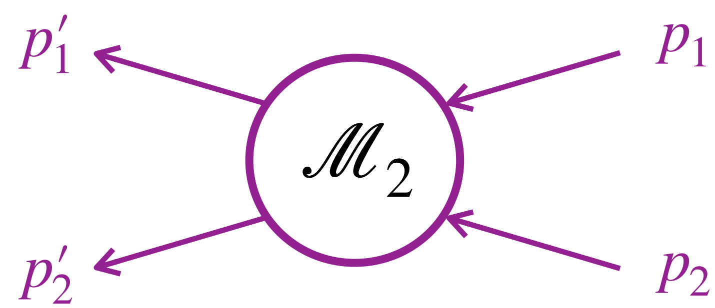

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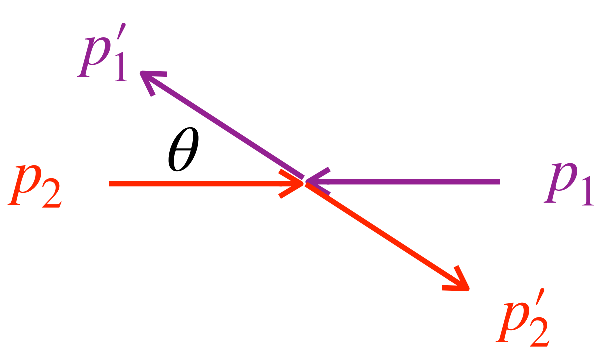
- Partial wave decomposition in CM frame



$$s = E^{*2} = 4(q^2 + m^2),\ t = -2q^2(1 - \cos\theta)$$

$$\mathcal{M}_2(s,t) = \sum_\ell (2\ell + 1)\, \mathcal{M}_2^{(\ell)}(s)\, P_\ell(\cos\theta)$$

Only even values of ℓ contribute for identical particles

Properties of $\mathscr{M}_2$

- Unitarity—holds in each partial wave (results here for identical particles)

$$S^\dagger S = 1 \;\Rightarrow\; \text{Im}(\mathscr{M}_2^{(\ell)}) = \mathscr{M}_2^{(\ell)*}\rho\mathscr{M}_2^{(\ell)} = \rho\,|\,\mathscr{M}_2^{(\ell)}|^2, \qquad \rho = \frac{q}{16\pi E^*} \text{ (phase space)}$$

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- Solve unitarity constraint in terms of an arbitrary, real K matrix

$$\text{Im}\left[1/\mathcal{M}_2^{(\ell)}\right] = -\rho \;\Rightarrow\; 1/\mathcal{M}_2^{(\ell)} \equiv 1/\mathcal{K}_2^{(\ell)} - i\rho \;\Rightarrow\; \mathcal{M}_2^{(\ell)} = \mathcal{K}_2^{(\ell)}\frac{1}{1 - i\rho\mathcal{K}_2^{(\ell)}}$$

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- Parametrize $\mathcal{K}_2$ using (real) phase shifts

$$\mathcal{K}_2^{(\ell)} \equiv \frac{1}{\rho}\tan\delta_\ell = \frac{16\pi E^*}{q\cot\delta_\ell} \;\Rightarrow\; \mathcal{M}_2^{(\ell)} = \frac{1}{\rho}e^{i\delta}\sin\delta_\ell$$

Properties of $\mathcal{M}_2$

- Threshold behavior (QM)

$$\delta_\ell \sim q^{1+2\ell}\left[1+\mathcal{O}(q^2)\right] \;\Rightarrow\; \mathcal{K}_2^{(\ell)} \sim q^{2\ell}\left[1+\mathcal{O}(q^2)\right]$$

- Effective range expansion (ERE)

$$\frac{1}{\mathcal{K}_2^{(0)}} = \frac{1}{16\pi E_2}\left[-\frac{1}{a_0} + r_0\frac{q^2}{2} + P_0 r_0^3 q^4 + \ldots\right], \quad \frac{1}{\mathcal{K}_2^{(2)}} = -\frac{1}{16\pi E_2}\frac{1}{q^4}\frac{1}{a_2} + \ldots$$

  - a_0 is s-wave scattering length, related to threshold scattering amplitude

$$\mathcal{M}_2(q=0) = \mathcal{K}_2(q=0) = 32\pi m a_0$$

  - a_0 is the intercept on the r axis of the s-wave radial QM wavefunction with $q=0$, and can have any value: $-\infty < a_0 < \infty$
  - r_0 is the effective range (typically of order the range of the interaction), P_0 is the "shape parameter" (typically of order unity), and a_2 is the d-wave scattering "length"

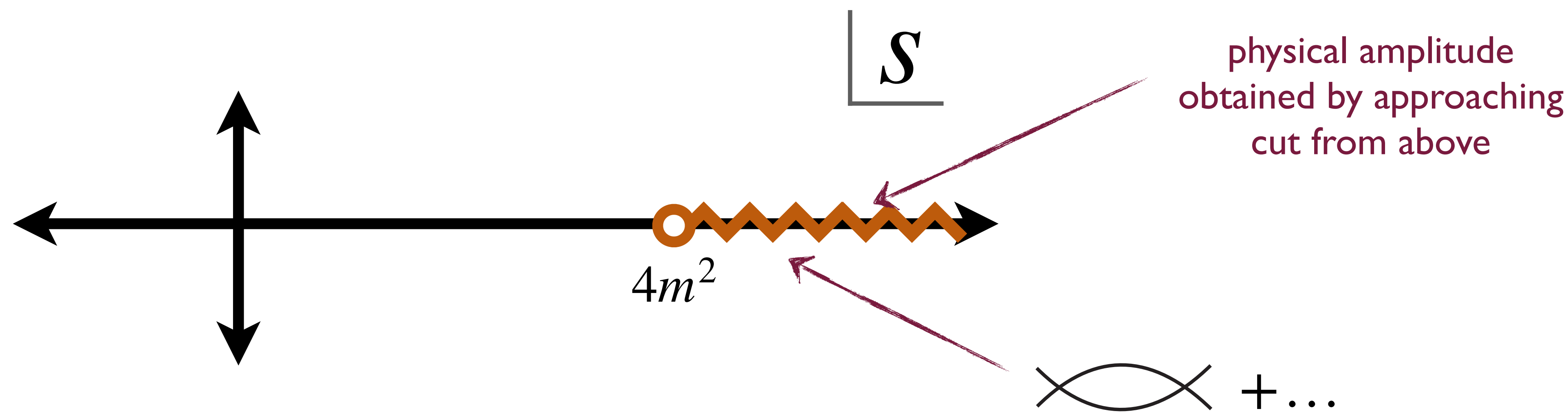
Properties of M_2

- Analytic structure: branch cut along real s axis above threshold, arising from unitarity

$$\mathcal{M}_2^{(\ell)} = \mathcal{K}_2^{(\ell)} + \mathcal{K}_2^{(\ell)} i\rho \mathcal{K}_2^{(\ell)} + \ldots, \qquad \rho = \frac{\sqrt{s-4m^2}}{32\pi\sqrt{s}}$$

gives rise to "right-hand cut"

canceled by factors in K_2

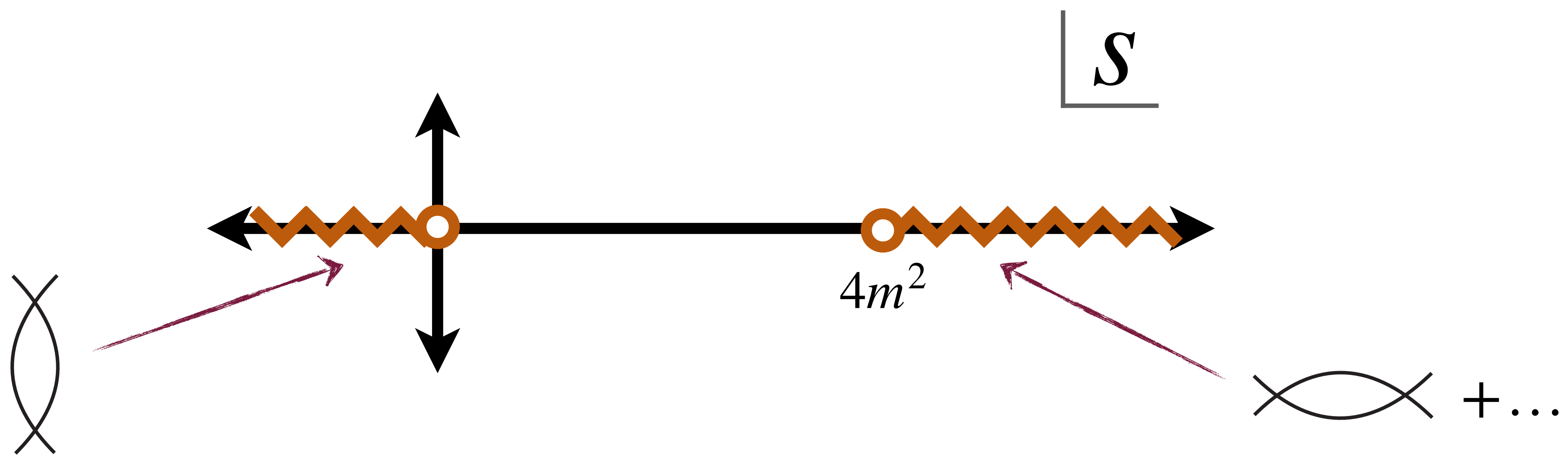


- $\mathcal{M}_2$ has two Riemann sheets, the top one being called the "physical sheet"
- $\mathcal{K}_2$ does not have the right-hand cut; it is analytic at threshold

Properties of M_2

- t- and u-channel exchanges lead to the "left-hand cut"

$$\mathcal{M}_2^{(\ell)} = \mathcal{K}_2^{(\ell)} + \mathcal{K}_2^{(\ell)} i\rho \mathcal{K}_2^{(\ell)} + \ldots, \qquad \rho = \frac{\sqrt{s-4m^2}}{32\pi\sqrt{s}}$$

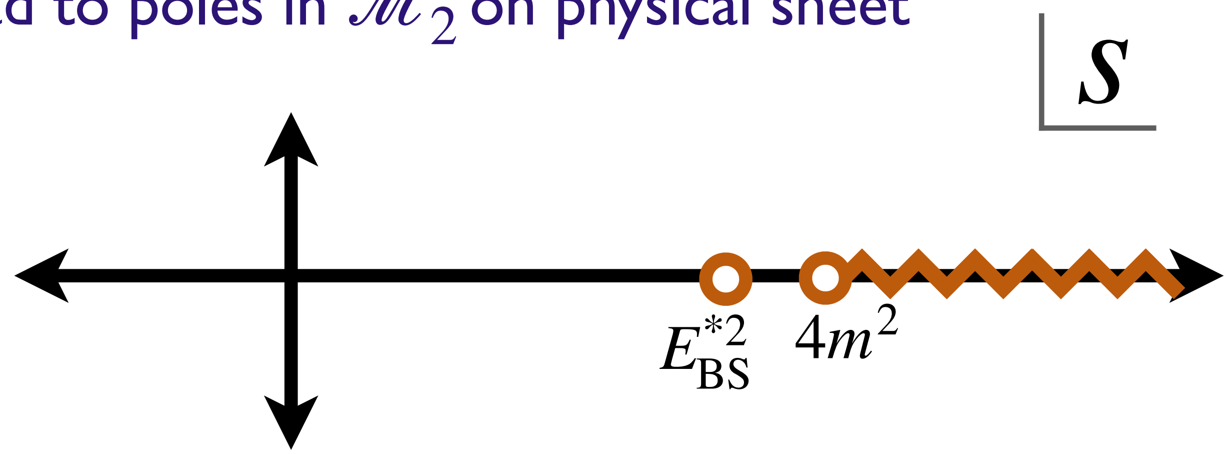


- For nondegenerate systems, the LH cut can lie close to threshold (e.g. $\Lambda\Lambda$ has LH cut due to pion exchange)

- LH cut invalidates standard QC2 and QC3 derivations

Bound states

- Bound states lead to poles in $\mathcal{M}_2$ on physical sheet



- $\mathcal{K}_2$ does not have a corresponding pole since ρ is nonzero below threshold

$$1/\mathcal{M}_2^{(\ell)} \equiv 1/\mathcal{K}_2^{(\ell)} - i\rho \text{ where } -i\rho = \frac{|q|}{16\pi E^*} \text{ with } E^{*2}_{\rm BS} = 4(m^2 - |q|^2)$$

- Bound state condition is thus

$$1/\mathcal{M}_2^{(\ell)} = \frac{1}{16\pi E^*}(q\cot\delta_\ell + |q|) = 0$$

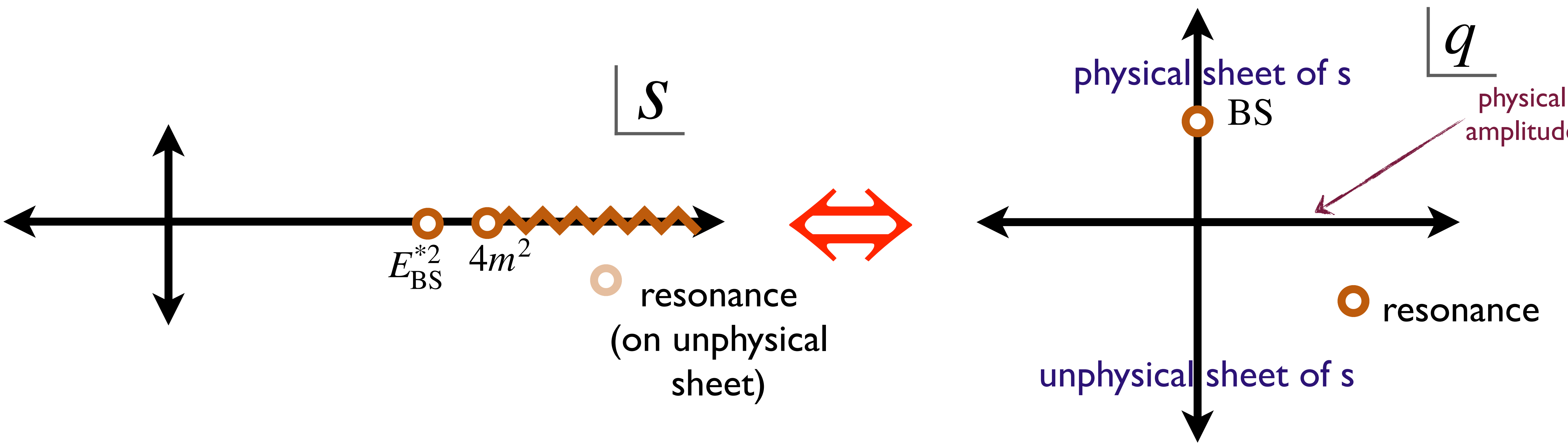
- If keep only the scattering length in the ERE, find bound state for $a_0 > 0$

$$q\cot\delta_0 = -1/a_0 \Rightarrow |q| = 1/a_0 \Rightarrow E^*_{BS} = 2\sqrt{m^2 - 1/a_0^2}$$

- Bound state at threshold in unitary limit $a_0 \to \infty$

Resonances

- Resonances lead to poles in $\mathcal{M}_2$ below the real axis on the second (unphysical) sheet
  - Cannot have poles on physical sheet aside from bound states due to causality
  - To display sheets it is better to use single-sheeted variable q



- Resonance with width $\Gamma = 1/\tau$ and mass M has pole at

$$E^* = M - i\Gamma/2 \;\Rightarrow\; s = M^2 + (\Gamma/2)^2 - iM\Gamma$$

- Leads to a bump in scattering cross-section $\propto |\mathcal{M}_2|^2$ as we saw earlier

Resonances

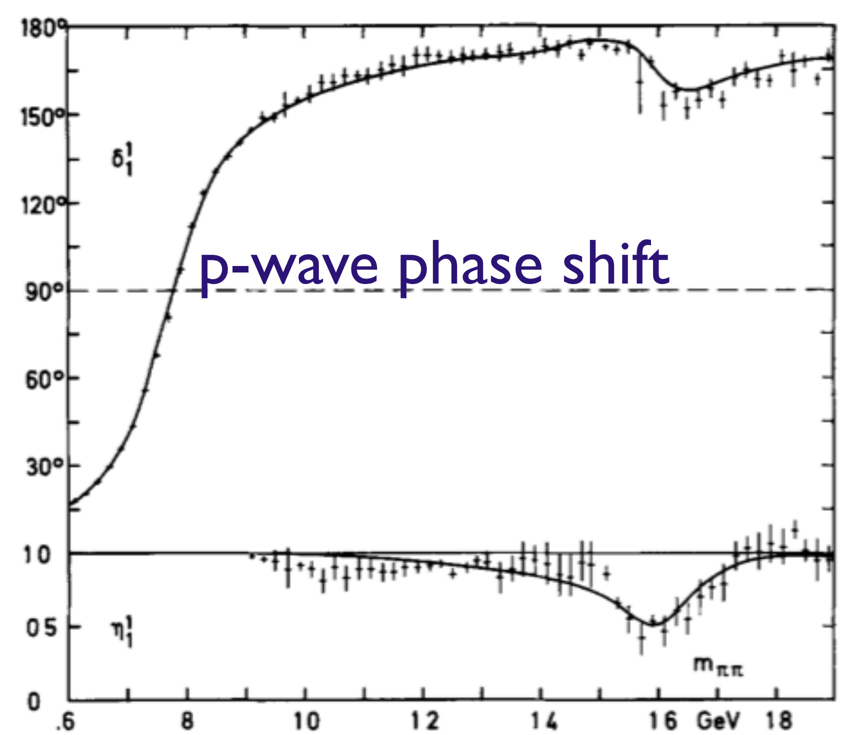
- Narrow s-wave resonances well described by Breit-Wigner form

$$\tan\delta_{\rm BW} = \frac{E^*\Gamma}{M^2 - E^{*2}} \;\Rightarrow\; \mathcal{M}_2 \propto \frac{1}{M^2 - E^{*2} - iE^*\Gamma}$$

- As E^* passes through M from below:
  - Phase shift rises rapidly through 90^0
  - $\mathcal{K}_2 \sim \tan\delta$ has a pole at M (i.e. on the real axis)

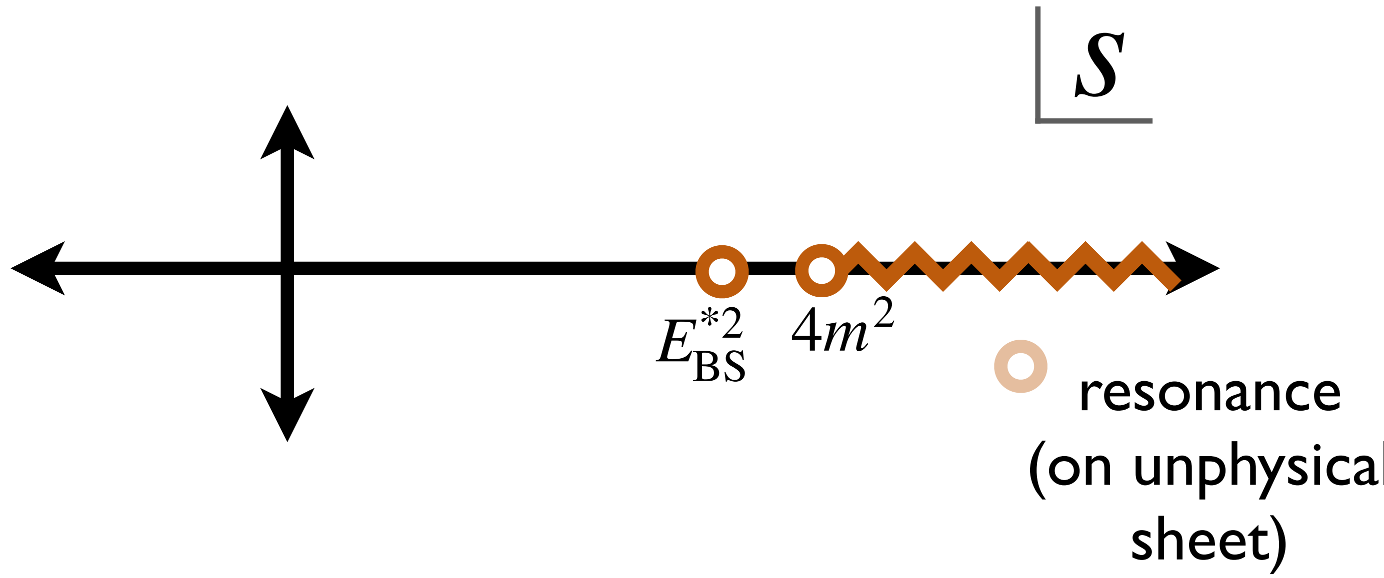
- Pole in $\mathcal{K}_2$ does not have any direct physical significance, but does play a role in the finite-volume analysis to follow

isospin=1



Hyams 1973

Resonances: unavoidable complication

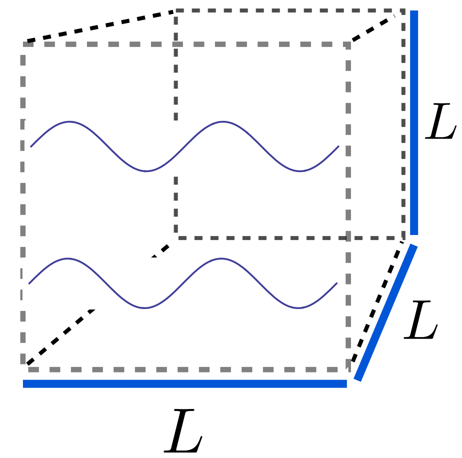


- Neither experiment, nor LQCD calculations, can directly access complex energies
- Thus, in order to study resonances, **both** methods have to parametrize the K matrices with an analytic form that can be continued into the complex plane
- Thus some parametrization dependence is unavoidable
- One should put as much physical knowledge as possible into the parametrization, while minimizing model dependence
- Input from the experimental analysis community can be helpful

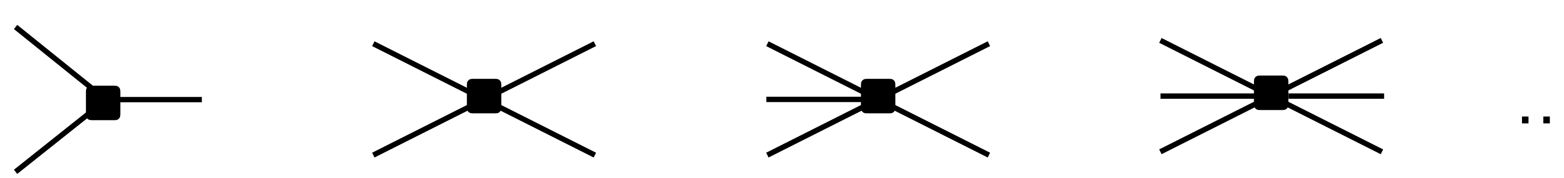
Deriving the two-particle QC

Following the method of [Kim, Sachrajda & SRS, 05]

Set-up



- Work in continuum (assume that LQCD can control discretization errors)
- Cubic box of size L with periodic BC, and infinite (*Minkowski*) time
  - Spatial loops are sums: $\frac{1}{L^3}\sum_{\vec{k}}$ $\vec{k} = \frac{2\pi}{L}\vec{n}$
  - Can easily generalize to other geometries and BC
- Consider identical particles with physical mass m (think of pions), interacting ***arbitrarily***—a generic (relativistic) effective field theory (RFT)
  - Work to all orders in perturbation theory with no assumptions about the size of coupling constants
  - Generalizations are known for nonidentical particles [Many authors] and to particles with spin [Briceño,14]



Methodology

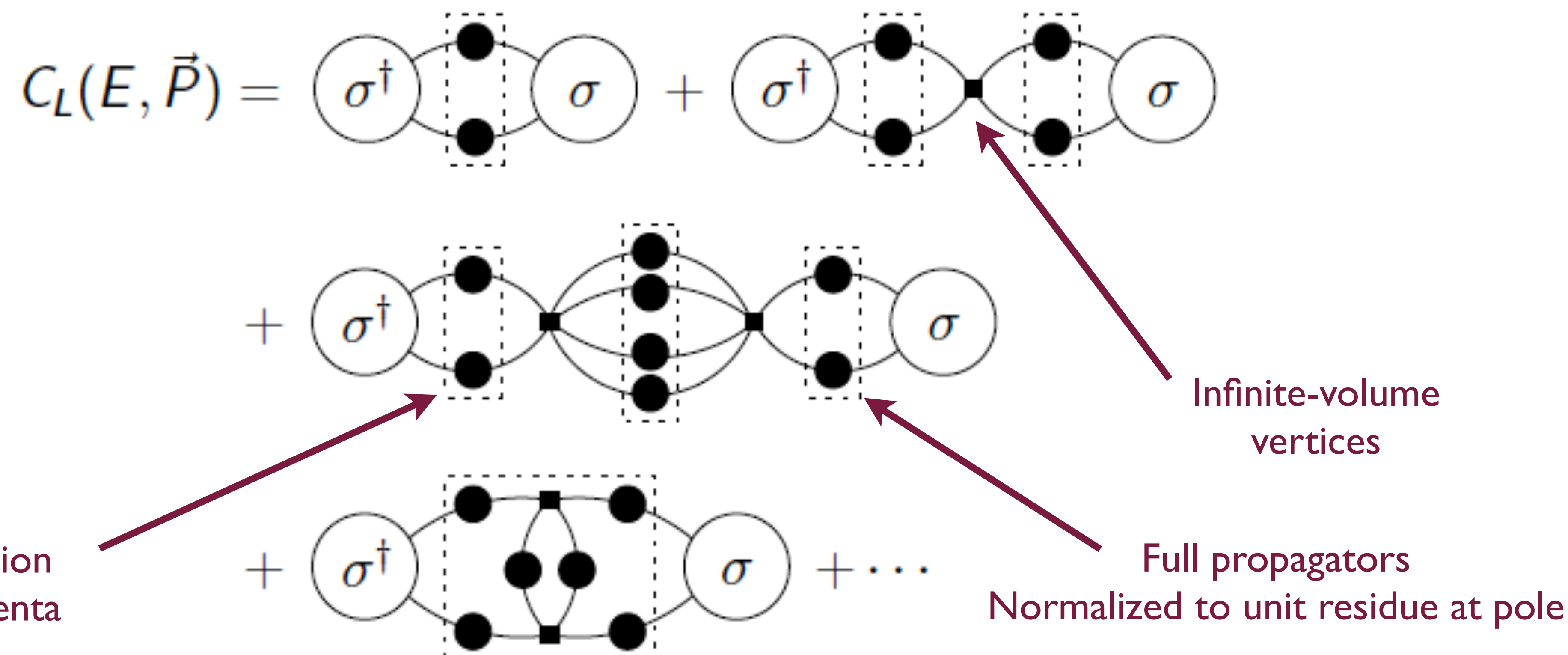
CM energy is $E^* = \sqrt{E^2 - P^2}$

- Calculate (for some $\boldsymbol{P} = 2\pi \boldsymbol{n}_P/L$)

$$C_L(E, \vec{P}) \equiv \int_L d^4x\, e^{iEt - i\vec{P}\cdot\vec{x}} \langle \Omega | T\left\{\sigma^\dagger(x)\sigma(0)\right\} | \Omega \rangle_L$$

- $\sigma \sim \pi^2$, e.g. $\sigma(\vec{x}, t) = \int_L d^3y\, \pi(\vec{x}+\vec{y}, t)\pi(\vec{x}-\vec{y}, t) e^{-i\vec{k}\cdot\vec{y}}$ $\qquad \pi(x) = \bar{u}(x)\gamma_5 d(x)$

- Poles in C_L occur at energies of finite-volume spectrum [Exercise]



Here I have assumed no odd-legged vertices—-not necessary for subsequent arguments, but simplifies diagrams

Key step 1

- Replace loop sums with integrals using Poisson summation formula where integrand is nonsingular
  - Drop exponentially suppressed terms ($e^{-ML}, e^{-(ML)^2}$, etc.) while keeping power-law dependence

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l}\neq\vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})$$

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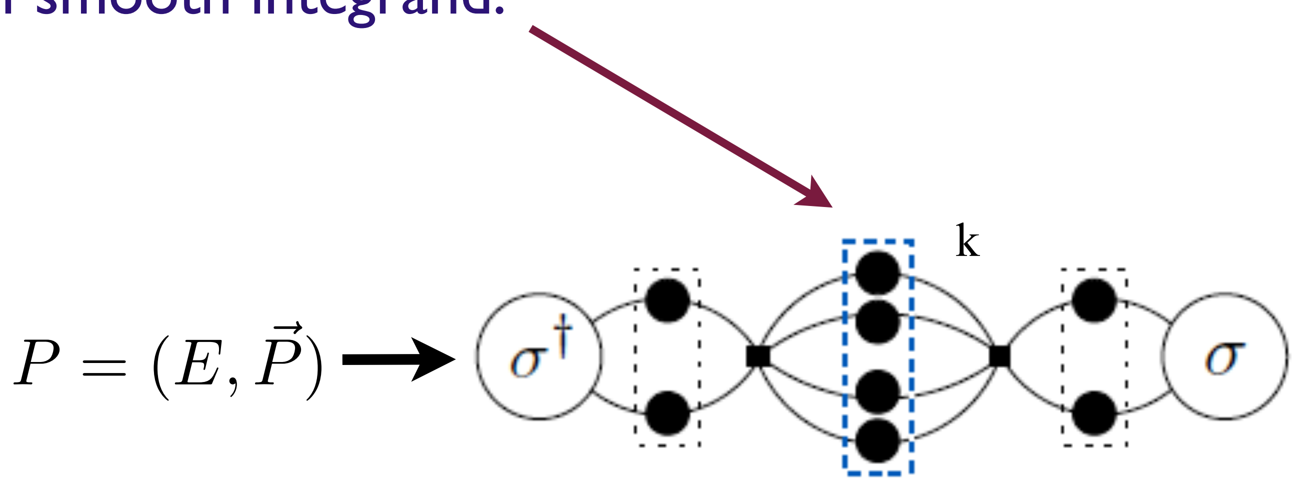
- Example of smooth integrand:



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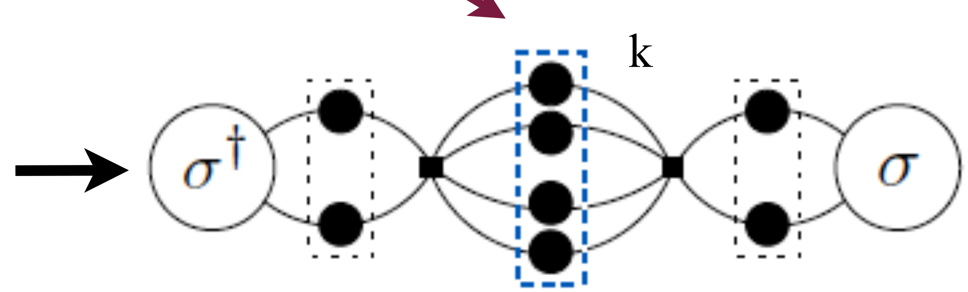
  - Drop exponentially suppressed terms ($e^{-ML}, e^{-(ML)^2}$, etc.) while keeping power-law dependence

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l}\neq\vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})$$

Exp. suppressed if g(k) is smooth and scale of derivatives of g is ~1/M

- Example of smooth integrand:

$P = (E, \vec{P})$



Key step 2

- Use "sum=integral + [sum-integral]" if integrand has pole, e.g. [KSS]

$$\frac{1}{2}\left(\int\frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}}-\int\frac{d^4k}{(2\pi)^4}\right)f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

symmetry factor

Key step 2

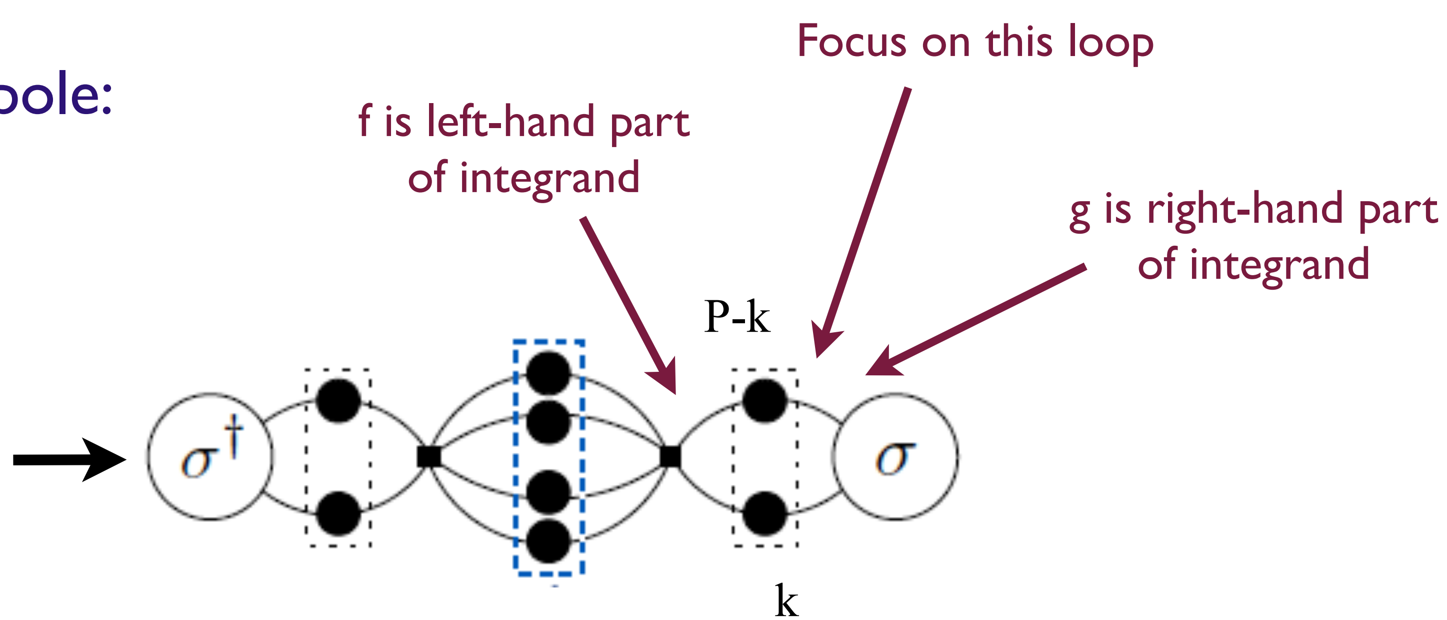
- Use "sum=integral + [sum-integral]" if integrand has pole, e.g. [KSS]

$$\frac{1}{2}\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon} g(k)$$

symmetry factor

- Example of pole:

$P = (E, \vec{P})$



Key step 2

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$$= \int d\Omega_{q^*}\, d\Omega_{q^{*\prime}}\, f^*(\hat{q}^*)\,\mathcal{F}(q^*,q^{*\prime})\,g^*(\hat{q}^{*\prime}) \quad + \text{exp. suppressed}$$

symmetry factor

$\mathbf{q}^*$ is relative momentum of pair on left in CM

Kinematic function

f & g evaluated for ON-SHELL momenta
Depend only on direction in CM

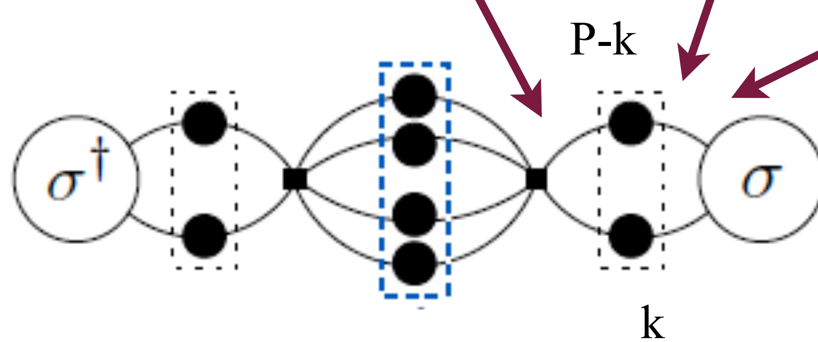
- Example of pole:

Focus on this loop

f is left-hand part of integrand

g is right-hand part of integrand

$P = (E, \vec{P}) \longrightarrow$



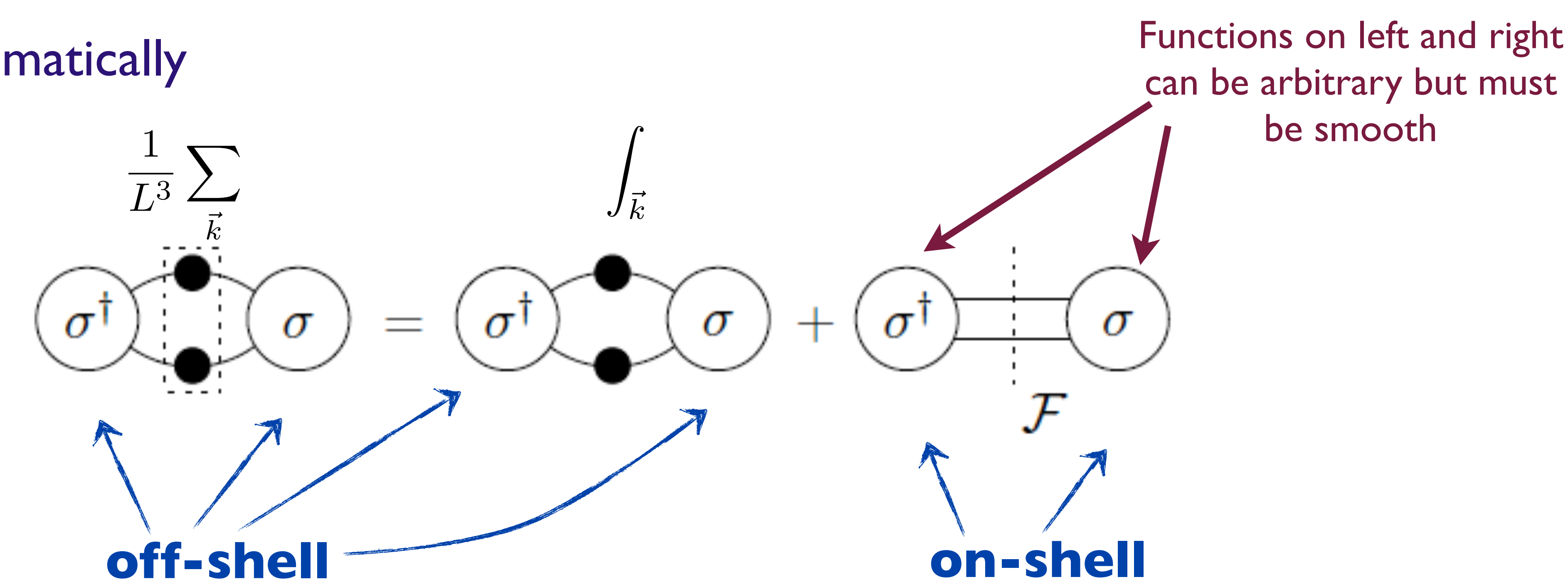
Key step 2

- Use "sum=integral + [sum-integral]" where integrand has pole, with [KSS]

$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec k} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat q^*)\mathcal{F}\ \ (q^*, q^{*\prime})g^*(\hat q^{*\prime})$$

- Diagrammatically



A new type of "cut"

Variant of key step 2

- For generalization to 3 particles will use a PV prescription instead of iε

$$\frac{1}{2}\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \text{PV}\!\!\int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+\cancel{i\epsilon}}\frac{1}{(P-k)^2-m^2+\cancel{i\epsilon}}g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*)\mathcal{F}_{\text{PV}}(q^*, q^{*\prime})g^*(\hat{q}^{*\prime})$$

- Key properties of F_{PV} : real and no unitary cusp at threshold
- These properties are important for the derivation of three-particle QC

More detail on key step 2 [HS14]

$$\frac{1}{2}\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m_j^2+i\epsilon}\frac{1}{(P-k)^2-m_j^2+i\epsilon}g(k)$$

$$= \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right)\frac{f(\vec{k}^*)g(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k}+i\epsilon)} + \mathcal{O}(e^{-mL})$$

Smooth UV regulator
Equals unity on shell

Time integrals set k on shell
$\mathbf{k}^*$ is on-shell k boosted to CM

$$= \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right) f_{\ell' m'}\frac{\mathcal{Y}_{\ell' m'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k}+i\epsilon)} g_{\ell m} + \mathcal{O}(e^{-mL})$$

Decompose f & g into
spherical harmonics,
and evaluate with P-k on shell

$$\mathcal{Y}_{\ell m}(\vec{k}^*) = \sqrt{4\pi}\left(\frac{k^*}{q^*}\right)^{\ell} Y_{\ell m}(\hat{k}^*)$$

$$q^* = \sqrt{E^{*2}/4 - m^2}$$

$$\equiv f_{\ell' m'}\, F_{\ell' m';\ell m}(E, \vec{P}, L)\, g_{\ell m}$$

More convenient to use
this matrix form

- Thus power-law volume dependence enters through geometrical function:

$$F_{\ell' m';\ell m}(E, \vec{P}, L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right)\frac{\mathcal{Y}_{\ell' m'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k}+i\epsilon)}$$

More detail on key step 2 [HS14]

$$F_{\ell'm';\ell m}(E,\vec{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right) \frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\, h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k}+i\epsilon)}$$

- Similarly, the PV version is

$$F_{\text{PV};\ell'm';\ell m}(E,\vec{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \text{PV}\int \frac{d^3k}{(2\pi)^3}\right) \frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\, h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k})}$$

$$= F_{\ell'm';\ell m}(E,\vec{P},L) - i\delta_{\ell'\ell}\,\delta_{m'm}\frac{q^*}{16\pi E^*}$$

x=q*L/(2π)

$$\propto \left(\frac{2\pi}{L}\right)^{1+\ell+\ell'} \mathcal{Z}_{\ell',m';\ell,m}(x^2,\mathbf{P})$$

"Lüscher zeta function"

Kinematic functions

$Z_{4,0}$ & $Z_{6,0}$ for **P=0** [Luu & Savage, `11]

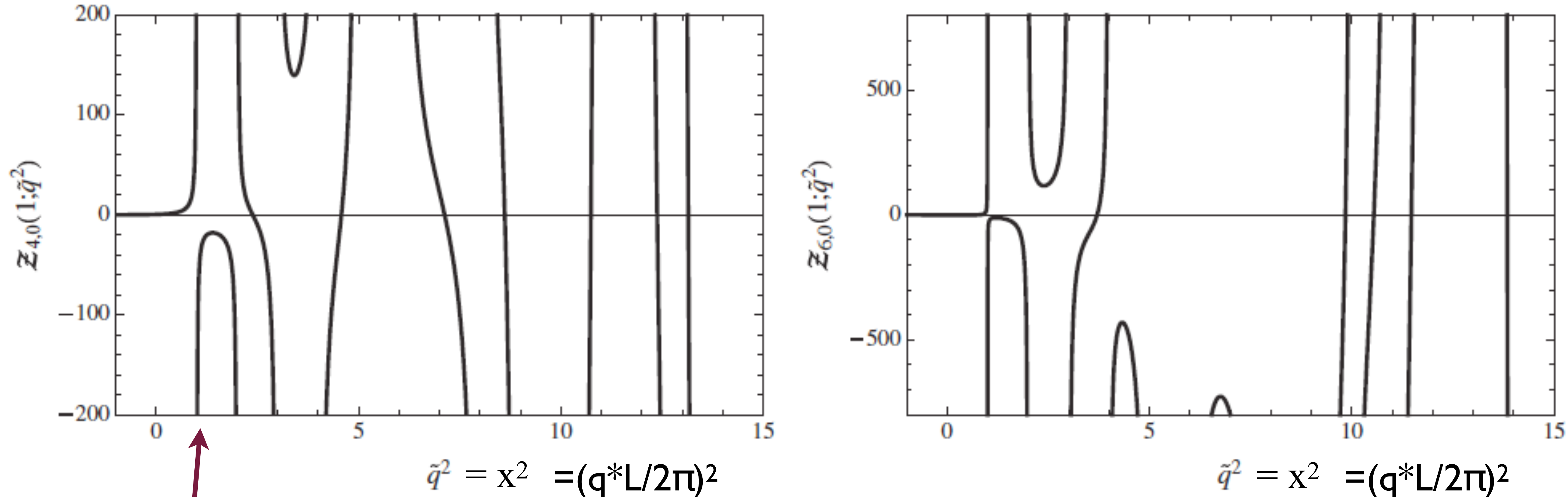


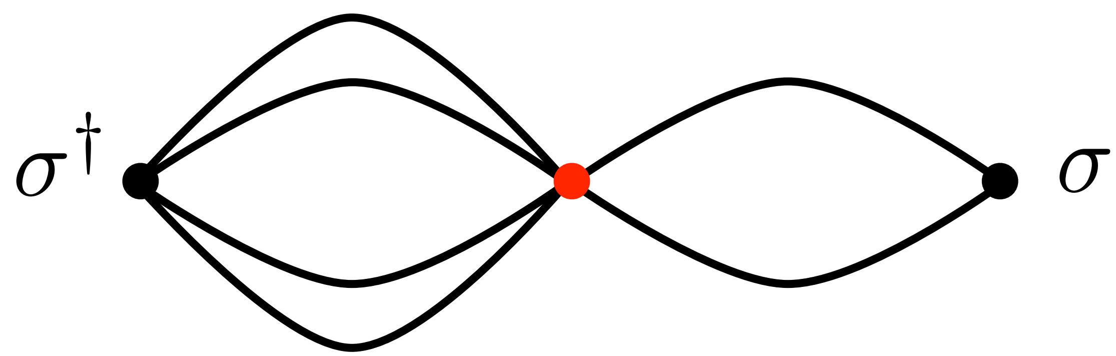
FIG. 29. The functions $Z_{4,0}(1;\tilde{q}^2)$ (left panel) and $Z_{6,0}(1;\tilde{q}^2)$ (right panel).

Divergences occur for values of E equal to the energy of two free particles in the box
[Exercise: why no divergence at x=0?]

Example:
$\mathbf{n}_1 = -\mathbf{n}_2 = (0,0,1)$
$\Rightarrow q^* = 2\pi/L \Rightarrow x=1$

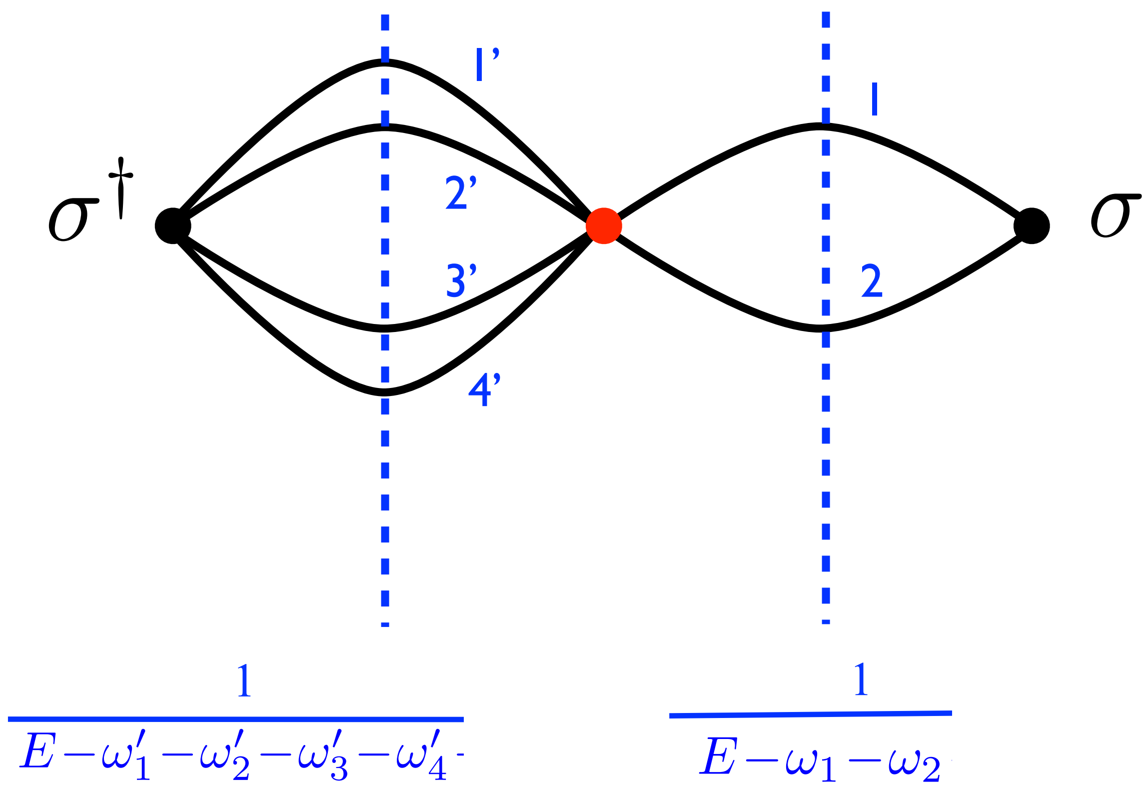
Key step 3

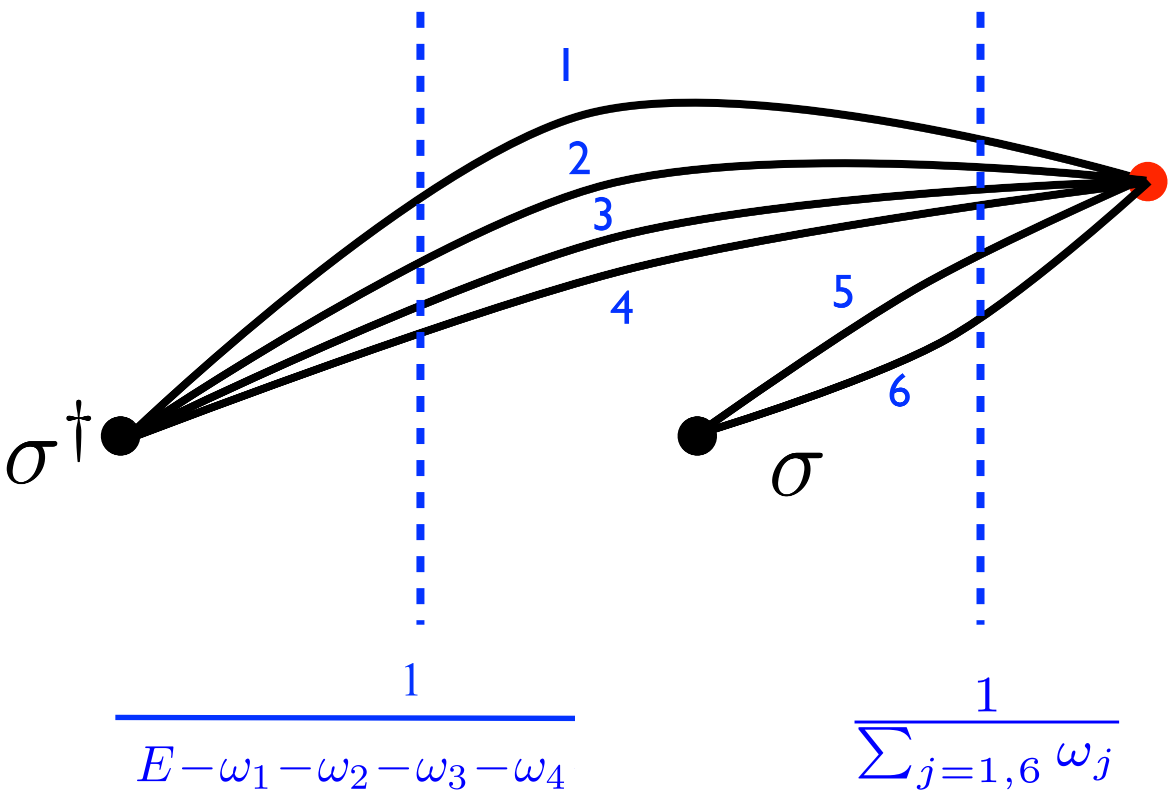
- Identify potential singularities using time-ordered PT (i.e. do k_0 integrals)
- Example (again assuming only even-legged vertices)



Key step 3

- 2 out of 6 time orderings:

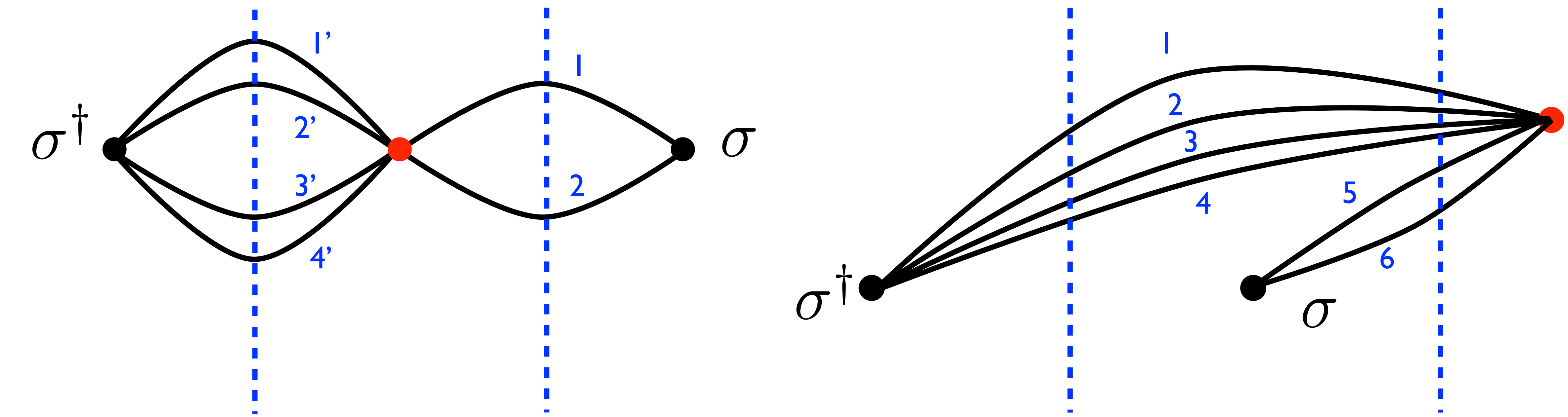




On-shell energy $\omega_j = \sqrt{\vec{k}_j^2 + M^2}$

Key step 3

- 2 out of 6 time orderings:

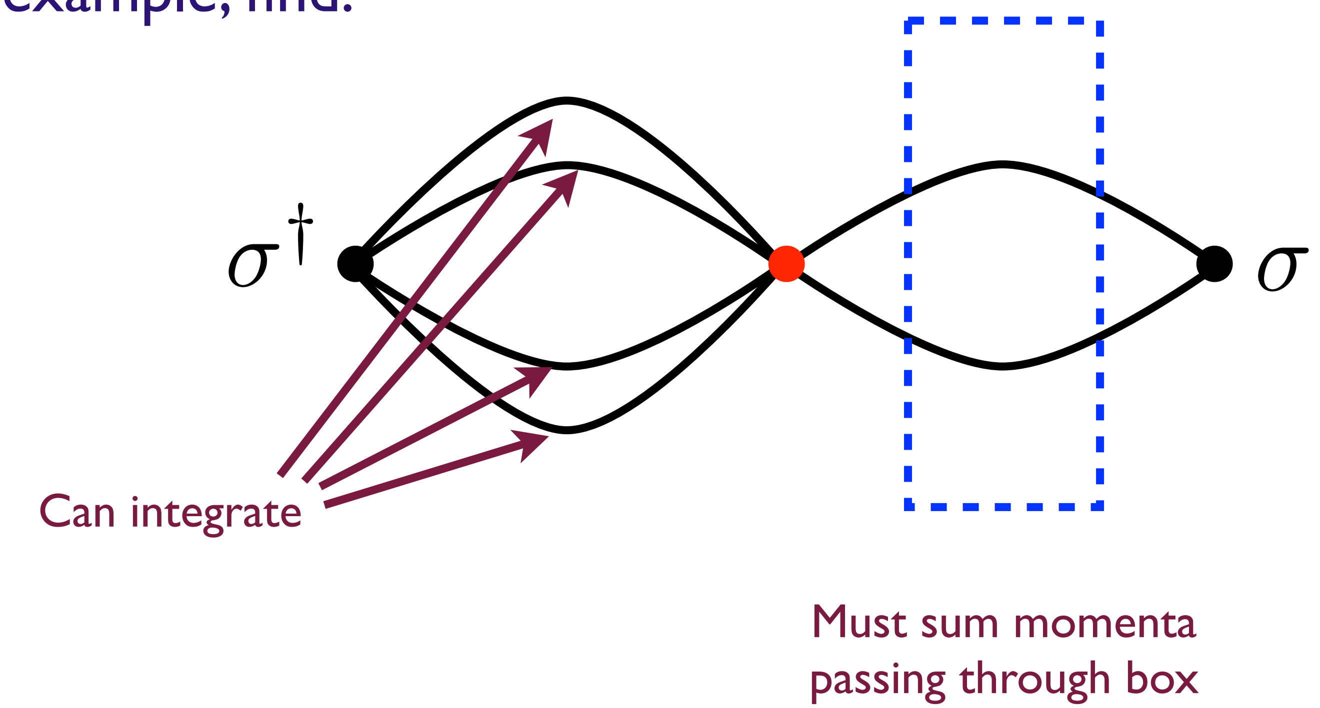


$$\frac{1}{E-\omega_1'-\omega_2'-\omega_3'-\omega_4'} \qquad \frac{1}{E-\omega_1-\omega_2} \qquad \frac{1}{E-\omega_1-\omega_2-\omega_3-\omega_4} \qquad \frac{1}{\sum_{j=1,6}\omega_j}$$

- If restrict 0 < E*< 4M (M < E* < 3M if have odd-legged vertices) then only 2-particle “cuts” have singularities, and these occur only when both particles go simultaneously on shell

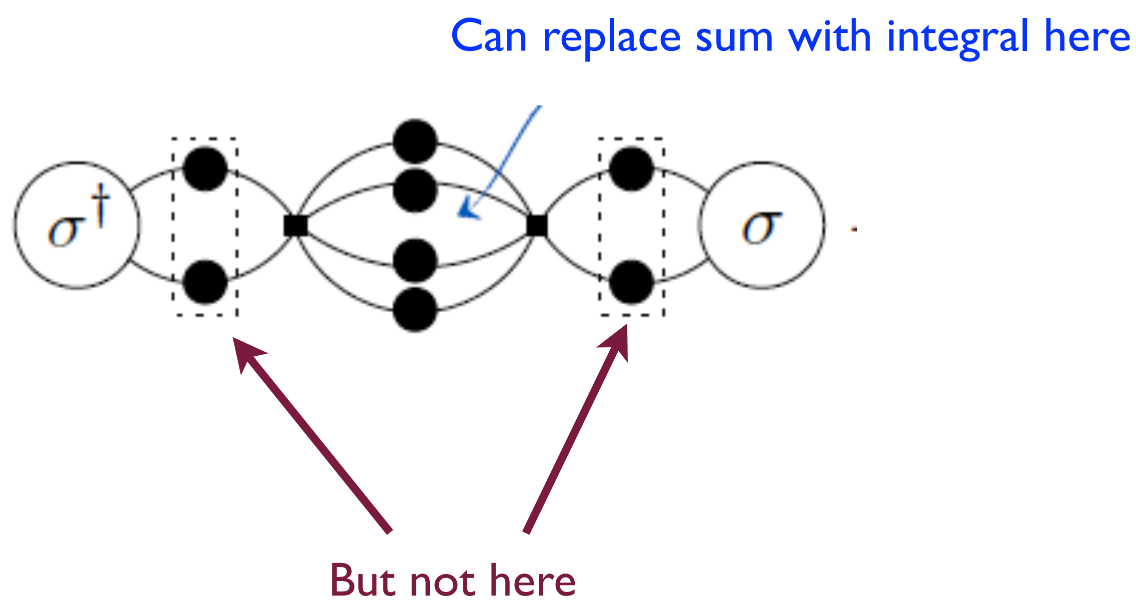
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated
- In our example, find:



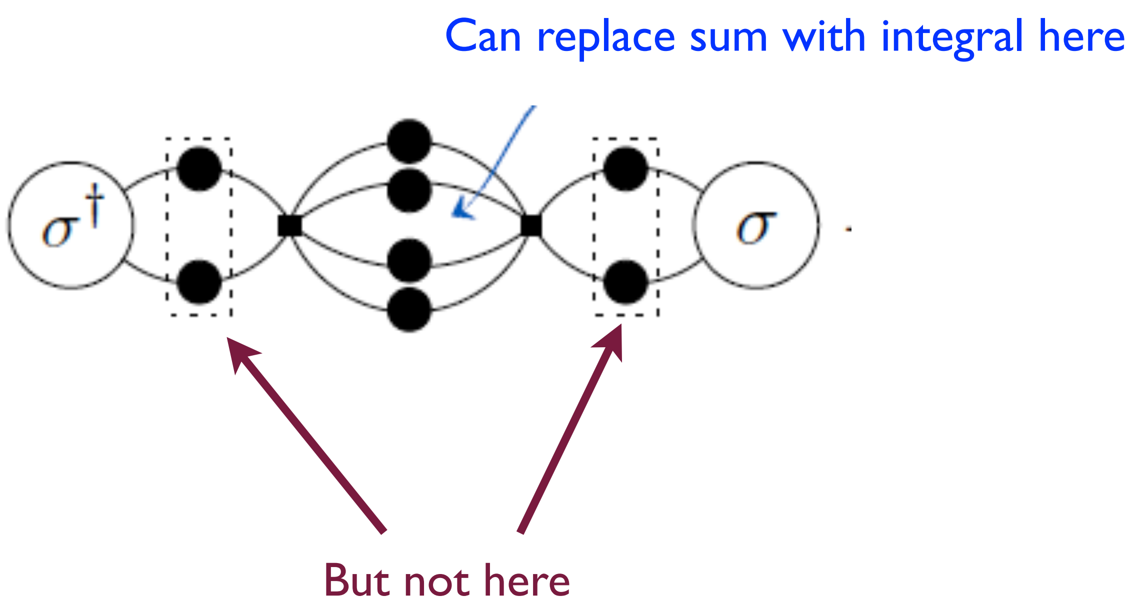
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated
- Another example:



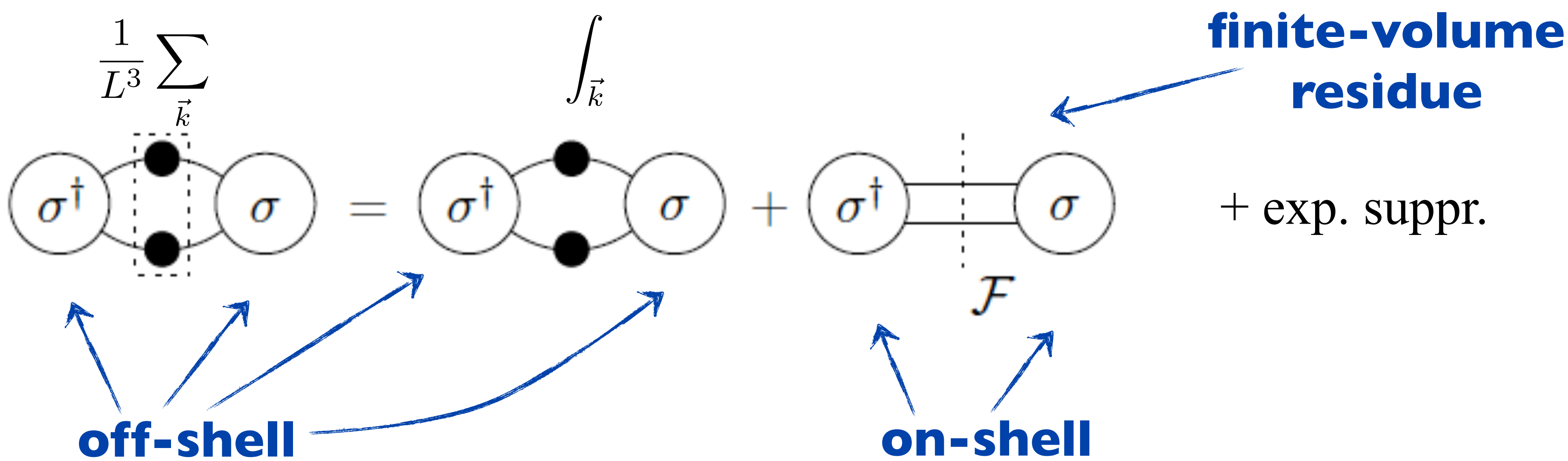
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated
- Another example:



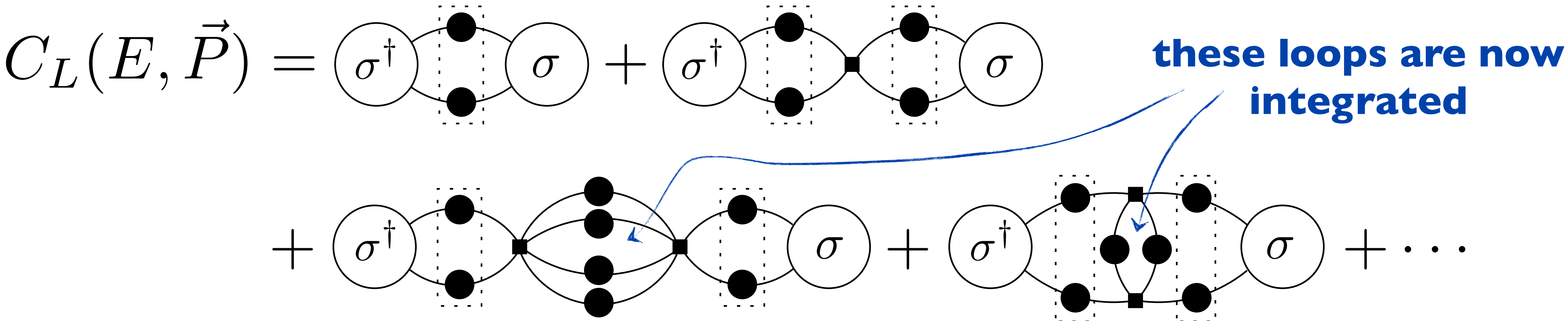
- Then repeatedly use sum=integral + "sum-integral" to simplify

Summary: the key "move"

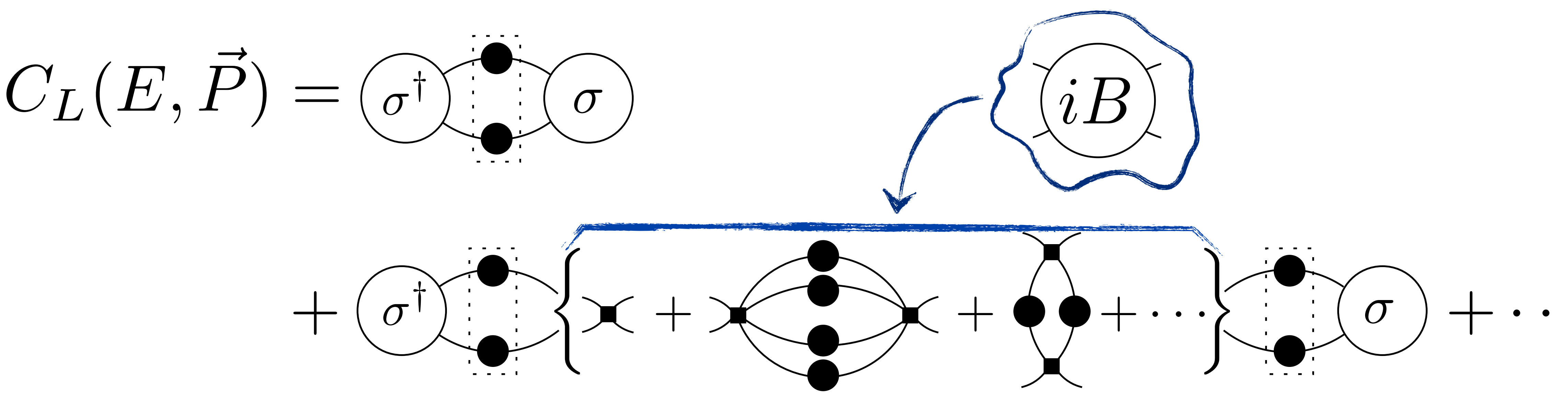


A new type of "cut"

- Apply previous analysis to 2-particle correlator ($0 < E^* < 4M$)

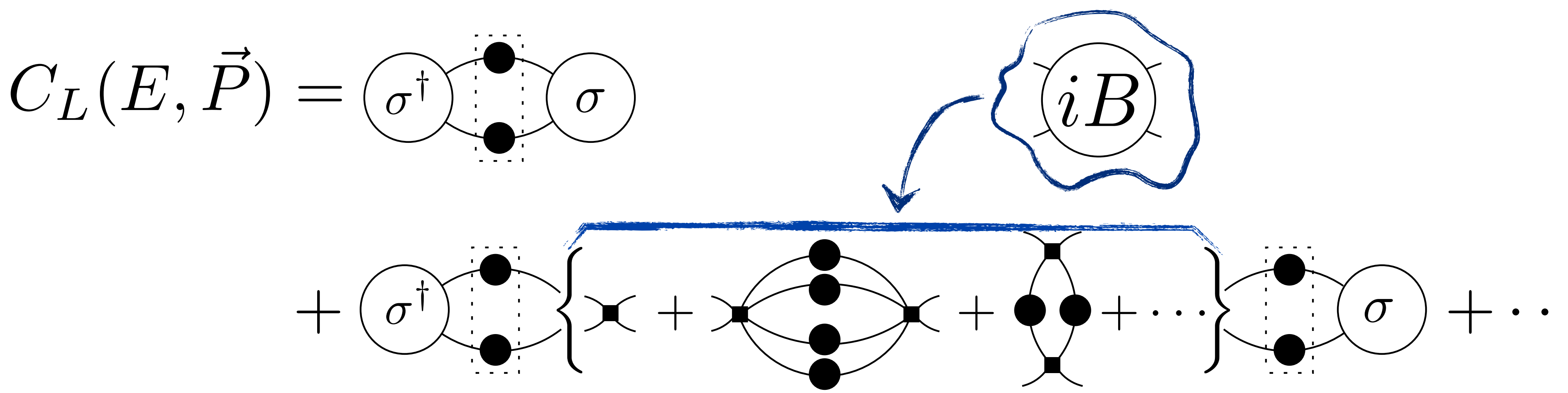


- Collect terms into infinite-volume Bethe-Salpeter kernels

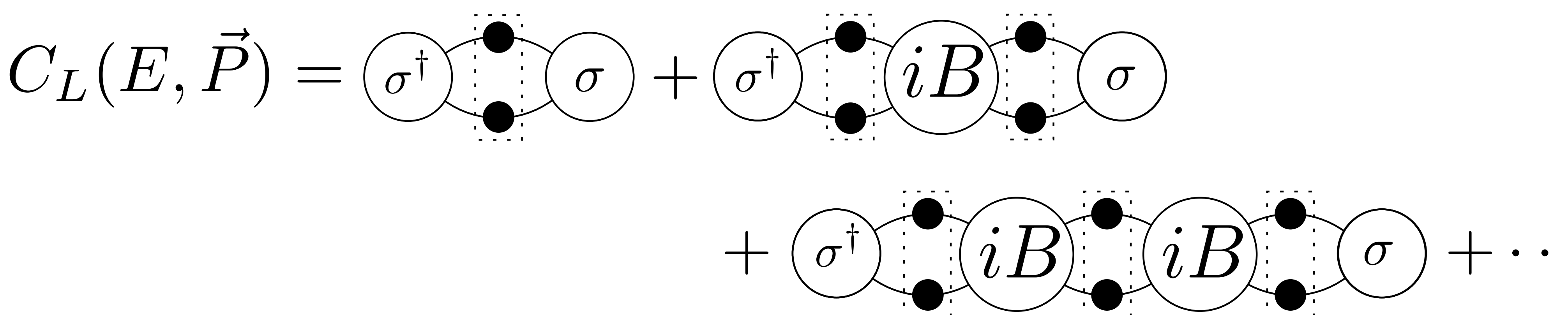


B-S kernel: 2-particle irreducible in the s-channel, i.e. no 2-particle cuts

- Apply previous analysis to 2-particle correlator
- Collect terms into infinite-volume Bethe-Salpeter kernels

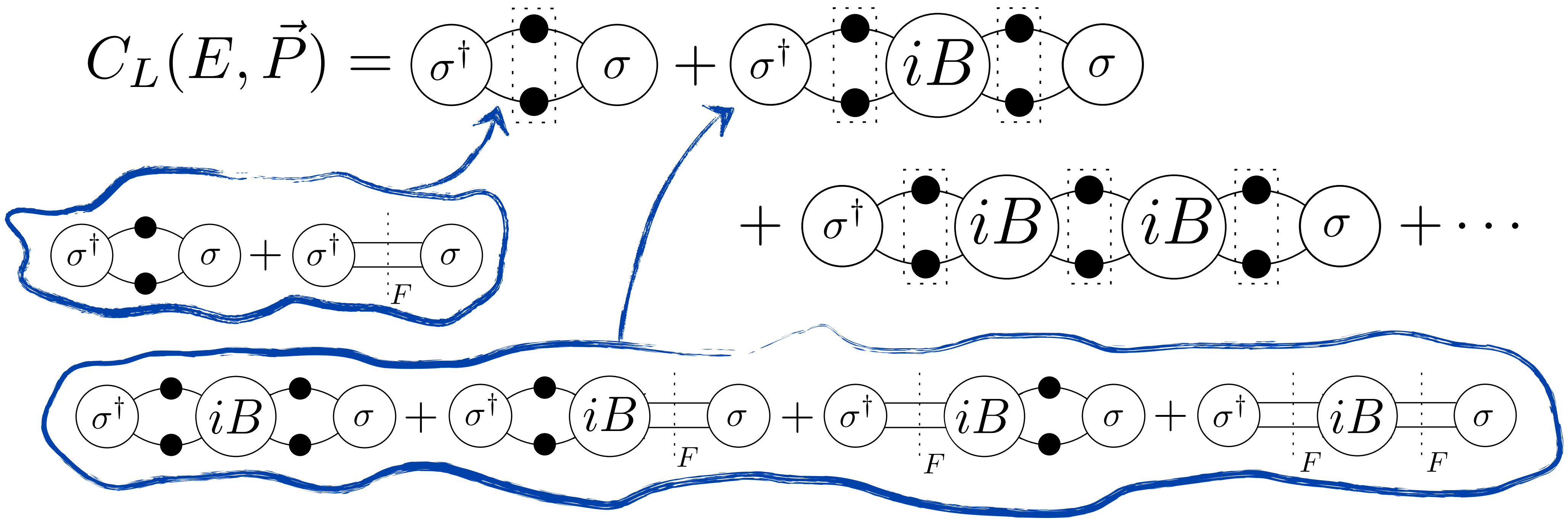


- Leading to

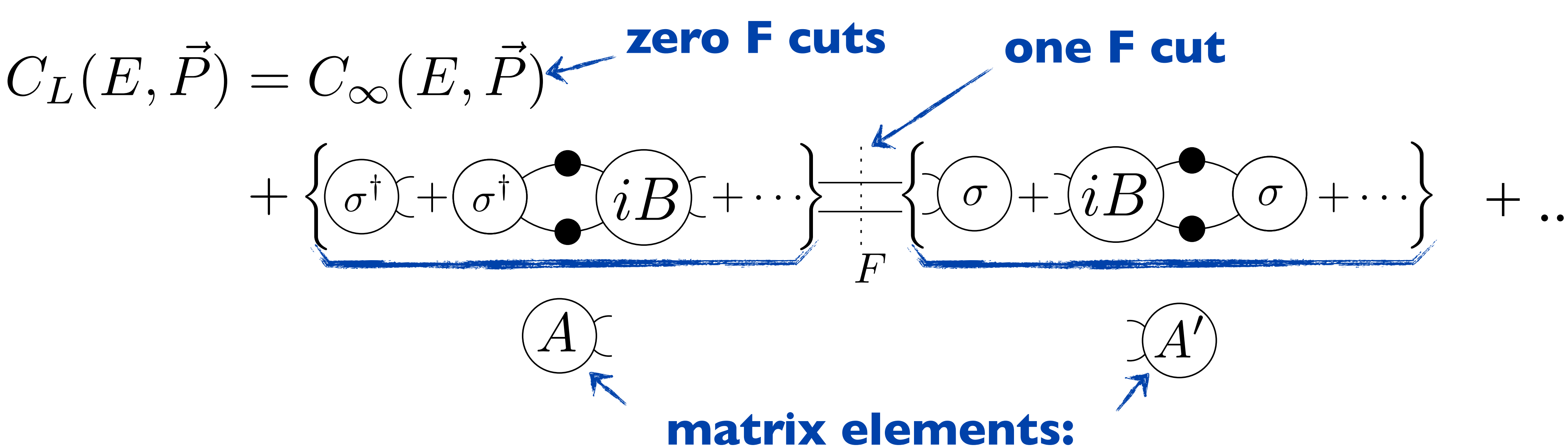


Similar structure to NREFT bubble-chain (e.g. in two nucleon system)

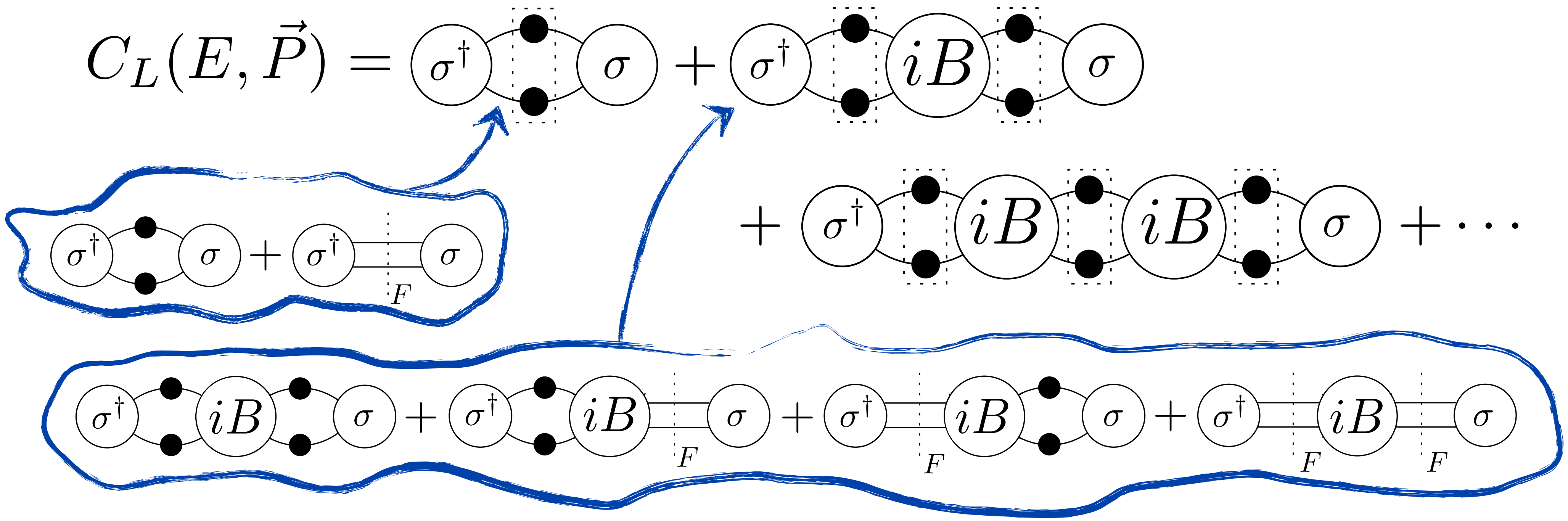
- Next use sum identity



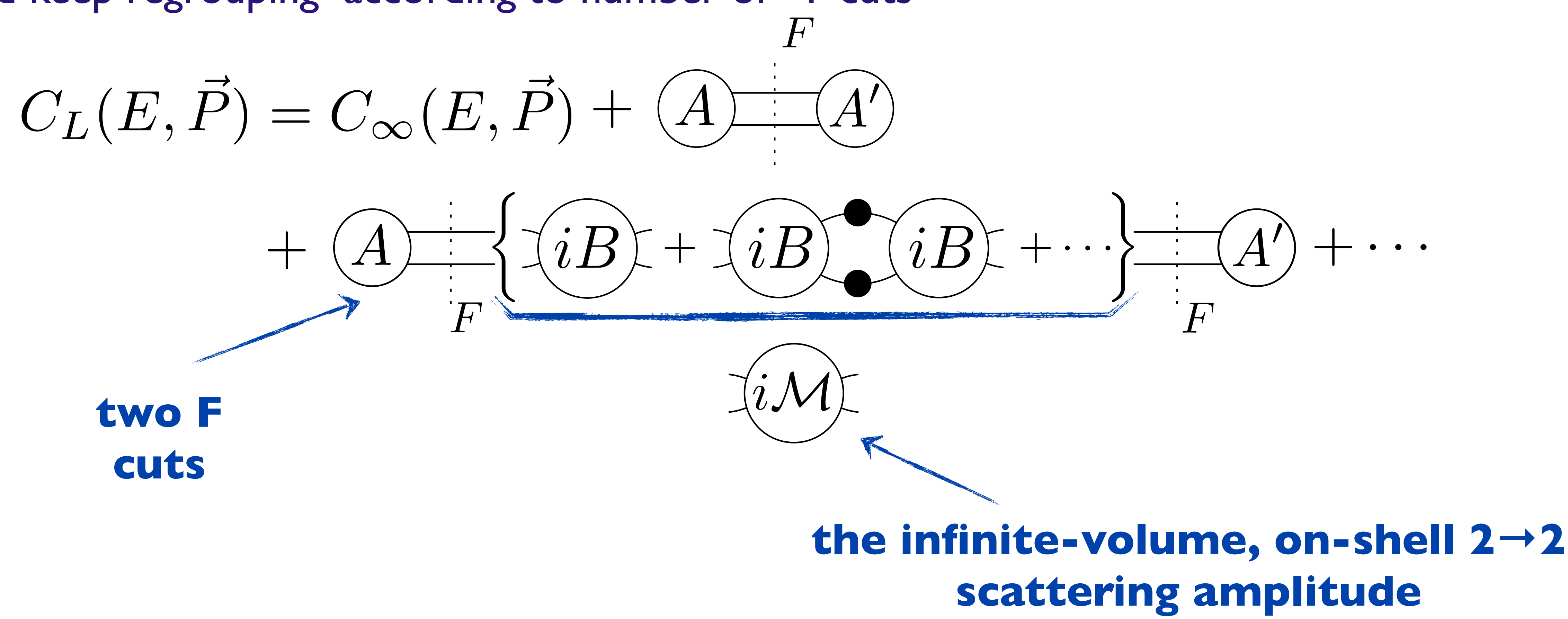
- And regroup according to number of "F cuts"



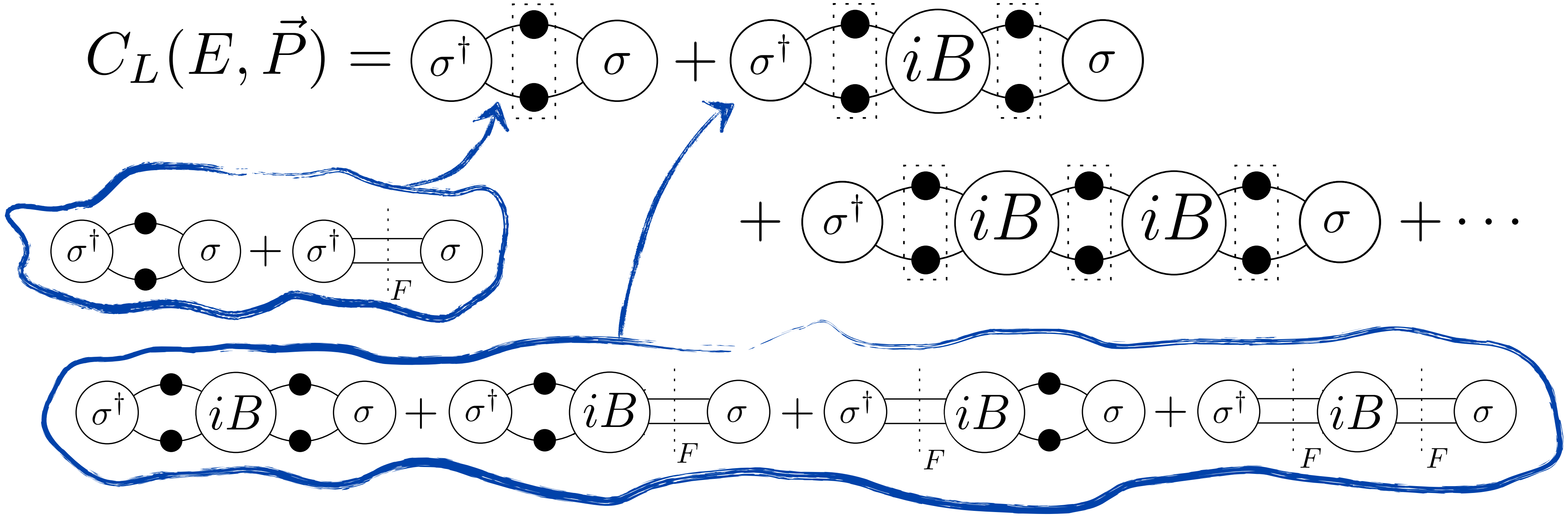
- Next use sum identity



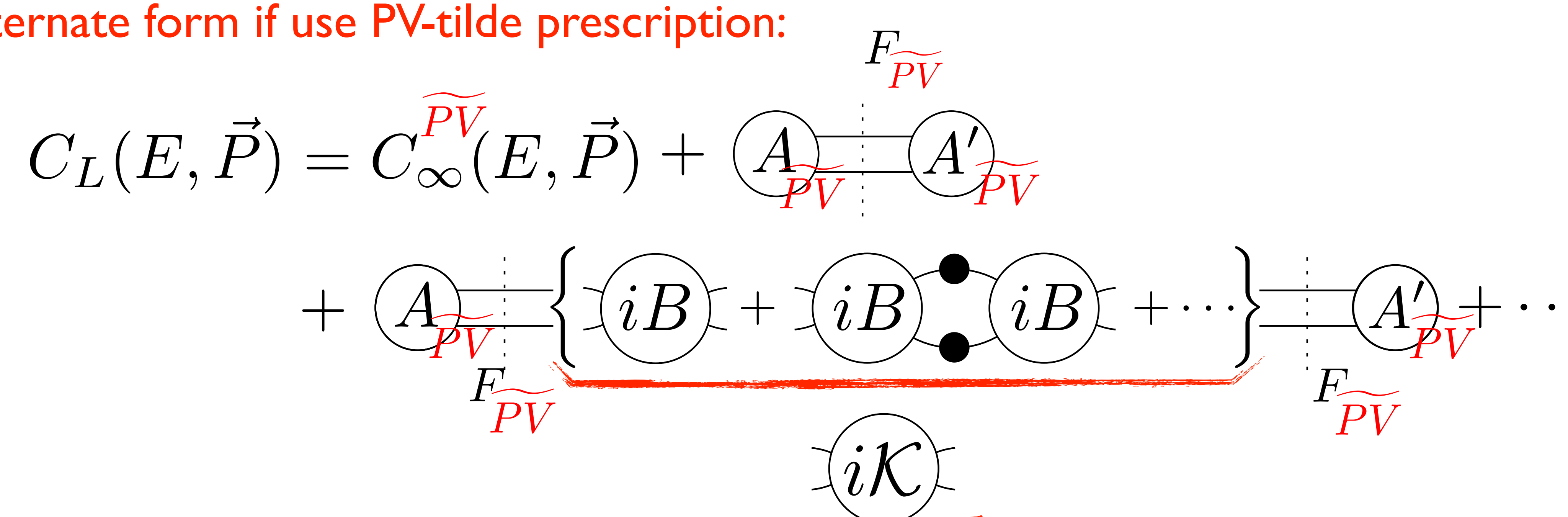
- And keep regrouping according to number of "F cuts"



- Next use sum identity



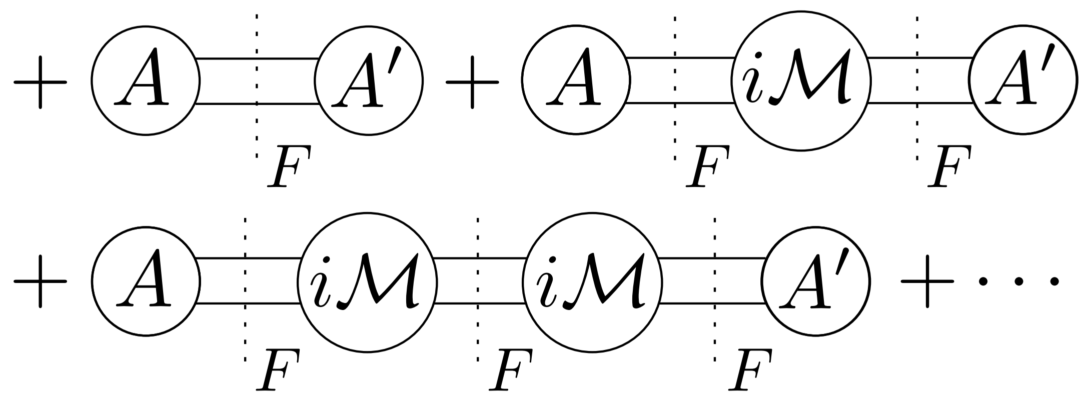
- Alternate form if use PV-tilde prescription:



the infinite-volume, on-shell 2→2 K-matrix

- Final result:

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$

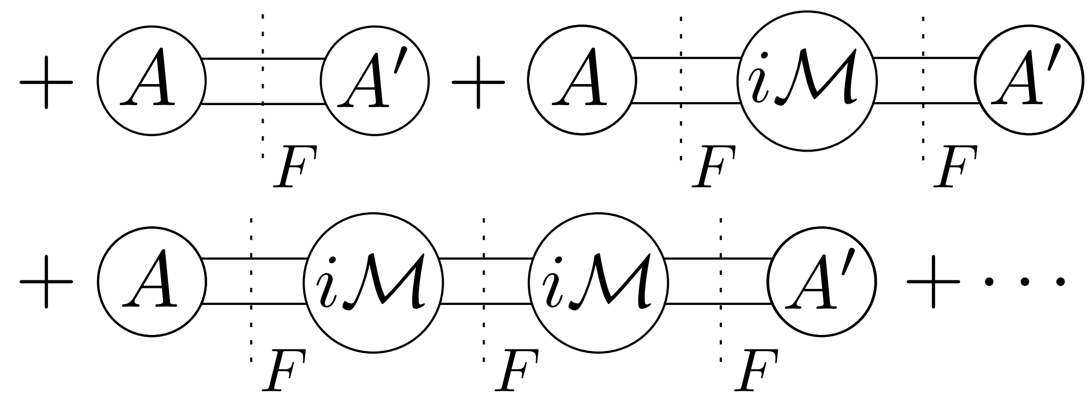


- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + \sum_{n=0}^{\infty} A' iF[i\mathcal{M}_{2\to 2} iF]^n A$$

- Correlator is expressed in terms of infinite-volume, physical quantities and kinematic functions encoding the finite-volume effects

- Final result:

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$



- $$C_L(E,\overrightarrow{P}) = C_\infty(E,\overrightarrow{P}) + \sum_{n=0}^{\infty} A' iF\,[i\mathcal{M}_2\, iF]^n A$$

- $$C_L(E,\overrightarrow{P}) = C_\infty(E,\overrightarrow{P}) + A' iF \frac{1}{1+\mathcal{M}_2 F} A$$

no poles, only cuts

no poles, only cuts

matrices in l,m space

⇒ Poles in C_L occur when

$$\det\left[F(E,\overrightarrow{P},L)^{-1} + \mathcal{M}_2(E^*)\right] = 0$$

2-particle quantization condition

- At fixed L & $\boldsymbol{P}$, the finite-volume spectrum $E_1, E_2, \ldots$ is given by solutions of

$$\det\left[F(E, \vec{P}, L)^{-1} + \mathscr{M}_2(E^*)\right] = 0$$

For $\boldsymbol{P} = 0$ this equivalent to original result by [Lüscher]

Generalization to moving frame first obtained using RQM by [Rummukainen & Gottlieb]

2-particle quantization condition

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Generalization to moving frame first obtained using RQM by [Rummukainen & Gottlieb]

- F and $\mathcal{M}_2$ are matrices in ℓ, m space:
  - $\mathcal{M}_2$ is diagonal; while F is off-diagonal, since the box violates rotation symmetry
- QC separates finite-volume (F) and infinite-volume quantities ($\mathcal{M}_2$)
- If $\mathcal{M}_2$ vanishes, solutions are free two-particle energies due to poles in F
- Each spectral energy gives information about all partial waves of $\mathcal{M}_2(E^*)$

2-particle quantization condition

- Equivalent form, obtained by using PV prescription throughout derivation, is

$$\det\left[F_{PV}(E,\vec{P},L)^{-1}+\mathscr{K}_2(E^*)\right]=0$$

- I prefer this as both $\mathscr{K}_2$, $F_{\rm PV}$ are real
- $\mathscr{K}_2$ contains the same information as $\mathscr{M}_2$, but is real and smooth (no threshold branch points)
- These differences are irrelevant for the two-particle QC—the two QCs are identical—but turn out to be important for the three-particle QC
- Beware when reading the literature, as each collaboration uses different notation for what I call F: sometimes B (box function), sometimes M

Summary of Lecture 1

Summary of Lecture 1

- Resonances are ubiquitous and mysterious in QCD
    - Usually decay to more than 2 particles
- Key issue is relating finite-volume spectrum to scattering amplitudes (or K matrices)
    - QC2 provides a very general, model-independent tool to do so

Thank you!
Questions?

Backup Slides

Examples of resonances

- Example 3: scalar, isoscalars—possible p-wave $q\bar{q}$ states

[PDG]

$f_0(500)$ [g] $I^G(J^{PC}) = 0^+(0^{++})$

Mass (T-Matrix Pole $\sqrt{s}$) = (400–550)$-i$(200–350) MeV
Mass (Breit-Wigner) = (400–550) MeV
Full width (Breit-Wigner) = (400–700) MeV

$f_0(500)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$\pi\pi$	seen	–
$\gamma\gamma$	seen	–

$f_0(980)$ [j] $I^G(J^{PC}) = 0^+(0^{++})$

Mass $m = 990 \pm 20$ MeV
Full width $\Gamma = 10$ to 100 MeV

$f_0(980)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$\pi\pi$	seen	476
$K\overline{K}$	seen	36
$\gamma\gamma$	seen	495

Examples of resonances

- Example 3: scalar, isoscalars—possible p-wave $q\bar{q}$ states

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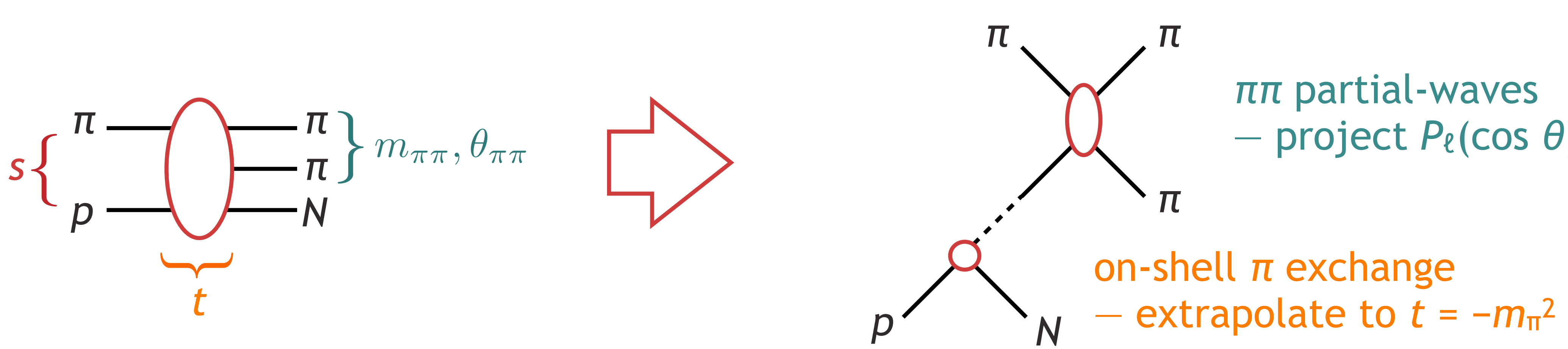
- Large uncertainties because analyses are difficult

extract from charged pion beams on nucleon targets

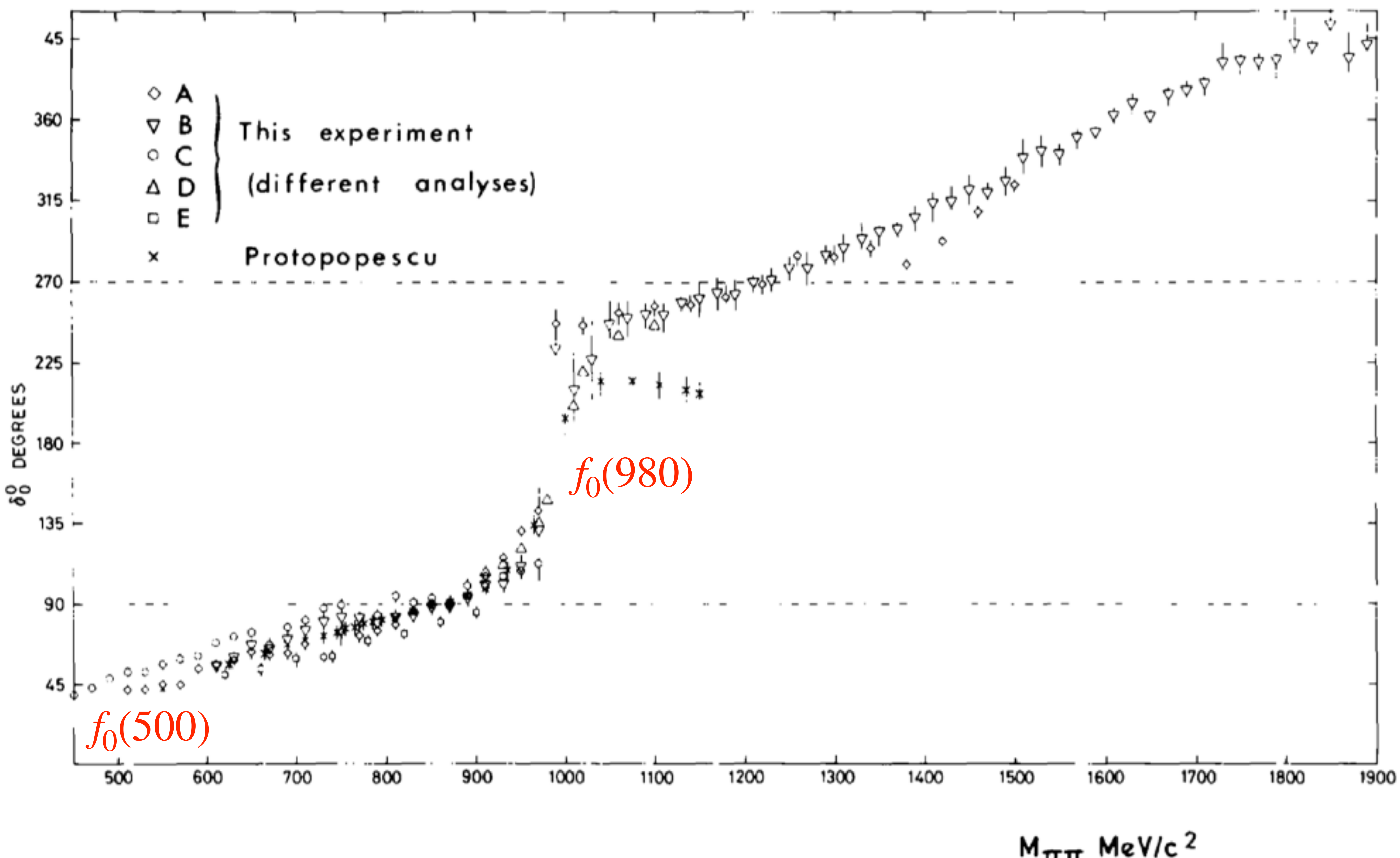


[Figure from HMI slides of Jo Dudek]

Examples of resonances

- Example 3: scalar, isoscalars—possible p-wave $q\bar{q}$ states
  - Extract the phase shift from complicated amplitude analysis

isospin=0

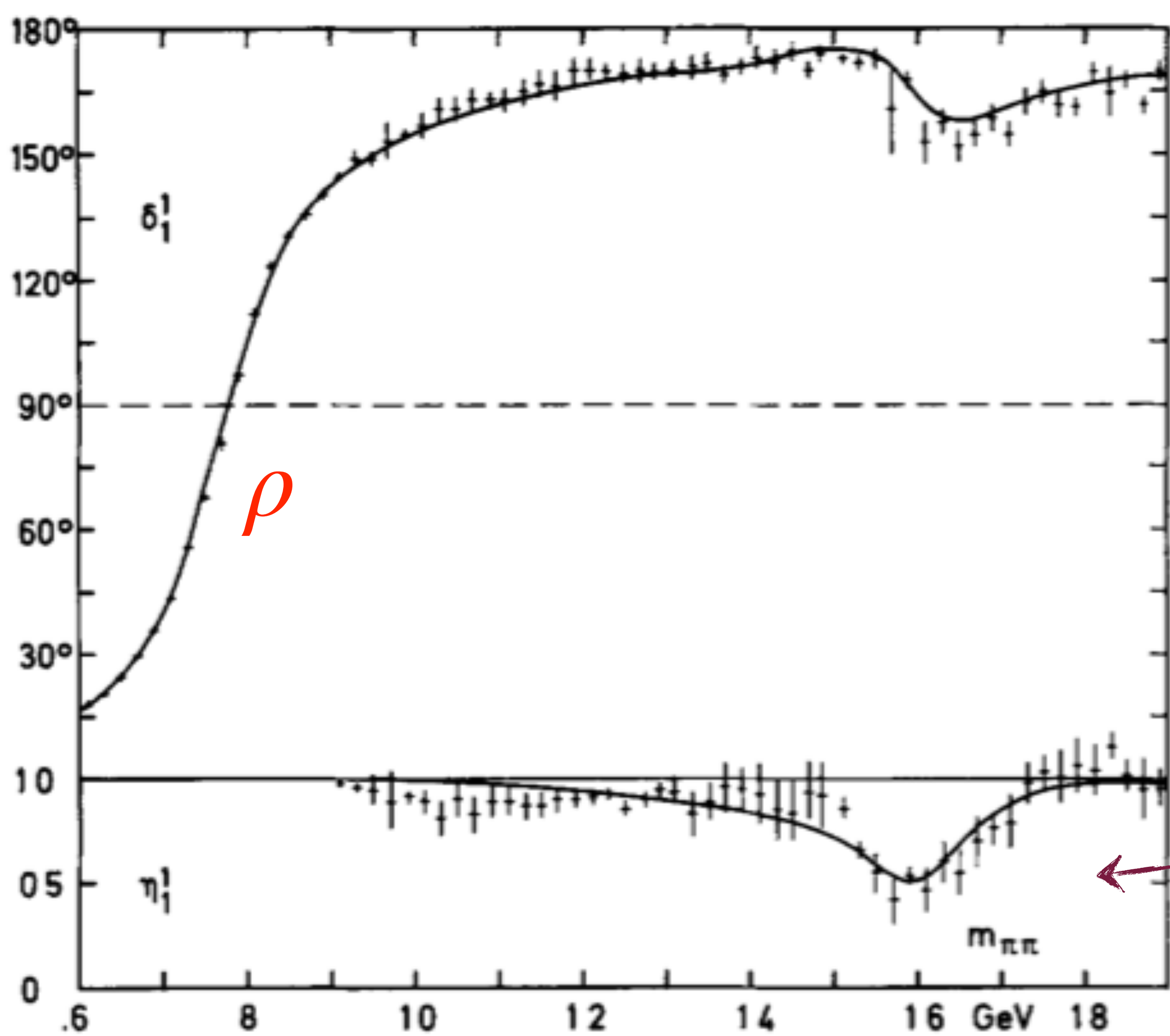


Grayer 1974

Aside on inelasticity

- Phase shift in $I=J=1$ ππ channel

isospin=1



$1 - |\eta|^2$
gives probability for
scattering into any final state
other than ππ,
e.g. KK-bar, ηη, 4π
Becomes nonzero above
1 GeV

Hyams 1973

Examples of resonances

- Example 4: Roper (excited nucleon)

[PDG]

$N(1440)\ 1/2^+$ $I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$

Re(pole position) = 1360 to 1380 (≈ 1370) MeV
−2Im(pole position) = 160 to 190 (≈ 175) MeV
Breit-Wigner mass = 1410 to 1470 (≈ 1440) MeV
Breit-Wigner full width = 250 to 450 (≈ 350) MeV

$N(1440)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$N\pi$	55–75 %	398
$N\eta$	<1 %	†
$N\pi\pi$	17–50 %	347
$\Delta(1232)\pi$, P-wave	6–27 %	147
$N\sigma$	11–23 %	–
$p\gamma$, helicity=1/2	0.035–0.048 %	414
$n\gamma$, helicity=1/2	0.02–0.04 %	413

- Extracted from amplitude analysis of πN scattering
- Lighter than expected from quark model for a radial excitation

Examples of resonances

- Example 5: $Z_c(3900)$—a nonstandard meson

$Z_c(3900)$ $\qquad I^G(J^{PC}) = 1^+(1^{+-})$

[PDG]

Mass $m = 3887.2 \pm 2.3$ MeV $\quad (S = 1.6)$
Full width $\Gamma = 28.2 \pm 2.6$ MeV

$Z_c(3900)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$J/\psi\pi$	seen	699
$h_c\pi^\pm$	not seen	318
$\eta_c\pi^+\pi^-$	not seen	759
$(D\overline{D}^*)^\pm$	seen	–
D^0D^{*-} + c.c.	seen	153
D^-D^{*0} + c.c.	seen	144
$\omega\pi^\pm$	not seen	1862
$J/\psi\eta$	not seen	510
D^+D^{*-} + c.c	seen	–
$D^0\overline{D}^{*0}$ + c.c	seen	–

$\rho\eta_c$ (now seen at 4.2σ significance, [BESIII])

Examples of resonances

- Example 5: $Z_c(3900)$—a nonstandard meson

$Z_c(3900)$ $\quad I^G(J^{PC}) = 1^+(1^{+-})$ [PDG]

Mass $m = 3887.2 \pm 2.3$ MeV (S = 1.6)
Full width $\Gamma = 28.2 \pm 2.6$ MeV

$Z_c(3900)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$J/\psi\pi$	seen	699
$h_c\pi^\pm$	not seen	318
$\eta_c\pi^+\pi^-$	not seen	759
$(D\overline{D}^*)^\pm$	seen	–
$D^0 D^{*-}$ + c.c.	seen	153
$D^- D^{*0}$ + c.c.	seen	144
$\omega\pi^\pm$	not seen	1862
$J/\psi\eta$	not seen	510
$D^+ D^{*-}$ + c.c	seen	–
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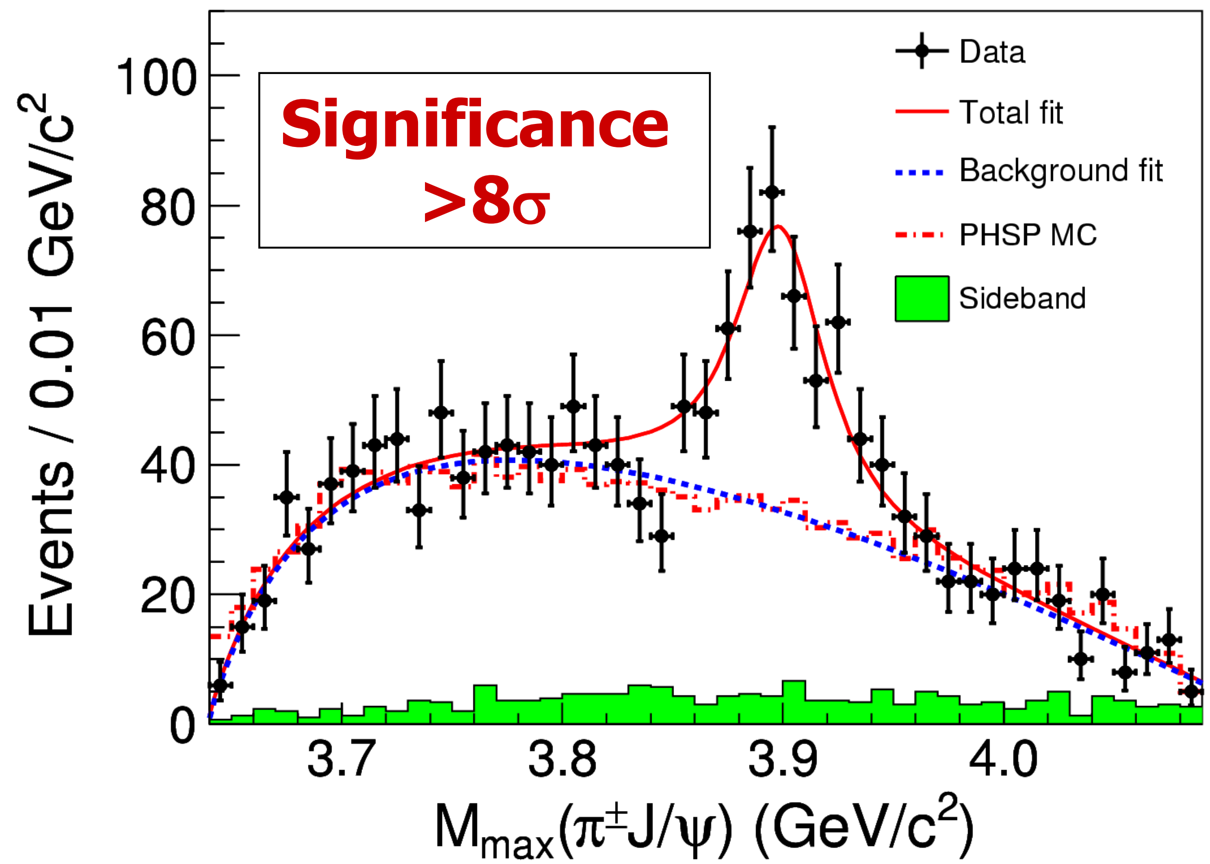
Observed by BESIII, Belle, CLEO-c in 2013

$$e^+e^- \to \pi^\pm Z_c^\mp$$



[BESIII, talk at Lattice 2019 by C.Yuan]

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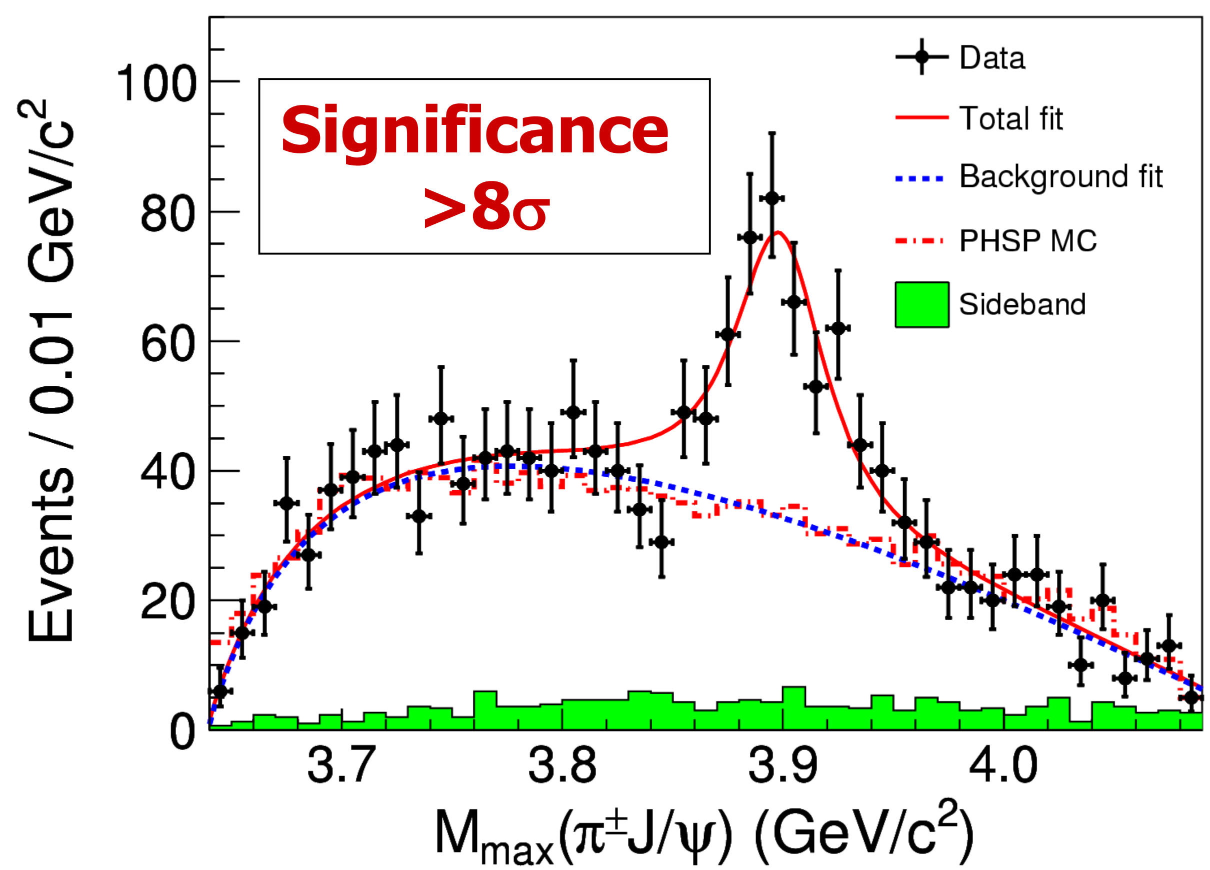

[BESIII, talk at Lattice 2019 by C.Yuan]

- Z_c^+ quark composition: $c\bar{c}u\bar{d}$

Examples of resonances

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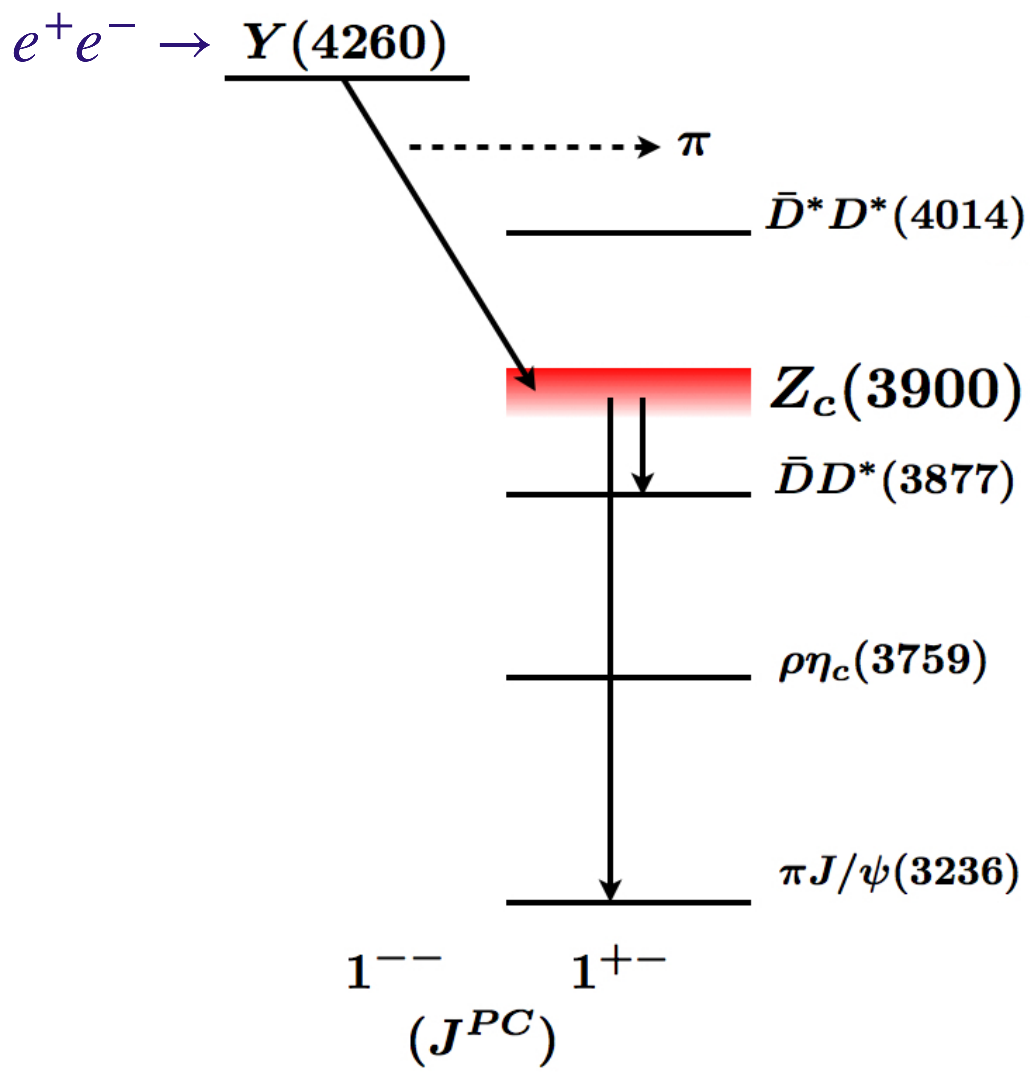
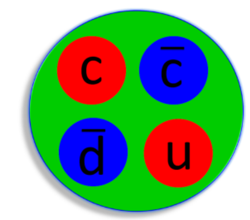
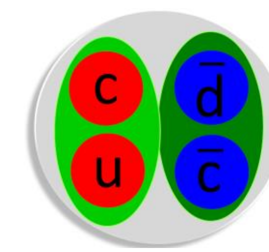


[Ikeda et al., 1602.03465]

- Z_c^+ quark composition: $c\bar{c}u\bar{d}$
- Possible interpretations:
  - Tetraquark



  - Molecule



  - Threshold enhancement—supported by HALQCD study [1602.03465]

G parity

- G parity will come up occasionally in the remaining lectures, so here is a reminder
  - $G = C\, e^{i\pi I_y}$ is an exact symmetry of isosymmetric QCD, and an approximate symmetry of real QCD
  - Eigenstates of G: $\pi(-1),\ \eta(+1),\ \rho(+1),\ \omega(-1),\ \ldots$
- Relevance for what follows:
  - Restricts decay channels, e.g. $\rho \to \pi\pi,\ \omega \to \pi\pi\pi\ (\eta \to \pi\pi$ forbidden by parity)
  - No interactions involving an odd number of pions, e.g.

$$\pi\pi \leftrightarrow 4\pi, \quad \pi\pi \not\leftrightarrow 3\pi$$