Multiparticle scattering
Outline

☐ Lecture 1
  • Motivation/Background/Overview
  • Deriving the two-particle quantization condition (QC2)

☐ Lecture 2
  • Applying the QC2, in brief
  • Deriving the three-particle quantization condition for identical scalars (QC3)

☐ Lecture 3
  • Status of three-particle formalism
  • Applications of QC3
  • Outlook
Main references for this lecture
[Full list of references at end of lecture 3]

• Briceño, Dudek & Young, “Scattering processes & resonances from LQCD,” 1706.06223, RMP 18

• Hansen & SS, “LQCD & three-particle decays of resonances,” 1901.00483, ARNPS 20

• Lectures by Dudek, Hansen & Meyer at HMI Institute on “Scattering from the lattice: applications to phenomenology and beyond,” May 2018, https://indico.cern.ch/event/690702/


• Kim, Sachrajda & SS, hep-lat/0507006, NPB 2015 (direct derivation in QFT of QC2)
Outline for Lecture 1

- Background: hadronic resonances
- Further motivation for studying multiparticle states
- Some scattering basics
- Derivation of QC2 = “Lüscher quantization condition”
Background: hadronic resonances
Stable hadrons in isosymmetric QCD

- QCD with $m_u=m_d$, and no EM (or weak) interactions
  - Theory studied in majority of LQCD simulations
  - Differs from real world at $\sim 1\%$ level
Stable hadrons in isosymmetric QCD

- QCD with $m_u=m_d$, and no EM (or weak) interactions
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- Mesons

$\begin{align*}
\text{meson} & \quad \text{red} \quad \text{anti-red} \\
\end{align*}$
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  - Mesons composed of light quarks:
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  - Mesons composed of light quarks: $\pi(q\bar{q}), K(q\bar{s}), \eta(q\bar{q})$
  - Including heavy quarks: $D(c\bar{q}), D_s(c\bar{s}), B(b\bar{q}), B^*(q\bar{b}), B_s(s\bar{b}), B^*_s(s\bar{b}), B_c(c\bar{b})$
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- Baryons
  - Baryons composed of light quarks:
Stable hadrons in isosymmetric QCD

- **QCD** with \( m_u = m_d \), and no EM (or weak) interactions
  - Theory studied in majority of LQCD simulations
  - Differs from real world at \(~1\%\) level

- **Mesons**
  - Mesons composed of light quarks: \( \pi(q\bar{q}), K(q\bar{s}), \eta(q\bar{q}) \)
  - Including heavy quarks: \( D(c\bar{q}), D_s(c\bar{s}), B(b\bar{q}), B^*(q\bar{b}), B_s(s\bar{b}), B_s^*(s\bar{b}), B_c(c\bar{b}) \)

- **Baryons**
  - Baryons composed of light quarks: \( N(qqq), \Lambda(qqs), \Sigma(qqs), \Xi(qss), \Omega(sss) \)
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  - Including heavy quarks: $\Lambda_c(qqc), \ldots, \Xi_{cc}(qcc), \ldots, \Lambda_{b}(qqb), \ldots$
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- Relatively short list has been the focus of most LQCD calculations to date
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Stable hadrons in isosymmetric QCD

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---

**Figure from ref. (174).** The horizontal lines are the experimental values, and the grey shaded regions represent the experimental error. The calculated precision for the quantities with labels in blue shaded boxes is better than that of current measurements.

**FIG. 10**

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[Image: Kronfeld, 1203.1204]

---

S. Sharpe, “Multiparticle Scattering”, Lecture 1, 7/18/2023, Bad Honnef Summer School
Relatively short list has been the focus of most LQCD calculations to date

\[ \pi(q\bar{q}), K(q\bar{s}), \eta(q\bar{q}) \quad D(c\bar{q}), D_s(c\bar{s}), B(b\bar{q}), B^*(q\bar{b}), B_s(s\bar{b}), B_s^*(s\bar{b}), B_c(c\bar{b}) \]

\[ N(qqq), \Lambda(qqs), \Sigma(qqs), \Xi(qss), \Omega(sss) \quad \Lambda_c(qqc), \ldots, \Xi_{cc}(qcc), \ldots, \Lambda_b(qqb), \ldots \]
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\end{align*}
\]

\[\begin{array}{c}
\text{M}_H \\
\text{M}_B - 4000
\end{array}\]

All other states are resonances!

[Kronfeld, 1203.1204]
Plethora of resonances

- Most hadrons are resonances!
Cornucopia of exotics

62 new hadrons at the LHC

[I. Danilkin, talk at INT workshop, March 23]
Examples of resonances

- Most hadrons are resonances!
- Very short lived, with decays into 2, 3, … stable hadrons
Examples of resonances

- Most hadrons are resonances!
- Very short lived, with decays into 2, 3, ... stable hadrons

  - Example 1: single-channel decay of s-wave spin-triplet q q-bar state:

    \[ I^G J^{PC} = 1^+ 1^- : \rho \to \pi \pi, \quad M_\rho \approx 775 \text{MeV}, \quad \Gamma_\rho \approx 150 \text{MeV} \quad (\tau = 4 \times 10^{-23} \text{s}) \]

- Many production mechanisms, e.g. \[ \tau^- \to \pi^- \pi^0 \nu_\tau \]

[CLEFT collab., hep-ex/9910046]
\[ e^+ e^- \to \tau^+ \tau^- + X \]

\( \rho \) is produced by the vector part of the weak current \( \bar{u} \gamma^\mu d \)

Fitting the spectrum involves models & uncertainties
Examples of resonances

- Example 2: multi-channel decay of p-wave $q\bar{q}$ state:

```
pdg summary entry

$\alpha_2(1320)$

$J^G(JPC) = 1^- (2++)$

Mass $m = 1318.3^{+0.5}_{-0.6}$ MeV
Full width $\Gamma = 107 \pm 5$ MeV

<table>
<thead>
<tr>
<th>$\alpha_2(1320)$ DECAY MODES</th>
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</tr>
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<tbody>
<tr>
<td>$3\pi$</td>
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<tr>
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<tr>
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\[ a_2(1320) \]

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same ‘bump’ appears in multiple different processes …

… due to same $a_2$ resonance

\[ \pi \text{ Pb} \to \pi \rho \text{ Pb} \]
\[ \gamma\gamma \to \pi \eta \]
\[ \pi p \to K\bar{K} p \]

COMPASS
Belle
CERN SPS

[Figures from HMI slides of Jo Dudek]
Lessons

- Extracting resonance parameters from experiment is indirect & challenging
  - Resonance is defined as a pole in a scattering amplitude—not directly accessible
- Typical resonances have multiple decay channels, each involving 2 or 3 (or more) particles
- Quark model (or other models) fails to explain presence or properties of an increasing number of resonances
  - X, Y, Z resonances, glueballs, hybrids, tetraquarks, pentaquark, …
- Resonances are a largely unexplored frontier in our attempts to understand hadronic physics (i.e. the properties of a strongly-coupled QFT) from first principles
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A major challenge for LQCD!
How can LQCD help?

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  - Methods for indirectly accessing scattering amplitudes must be developed (the main topic of these lectures)
  - LQCD has advantage of being able to turn off electroweak interactions
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- LQCD calculations must use large bases of operators to allow understanding of structure of hadrons—any input is useful!
- Varying the quark masses can provide additional useful information
Personal note

- As a grad student I used the MIT bag model to predict the masses of “hybrid” mesons—resonances of the form: quark + antiquark + “constituent gluon”
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HYBRIDS: MIXED STATES OF QUARKS AND GLUONS*

Michael CHANOWITZ and Stephen SHARPE

Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720, USA
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Submitted for publication

MEIKTONS: MIXED STATES OF QUARKS AND GLUONS

Michael Chanowitz and Stephen Sharpe

August 1982

RECEIVED

BERKELEY LABORATORY
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  Noise!

• There are now increasingly sophisticated calculations of hybrid meson properties, and these will eventually be based on the formalism I will describe in these lectures
• **Fundamental issue:**
  
  • LQCD simulations are done in finite volumes, with imaginary time
  
  • Experiments are done in infinite volume in real time

---

**How do we connect?**
Fundamental Issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box.
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?

\[
\begin{align*}
E_0(L) \\
E_1(L) \\
E_2(L)
\end{align*}
\]

\[iM_{n \rightarrow m}\]

Discrete energy spectrum

Scattering amplitudes
Fundamental Issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?

N.B. This is a finite volume QFT problem (can ignore lattice spacing)
Further motivations for studying multiparticle states
Motivations

• Calculating electroweak decay and transition amplitudes for processes involving multiple particles

• Determining NN and NNN interactions as input for predicting properties of nuclei and nuclear matter
  
  • NNN interactions needed as input for EFT treatments of large nuclei, and for the neutron-star equation of state

• $\pi\pi\pi, \pi K \bar{K}, \ldots$ interactions needed as input to study pion & kaon condensation
Electroweak decays

e.g. $K \rightarrow \pi\pi\pi\pi$ decay amplitudes

- Does the SM reproduce the observed CP violation in $K \rightarrow \pi\pi\pi\pi$ decays?
- Formalism to study this now exists [Hansen, Romero-López, SRS, 2021]
A more distant motivation

**Observation of CP violation in charm decays**

LHCb collaboration†

**Abstract**

A search for charge-parity (CP) violation in \( D^0 \rightarrow K^- K^+ \) and \( D^0 \rightarrow \pi^- \pi^+ \) decays is reported, using \( pp \) collision data corresponding to an integrated luminosity of 6 fb\(^{-1} \) collected at a center-of-mass energy of 13 TeV with the LHCb detector. The flavor of the charm meson is inferred from the charge of the pion in \( D^*(2010)^+ \rightarrow D^0 \pi^+ \) decays or from the charge of the muon in \( \bar{B} \rightarrow D^0 \mu^- \bar{\nu}_\mu X \) decays. The difference between the \( CP \) asymmetries in \( D^0 \rightarrow K^- K^+ \) and \( D^0 \rightarrow \pi^- \pi^+ \) decays is measured to be \( \Delta A_{CP} = [-18.2 \pm 3.2 \text{ (stat.)} \pm 0.9 \text{ (syst.)}] \times 10^{-4} \) for \( \pi^- \)-tagged decays and \( \Delta A_{CP} = [-9 \pm 8 \text{ (stat.)} \pm 5 \text{ (syst.)}] \times 10^{-4} \) for \( \mu^- \)-tagged \( D^0 \) mesons. Combining these with previous LHCb results leads to

\[
\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4},
\]

where the uncertainty includes both statistical and systematic contributions. The measured value differs from zero by more than five standard deviations. This is the first observation of \( CP \) violation in the decay of charm hadrons.
A more distant motivation

- Calculating CP-violation in $D \to \pi \pi, K\bar{K}$ in the Standard Model
- Finite-volume state is a mix of $2\pi, K\bar{K}, \eta\eta, 4\pi, 6\pi, \ldots$
- Need 4 (or more) particles in the box!
Scattering basics (infinite-volume)
Recall some details of the simplest scattering process: \(2 \rightarrow 2\)

- We will mainly discuss spinless particles in these lectures, e.g. pions
- We will consider both identical particles, e.g. \(\pi^+\pi^+\), and nonidentical, e.g. \(\pi^+K^+\)

Scattering amplitude related to the S matrix

\[
S = 1 + iT \quad \langle f \mid T \mid i \rangle = (2\pi)^4 \delta^4(P_f - P_i) \mathcal{M}_{fi}
\]

In a given theory, can calculate in perturbation theory (PT), e.g. in \(\Phi^4\) theory

\[
i \mathcal{M}_2 = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1.png} + \includegraphics[width=0.2\textwidth]{diagram2.png} + \includegraphics[width=0.2\textwidth]{diagram3.png} + \includegraphics[width=0.2\textwidth]{diagram4.png} + \includegraphics[width=0.2\textwidth]{diagram5.png} + \ldots
\end{array}
\]
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\hline
\hline
\hline
\end{array} + \begin{array}{c}
\hline
\hline
\hline
\end{array} + \begin{array}{c}
\hline
\hline
\hline
\end{array} + \begin{array}{c}
\hline
\hline
\hline
\end{array} + \begin{array}{c}
\hline
\hline
\hline
\end{array} + \ldots
\]

- We will not assume a particular theory, e.g. ChPT or $\phi^4$; instead we use a generic relativistic QFT, with all possible vertices, and work to all orders in PT
Properties of $M_2$

- Poincaré invariance $\Rightarrow M_2$ depends on the two independent Mandelstam variables

\[
\begin{align*}
M_2 &= M_2(s,t), \quad s = (p_1 + p_2)^2, \quad t = (p_1 - p'_1)^2, \quad u = (p_1 - p'_2)^2 = 4m^2 - s - t
\end{align*}
\]

\[
\begin{array}{c}
p_1' \\
M_2 \\
p_2'
\end{array}
\]
Properties of $\mathcal{M}_2$

- Poincaré invariance $\Rightarrow \mathcal{M}_2$ depends on the two independent Mandelstam variables

\[ \mathcal{M}_2 = \mathcal{M}_2(s, t), \quad s = (p_1 + p_2)^2, \quad t = (p_1 - p_1')^2, \quad u = (p_1 - p_2')^2 = 4m^2 - s - t \]

- Partial wave decomposition in CM frame

\[ s = E^{*2} = 4(q^2 + m^2), \quad t = -2q^2(1 - \cos \theta) \]

\[ \mathcal{M}_2(s, t) = \sum_{\ell} (2\ell + 1) \mathcal{M}^{(\ell)}(s) P_{\ell}(\cos \theta) \]

Only even values of $\ell$ contribute for identical particles.
Properties of $M_2$

- Unitarity—holds in each partial wave (results here for identical particles)

\[ S^\dagger S = 1 \Rightarrow \text{Im}(M_2^{(\ell)}) = M_2^{(\ell)*} \rho M_2^{(\ell)} = \rho |M_2^{(\ell)}|^2, \quad \rho = \frac{q}{16\pi E^*} \text{ (phase space)} \]
Properties of $\mathcal{M}_2$

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- Solve unitarity constraint in terms of an arbitrary, real $K$ matrix

$$\text{Im} \left[ \frac{1}{\mathcal{M}_2^{(\ell)}} \right] = -\rho \Rightarrow \frac{1}{\mathcal{M}_2^{(\ell)}} \equiv \frac{1}{\mathcal{K}_2^{(\ell)}} - i\rho \Rightarrow \mathcal{M}_2^{(\ell)} = \mathcal{K}_2^{(\ell)} \frac{1}{1 - i\rho \mathcal{K}_2^{(\ell)}}$$
Properties of $\mathcal{M}_2$

- **Unitarity**—holds in each partial wave (results here for identical particles)

\[
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\]

- **Solve unitarity constraint in terms of an arbitrary, real $K$ matrix**

\[
\text{Im} \left[ \frac{1}{\mathcal{M}_2^{(\ell)}} \right] = -\rho \Rightarrow \frac{1}{\mathcal{M}_2^{(\ell)}} \equiv \frac{1}{\mathcal{K}_2^{(\ell)}} - i\rho \Rightarrow \mathcal{M}_2^{(\ell)} = \mathcal{K}_2^{(\ell)} \frac{1}{1 - i\rho \mathcal{K}_2^{(\ell)}}
\]

- **Parametrize $\mathcal{K}_2$ using (real) phase shifts**

\[
\mathcal{K}_2^{(\ell)} \equiv \frac{1}{\rho} \tan \delta_\ell = \frac{16\pi E^*}{q \cot \delta_\ell} \Rightarrow \mathcal{M}_2^{(\ell)} = \frac{1}{\rho} e^{i\delta} \sin \delta_\ell
\]
Properties of $\mathcal{M}_2$

- **Threshold behavior (QM)**

  $$\delta_\ell \sim q^{1+2\ell} \left[ 1 + \mathcal{O}(q^2) \right] \Rightarrow \mathcal{K}_2^{(\ell)} \sim q^{2\ell} \left[ 1 + \mathcal{O}(q^2) \right]$$

- **Effective range expansion (ERE)**

  $$\frac{1}{\mathcal{K}_2^{(0)}} = \frac{1}{16\pi E_2} \left[ -\frac{1}{a_0} + r_0 \frac{q^2}{2} + P_0 r_0^3 q^4 + \ldots \right], \quad \frac{1}{\mathcal{K}_2^{(2)}} = -\frac{1}{16\pi E_2} \frac{1}{q^4} \frac{1}{a_2} + \ldots$$

- $a_0$ is s-wave scattering length, related to threshold scattering amplitude

  $$\mathcal{M}_2(q = 0) = \mathcal{K}_2(q = 0) = 32\pi m a_0$$

- $a_0$ is the intercept on the $r$ axis of the s-wave radial QM wavefunction with $q = 0$, and can have any value: $-\infty < a_0 < \infty$

- $r_0$ is the effective range (typically of order the range of the interaction), $P_0$ is the “shape parameter” (typically of order unity), and $a_2$ is the d-wave scattering “length”
Properties of $M_2$

- Analytic structure: branch cut along real s axis above threshold, arising from unitarity

$$M_2^{(ℓ)} = H_2^{(ℓ)} + H_2^{(ℓ)} iρH_2^{(ℓ)} + \ldots, \quad \rho = \frac{\sqrt{s - 4m^2}}{32\pi\sqrt{s}}$$

- $M_2$ has two Riemann sheets, the top one being called the “physical sheet”

- $H_2$ does not have the right-hand cut; it is analytic at threshold

S. Sharpe, “Multiparticle Scattering”, Lecture 1, 7/18/2023, Bad Honnef Summer School
Properties of $M_2$

- $t$- and $u$-channel exchanges lead to the “left-hand cut”

\[ M_2^{(\ell)} = \mathcal{H}_2^{(\ell)} + \mathcal{H}_2^{(\ell)} i\rho \mathcal{H}_2^{(\ell)} + \ldots \]

\[ \rho = \frac{\sqrt{s - 4m^2}}{32\pi\sqrt{s}} \]

- For nondegenerate systems, the LH cut can lie close to threshold (e.g. $\Lambda\Lambda$ has LH cut due to pion exchange)

- LH cut invalidates standard QC2 and QC3 derivations
Bound states

- Bound states lead to poles in $\mathcal{M}_2$ on physical sheet

- $\mathcal{K}_2$ does not have a corresponding pole since $\rho$ is nonzero below threshold

$$1/\mathcal{M}_2^{(\ell)} \equiv 1/\mathcal{K}_2^{(\ell)} - i\rho \text{ where } -i\rho = \frac{|q|}{16\pi E^*} \text{ with } E_{BS}^* = 4(m^2 - |q|^2)$$

- Bound state condition is thus

$$1/\mathcal{M}_2^{(\ell)} = \frac{1}{16\pi E^*}(q \cot \delta_\ell + |q|) = 0$$

- If keep only the scattering length in the ERE, find bound state for $a_0 > 0$

$$q \cot \delta_0 = -1/a_0 \Rightarrow |q| = 1/a_0 \Rightarrow E_{BS}^* = 2\sqrt{m^2 - 1/a_0^2}$$

- Bound state at threshold in unitary limit $a_0 \to \infty$
**Resonances**

- Resonances lead to poles in $\mathcal{M}_2$ below the real axis on the second (unphysical) sheet.
  - Cannot have poles on physical sheet aside from bound states due to causality.
  - To display sheets it is better to use single-sheeted variable $q$.

  - Resonance with width $\Gamma = 1/\tau$ and mass $M$ has pole at

$$E^* = M - i\Gamma/2 \Rightarrow s = M^2 + (\Gamma/2)^2 - iM\Gamma$$

- Leads to a bump in scattering cross-section $\propto |\mathcal{M}_2|^2$ as we saw earlier.
Resonances

• Narrow s-wave resonances well described by Breit-Wigner form

\[ \tan \delta_{\text{BW}} = \frac{E^* \Gamma}{M^2 - E^{*2}} \Rightarrow M_2 \propto \frac{1}{M^2 - E^{*2} - iE^*\Gamma} \]

• As \( E^* \) passes through \( M \) from below:
  - Phase shift rises rapidly through 90°
  - \( \mathcal{K}_2 \sim \tan \delta \) has a pole at \( M \) (i.e. on the real axis)

• Pole in \( \mathcal{K}_2 \) does not have any direct physical significance, but does play a role in the finite-volume analysis to follow

\[ \text{isospin}=1 \]

\( \begin{align*}
\text{p-wave phase shift} \\
\text{Hyams 1973}
\end{align*} \]
Resonances: unavoidable complication

- Neither experiment, nor LQCD calculations, can directly access complex energies.
- Thus, in order to study resonances, both methods have to parametrize the $K$ matrices with an analytic form that can be continued into the complex plane.
- Thus some parametrization dependence is unavoidable.
- One should put as much physical knowledge as possible into the parametrization, while minimizing model dependence.
- Input from the experimental analysis community can be helpful.
Deriving the two-particle QC

Following the method of [Kim, Sachrajda & SRS, 05]
Set-up

- Work in continuum (assume that LQCD can control discretization errors)

- Cubic box of size L with periodic BC, and infinite (Minkowski) time
  - Spatial loops are sums: \( \frac{1}{L^3} \sum_k \vec{k} \)
  - Can easily generalize to other geometries and BC
  - Consider identical particles with physical mass m (think of pions), interacting arbitrarily—a generic (relativistic) effective field theory (RFT)
    - Work to all orders in perturbation theory with no assumptions about the size of coupling constants
    - Generalizations are known for nonidentical particles [Many authors] and to particles with spin [Briceño, 14]
Methodology

- Calculate (for some $P = 2\pi n_P / L$)

$$C_L(E, \vec{P}) \equiv \int_L d^4 x \, e^{iEt - i\vec{P} \cdot \vec{x}} \langle \Omega | T \{ \sigma^\dagger(x) \sigma(0) \} | \Omega \rangle_L$$

- $\sigma \sim \pi^2$, e.g.

$$\sigma(\vec{x}, t) = \int_L d^3 y \, \pi(\vec{x} + \vec{y}, t) \pi(\vec{x} - \vec{y}, t) e^{-i\vec{k} \cdot \vec{y}}$$

- Poles in $C_L$ occur at energies of finite-volume spectrum [Exercise]

Here I have assumed no odd-legged vertices—-not necessary for subsequent arguments, but simplifies diagrams

CM energy is

$$E^* = \sqrt{E^2 - P^2}$$

Boxes indicated summation over finite-volume momenta

Infinite-volume vertices

Full propagators Normalized to unit residue at pole
Key step 1

- Replace loop sums with integrals using Poisson summation formula where integrand is nonsingular

  - Drop exponentially suppressed terms (\(e^{-ML}, e^{-(ML)^2}\), etc.) while keeping power-law dependence

\[
\frac{1}{L^3} \sum_k g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l} \neq \vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{l} \cdot \vec{k}} g(\vec{k})
\]
Key step 1

- Replace loop sums with integrals using Poisson summation formula where integrand is nonsingular
  
  - Drop exponentially suppressed terms \( (e^{-ML}, e^{-(ML)^2}, \text{etc.}) \) while keeping power-law dependence

\[
\frac{1}{L^3} \sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{l \neq 0} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{l} \cdot \vec{k}} g(\vec{k})
\]

- Example of smooth integrand:
Key step 1

- Replace loop sums with integrals using Poisson summation formula where integrand is nonsingular
  - Drop exponentially suppressed terms \( e^{-ML}, e^{-(ML)^2} \), etc., while keeping power-law dependence
    \[
    \frac{1}{L^3} \sum \overrightarrow{k} g(\overrightarrow{k}) = \int \frac{d^3k}{(2\pi)^3} g(\overrightarrow{k}) + \sum_{\overrightarrow{l} \neq \overrightarrow{0}} \int \frac{d^3k}{(2\pi)^3} e^{i\overrightarrow{l}\cdot\overrightarrow{k}} g(\overrightarrow{k})
    \]
    Exp. suppressed if \( g(\overrightarrow{k}) \) is smooth and scale of derivatives of \( g \) is \( \sim 1/M \)

- Example of smooth integrand:
Key step 2

- Use “sum=integral + [sum-integral]” if integrand has pole, e.g. [KSS]

\[
\frac{1}{2} \left( \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(P-k)^2 - m^2 + i\epsilon} g(k)
\]

symmetry factor
Key step 2

- Use “sum=integral + [sum-integral]” if integrand has pole, e.g. [KSS]

\[
\frac{1}{2} \left( \int \frac{d k_0}{2 \pi} \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^4 k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + i\epsilon} g(k)
\]

Symmetry factor

- Example of pole:

\[ P = (E, \vec{P}) \]

Focus on this loop

f is left-hand part of integrand

g is right-hand part of integrand

\[ P-k \]

S. Sharpe, “Multiparticle Scattering”, Lecture 1, 7/18/2023, Bad Honnef Summer School
Key step 2

- Use “sum=integral + [sum-integral]” if integrand has pole, e.g. [KSS]

\[
\frac{1}{2} \left( \int \frac{d k_0}{2\pi} \frac{1}{L^3} \sum_k - \int \frac{d^4 k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + i\epsilon} g(k) \frac{1}{(P-k)^2 - m^2 + i\epsilon} \]

- Example of pole:

- Focus on this loop

\[P = (E, \vec{P})\]

- \(q^*\) is relative momentum of pair on left in CM

- Kinematic function

- \(f & g\) evaluated for ON-SHELL momenta

- Symmetry factor

- Depend only on direction in CM

- \(f\) is left-hand part of integrand

- \(g\) is right-hand part of integrand
Key step 2

- Use “sum=integral + [sum-integral]” where integrand has pole, with [KSS]

\[
\left( \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(P - k)^2 - m^2 + i\epsilon} g(k)
\]

\[
= \int d\Omega_{\vec{q}^*} d\Omega_{\vec{q}^{*\prime}} f^*(\hat{\vec{q}}^*) \mathcal{F} (q^*, q^{*\prime}) g^*(\hat{\vec{q}}^{*\prime})
\]

- Diagrammatically

Functions on left and right can be arbitrary but must be smooth

A new type of “cut”
Variant of key step 2

• For generalization to 3 particles will use a PV prescription instead of $i\epsilon$

\[
\frac{1}{2} \left( \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_k - \int \frac{d^4k}{(2\pi)^4} \right) \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(P - k)^2 - m^2 + i\epsilon} g(k) = \int d\Omega_{q^*} d\Omega_{q^{*'}} f^*(\hat{q}^*) F_{PV} (q^*, q^{*'}) g^*(\hat{q}^{*'})
\]

• Key properties of $F_{PV}$: real and no unitary cusp at threshold

• These properties are important for the derivation of three-particle QC
More detail on key step 2 \cite{HS14}

\[
\frac{1}{2} \left( \int \frac{dk_0}{2\pi} \sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m_j^2 + i\epsilon} \frac{1}{(P - k)^2 - m_j^2 + i\epsilon} g(k)
\]

\[
= \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3} \right) f(\vec{k}^*) g(\vec{k}^*) h(\vec{k}) \frac{1}{2\omega_k 2\omega_{P-k}(E - \omega_k - \omega_{P-k} + i\epsilon)} + \mathcal{O}(e^{-mL})
\]

\[
= \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3} \right) f_{\ell'm'} Y_{\ell'm'}(\vec{k}^*) Y_{\ell'm}^*(\vec{k}^*) h(\vec{k}) \frac{1}{2\omega_k 2\omega_{P-k}(E - \omega_k - \omega_{P-k} + i\epsilon)} g_{\ell'm} + \mathcal{O}(e^{-mL})
\]

\[
\equiv f_{\ell'm'} F_{\ell'm';\ell'm}(E, \vec{P}, L) g_{\ell'm}
\]

More convenient to use this matrix form

\[
F_{\ell'm';\ell'm}(E, \vec{P}, L) = \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3} \right) f_{\ell'm'} Y_{\ell'm'}(\vec{k}^*) Y_{\ell'm}^*(\vec{k}^*) h(\vec{k}) \frac{1}{2\omega_k 2\omega_{P-k}(E - \omega_k - \omega_{P-k} + i\epsilon)}
\]

\[
Y_{\ell'm}(\vec{k}^*) = \sqrt{4\pi} \left( \frac{k^*}{q^*} \right)^\ell Y_{\ell'm}(\hat{k}^*)
\]

\[
q^* = \sqrt{E^*^2/4 - m^2}
\]

\begin{itemize}
  \item Thus power-law volume dependence enters through geometrical function:
\end{itemize}
More detail on key step 2 \[\text{[HS14]}\]

\[
F_{\ell' m'; \ell m}(E, \vec{P}, L) = \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3} \right) \frac{\gamma_{\ell' m}(\vec{k}^*) \gamma_{\ell m}(\vec{k}^*) h(\vec{k})}{2\omega_k 2\omega_{P-k} (E - \omega_k - \omega_{P-k} + i\epsilon)}
\]

• Similarly, the PV version is

\[
F_{PV; \ell' m'; \ell m}(E, \vec{P}, L) = \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \text{PV} \int \frac{d^3k}{(2\pi)^3} \right) \frac{\gamma_{\ell' m}(\vec{k}^*) \gamma_{\ell m}(\vec{k}^*) h(\vec{k})}{2\omega_k 2\omega_{P-k} (E - \omega_k - \omega_{P-k})}
\]

\[= F_{\ell' m'; \ell m}(E, \vec{P}, L) - i\delta_{\ell' \ell} \delta_{m'm} \frac{q^*}{16\pi E^*} \]

\[\propto \left( \frac{2\pi}{L} \right)^{1+\ell+\ell'} \mathcal{Z}_{\ell', m'; \ell, m}(x^2, \vec{P}) \]

“Lüscher zeta function”
Kinematic functions

\[ Z_{4,0} \text{ and } Z_{6,0} \text{ for } P=0 \]

[Luu & Savage, '11]

Divergences occur for values of \( E \) equal to the energy of two free particles in the box

[Exercise: why no divergence at \( x=0 \)?]

Example:
\[ n_1 = -n_2 = (0,0,1) \]
\[ \Rightarrow q^* = 2\pi/L \Rightarrow x=1 \]
Key step 3

- Identify potential singularities using time-ordered PT (i.e. do $k_0$ integrals)
- Example (again assuming only even-legged vertices)
Key step 3

- 2 out of 6 time orderings:

\[
\begin{align*}
\sigma^\dagger & \quad 2' \quad 3' \quad 4' \\
\frac{E - \omega_1' - \omega_2' - \omega_3' - \omega_4'}{E - \omega_1 - \omega_2} & \quad 1 \\
\end{align*}
\]

- On-shell energy:

\[
\omega_j = \sqrt{k_j^2 + M^2}
\]

\[
\sum_{j=1,6} \omega_j
\]
Key step 3

- 2 out of 6 time orderings:

![Diagram showing two time orderings with vertices labeled 1, 2, 3, 4, 5, 6 and expressions for energy denominators involving 
\( E - \omega_1 \), \( E - \omega_2 \), \( E - \omega_1 - \omega_2 \), and \( \sum_{j=1,6} \omega_j \).]

- If restrict \( 0 < E^* < 4M \) (\( M < E^* < 3M \) if have odd-legged vertices) then only 2-particle “cuts” have singularities, and these occur only when both particles go simultaneously on shell.
Combining key steps 1-3

• For each diagram, determine which momenta must be summed, and which can be integrated

• In our example, find:

![Diagram showing momenta to sum and integrate](image)

Can integrate

Must sum momenta passing through box
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated.

- Another example:

  Can replace sum with integral here

But not here
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated.

- Another example:

  ![Diagram](image)

  Can replace sum with integral here

  But not here

- Then repeatedly use \( \text{sum} = \text{integral} + \text{“sum-integral”} \) to simplify.
Summary: the key “move”

\[ \frac{1}{L^3} \sum_k \sigma^+ \sigma = \int_k \sigma^+ \sigma + \sigma^+ \sigma + \text{finite-volume residue} + \text{exp. suppr.} \]

A new type of “cut”
• Apply previous analysis to 2-particle correlator \((0 < E^* < 4M)\)

\[
C_L(E, \vec{P}) = \sigma^{\dagger} \sigma + \sigma^{\dagger} \sigma + \cdots
\]

\[
\sigma^{\dagger} \sigma + \sigma^{\dagger} \sigma + \cdots
\]

These loops are now integrated

• Collect terms into infinite-volume Bethe-Salpeter kernels

\[
C_L(E, \vec{P}) = \sigma^{\dagger} \sigma
\]

\[
\sigma^{\dagger} \sigma \quad \text{iB}
\]

\[
\sigma^{\dagger} \sigma \quad \{ + \cdots \}
\]

B-S kernel: 2-particle irreducible in the s-channel, i.e. no 2-particle cuts
Apply previous analysis to 2-particle correlator

Collect terms into infinite-volume Bethe-Salpeter kernels

\[ C_L(E, \vec{P}) = \sigma^\dagger \begin{array}{c}
\end{array} \sigma \]

\[ + \sigma^\dagger \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \cdots \]

Leading to

\[ C_L(E, \vec{P}) = \sigma^\dagger \begin{array}{c}
\end{array} \sigma + \sigma^\dagger \begin{array}{c}
\end{array} iB \begin{array}{c}
\end{array} + \cdots \]

\[ + \sigma^\dagger \begin{array}{c}
\end{array} iB iB iB \begin{array}{c}
\end{array} \]

Similar structure to NREFT bubble-chain (e.g. in two nucleon system)
Next use sum identity

\[ C_L(E, \bar{P}) = \sigma^\dagger \sigma + \sigma^\dagger iB \sigma + \sigma^\dagger \sigma \]

And regroup according to number of “F cuts”

\[ C_L(E, \bar{P}) = C_\infty(E, \bar{P}) \]

zero F cuts

one F cut

matrix elements:
• Next use sum identity

\[ C_L(E, \vec{P}) = \sigma^\dagger \sigma + \sigma^\dagger iB \sigma + \cdots \]

\[ C(L, \vec{P}) = C_{\infty}(E, \vec{P}) + \overset{F}{A} \overset{A'}{A'} \]

\[ + \overset{F}{A} \left\{ iB + iB + iB + \cdots \right\} \overset{F}{A'} + \cdots \]

• And keep regrouping according to number of “F cuts”

\[ iM \]

two F cuts

the infinite-volume, on-shell 2→2 scattering amplitude
• Next use sum identity

\[ C_L(E, \vec{P}) = \sigma^\dagger \sigma + \sigma^\dagger iB F \sigma + \cdots \]

• Alternate form if use PV-tilde prescription:

\[ C_L(E, \vec{P}) = C_{\infty}^{PV}(E, \vec{P}) + \frac{F_{\overline{PV}}}{F_{PV}} \]

\[ \left\{ A_{PV} \right\}_{F_{PV}} + \cdots \]

\[ iK_{PV} \]

the infinite-volume, on-shell 2→2 K-matrix
Final result:

\[ C_L(E, \vec{P}) = C_\infty(E, \vec{P}) \]

\[ + \begin{array}{c}
\begin{array}{c}
A \\
F
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
A' \\
F
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
A \\
F
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
i \mathcal{M} \\
F
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
A' \\
F
\end{array}
\end{array} + \cdots
\end{array} \]

\[ C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' i F [i \mathcal{M}_{2 \to 2} i F]^n A \]

- Correlator is expressed in terms of infinite-volume, physical quantities and kinematic functions encoding the finite-volume effects.
- Final result:

\[ C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A' + iM + iM + A' + \cdots \]

- \[ C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A'iF[iM_2iF]^nA \]

- \[ C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'iF \frac{1}{1 + M_2F}A \]

\[ \Rightarrow \text{Poles in } C_L \text{ occur when} \]

\[ \det \left[ F(E, \vec{P}, L)^{-1} + M_2(E^*) \right] = 0 \]
2-particle quantization condition

- At fixed $L$ & $P$, the finite-volume spectrum $E_1, E_2, \ldots$ is given by solutions of

$$\det \left[ F(E, \vec{P}, L)^{-1} + M_2(E^*) \right] = 0$$

For $P = 0$ this equivalent to original result by [Lüscher]

Generalization to moving frame first obtained using RQM by [Rummukainen & Gottlieb]
2-particle quantization condition

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For $P = 0$ this equivalent to original result by [Lüscher]

Generalization to moving frame first obtained using RQM by [Rummukainen & Gottlieb]

- $F$ and $M_2$ are matrices in $\ell, m$ space:
  - $M_2$ is diagonal; while $F$ is off-diagonal, since the box violates rotation symmetry

- QC separates finite-volume ($F$) and infinite-volume quantities ($M_2$)

- If $M_2$ vanishes, solutions are free two-particle energies due to poles in $F$

- Each spectral energy gives information about all partial waves of $M_2(E^*)$
2-particle quantization condition

- Equivalent form, obtained by using PV prescription throughout derivation, is

\[
\det \left[ F_{PV}(E, \vec{P}, L)^{-1} + \mathcal{K}_2(E^*) \right] = 0
\]

- I prefer this as both \( \mathcal{K}_2, F_{PV} \) are real

- \( \mathcal{K}_2 \) contains the same information as \( M_2 \), but is real and smooth (no threshold branch points)

- These differences are irrelevant for the two-particle QC—the two QCs are identical—but turn out to be important for the three-particle QC

- Beware when reading the literature, as each collaboration uses different notation for what I call \( F \): sometimes \( B \) (box function), sometimes \( M \)
Summary of Lecture 1
Summary of Lecture 1

- Resonances are ubiquitous and mysterious in QCD
  - Usually decay to more than 2 particles
- Key issue is relating finite-volume spectrum to scattering amplitudes (or K matrices)
  - QC2 provides a very general, model-independent tool to do so
Thank you!
Questions?
Backup Slides
Examples of resonances

- Example 3: scalar, isoscalars—possible p-wave $q\bar{q}$ states

**$f_0(500)$**

$I^G(J^{PC}) = 0^+(0++)$

- Mass (T-Matrix Pole $\sqrt{s}$) = (400–550)–$i$(200–350) MeV
- Mass (Breit-Wigner) = (400–550) MeV
- Full width (Breit-Wigner) = (400–700) MeV

<table>
<thead>
<tr>
<th>$f_0(500)$ DECAY MODES</th>
<th>Fraction ($\Gamma_i/\Gamma$)</th>
<th>$p$ (MeV/c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi\pi$</td>
<td>seen</td>
<td>–</td>
</tr>
<tr>
<td>$\gamma\gamma$</td>
<td>seen</td>
<td>–</td>
</tr>
</tbody>
</table>

**$f_0(980)$**

$I^G(J^{PC}) = 0^+(0++)$

- Mass $m = 990 \pm 20$ MeV
- Full width $\Gamma = 10$ to 100 MeV

<table>
<thead>
<tr>
<th>$f_0(980)$ DECAY MODES</th>
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<tbody>
<tr>
<td>$\pi\pi$</td>
<td>seen</td>
<td>476</td>
</tr>
<tr>
<td>$K\bar{K}$</td>
<td>seen</td>
<td>36</td>
</tr>
<tr>
<td>$\gamma\gamma$</td>
<td>seen</td>
<td>495</td>
</tr>
</tbody>
</table>
Examples of resonances

- Example 3: scalar, isoscalars—possible p-wave $q\bar{q}$ states

**$f_0(500)$**

$\frac{\Gamma}{G(J^{PC}) = 0^+(0^+)}$

- Mass (T-Matrix Pole $\sqrt{s}$) = (400–550)–i(200–350) MeV
- Mass (Breit-Wigner) = (400–550) MeV
- Full width (Breit-Wigner) = (400–700) MeV

<table>
<thead>
<tr>
<th>$f_0(500)$ DECAY MODES</th>
<th>Fraction ($\Gamma_i/\Gamma$)</th>
<th>$p$ (MeV/c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi\pi$</td>
<td>seen</td>
<td>–</td>
</tr>
<tr>
<td>$\gamma\gamma$</td>
<td>seen</td>
<td>–</td>
</tr>
</tbody>
</table>

**$f_0(980)$**

$\frac{\Gamma}{G(J^{PC}) = 0^+(0^+)}$

- Mass $m = 990 \pm 20$ MeV
- Full width $\Gamma = 10$ to 100 MeV

<table>
<thead>
<tr>
<th>$f_0(980)$ DECAY MODES</th>
<th>Fraction ($\Gamma_i/\Gamma$)</th>
<th>$p$ (MeV/c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi\pi$</td>
<td>seen</td>
<td>476</td>
</tr>
<tr>
<td>$\bar{K}K$</td>
<td>seen</td>
<td>36</td>
</tr>
<tr>
<td>$\gamma\gamma$</td>
<td>seen</td>
<td>495</td>
</tr>
</tbody>
</table>

- Large uncertainties because analyses are difficult

extract from charged pion beams on nucleon targets

\[ s \left\{ \begin{array}{c} \pi \\ p \end{array} \right\} m_{\pi\pi}, \theta_{\pi\pi} \quad \rightarrow \quad \pi\pi \text{ partial-waves} - \text{project } P_t(\cos \theta) \]

on-shell $\pi$ exchange

- extrapolate to $t = -m_{\pi}^2$

[Figure from HMI slides of Jo Dudek]
Examples of resonances

- Example 3: scalar, isoscalars—possible p-wave $q\bar{q}$ states
- Extract the phase shift from complicated amplitude analysis

**isospin = 0**

![Graph showing examples of resonances with $f_0(500)$ and $f_0(980)$]
Aside on inelasticity

- Phase shift in $I=J=1$ $\pi\pi$ channel

**isospin=1**

![Graph showing phase shifts and mass spectra](image)

- $\rho$ phase shift in $I=J=1$ $\pi\pi$ channel

The graph shows $\delta_1$ and $\eta_1$ as functions of $m_{\pi\pi}$.

$$1 - |\eta|^2$$

gives probability for scattering into any final state other than $\pi\pi$.

E.g.: KK-bar, $\eta\eta$, $4\pi$

Becomes nonzero above 1 GeV

*Hyams 1973*
Examples of resonances

- Example 4: Roper (excited nucleon)

\[ N(1440) \, 1/2^+ \]

\[ \Gamma(J^P) = \frac{1}{2}(1^+) \]

- Re(pole position) = 1360 to 1380 (≈ 1370) MeV
- \(-2\text{Im}(\text{pole position}) = 160\) to 190 (≈ 175) MeV
- Breit-Wigner mass = 1410 to 1470 (≈ 1440) MeV
- Breit-Wigner full width = 250 to 450 (≈ 350) MeV

<table>
<thead>
<tr>
<th>[ N(1440) ] DECAY MODES</th>
<th>Fraction (\Gamma_i/\Gamma)</th>
<th>(p) (MeV/c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N\pi )</td>
<td>55–75 %</td>
<td>398</td>
</tr>
<tr>
<td>( N\eta )</td>
<td>&lt;1 %</td>
<td>(\dagger)</td>
</tr>
<tr>
<td>( N\pi\pi )</td>
<td>17–50 %</td>
<td>347</td>
</tr>
<tr>
<td>( \Delta(1232)\pi, P)-wave</td>
<td>6–27 %</td>
<td>147</td>
</tr>
<tr>
<td>( N\sigma )</td>
<td>11–23 %</td>
<td>–</td>
</tr>
<tr>
<td>( p\gamma), helicity=1/2</td>
<td>0.035–0.048 %</td>
<td>414</td>
</tr>
<tr>
<td>( n\gamma), helicity=1/2</td>
<td>0.02–0.04 %</td>
<td>413</td>
</tr>
</tbody>
</table>

- Extracted from amplitude analysis of \(\pi\)\(N\) scattering
- Lighter than expected from quark model for a radial excitation
Examples of resonances

- **Example 5: $Z_c(3900)$**—a nonstandard meson

\[ Z_c(3900) \]

\[ I^G(J^{PC}) = 1^+(1+--) \]

Mass \( m = 3887.2 \pm 2.3 \text{ MeV} \) \( (S = 1.6) \)

Full width \( \Gamma = 28.2 \pm 2.6 \text{ MeV} \)

\[ \text{[PDG]} \]

**$Z_c(3900)$ DECAY MODES**

<table>
<thead>
<tr>
<th>Decay Mode</th>
<th>Fraction ( \Gamma_i/\Gamma )</th>
<th>( \rho ) (MeV/c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J/\psi \pi )</td>
<td>seen</td>
<td>699</td>
</tr>
<tr>
<td>( h_c \pi )</td>
<td>not seen</td>
<td>318</td>
</tr>
<tr>
<td>( \eta_c \pi^+ \pi^- )</td>
<td>not seen</td>
<td>759</td>
</tr>
<tr>
<td>( (D \bar{D}^*) )</td>
<td>seen</td>
<td>-</td>
</tr>
<tr>
<td>( D^0 D^{*-} + \text{c.c.} )</td>
<td>seen</td>
<td>153</td>
</tr>
<tr>
<td>( D^- D^{*-0} + \text{c.c.} )</td>
<td>seen</td>
<td>144</td>
</tr>
<tr>
<td>( \omega \pi^\pm )</td>
<td>not seen</td>
<td>1862</td>
</tr>
<tr>
<td>( J/\psi\eta )</td>
<td>not seen</td>
<td>510</td>
</tr>
<tr>
<td>( D^+ D^{*-} + \text{c.c.} )</td>
<td>seen</td>
<td>-</td>
</tr>
<tr>
<td>( D^0\bar{D}^{*-0} + \text{c.c.} )</td>
<td>seen</td>
<td>-</td>
</tr>
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\( \rho \eta_c \) (now seen at 4.2\sigma significance, \[\text{[BESIII]}\])
Examples of resonances

- Example 5: $Z_c(3900)$—a nonstandard meson

$Z_c(3900)$

$\Gamma^G(J^{PC}) = 1^+(1^-)$

Mass $m = 3887.2 \pm 2.3$ MeV \quad (S = 1.6)

Full width $\Gamma = 28.2 \pm 2.6$ MeV

PDG

$\pi^+\pi^- \chi_c(1S)$ seen

$\pi^0\eta_c$ not seen

$\eta_c\pi^0$ not seen

$J/\psi\pi^+$ seen

$J/\psi\eta$ not seen

$D^+D^{*+} + c.c.$ seen

$D^-D^{*0} + c.c.$ seen

$\omega\pi^+$ not seen

$D^+D^{*-} + c.c.$ seen

$D^0D^{*-0} + c.c.$ seen

$\rho\eta_c$ (now seen at 4.2$\sigma$ significance, [BESIII])

Observed by BESIII, Belle, CLEO-c in 2013

$e^+e^- \rightarrow \pi^\pm Z_c^\mp$

Significance $>8\sigma$

[BESIII, talk at Lattice 2019 by C.Yuan]
Examples of resonances

- Example 5: $Z_c(3900)$—a nonstandard meson

$Z_c(3900)$

$$I^G(J^{PC}) = 1^+(1^-)$$

Mass $m = 3887.2 \pm 2.3$ MeV ($S = 1.6$)

Full width $\Gamma = 28.2 \pm 2.6$ MeV

$Z_c(3900)$ DECAY MODES

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</tr>
<tr>
<td>$(D D^*)^+$</td>
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</table>

$\rho \eta_c$ (now seen at 4.2$\sigma$ significance, [BESIII])

Observed by BESIII, Belle, CLEO-c in 2013

$$e^+e^- \rightarrow \pi^\pm Z_c^\mp$$

Significance $>8\sigma$

![Graph showing decay modes and significance](image)

$Z_c^+$ quark composition: $c\bar{c}u\bar{d}$

[Citation: M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018) and 2019 update]

S. Sharpe, “Multiparticle Scattering”, Lecture 1, 7/18/2023, Bad Honnef Summer School
Examples of resonances

- Example 5: \(Z_c(3900)\)—a nonstandard meson

\[ e^+e^- \rightarrow Y(4260) \rightarrow \pi \rightarrow \bar{D}^*D^*(4014) \]

\[ Z_c(3900) \rightarrow \bar{D}D^*(3877) \]

\[ \rho\eta_c(3759) \rightarrow \pi J/\psi(3236) \]

\[ 1^{--} \rightarrow 1^{+-} \quad (J^{PC}) \]

- \(Z_c^+\) quark composition: \(c\bar{c}u\bar{d}\)

- Possible interpretations:
  - Tetraquark
  - Molecule

- Threshold enhancement—supported by HALQCD study [1602.03465]

\[ \text{[Ikeda et al., 1602.03465]} \]
G parity

- G parity will come up occasionally in the remaining lectures, so here is a reminder.
  - \( G = C e^{i\pi I_y} \) is an exact symmetry of isosymmetric QCD, and an approximate symmetry of real QCD.
  - Eigenstates of G: \( \pi(-1), \eta(+1), \rho(+1), \omega(-1), \ldots \)
- Relevance for what follows:
  - Restricts decay channels, e.g. \( \rho \to \pi\pi, \omega \to \pi\pi\pi \) (\( \eta \to \pi\pi \) forbidden by parity).
  - No interactions involving an odd number of pions, e.g.
    \[
    \pi\pi \leftrightarrow 4\pi, \quad \pi\pi \leftrightarrow 3\pi
    \]