Graduate QM (PHYS 517) Handout 11/30/10
Resonance decay in WKB approximation

For further reading see B.R. Holstein, Am. Journ. Phys. 64,1061 (1996), *"Understanding alpha decay."*

1 Matching formulae recap

We will need the matching formulae for both types of Airy functions. These are obtained from the asymptotic expansions:

$$\text{Ai}: \frac{1}{|z|^{1/4}} \sin\left(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}\right) \quad \leftrightarrow \quad \frac{1}{2}\frac{1}{z^{1/4}} \exp\left(-\frac{2}{3}z^{3/2}\right) \tag{1}$$

$$\text{Bi}: \frac{1}{|z|^{1/4}} \cos\left(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}\right) \quad \leftrightarrow \quad \frac{1}{z^{1/4}} \exp\left(-\frac{2}{3}z^{3/2}\right). \tag{2}$$

Note the change from sin to cos and the lack of the 1/2 on the right-hand side of the Bi matching. These imply the following matching of WKB forms. First, from Ai matching at a turning point $x = a$:

$$\frac{1}{\sqrt{k(x)}} \sin\left(\int_x^a k(x')dx' + \frac{\pi}{4}\right) \quad \leftrightarrow \quad \frac{1}{2\sqrt{\kappa(x)}} \exp\left(-\int_a^x \kappa(x')dx'\right) \tag{3}$$

Second, from Bi matching at turning point $x = b$, with the potential decreasing as x increases:

$$\frac{1}{2\sqrt{\kappa(x)}} \exp\left(\int_z^b \kappa(x')dx'\right) \quad \leftrightarrow \quad \frac{1}{\sqrt{k(x)}} \cos\left(\int_b^x k(x')dx' + \frac{\pi}{4}\right) \tag{4}$$

$$= \quad \frac{1}{\sqrt{k(x)}} \sin\left(\int_b^x k(x')dx' + \frac{3\pi}{4}\right). \tag{5}$$

Here, as usual,

$$\hbar^2 k(x)^2 = 2m[E - V(x)], \qquad \hbar^2 \kappa(x)^2 = 2m[V(x) - E].$$

2 Example of a resonance

We are imagining solving an s-wave problem in 3-dimensions, so x (usually called r) runs from $0 - \infty$, and $u(0) = 0$. We consider the potential barrier

$$V(x) = \begin{cases} V_0 & a < x < b \\ 0 & \text{elsewhere} \end{cases}$$

We express energies relative to V_0, and distances in units of $L = \hbar/\sqrt{2mV_0}$. As shown in the accompanying notebook `resonanceall.nb`, if we choose $a = \pi L$ and $b = 3\pi L$, then there is a sharp resonance at $E_res \approx 0.542409 V_0$. The resonance shows up as a peak in the ratio of

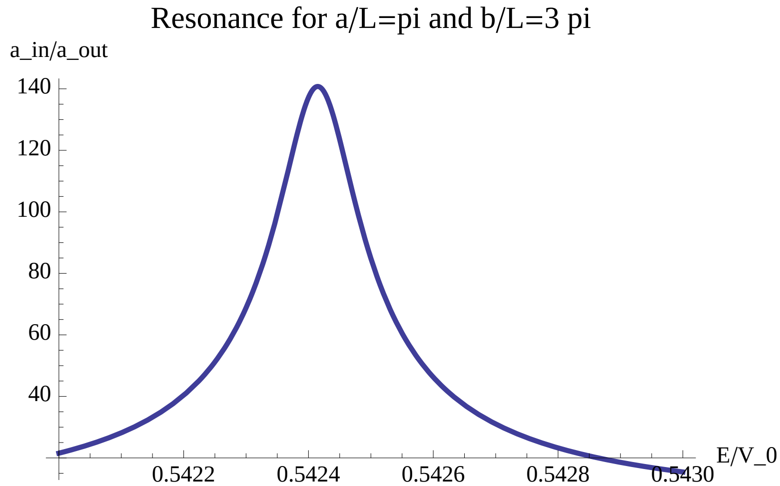

Figure 1: Resonance peak for potential barrier.

the "inside" to "outside" amplitudes of the stationary solutions. More precisely, if we write these solutions as

$$u_k(x) = \begin{cases} A\sin(kr) & r < a \\ \sin(kr+\phi) & r > b \end{cases}$$

[We will not need the (known) form in the barrier region.] Then the inside/outside ratio is $|A|$.

A plot of $|A|$ versus E/V_0 is shown in Fig. 1, displaying a sharp peak. On form of the wavefunctions as one approaches resonance from below are shown in Figs. 2 and 3. Particularly important for today's lecture is that the form of the inside wavefunction changes very little as one scans across the resonance. It is only the amplitude which changes substantially.

3 General resonance

We now consider a general barrier, the only property of which we require is that it leads to some number of "almost bound states", i.e. resonances. We first want to determine approximately the resonance energies and the form of the wavefunctions for all x.

Using the WKB approximation, an eigenstate with energy E_k is given in the three different regions ("inside," "barrier," and "outside") by

$$0 < z < a: \qquad u_k(z) \approx \frac{1}{\sqrt{k(x)}}\sin\left[\int_0^z k(z')dz'\right]$$

$$a < z < b: \qquad u_k(z) \approx \frac{1}{\sqrt{\kappa(x)}}\left\{A_+\exp\left[+\int_a^z \kappa(z')dz'\right] + A_-\exp\left[-\int_a^z \kappa(z')dz'\right]\right\}$$

$$b < z: \qquad u_k(z) \approx \frac{C}{\sqrt{k(x)}}\sin\left[\int_b^z k(z')dz' + \phi_k\right]$$

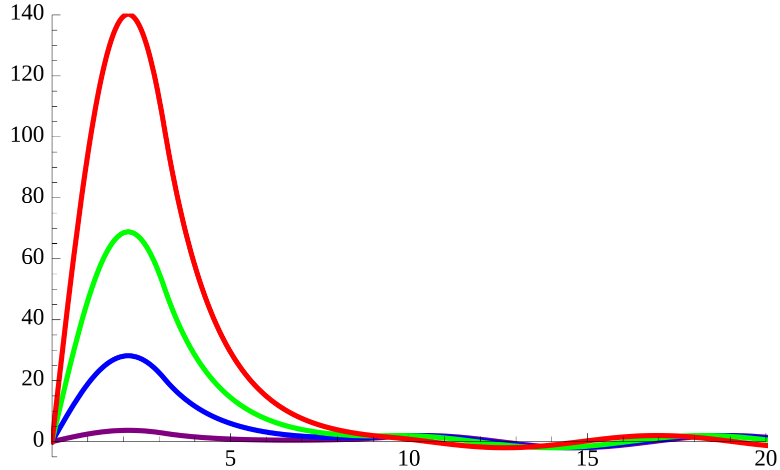

Figure 2: Stationary state wavefunctions as one approaches resonance from below. Horizontal axis is x/L. Normalization is chosen so that the outside wavefunctions ($x/L > 3\pi$) is essentially the same for all energies shown, and is of $O(1)$. Energies are $E/V_0 = 0.54$ (purple), 0.5421 (blue), 0.5423 (green) and 0.542409 (red), the latter being the resonant energy.

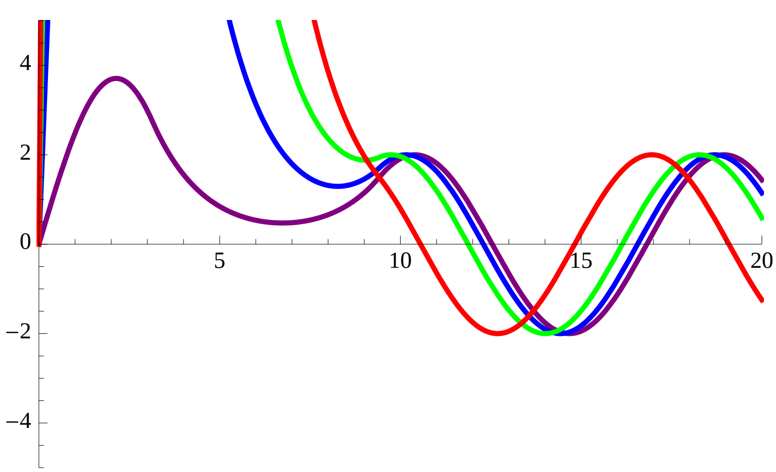

Figure 3: As for Fig. 2, except using a horizontal scale so the outside wavefunctions are visible.

with a and b the classical turning points. Note that we have implemented the BC $u(0) = 0$, and that we can choose (and have chosen) the wavefunction to be real.

The four constants $A_\pm$, C and ϕ are determined by the 4 matching conditions at the 2 boundaries. We use the WKB connection formulae, generalized to include both Airy functions.

It is useful to rewrite the inner and barrier forms:

$$0 < z < a: \qquad u_k(z) \approx -\frac{1}{\sqrt{k(x)}} \sin\left[\int_z^a k(z')dz' - \int_0^a k(z')dz'\right]$$

$$a < z < b: \qquad u_k(z) \approx \frac{1}{\sqrt{\kappa(x)}} \left\{A_+ e^\sigma \exp\left[-\int_z^b \kappa(z')dz'\right] + A_- e^- \sigma \exp\left[\int_z^b \kappa(z')dz'\right]\right\},$$

with

$$\sigma = \int_a^b \kappa(z')dz',$$

the WKB exponent in barrier.

The analysis described in class leads to the resonance condition (which applies when $A_+ = 0$)

$$\int_0^a k(z')dz' = n\pi - \pi/4, \qquad n = 1, 2, 3, \ldots$$

for which energies the outside amplitude and phase are

$$C = \frac{1}{2} e^{-\sigma}, \qquad \phi = 3\pi/4.$$