Outline

Lecture 1
- Motivation/Background/Overview
- Deriving the two-particle quantization condition (QC2)

Lecture 2
- Applying the QC2, in brief
- Deriving the three-particle quantization condition for identical scalars (QC3)

Lecture 3
- Status of three-particle formalism
- Applications of QC3
- Outlook
Main references for this lecture

[Full list of references at end of lecture 3]

- Briceño, Dudek & Young, “Scattering processes & resonances from LQCD,” 1706.06223, RMP 18
- Hansen & SS, “LQCD & three-particle decays of resonances,” 1901.00483, ARNPS 20
- Lectures by Dudek, Hansen & Meyer at HMI Institute on “Scattering from the lattice: applications to phenomenology and beyond,” May 2018, https://indico.cern.ch/event/690702/
- Kim, Sachrajda & SS, hep-lat/0507006, NPB 2015 (direct derivation in QFT of QC2)
Outline for Lecture 1

- Background: hadronic resonances
- Further motivation for studying multiparticle states
- Some scattering basics
- Derivation of QC2 = “Lüscher quantization condition”
Background: hadronic resonances
Stable hadrons in isosymmetric QCD

- QCD with $m_u = m_d$, and no EM (or weak) interactions
  - Theory studied in majority of LQCD simulations
  - Differs from real world at ~1% level
Stable hadrons in isosymmetric QCD

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Stable hadrons in isosymmetric QCD

- Relatively short list has been the focus of most LQCD calculations to date
Stable hadrons in isosymmetric QCD

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[Image: A graph showing the mass spectrum of hadrons, with labels for different hadron types such as D, B, B_s, B_s^*, B_c, B_c^*, and others. The graph includes data points and error bars.]
Stable hadrons in isosymmetric QCD

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\[ \pi(q\bar{q}), K(q\bar{s}), \eta(q\bar{q}) \]
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\[ M_{H} \]
\[ M_{B}-4000 \]}
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$$\pi(q\bar{q}), K(q\bar{s}), \eta(q\bar{q}) \quad D(c\bar{q}), D_s(c\bar{s}), B(b\bar{q}), B^*(q\bar{b}), B_s(s\bar{b}), B_s^*(s\bar{b}), B_c(c\bar{b})$$

$$N(qqq), \Lambda(qqs), \Sigma(qqs), \Xi(qss), \Omega(sss) \quad \Lambda_c(qqc), ..., \Xi_{cc}(qcc), ..., \Lambda_b(qqb), ...$$

![Stable hadrons in isosymmetric QCD](image)
Stable hadrons in isosymmetric QCD

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  \[ \pi(q\bar{q}), K(q\bar{s}), \eta(q\bar{q}), \quad D(c\bar{q}), D_s(c\bar{s}), B(b\bar{q}), B^*(q\bar{b}), B_s(s\bar{b}), B^*_s(s\bar{b}), B_c(c\bar{b}) \]
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![Diagram of hadron masses](Kronfeld, 1203.1204)

S. Sharpe, “Multiparticle Scattering”, Lecture 1, 7/18/2023, Bad Honnef Summer School
Plethora of resonances

- Most hadrons are resonances!

### pdg meson listings

<table>
<thead>
<tr>
<th>LIGHT UNFLAVORED (S = C = B = 0)</th>
<th>STRANGE (S = ±1, C = B = 0)</th>
<th>CHARMED, STRANGE (C = S = ±1)</th>
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<th>pdg meson listings</th>
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<tbody>
<tr>
<td>( \pi^+ )</td>
<td>( \rho_3(1690)^+ )</td>
<td>( D_s^-(2010)^- )</td>
<td>( \eta_c(1S) )</td>
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<td>( \pi^0 )</td>
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<td>( D_s^+(2010)^+ )</td>
<td>( J/\psi(1S) )</td>
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<tr>
<td>( \eta )</td>
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<td>( D_s^0(2008)^0 )</td>
<td>( X_{c0}(1P) )</td>
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<td>( D_1(2420)^+ )</td>
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<td>( X_{c2}(1P) )</td>
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<td>( \phi(980)^0 )</td>
<td>( f_1(1250)^0, f_2(1270)^0 )</td>
<td>( D_4(2420)^0 )</td>
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<td>( \psi(3770) )</td>
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<td>( \rho(2450)^+ )</td>
<td>( D_{17}(2420)^0 )</td>
<td>( X(4250)^0 )</td>
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</tbody>
</table>

Stable under strong ints

S. Sharpe, “Multiparticle Scattering”, Lecture 1, 7/18/2023, Bad Honnef Summer School
Cornucopia of exotics

62 new hadrons at the LHC

[I. Danilkin, talk at INT workshop, March 23]

+ data from Babar, Belle, COMPASS, …
Examples of resonances

• Most hadrons are resonances!

• Very short lived, with decays into 2, 3, … stable hadrons
Examples of resonances

- Most hadrons are resonances!
- Very short lived, with decays into 2, 3, … stable hadrons
  - Example 1: single-channel decay of s-wave spin-triplet q q-bar state:
    \[ I^G J^{PC} = 1^+1^- : \rho \rightarrow \pi\pi, \ M_\rho \approx 775\text{MeV}, \ \Gamma_\rho \approx 150\text{MeV} \ (\tau = 4 \times 10^{-23}\text{s}) \]
- Many production mechanisms, e.g. \( \tau^- \rightarrow \pi^-\pi^0\nu_\tau \)

[CLEO collab., hep-ex/9910046]
\[ e^+e^- \rightarrow \tau^+\tau^- + X \]

\( \rho \) is produced by the vector part of the weak current \( \bar{u}\gamma^\mu d \)

Fitting the spectrum involves models & uncertainties
Examples of resonances

- Example 2: multi-channel decay of p-wave $q\bar{q}$ state:

  **pdg summary entry**

  $a_2(1320)$

  $J^G(J^{PC}) = 1^-(2^{++})$

  Mass $m = 1318.3^{+0.5}_{-0.6}$ MeV
  Full width $\Gamma = 107 \pm 5$ MeV

<table>
<thead>
<tr>
<th>$a_2(1320)$ DECAY MODES</th>
<th>Fraction ($\Gamma_f/\Gamma$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3\pi$</td>
<td>(70.1 ± 2.7 ) %</td>
</tr>
<tr>
<td>$\eta\pi$</td>
<td>(14.5 ± 1.2 ) %</td>
</tr>
<tr>
<td>$\omega\pi\pi$</td>
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</tr>
<tr>
<td>$KK$</td>
<td>( 4.9 ± 0.8 ) %</td>
</tr>
<tr>
<td>$\eta'(958)\pi$</td>
<td>( 5.5 ± 0.9 ) x $10^{-3}$</td>
</tr>
<tr>
<td>$\pi^\pm\gamma$</td>
<td>( 2.91±0.27) x $10^{-3}$</td>
</tr>
<tr>
<td>$\gamma\gamma$</td>
<td>( 9.4 ± 0.7 ) x $10^{-6}$</td>
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Examples of resonances

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  same ‘bump’ appears in multiple different processes ...

  ... due to same $a_2$ resonance

  $\pi$ Pb $\rightarrow$ $\pi$ $\rho$ Pb
  $\gamma\gamma$ $\rightarrow$ $\pi\eta$
  $\pi p$ $\rightarrow$ $K\bar{K} p$

  [Figures from HMI slides of Jo Dudek]
Lessons

- Extracting resonance parameters from experiment is indirect & challenging
  - Resonance is defined as a pole in a scattering amplitude—not directly accessible

- Typical resonances have multiple decay channels, each involving 2 or 3 (or more) particles

- Quark model (or other models) fails to explain presence or properties of an increasing number of resonances
  - X, Y, Z resonances, glueballs, hybrids, tetraquarks, pentaquark, …

- Resonances are a largely unexplored frontier in our attempts to understand hadronic physics (i.e. the properties of a strongly-coupled QFT) from first principles
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A major challenge for LQCD!
How can LQCD help?

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  - Resonance is defined as a pole in a scattering amplitude—not directly accessible
  - Methods for indirectly accessing scattering amplitudes must be developed (the main topic of these lectures)
  - LQCD has advantage of being able to turn off electroweak interactions
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- Typical resonances have multiple decay channels, each involving 2 or 3 (or more) particles
  - QCD calculations must deal with multiple channels of multiparticle states

- Quark model fails to explain presence or properties of an increasing number of resonances
  - X, Y, Z resonances, glueballs, hybrids, tetraquarks, pentaquark, …
  - LQCD calculations must use large bases of operators to allow understanding of structure of hadrons—any input is useful!
  - Varying the quark masses can provide additional useful information
Personal note

- As a grad student I used the MIT bag model to predict the masses of “hybrid” mesons—resonances of the form: quark + antiquark + “constituent gluon”
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**HYBRIDS: MIXED STATES OF QUARKS AND GLUONS**

Michael CHANOWITZ and Stephen SHARPE

*Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720, USA*
Personal note

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submitted for publication

MEIKTONS: MIXED STATES OF QUARKS AND GLUONS

Michael Chanowitz and Stephen Sharpe

August 1982

RECEIVED
LAWRENCE
BERKELEY LABORATORY
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  Michael Chanowitz and Stephen Sharpe

  August 1982

  ReceiVed

  Lawrence Berkeley Laboratory

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    Noise!

- There are now increasingly sophisticated calculations of hybrid meson properties, and these will eventually be based on the formalism I will describe in these lectures
Fundamental issue:

- LQCD simulations are done in finite volumes, with imaginary time
- Experiments are done in infinite volume in real time

How do we connect?
Fundamental Issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?

\[ E_0(L) \]

\[ E_1(L) \]

\[ E_2(L) \]

Discrete energy spectrum

\[ iM_{n \rightarrow m} \]

Scattering amplitudes
Fundamental Issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box.
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?

N.B. This is a finite volume QFT problem (can ignore lattice spacing).
Further motivations for studying multiparticle states
Motivations

- Calculating electroweak decay and transition amplitudes for processes involving multiple particles
- Determining NN and NNN interactions as input for predicting properties of nuclei and nuclear matter
  - NNN interactions needed as input for EFT treatments of large nuclei, and for the neutron-star equation of state
  - $\pi\pi\pi, \pi K \bar{K}, \ldots$ interactions needed as input to study pion & kaon condensation
Electroweak decays
e.g. $K \rightarrow \pi \pi \pi \pi$ decay amplitudes

- Does the SM reproduce the observed CP violation in $K \rightarrow \pi \pi \pi \pi$ decays?
- Formalism to study this now exists [Hansen, Romero-López, SRS, 2021]
A more distant motivation

Observation of $CP$ violation in charm decays

LHCb collaboration†

Abstract

A search for charge-parity ($CP$) violation in $D^0 \rightarrow K^-K^+$ and $D^0 \rightarrow \pi^-\pi^+$ decays is reported, using $pp$ collision data corresponding to an integrated luminosity of 6 fb$^{-1}$ collected at a center-of-mass energy of 13 TeV with the LHCb detector. The flavor of the charm meson is inferred from the charge of the pion in $D^*(2010)^+ \rightarrow D^0\pi^+$ decays or from the charge of the muon in $\bar{B} \rightarrow D^0\mu^-\bar{\nu}_\mu X$ decays. The difference between the $CP$ asymmetries in $D^0 \rightarrow K^-K^+$ and $D^0 \rightarrow \pi^-\pi^+$ decays is measured to be $\Delta A_{CP} = [-18.2 \pm 3.2\,(\text{stat.}) \pm 0.9\,(\text{syst.})] \times 10^{-4}$ for $\pi$-tagged and $\Delta A_{CP} = [-9 \pm 8\,(\text{stat.}) \pm 5\,(\text{syst.})] \times 10^{-4}$ for $\mu$-tagged $D^0$ mesons. Combining these with previous LHCb results leads to

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4},$$

where the uncertainty includes both statistical and systematic contributions. The measured value differs from zero by more than five standard deviations. This is the first observation of $CP$ violation in the decay of charm hadrons.

5.3σ effect

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A more distant motivation

- Calculating CP-violation in $D \to \pi\pi, K\bar{K}$ in the Standard Model
- Finite-volume state is a mix of $2\pi, K\bar{K}, \eta\eta, 4\pi, 6\pi, \ldots$
- Need 4 (or more) particles in the box!
Scattering basics
(infinite-volume)
\[ M_2 \]

- Recall some details of the simplest scattering process: \( 2 \rightarrow 2 \)
- We will mainly discuss spinless particles in these lectures, e.g. pions
- We will consider both identical particles, e.g. \( \pi^+ \pi^+ \), and nonidentical, e.g. \( \pi^+ K^+ \)
- Scattering amplitude related to the S matrix

\[
S = 1 + iT \quad \langle f | T | i \rangle = (2\pi)^4 \delta^4(P_f - P_i)M_{fi}
\]

- In a given theory, can calculate in perturbation theory (PT), e.g. in \( \Phi^4 \) theory

\[
iM_2 = \quad + \quad + \quad + \quad + \quad + \ldots
\]
M_2

- Recall some details of the simplest scattering process: 2 → 2
  - We will mainly discuss spinless particles in these lectures, e.g. pions
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- Scattering amplitude related to the S matrix

\[
S = 1 + iT \\
\langle f \mid T \mid i \rangle = (2\pi)^4 \delta^4(P_f - P_i) \mathcal{M}_{fi}
\]

- In a given theory, can calculate in perturbation theory (PT), e.g. in \( \Phi^4 \) theory

\[
i\mathcal{M}_2 = \begin{array}{cccc}
\hline
\hline
& \hline
\end{array} + \begin{array}{c}
\hline
\hline
& \hline
\end{array} + \begin{array}{c}
\hline
\hline
& \hline
\end{array} + \begin{array}{c}
\hline
\hline
& \hline
\end{array} + \begin{array}{c}
\hline
\hline
& \hline
\end{array} + \ldots
\]

- We will not assume a particular theory, e.g. ChPT or \( \Phi^4 \); instead we use a generic relativistic QFT, with all possible vertices, and work to all orders in PT
Properties of $M_2$

- Poincaré invariance $\Rightarrow M_2$ depends on the two independent Mandelstam variables

\[ M_2 = M_2(s, t), \quad s = (p_1 + p_2)^2, \quad t = (p_1 - p'_1)^2, \quad u = (p_1 - p'_2)^2 = 4m^2 - s - t \]
Properties of $M_2$

- Poincaré invariance $\Rightarrow M_2$ depends on the two independent Mandelstam variables

$$M_2 = M_2(s, t), \quad s = (p_1 + p_2)^2, \quad t = (p_1 - p_1')^2, \quad u = (p_1 - p_2')^2 = 4m^2 - s - t$$

- Partial wave decomposition in CM frame

$$s = E^{*2} = 4(q^2 + m^2), \quad t = -2q^2(1 - \cos \theta)$$

$$M_2(s, t) = \sum_{\ell} (2\ell + 1) M_2^{(\ell)}(s) P_\ell(\cos \theta)$$

Only even values of $\ell$ contribute for identical particles
Properties of $\mathcal{M}_2$

- Unitarity—holds in each partial wave (results here for identical particles)

\[ S^\dagger S = 1 \Rightarrow \text{Im}(\mathcal{M}_2^{(\ell)}) = \mathcal{M}_2^{(\ell)^*} \rho \mathcal{M}_2^{(\ell)} = \rho |\mathcal{M}_2^{(\ell)}|^2, \quad \rho = \frac{q}{16\pi E^*} \text{ (phase space)} \]
Properties of $\mathcal{M}_2$

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- Solve unitarity constraint in terms of an arbitrary, real $K$ matrix

\[ \text{Im} \left[ \frac{1}{\mathcal{M}_2^{(\ell)}} \right] = -\rho \Rightarrow \frac{1}{\mathcal{M}_2^{(\ell)}} \equiv \frac{1}{\mathcal{K}_2^{(\ell)}} - i\rho \Rightarrow \mathcal{M}_2^{(\ell)} = \mathcal{K}_2^{(\ell)} \frac{1}{1 - i\rho \mathcal{K}_2^{(\ell)}} \]
Properties of $\mathcal{M}_2$

- **Unitarity**—holds in each partial wave (results here for identical particles)

\[ S^\dagger S = 1 \Rightarrow \text{Im}(\mathcal{M}_2^{(\ell)}) = \mathcal{M}_2^{(\ell)*} \rho \mathcal{M}_2^{(\ell)} = \rho |\mathcal{M}_2^{(\ell)}|^2, \quad \rho = \frac{q}{16\pi E^*} \text{ (phase space)} \]

- Solve unitarity constraint in terms of an arbitrary, real $K$ matrix

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- Parametrize $K_2$ using (real) phase shifts

\[ \mathcal{K}_2^{(\ell)} \equiv \frac{1}{\rho} \tan \delta_\ell = \frac{16\pi E^*}{q \cot \delta_\ell} \Rightarrow \mathcal{M}_2 = \frac{1}{\rho} e^{i\delta} \sin \delta_\ell \]
Properties of $M_2$

- **Threshold behavior (QM)**

$$\delta_\ell \sim q^{1+2\ell} \left[ 1 + \mathcal{O}(q^2) \right] \Rightarrow \mathcal{H}_2^{(\ell)} \sim q^{2\ell} \left[ 1 + \mathcal{O}(q^2) \right]$$

- **Effective range expansion (ERE)**

$$\frac{1}{\mathcal{K}_2^{(0)}} = \frac{1}{16\pi E_2} \left[ -\frac{1}{a_0} + r_0 \frac{q^2}{2} + P_0 r_0^3 q^4 + \ldots \right], \quad \frac{1}{\mathcal{K}_2^{(2)}} = -\frac{1}{16\pi E_2} \frac{1}{q^4} \frac{1}{a_2^5} + \ldots$$

- $a_0$ is s-wave scattering length, related to threshold scattering amplitude

$$M_2(q = 0) = \mathcal{H}_2(q = 0) = 32\pi ma_0$$

- $a_0$ is the intercept of the s-wave radial QM wavefunction at $q=0$ on the $r$ axis, and can have any value: $-\infty < a_0 < +\infty$

- $r_0$ is the effective range (typically of order the range of the interaction), $P_0$ is the “shape parameter” (typically of order unity), and $a_2$ is the d-wave scattering length
Properties of $M_2$

- Analytic structure: branch cut along real $s$ axis above threshold, arising from unitarity

\[ M_2^{(\ell)} = \mathcal{K}_2^{(\ell)} + \mathcal{K}_2^{(\ell)} i \rho \mathcal{K}_2^{(\ell)} + \ldots, \quad \rho = \frac{\sqrt{s - 4m^2}}{32\pi \sqrt{s}} \]

- $M_2$ has two Riemann sheets, the top one being called the “physical sheet”
- $K_2$ does not have the right-hand cut; it is analytic at threshold
Properties of $M_2$

- t- and u-channel exchanges lead to the “left-hand cut”

$$M_2^{(\ell)} = \mathcal{H}_2^{(\ell)} + \mathcal{H}_2^{(\ell)}i\rho \mathcal{H}_2^{(\ell)} + \ldots,$$

$$\rho = \frac{\sqrt{s - 4m^2}}{32\pi \sqrt{s}}$$

- For nondegenerate systems, the LH cut can lie close to threshold (e.g. $\Lambda\Lambda$ has LH cut due to pion exchange)

- LH cut invalidates standard QC2 and QC3 derivations
Bound states

• Bound states lead to poles in $M_2$ on physical sheet

\[ \frac{1}{M_2^{(\ell)}} \equiv \frac{1}{K_2^{(\ell)}} - i\rho \quad \text{where} \quad -i\rho = \frac{|q|}{16\pi E^*} \quad \text{with} \quad E_{BS}^* = 4(m^2 - |q|^2) \]

• $K_2$ does not have a corresponding pole since $\rho$ is nonzero below threshold

• Bound state condition is thus

\[ \frac{1}{M_2^{(\ell)}} = \frac{1}{16\pi E^*}(q \cot \delta_\ell + |q|) = 0 \]

• If keep only the scattering length in the ERE, find bound state for $a_0 > 0$

\[ q \cot \delta_0 = -\frac{1}{a_0} \quad \Rightarrow \quad |q| = \frac{1}{a_0} \quad \Rightarrow \quad E_{BS}^* = 2\sqrt{m^2 - 1/a_0^2} \]

• Bound state at threshold in unitary limit $a_0 \to \infty$
Resonances

- Resonances lead to poles in $M_2$ below the real axis on the second (unphysical) sheet
  - Cannot have poles on physical sheet aside from bound states due to causality
  - To display sheets it is better to use single-sheeted variable $q$

- Resonance with width $\Gamma = 1/\tau$ and mass $M$ has pole at

$$E^* = M - i\Gamma/2 \implies s = M^2 + (\Gamma/2)^2 - iM\Gamma$$

- Leads to a bump in scattering cross-section $\propto |M_2|^2$ as we saw earlier

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Resonances

- Narrow s-wave resonances well described by Breit-Wigner form

\[
\tan \delta_{\text{BW}} = \frac{E^* \Gamma}{M^2 - E^{*2}} \Rightarrow \mathcal{M}_2 \propto \frac{1}{M^2 - E^{*2} - iE^*\Gamma}
\]

- As \( E^* \) passes through \( M \) from below:
  - Phase shift rises rapidly through 90°
  - \( K_2 \sim \tan \delta \) has a pole at \( M \) (i.e. on the real axis)

- Pole in \( K_2 \) does not have any direct physical significance, but does play a role in the finite-volume analysis to follow
Resonances: unavoidable complication

- Neither experiment, nor LQCD calculations, can directly access complex energies.
- Thus, in order to study resonances, both methods have to parametrize the $K$ matrices with an analytic form that can be continued into the complex plane.
- Thus some parametrization dependence is unavoidable.
- One should put as much physical knowledge as possible into the parametrization, while minimizing model dependence.
- Input from the experimental analysis community can be helpful.

$E_{BS}^* - 4m^2$ resonance (on unphysical sheet)
Deriving the two-particle QC

Following the method of [Kim, Sachrajda & SRS, 05]
Set-up

- Work in continuum (assume that LQCD can control discretization errors)

- Cubic box of size $L$ with periodic BC, and infinite ($Minkowski$) time
  
  - Spatial loops are sums: $\frac{1}{L^3} \sum \vec{k}$
  
  - Can easily generalize to other geometries and BC

- Consider identical particles with physical mass $m$ (think of pions), interacting arbitrarily—a generic (relativistic) effective field theory (RFT)

  - Work to all orders in perturbation theory with no assumptions about the size of coupling constants

  - Generalizations are known for nonidentical particles [Many authors] and to particles with spin [Bric14]
Methodology

• Calculate (for some $P = 2\pi n_p / L$)

$$C_L(E, \vec{P}) \equiv \int_{L} d^4x e^{iE t - i\vec{P} \cdot \vec{x}} \langle \Omega | T \{ \sigma^\dagger(x) \sigma(0) \} | \Omega \rangle_L$$

• $\sigma \sim \pi^2$, e.g.  

$$\sigma(\vec{x}, t) = \int_{L} d^3y \pi(\vec{x} + \vec{y}, t) \pi(\vec{x} - \vec{y}, t) e^{-i\vec{k} \cdot \vec{y}}$$

• Poles in $C_L$ occur at energies of finite-volume spectrum [Exercise]

Here I have assumed no odd-legged vertices—-not necessary for subsequent arguments, but simplifies diagrams
Key step 1

• Replace loop sums with integrals using Poisson summation formula where integrand is nonsingular

• Drop exponentially suppressed terms (e^{-ML}, e^{-(ML)^2}, etc.) while keeping power-law dependence

\[ \frac{1}{L^3} \sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l} \neq \vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{l} \cdot \vec{k}} g(\vec{k}) \]
Key step 1

- Replace loop sums with integrals using Poisson summation formula where integrand is nonsingular
  
  - Drop exponentially suppressed terms ($e^{-ML}$, $e^{-(ML)^2}$, etc.) while keeping power-law dependence

\[
\frac{1}{L^3} \sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l} \neq \vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{l} \cdot \vec{k}} g(\vec{k})
\]

- Example of smooth integrand:
Key step 1

- Replace loop sums with integrals using Poisson summation formula where integrand is nonsingular
  - Drop exponentially suppressed terms ($e^{-ML}$, $e^{-(ML)^2}$, etc.) while keeping power-law dependence
    \[
    \frac{1}{L^3} \sum_\vec{k} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{l \neq 0} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l} \cdot \vec{k}} g(\vec{k})
    \]
    Exp. suppressed if $g(k)$ is smooth and scale of derivatives of $g$ is $\sim l/M$

- Example of smooth integrand:

\[
P = (E, \vec{P}) \rightarrow \sigma^\dagger \rightarrow \sigma
\]
Key step 2

- Use “sum=integral + [sum-integral]” if integrand has pole, e.g. \[KSS\]

$$\frac{1}{2} \left( \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum \overline{k} - \int \frac{d^4 k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(P - k)^2 - m^2 + i\epsilon} g(k)$$

\[\text{symmetry factor}\]
Key step 2

- Use “sum=integral + [sum-integral]” if integrand has pole, e.g. [KSS]

\[
\frac{1}{2} \left( \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(P - k)^2 - m^2 + i\epsilon} g(k)
\]

- Example of pole:

\[ P = (E, \vec{P}) \rightarrow \sigma^\uparrow \]

\[ \sigma \]

f is left-hand part of integrand

g is right-hand part of integrand

Focus on this loop
Key step 2

- Use “sum=integral + [sum-integral]” if integrand has pole, e.g. [KSS]

\[
\frac{1}{2} \left( \int \frac{d k_0}{2 \pi} \frac{1}{L^3} \sum_k - \int \frac{d^4 k}{(2 \pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + i \epsilon} \frac{1}{(P - k)^2 - m^2 + i \epsilon} g(k)
\]

\[
= \int d \Omega_{q^*} d \Omega_{q^{*'}} f^* (\hat{q}^*) F (q^*, q^{*'}) g^* (\hat{q}^{*'}) + \text{exp. suppressed}
\]

- Example of pole:

- **Symmetry factor**

- **q^* is relative momentum of pair on left in CM**

- **Kinematic function**

- **f & g evaluated for ON-SHELL momenta**
  - Depend only on direction in CM

- **Focus on this loop**

- **f is left-hand part of integrand**

- **g is right-hand part of integrand**

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Key step 2

- Use “sum=integral + [sum-integral]” where integrand has pole, with [KSS]

\[
\left( \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_k - \int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(P - k)^2 - m^2 + i\epsilon} g(k)
\]

\[
= \int d\Omega_q d\Omega_{q^*} f^*(\hat{q}^*) \mathcal{F} (q^*, q^{*'}) g^*(\hat{q}'^*)
\]

- Diagrammatically

Functions on left and right can be arbitrary but must be smooth

A new type of “cut”
Variant of key step 2

- For generalization to 3 particles will use a PV prescription instead of $i\varepsilon$

$$\frac{1}{2} \left( \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_k \left[ \frac{PV}{\frac{d^4k}{(2\pi)^4}} \right] f(k) \frac{1}{k^2 - m^2 + i\varepsilon} \frac{1}{(P-k)^2 - m^2 + i\varepsilon} \right) = \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^* (\hat{q}^*) \mathcal{F}_{PV} (q^*, q^{*\prime}) g^* (\hat{q}^{*\prime})$$

- Key properties of $F_{PV}$: real and no unitary cusp at threshold

- These properties are important for the derivation of three-particle QC
More detail on key step 2 [HS14]

\[
\frac{1}{2} \left( \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m_j^2 + i\epsilon} (P - k)^2 - m_j^2 + i\epsilon g(k)
\]

\[
= \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3} \right) f(\vec{k}^\ast) g(\vec{k}^\ast) h(\vec{k}) \frac{1}{2\omega_k 2\omega_{P-k}(E - \omega_k - \omega_{P-k} + i\epsilon)} + \mathcal{O}(e^{-mL})
\]

\[
= \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3} \right) f_{\ell'm'} Y_{\ell'm'}(\vec{k}^\ast) \frac{\mathcal{Y}_{\ell'm}(\vec{k}^\ast) Y_{\ell'm}(\vec{k}^\ast)}{2\omega_k 2\omega_{P-k}(E - \omega_k - \omega_{P-k} + i\epsilon)} h(\vec{k}) g_{\ell'm} + \mathcal{O}(e^{-mL})
\]

\[
\equiv f_{\ell'm'}^\ast F_{\ell'm';\ell'm}(E, \vec{P}, L) g_{\ell'm}
\]

More convenient to use this matrix form

\[
F_{\ell'm';\ell'm}(E, \vec{P}, L) = \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3} \right) f_{\ell'm'} Y_{\ell'm'}(\vec{k}^\ast) \frac{\mathcal{Y}_{\ell'm}(\vec{k}^\ast) Y_{\ell'm}(\vec{k}^\ast)}{2\omega_k 2\omega_{P-k}(E - \omega_k - \omega_{P-k} + i\epsilon)} h(\vec{k})
\]

- Thus power-law volume dependence enters through geometrical function:

\[
q^\ast = \sqrt{E^*2/4 - m^2}
\]

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More detail on key step 2 [HS14]

\[ F_{\ell' m'; \ell m}(E, \vec{P}, L) = \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3} \right) \frac{\mathcal{Y}_{\ell' m}(\vec{k}^*) \mathcal{Y}^*_{\ell m}(\vec{k}^*) h(\vec{k})}{2\omega_k 2\omega_{P-k}(E - \omega_k - \omega_{P-k} + i\epsilon)} \]

- Similarly, the PV version is

\[ F_{PV; \ell' m'; \ell m}(E, \vec{P}, L) = \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \text{PV} \int \frac{d^3k}{(2\pi)^3} \right) \frac{\mathcal{Y}_{\ell' m}(\vec{k}^*) \mathcal{Y}^*_{\ell m}(\vec{k}^*) h(\vec{k})}{2\omega_k 2\omega_{P-k}(E - \omega_k - \omega_{P-k})} \]

\[ = F_{\ell' m'; \ell m}(E, \vec{P}, L) - i\delta_{\ell' \ell} \delta_{m'm} \frac{q^*}{16\pi E^*} \]

\[ \propto \left( \frac{2\pi}{L} \right)^{1+\ell+\ell'} \mathcal{Z}_{\ell', m'; \ell m}(x^2, \vec{P}) \]

x=q*/(2\pi)  

"Lüscher zeta function"
Kinematic functions

\[ Z_{4,0} \text{ and } Z_{6,0} \text{ for } P=0 \]  
[Luu & Savage, `11]

\[ q^2 = x^2 = (q^*L/2\pi)^2 \]

Divergences occur for values of \( E \) equal to the energy of two free particles in the box 
[Exercise: why no divergence at \( x=0 \)!]

Example:
\[ n_1 = -n_2 = (0,0,1) \]
\[ \Rightarrow q^* = 2\pi/L \Rightarrow x = 1 \]
Key step 3

- Identify potential singularities using time-ordered PT (i.e. do $k_0$ integrals)
- Example (again assuming only even-legged vertices)
Key step 3

- 2 out of 6 time orderings:

\[
\sigma^\dagger \rightarrow \sigma
\]

\[
\frac{1}{E - \omega_1' - \omega_2' - \omega_3' - \omega_4'} \quad \frac{1}{E - \omega_1 - \omega_2}
\]

On-shell energy \( \omega_j = \sqrt{k_j^2 + M^2} \)
Key step 3

- 2 out of 6 time orderings:

If restrict $0 < E^* < 4M$ ($M < E^* < 3M$ if have odd-legged vertices) then only 2-particle “cuts” have singularities, and these occur only when both particles go simultaneously on shell.
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated.
- In our example, find:

\[ \sigma^\dagger \]

Can integrate

\[ \sigma \]

Must sum momenta passing through box
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated.

- Another example:

  Can replace sum with integral here

  But not here
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated.

- Another example:

  
  Can replace sum with integral here

  ![Diagram](image)

  But not here

- Then repeatedly use $\text{sum} = \text{integral} + \text{“sum-integral”}$ to simplify.
Summary: the key “move”

\[
\frac{1}{L^3} \sum_k \quad \int_k \quad \text{finite-volume residue}
\]

\[
\sigma^\dagger \quad \sigma \quad = \quad \sigma^\dagger \quad \sigma \quad + \quad \sigma^\dagger \quad \sigma
\]

on-shell

off-shell

A new type of “cut”

+ exp. suppr.
• Apply previous analysis to 2-particle correlator ($0 < E^* < 4M$)

\[ C_L(E, \vec{P}) = \sigma^\dagger \sigma + \sigma^\dagger \sigma + \sigma^\dagger \sigma + \cdots \]

these loops are now integrated

• Collect terms into infinite-volume Bethe-Salpeter kernels

\[ C_L(E, \vec{P}) = \sigma^\dagger \sigma + \{ \sigma^\dagger \sigma + \cdots \} \]

B-S kernel: 2-particle irreducible in the s-channel, i.e. no 2-particle cuts
• Apply previous analysis to 2-particle correlator

• Collect terms into infinite-volume Bethe-Salpeter kernels

\[ C_L(E, \vec{P}) = \sigma^\dagger \quad \begin{array}{c} \bullet \bullet \sigma \\ \sigma \end{array} \]

\[ + \sigma^\dagger \quad \begin{array}{c} \bullet \bullet \sigma \\ \sigma \end{array} \quad + \quad \begin{array}{c} \bullet \bullet \sigma \\ \sigma \end{array} \quad + \quad \begin{array}{c} \bullet \bullet \sigma \\ \sigma \end{array} \quad + \ldots \]

• Leading to

\[ C_L(E, \vec{P}) = \sigma^\dagger \quad \begin{array}{c} \bullet \bullet \sigma \\ \sigma \end{array} \quad + \quad \sigma^\dagger \quad \begin{array}{c} \bullet \bullet \sigma \\ \sigma \end{array} \quad iB \quad \begin{array}{c} \bullet \bullet \sigma \\ \sigma \end{array} \]

\[ + \sigma^\dagger \quad \begin{array}{c} \bullet \bullet \sigma \\ \sigma \end{array} \quad iB \quad \begin{array}{c} \bullet \bullet \sigma \\ \sigma \end{array} \quad iB \quad \begin{array}{c} \bullet \bullet \sigma \\ \sigma \end{array} \quad + \ldots \]

Similar structure to NREFT bubble-chain (e.g. in two nucleon system)
Next use sum identity

\[ C_L(E, \vec{P}) = \sigma^\dagger \sigma + \sigma^\dagger iB \sigma + \cdots + \sigma^\dagger iB iB \sigma + \cdots \]

And regroup according to number of “F cuts”

\[ C_L(E, \vec{P}) = C_\infty(E, \vec{P}) \]

\[ + \left\{ \sigma^\dagger iB + \cdots \right\} \]

\[ + \cdots \]

\[ + \cdots \]

matrix elements:
Next use sum identity

\[ C_L(E, \vec{P}) = \sigma \text{†} \sigma + \sigma \text{†} iB \sigma + \sigma \text{†} iB \sigma + \cdots \]

And keep regrouping according to number of “F cuts”

\[ C_L(E, \vec{P}) = C_{\infty}(E, \vec{P}) + \sum_{F}^{} \left( iB + iB + \cdots \right) A' + \cdots \]

**two F cuts**

**the infinite-volume, on-shell 2→2 scattering amplitude**
Next use sum identity

\[
C_L(E, \vec{P}) = \sigma + \sigma^{\dagger} \sum_i \left( \sigma + iB \right) + \cdots
\]

Alternate form if use PV-tilde prescription:

\[
C_L(E, \vec{P}) = C_{\infty}^{\text{PV}}(E, \vec{P}) + \left( A_{\text{PV}} + A'_{\text{PV}} \right) + \left( iB_{\text{PV}} + iB_{\text{PV}} + \cdots \right) + \cdots
\]

the infinite-volume, on-shell \( 2 \rightarrow 2 \) K-matrix
• Final result:

\[ C_L(E, \vec{P}) = C_\infty(E, \vec{P}) \]

\[ + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A'
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
i\mathcal{M}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A'
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
i\mathcal{M}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{array}\end{array} + \cdots \]

\[ C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A'iF[i\mathcal{M}_{2\rightarrow2}iF]^n A \]

• Correlator is expressed in terms of infinite-volume, physical quantities and kinematic functions encoding the finite-volume effects
- Final result:

\[ C_L(E, \vec{P}) = C_\infty(E, \vec{P}) \]
\[ + \left( A F A' \right) + \left( A F iM A' \right) \]
\[ + \left( A F iM iM A' \right) + \cdots \]

- \( C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' iF [iM_2 iF]^n A \)

- \( C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' iF \frac{1}{1 + M_2 F} A \)

\( \Rightarrow \) Poles in \( C_L \) occur when

\[ \text{det} \left[ F(E, \vec{P}, L)^{-1} + M_2(E^*) \right] = 0 \]
2-particle quantization condition

- At fixed $L$ & $P$, the finite-volume spectrum $E_1, E_2, ...$ is given by solutions of

$$\det \left[ F(E, \vec{P}, L)^{-1} + M_2(E^*) \right] = 0$$

For $P=0$ this equivalent to original result by [Lüscher]

Generalization to moving frame first obtained using RQM by [Rummukainen & Gottlieb]
2-particle quantization condition

- At fixed $L$ & $\mathbf{P}$, the finite-volume spectrum $E_1, E_2, \ldots$ is given by solutions of

$$\det \left[ F(E, \mathbf{P}, L)^{-1} + M_2(E^*) \right] = 0$$

For $\mathbf{P}=0$ this equivalent to original result by [Lüscher]

Generalization to moving frame first obtained using RQM by [Rummukainen & Gottlieb]

- $F$ and $M_2$ are matrices in $l,m$ space:
  - $M_2$ is diagonal; while $F$ is off-diagonal, since the box violates rotation symmetry
  - QC separates finite-volume ($F$) and infinite-volume quantities ($M_2$)
  - If $M_2$ vanishes, solutions are free two-particle energies due to poles in $F$
  - Each spectral energy gives information about all partial waves of $M_2(E^*)$
2-particle quantization condition

- Equivalent form, obtained by using PV prescription throughout derivation, is

\[
\det \left[ \mathcal{F}_{PV}(E, \vec{P}, L)^{-1} + \mathcal{K}_2(E^*) \right] = 0
\]

- I prefer this as both \(K_2\), \(F_{PV}\) are real

- \(K_2\) contains the same information as \(M_2\), but is real and smooth (no threshold branch points)

- These differences are irrelevant for the two-particle QC—the two QCs are identical—but turn out to be important for the three-particle QC

- Beware when reading the literature, as each collaboration uses different notation for what I call \(F\): sometimes \(B\) (box function), sometimes \(M\)
Summary of Lecture 1
Summary of Lecture 1

- Resonances are ubiquitous and mysterious in QCD
  - Usually decay to more than 2 particles
- Key issue is relating finite-volume spectrum to scattering amplitudes (or K matrices)
  - QC2 provides a very general, model-independent tool to do so
Thank you!

Questions?
Backup Slides
Examples of resonances

- Example 3: scalar, isoscalars—possible p-wave $q\bar{q}$ states

**[PDG]**

- $f_0(500)$
  - $I^G(J^{PC}) = 0^+(0++)$
  - Mass (T-Matrix Pole $\sqrt{s}$) = (400–550) – $i$(200–350) MeV
  - Mass (Breit-Wigner) = (400–550) MeV
  - Full width (Breit-Wigner) = (400–700) MeV

<table>
<thead>
<tr>
<th>$f_0(500)$ DECAY MODES</th>
<th>Fraction ($\Gamma_i/\Gamma$)</th>
<th>$\rho$ (MeV/c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi\pi$</td>
<td>seen</td>
<td>–</td>
</tr>
<tr>
<td>$\gamma\gamma$</td>
<td>seen</td>
<td>–</td>
</tr>
</tbody>
</table>

- $f_0(980)$
  - $I^G(J^{PC}) = 0^+(0++)$
  - Mass $m = 990 \pm 20$ MeV
  - Full width $\Gamma = 10$ to 100 MeV

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<tr>
<th>$f_0(980)$ DECAY MODES</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\pi\pi$</td>
<td>seen</td>
<td>476</td>
</tr>
<tr>
<td>$K\bar{K}$</td>
<td>seen</td>
<td>36</td>
</tr>
<tr>
<td>$\gamma\gamma$</td>
<td>seen</td>
<td>495</td>
</tr>
</tbody>
</table>
Examples of resonances

- Example 3: scalar, isoscalars—possible p-wave $q\bar{q}$ states

\[ f_0(500) \] 
\[ I^G(JPC) = 0^+(0++) \]
Mass (T-Matrix Pole $\sqrt{s}$) = (400–550)\(\pm\)i(200–350) MeV
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\[ f_0(980) \] 
\[ I^G(JPC) = 0^+(0++) \]
Mass $m = 990 \pm 20$ MeV
Full width $\Gamma = 10$ to 100 MeV

- Large uncertainties because analyses are difficult
  extract from charged pion beams on nucleon targets

\[ s \left\{ \begin{array}{l}
\pi \\
p \\
\pi \\
N \\
\end{array} \right\} m_{\pi\pi}, \theta_{\pi\pi} \]
\[ \pi \pi \text{ partial-waves} \]
- project $P_\ell(\cos \theta)$
- on-shell $\pi$ exchange
- extrapolate to $t = -m_{\pi}^2$

[Figure from HMI slides of Jo Dudek]
Examples of resonances

- Example 3: scalar, isoscalars—possible p-wave $q\bar{q}$ states
- Extract the phase shift from complicated amplitude analysis

**isospin = 0**

![Graph showing examples of resonances](image)

- $f_0(500)$
- $f_0(980)$

Logos and imagery: Grayer 1974
Aside on inelasticity

- Phase shift in $I=J=1$ $\pi\pi$ channel

**isospin=1**

\[ 1 - |\eta|^2 \]

gives probability for scattering into any final state other than $\pi\pi$, e.g. KK-bar, $\eta\eta$, $4\pi$

Becomes nonzero above 1 GeV

\[ \text{Hyams 1973} \]
Examples of resonances

- Example 4: Roper (excited nucleon)

\[ N(1440) 1/2^+ \]

\[ I(J^P) = \frac{1}{2}(1^+) \]

Re(pole position) = 1360 to 1380 (\(\approx\) 1370) MeV

\(-2\text{Im}(\text{pole position}) = 160 \text{ to } 190 \ (\approx 175) \text{ MeV} \)

Breit-Wigner mass = 1410 to 1470 (\(\approx\) 1440) MeV

Breit-Wigner full width = 250 to 450 (\(\approx\) 350) MeV

\[ \begin{array}{ccc}
N \pi & 55\text{–}75 \ % & 398 \\
N \eta & <1 \ % & 1 \\
N \pi \pi & 17\text{–}50 \ % & 347 \\
\Delta(1232) \pi, \ P\text{-wave} & 6\text{–}27 \ % & 147 \\
N \sigma & 11\text{–}23 \ % & – \\
p \gamma, \ \text{helicity}=1/2 & 0.035\text{–}0.048 \ % & 414 \\
n \gamma, \ \text{helicity}=1/2 & 0.02\text{–}0.04 \ % & 413 \\
\end{array} \]

- Extracted from amplitude analysis of \(\pi N\) scattering

- Lighter than expected from quark model for a radial excitation
Examples of resonances

- Example 5: $Z_c(3900)$—a nonstandard meson

$Z_c(3900)$

\[ J^G(J^{PC}) = 1^+(1^+) \]

- Mass $m = 3887.2 \pm 2.3$ MeV (S = 1.6)
- Full width $\Gamma = 28.2 \pm 2.6$ MeV

$Z_c(3900)$ DECAY MODES

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<td>not seen</td>
<td>318</td>
</tr>
<tr>
<td>$\eta_c \pi^+ \pi^-$</td>
<td>not seen</td>
<td>759</td>
</tr>
<tr>
<td>$(D^0 D^*)^-$</td>
<td>seen</td>
<td>-</td>
</tr>
<tr>
<td>$D^- D^{*0} + c.c.$</td>
<td>seen</td>
<td>153</td>
</tr>
<tr>
<td>$D^+ D^{-} + c.c.$</td>
<td>seen</td>
<td>144</td>
</tr>
<tr>
<td>$\omega \pi^\pm$</td>
<td>not seen</td>
<td>1862</td>
</tr>
<tr>
<td>$J/\psi \eta$</td>
<td>not seen</td>
<td>510</td>
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$\rho \eta_c$ (now seen at 4.2\sigma significance, [BESIII])
Examples of resonances

- **Example 5: $Z_c(3900)$—a nonstandard meson**

$$I^G(J^{PC}) = 1^+(1^-)$$

[Zc(3900)]

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$\rho\eta_c$ (now seen at 4.2$\sigma$ significance, [BESIII])

**Significance >8$\sigma$**

Observed by BESIII, Belle, CLEO-c in 2013

$$e^+e^- \rightarrow \pi^\pm Z_c^\mp$$

[BESIII, talk at Lattice 2019 by C. Yuan]
Examples of resonances

- Example 5: $Z_c(3900)$—a nonstandard meson

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$J^G(J^{PC}) = 1^+(1^-^-)$

Mass $m = 3887.2 \pm 2.3$ MeV \ (S = 1.6)
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$\rho \eta_c$ (now seen at 4.2\sigma significance, [BESIII])

Observed by BESIII, Belle, CLEO-c in 2013

$e^+e^- \rightarrow \pi^\pm Z_c^\mp$

Significance $>8\sigma$

[BESIII, talk at Lattice 2019 by C. Yuan]

- $Z_c^+$ quark composition: $c\bar{c}ud\bar{d}$
Examples of resonances

- Example 5: \(Z_c(3900)\)—a nonstandard meson

\[
e^{+}e^{-} \rightarrow Y(4260) \rightarrow \pi \rightarrow D^{*}D^{*}(4014) \rightarrow Z_c(3900) \rightarrow \bar{D}D^{*}(3877) \rightarrow \rho \eta_{c}(3759) \rightarrow \pi J/\psi(3236) \rightarrow 1^{--} 1^{+-} (J^{PC})\]

- \(Z_c^+\) quark composition: \(\bar{c}u\bar{d}\)

- Possible interpretations:
  - Tetraquark
  - Molecule

- Threshold enhancement—supported by HALQCD study [1602.03465]

[Ikeda et al., 1602.03465]
G parity

- G parity will come up occasionally in the remaining lectures, so here is a reminder.

- \( G = C e^{i\pi I_y} \) is an exact symmetry of isosymmetric QCD, and an approximate symmetry of real QCD.

- Eigenstates of G: \( \pi(-1), \eta(+1), \rho(+1), \omega(-1), \ldots \)

- Relevance for what follows:
  - Restricts decay channels, e.g. \( \rho \to \pi\pi, \omega \to \pi\pi\pi (\eta \to \pi\pi \text{ forbidden by parity}) \)
  - No interactions involving an odd number of pions, e.g.

\[
\pi\pi \leftrightarrow 4\pi, \quad \pi\pi \leftrightarrow 3\pi
\]
3-particle scattering

- In a theory with a G-parity-like $Z_2$ symmetry only have $3\rightarrow 3$ scattering

\[ M_3 \sim \begin{array}{ccc}
    & \times & \\
    + & \times & \\
    + & \times & \\
    + & \ldots
\end{array} \]

- Difficult to measure experimentally, but well defined in QFT

- 3 particle finite-volume states are accessible to LQCD
3-particle scattering

- In a theory with a G-parity-like $Z_2$ symmetry only have $3\to 3$ scattering

\[ M_3 \sim \quad \text{diagram 1} + \quad \text{diagram 2} + \quad \text{diagram 3} + \ldots \]

- Difficult to measure experimentally, but well defined in QFT

- 3 particle finite-volume states are accessible to LQCD

- Without the $Z_2$ symmetry have $2\to 3$, $3\to 2$ & $3\to 3$ scattering, e.g.

\[ M_{23} \sim \quad \text{diagram 4} + \quad \text{diagram 5} + \ldots \]
3-particle scattering

- In a theory with a G-parity-like $Z_2$ symmetry only have $3\rightarrow 3$ scattering

$$M_3 \sim \begin{array}{c}
\begin{array}{c}
\times
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\times
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\times
\end{array}
\end{array} + \ldots
\end{array}$$

- Difficult to measure experimentally, but well defined in QFT
- 3 particle finite-volume states are accessible to LQCD

- Without the $Z_2$ symmetry have $2\rightarrow 3$, $3\rightarrow 2$ & $3\rightarrow 3$ scattering, e.g.

$$M_{23} \sim \begin{array}{c}
\begin{array}{c}
\times
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\times
\end{array}
\end{array} + \ldots
\end{array}$$

- Parametrizing these amplitudes in terms of real $K$ matrices is a nontrivial problem to which the methods I will describe provide, as a spinoff, one solution