

I. HOME WORK 424

HW 1. Due Oct.14

10 points each problem.

1

(Thornton, Marion, Classical dynamics 6.14.)

Find the shortest path between the (x, y, z) points $(0, -1, 0)$ and $(0, 1, 0)$ on the conical surface $z = 1 - \sqrt{x^2 + y^2}$.

Solution:

Let us use cylindrical coordinates: $x = r \sin \phi$; $y = r \cos \phi$ which gives us $z = 1 - r$.

Lagrangian has a form

$$L = \int ds = \int \sqrt{dr^2 + r^2 d\phi^2 + dz^2} = \int \sqrt{2 + r^2 \phi_r'^2} dr \quad (1.1)$$

$dL/d\phi = 0$.

Euler equation:

$$\begin{aligned} \frac{d}{dr} \frac{r^2 \phi'}{\sqrt{2 + r^2 \phi_r'^2}} &= 0 \\ r^2 \phi' &= C \sqrt{2 + r^2 \phi_r'^2} \\ \phi' &= \pm \frac{\sqrt{2}C}{r\sqrt{r^2 - C^2}} \\ \phi &= \pm \sqrt{2}C \int \frac{dr}{r\sqrt{r^2 - C^2}} + C_1; \end{aligned} \quad (1.2)$$

2

(Thornton, Marion, Classical dynamics 6.15)

Find the curve $y(x)$ that passes through the endpoints $(0, 0)$ and $(1, 1)$ and minimizes the functional

$$I[y] = \int_0^1 \left[\left(\frac{dy}{dx} \right)^2 - y^2 \right] dx$$

Solution:

$$f = (y')^2 - y^2 \quad (1.3)$$

Euler equation:

$$\begin{aligned} -2y(x) - 2y'' &= 0 \\ y &= C_1 \sin t + C_2 \cos t \end{aligned}$$

3

(Thornton, Marion, Classical dynamics 7.7)

A double pendulum consists of two simple pendule, with one pendulum suspended from the the blob of the other. The two pendule have equal lengths and bobs have equal masses. They are confined to move in the same plane. Find Lagrange's equations of motion for the system. Do not assume small angles.

Solution:

The Lagrangian has a form:

$$\begin{aligned}
 L &= ml^2\dot{\phi}_1^2 + \frac{m}{2}l^2\dot{\phi}_2^2 + ml^2\dot{\phi}_1\dot{\phi}_2 \cos(\phi_1 - \phi_2) + 2mgl \cos \phi_1 + mgl \cos \phi_2 \\
 \frac{\partial L}{\partial \phi_1} &= -2mgl \sin \phi_1 - ml^2\dot{\phi}_1\dot{\phi}_2 \sin(\phi_1 - \phi_2), \\
 \frac{\partial L}{\partial \phi_2} &= -mgl \sin \phi_1 + ml^2\dot{\phi}_1\dot{\phi}_2 \sin(\phi_1 - \phi_2), \\
 \frac{\partial L}{\partial \dot{\phi}_1} &= 2ml^2\dot{\phi}_1 + ml^2\dot{\phi}_2 \cos(\phi_1 - \phi_2) \\
 \frac{\partial L}{\partial \dot{\phi}_2} &= 2ml^2\dot{\phi}_1 + ml^2\dot{\phi}_2 \cos(\phi_1 - \phi_2)
 \end{aligned} \tag{1.4}$$

Lagrange equations:

$$\begin{aligned}
 -2mgl \sin \phi_1 - \frac{d}{dt}[2ml^2\dot{\phi}_1 + ml^2\dot{\phi}_2 \cos(\phi_1 - \phi_2)] &= 0 \\
 -mgl \sin \phi_2 - \frac{d}{dt}[2ml^2\dot{\phi}_2 + ml^2\dot{\phi}_2 \cos(\phi_1 - \phi_2)] &= 0
 \end{aligned} \tag{1.5}$$

4

(Thornton, Marion, Classical dynamics 7.15)

A pendulum consists of a mass m suspended by massless spring with unextended length b and spring constant k . Find Lagrange's equations of motion.

Solution:

$$\begin{aligned}
 x &= l \sin \phi; \dots\dots y = l \cos \phi \\
 \dot{x} &= \dot{l} \sin \phi + l\dot{\phi} \cos \phi; \dots \dot{y} = \dot{l} \cos \phi - l\dot{\phi} \sin \phi; \\
 L &= \frac{m}{2}(\dot{l}^2 + l^2\dot{\phi}^2) + mgl \cos \phi - k(l - b)^2 \\
 \frac{\partial L}{\partial l} &= m\dot{\phi}^2 + mg \cos \phi - 2k(l - b) \\
 \frac{\partial L}{\partial \dot{l}} &= m\dot{l} \\
 \frac{\partial L}{\partial \phi} &= -mgl \sin \phi \\
 \frac{\partial L}{\partial \dot{\phi}} &= ml^2\dot{\phi}
 \end{aligned} \tag{1.6}$$

Lagrange equations:

$$\begin{aligned}
 ml\dot{\phi}^2 + mg \cos \phi - 2k(l - b) - m\ddot{l} &= 0 \\
 -mgl \sin \phi - ml^2\ddot{\phi} &= 0
 \end{aligned} \tag{1.7}$$

5

A point particle with a mass m moves along a circle of radius l in a vertical plane under the influence of the gravity field (mathematical pendulum). Estimate the period of the pendulum if $E - 2mgl \ll 2mgl$

$$T = \sqrt{\frac{l}{g}} \ln \frac{E}{|E - 2mgl|}$$

6

Consider a spherical pendulum of a mass m and a length l . Write expressions for a. the Lagrangian in terms of generalized coordinates, b. the generalized momenta.

Solution:

$$L = \frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}ml^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 + mgl\cos\theta$$

$$p_\theta = ml^2\frac{d\theta}{dt}$$

$$p_\phi = ml^2\sin^2\theta\frac{d\phi}{dt} \tag{1.8}$$

HW 2. Due Oct 21

1. For what values of the angular momentum M is it possible to have finite orbits in the potential

$$U(r) = -\frac{\alpha \exp(-\frac{r^2}{R^2})}{r}?$$

Solution:

A finite orbit is possible if $U_{eff}(r) = U + \frac{M^2}{2mr^2}$ has a minimum which gives us $f(x) = aM^2/\alpha m$, where $f(x) = x(x+1)\exp(-x)$. This equation has real roots only when $aM^2/\alpha m$ is less than the maximum value of $f(x)$, ($x > 0$). It is equal to $B = (2 + \sqrt{5})\exp[-1/2(1 + \sqrt{5})]$. Thus a finite orbit is possible, provided

$$M^2 < B\alpha m/a$$

2. Find the deflection angle of fast particles ($E \gg V$) moving in a potential

$$U(r) = V \exp(-\frac{r^2}{R^2})$$

as a function of the impact parameter ρ .

Solution:

$$\Delta p = -\frac{\partial}{\partial \rho} \int_{-\infty}^{\infty} U(|\rho + vt|) dt = \frac{2V\sqrt{\pi}}{v} x e^{-x^2}$$

where $x = \rho/R$

3. Find the frequency of the small oscillations for particles moving in the following 1D potential:

$$U(x) = V \cos \alpha x - Fx$$

Solution:

$$U' = -V\alpha \sin \alpha x_0 - F = 0$$

$$\sin \alpha x_0 = -F/V\alpha$$

$$U'' = -V\alpha^2 \cos \alpha x_0$$

$$\omega^2 = \frac{V\alpha^2}{m} \sqrt{1 - \left(\frac{F}{V\alpha}\right)^2} \tag{1.9}$$

4. Consider the problem 4 in §5 in Landau and Lifshitz.

Find

- a. The equilibrium value of the generalized coordinate.
- b. The frequency of small oscillations.

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Solution:

a) The Lagrangian of the system has a form

$$L = ma^2[\dot{\theta}^2(1 + 2\sin^2 \theta) + \Omega^2 \sin^2 \theta + 2\Omega_0^2 \cos \theta]$$

where $\Omega_0^2 = 2g/a$. If $\Omega > \Omega_0$ the potential energy of the system,

$$U = -ma^2(\Omega^2 \sin^2 \theta + 2\Omega_0^2 \cos \theta)$$

has a minimum when $\cos \theta_0 = \Omega_0^2/\Omega^2$

b) Expanding U in the neighborhood of θ_0 and putting in the kinetic energy $1 + 2\sin^2 \theta = 1 + 2\sin^2 \theta_0 = 3 - 2(\Omega_0/\Omega)^2 = M/2ma^2$ we get

$$L = M\dot{x}^2/2 - kx^2$$

where $k = U''(\theta_0)$. Hence we get

$$\omega^2 = k/M = \Omega^2 \frac{1 - \Omega_0^4/\Omega^4}{3 - 2\Omega_0^4/\Omega^4}$$

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5.

The Lagrangian of a system has a form

$$L = \frac{1}{2}(m_1\dot{x}^2 + m_2\dot{y}^2 + \beta\dot{x}\dot{y}) - \frac{1}{2}(x^2 + y^2)$$

- a. Write the effective mass matrix
- b. Find the eigenfrequencies of the system.

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Solution:

Lagrange equations:

$$\begin{aligned} m_1\ddot{x} + \beta\ddot{y} + x &= 0 \\ m_2\ddot{y} + \beta\ddot{x} + y &= 0 \end{aligned}$$

(1.10)

$$(m_1\omega^2 - 1)(m_2\omega^2 - 1) - \beta^2\omega^4 = 0$$

$$\omega^2 = \frac{(m_1 + m_2) \pm \sqrt{(m_1 - m_2)^2 + 4\beta^2}}{2(\beta^2 - m_1m_2)}$$

HW 3. Due Oct 28

1.

- a. Determine the amplitude of oscillations of an oscillator described by the equation ($h \ll 1$)

$$\ddot{x} + \omega^2(1 + h \cos 2\omega t)x + \beta x^3 = 0$$

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b. Determine the amplitude of the third harmonic and show that it is much smaller than the first one.

Solution:

At small amplitudes a of oscillations the nonlinear term can be dropped, and we have a situation of parametric resonance when the amplitude grows with time. At finite amplitude there is a nonlinear correction to the frequency, and the system is driven out of the frequency interval where the resonance takes place. The saturation occurs when the nonlinear correction is equal to the width of the frequency interval where the parametric resonance takes place.

One can equate the half of the width of the interval of parametric resonance Eq.27.11 $h\omega_0/2$ to the frequency shift given by the Eq.28.13 in Landau and Lifshitz. $\omega_3 = 3\beta a^2/8\omega_0$ This will give us the value of a .

$$a^2 = \frac{4\omega_0^2}{3\beta} \quad (1.11)$$

Since the amplitude of the first harmonics is small, the amplitude of the second harmonics is given by Eq. 28.14 in Landau and Lifshitz.

$$a_{(3)} = \frac{a^3\beta}{32\omega^2} \ll a \quad (1.12)$$

2. Consider a pendulum whose point of suspension performs vertical oscillations with frequency $\Omega \gg \sqrt{g/l}$ and the amplitude a . Here l is the length of the pendulum.

$$L = \frac{1}{2}ml^2(\dot{\phi})^2 + mlaw^2 \cos(\omega t) \cos \phi + mgl \cos \phi \quad (1.13)$$

(See LL ch. 5 problem 3.

$$U_{eff} = -mgl \cos \phi + \frac{(mlaw^2)^2}{8ml^2\omega^2} (\sin \phi)^2 \quad (1.14)$$

(See LL ch. 30)

a. Find stable equilibrium states of the pendulum.

$\sin \phi = 0$

b. Find a frequency of small oscillations Ω near the equilibrium states

$$\Omega = \left[\frac{g}{l} - \frac{a^2\omega^2}{4l^2} \right]^{1/2} \quad (1.15)$$

c. Find the correction to the frequency which is quadratic in the amplitude of oscillation.

Expanding $U_{eff}(\phi)$ up to quadratic term in ϕ and using Eq. 28.14 in LL we get

$$\Omega^{(2)} = \frac{5 \left[\frac{g}{2l} - \frac{a^2\omega^2}{4l^2} \right]^2}{12\Omega^3} \phi^2 \quad (1.16)$$

3. Consider a system of three identical particles of mass m which are connected by identical springs with spring constants k and move on a circle (See Fig.1).

a. How many vibrational degrees of freedom does this system have?

Two

b. Find eigenfrequencies of the vibrations.

$$L = \frac{1}{2}((\dot{x}_1)^2 + (\dot{x}_2)^2 + (\dot{x}_3)^2) - \frac{k}{2}[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2] \quad (1.17)$$

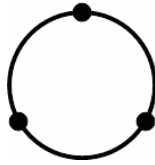


FIG. 1:

The three mass can rotate with a constant angular velocity. So $\omega_1 = 0$.
 $\omega_2 = \omega_3 = 3k/m$

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HW 4 Due Nov. 4.

1.
 A fast particle ($v \gg \sqrt{A/k^2m}$) of mass m is moving in the potential field

$$U(\mathbf{r}) = A(x^2 - y^2) \sin kz \quad (1.18)$$

at a small angle to the z-axis.

Describe the motion of the particle in xy -plane.

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 Solution:

In the first approximation $z = vt$. In the xy -plane there is a fast oscillating force acting on the particle

$$f_x = 2Ax \sin kvt; f_y = -2Ay \sin kvt$$

the corresponding effective potential (See Ch. 30 Landau) has a form

$$U_{eff}(x, y) = \frac{m\Omega^2}{2}(x^2 + y^2) \quad (1.19)$$

where $\Omega = A/mkv$.

Thus the particle performs harmonic oscillations in the xy plane with the frequency Ω . Note that $\Omega \ll kv$ and we can apply results of Ch. 30 to this problem. (BTW this problem is relevant to the accelerators physics.)

2.

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 Determine the principal moments of inertia of the following systems

a. masses m and $2m$ at the vertices of a triangle with the side length $2a$ and $4a$ (See Fig. 2a)

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 b. a sphere of radius R inside which there is a spherical cavity of radius r (See Fig2b.). (The density of the sphere is ρ .)

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 Solution:

The coordinates of the center of mass of the system are: (a, a) .

$$I_{zz} = 16ma^2; I_{xx} = 4(2 + \sqrt{2})ma^2; I_{yy} = 4(2 + \sqrt{2})ma^2 \quad (1.20)$$

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a sphere of radius R inside which there is a spherical cavity of radius r (See Fig2b.). (The density of the sphere is ρ .)

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 Solution:

The center of mass is a point on the axis of symmetry at a distance $(R - r)r^3/(R^3 - r^3)$ to the left of the center of the sphere. The body is a symmetric top. With respect of the axis of symmetry we have

$$I_z = \frac{4}{3}\pi\rho\frac{2}{5}(R^5 - r^5)$$

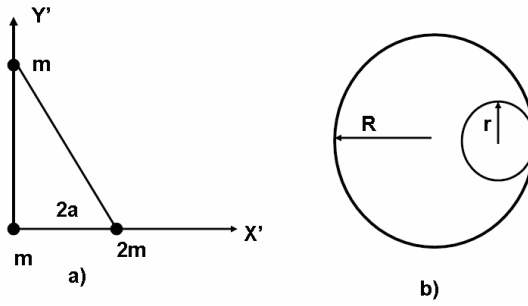


FIG. 2:

and with respect to any two perpendicular axis passing through the center of mass:

$$I_x = I_y = \frac{4}{3}\pi\rho\left[\frac{2}{5}(R^5 - r^5) - \frac{(R-r)^2 r^3 R^3}{R^3 - r^3}\right] \quad (1.21)$$

3.

Referring to Fig.3 find the minimum height h that will permits a spherical ball of radius r (which rolls without sliding) to maintain contact with the rail of the loop.

$$\begin{aligned} mg(h + 2R) &= mg(R - r) + mv^2/2 + I\omega^2/3; v = r\omega \\ \frac{mv^2}{R - r} &= mg \end{aligned} \quad (1.22)$$

4.

A solid homogeneous cylinder of radius r and mass m rolls without slipping on the inside of a stationary larger cylinder of radius R as shown in Fig.4.

- a. Write the Lagrangian in terms of θ .
- b. Lagrange equation of motion
- c. The period of small oscillations

Solution: The moment of inertia of the cylinder is $I = mr^2/2$,
the condition of rolling without slipping is $(R - r)\theta = r\phi$

$$\begin{aligned} L = T &= \frac{1}{2}m(R - r)^2\dot{\theta}^2 + \frac{1}{4}mr^2\dot{\phi}^2 + mg(R - r)\cos\theta = \\ &= \frac{3}{4}m(R - r)^2\dot{\theta}^2 + mg(R - r)\cos\theta \end{aligned} \quad (1.23)$$

- b. Lagrange equation of motion

$$\ddot{\theta} + \frac{2}{3}\frac{g}{R - r}\sin\theta = 0 \quad (1.24)$$

- c. The period of small oscillations

$$T = \pi\sqrt{\frac{6(R - r)}{g}} \quad (1.25)$$

HW 5 Due Now. 20.

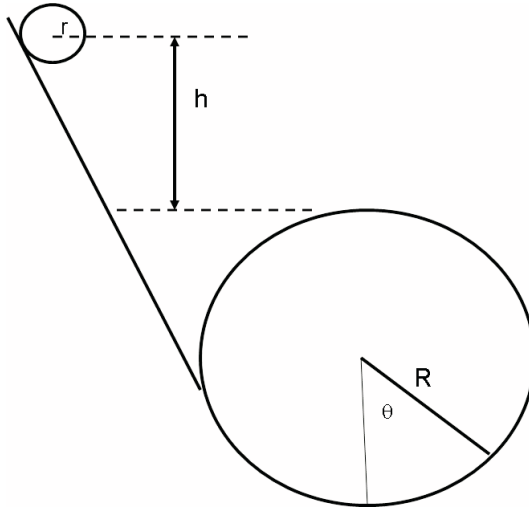


FIG. 3:

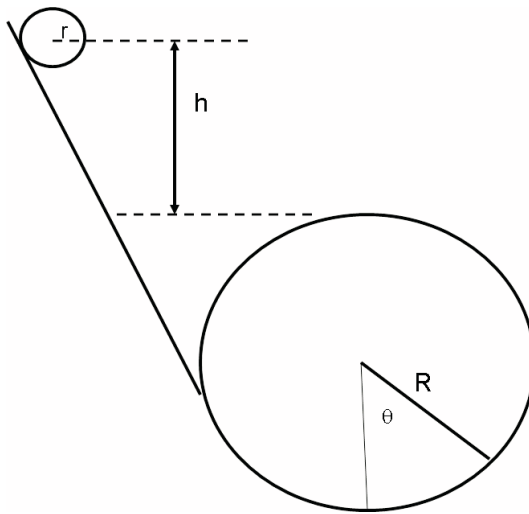


FIG. 4:

1. Find the canonical transformation produced by the generating function

$$F = (\mathbf{rP}) + (\delta\mathbf{a}[\mathbf{rP}])$$

What is the physical meaning of the transformation?

Solution:

$$\mathbf{p} = \frac{dF}{d\mathbf{r}} = \mathbf{P}; \mathbf{R} = \frac{dF}{d\mathbf{P}} = [\delta\mathbf{ar}] \tag{1.26}$$

It is a rotation of the system of co-ordinates.

2. Prove that the following transformation is canonical

$$\begin{aligned} Q &= \ln(1 + q^{1/2} \cos p) \\ P &= 2(1 + q^{1/2} \cos p)q^{1/2} \sin p \end{aligned} \tag{1.27}$$

Solution:

$$[Q, Q] = 0; [P, P] = 0, [Q, P]_{p,q} = 1$$

3. Suppose the Hamiltonian of a system has a form $H_0(p, q) + bq \sin(\omega t)$. Find a generation function a the canonical transformation which leads to a Hamiltonian $H_0(P, Q)$.

Solution:

$$\begin{aligned} H' - H_0 &= \frac{\partial F}{\partial t}; \\ F &= qP - \frac{bq}{\omega} \cos(\omega t) \end{aligned} \tag{1.28}$$

The transformation has a form

$$\begin{aligned} p &= \frac{\partial F}{\partial q} = P - \frac{b}{\omega} \cos(\omega t) \\ Q &= q = \partial F / \partial P \end{aligned} \tag{1.29}$$

4. Determine Poisson brackets formed from the components of the angular momentum M_i , where $i = x, y, x$.

Solution: $\mathbf{M} = \mathbf{r} \times \mathbf{p}$; $[M_x, M_y] = -M_z$; $[M_y, M_z] = -M_x$; $[M_z, M_x] = -M_y$

HW Due Dec.3

1. Prove that $[F(|\mathbf{r}|), M_z] = 0$. Here M_z is z -component of the angular momentum, and $F(|\mathbf{r}|)$ is an arbitrary function of the modulus of the coordinate.

2. Consider a uniform hemisphere which lies on a smooth horizontal surface in the field of gravity. The center of mass lies at the distance $3R/8$ from the center of the sphere, and the moment of inertia is $I = 83mR^2/320$. Here R and m are the radius and the mass of the sphere. Write Lagrange equation of motion.

Solution:

$$z = R - \frac{3}{8}R \cos \phi$$

$$L = \frac{I\dot{\phi}^2}{2} + \frac{1}{2}\dot{z}^2 - mgz \tag{1.30}$$

3. What is the meaning of the canonical transformation $\Phi(q, P) = \alpha qP$?

$$p = \alpha P; Q = \alpha q.$$

4.

Determine the Hamiltonian of the system if the Lagrangian has a form

$$L = \dot{x}^2/2 - ax^2 + bx^4 + cx^2\dot{x}^2.$$

Write Hamiltonian equations of motion.

Solution:

$$p = \dot{x}(1 + 2cx^2) \tag{1.31}$$

$$H = p\dot{x} - L = \frac{p^2}{2(1 + 2cx^2)} - ax^2 - bx^4 \tag{1.32}$$

5.

Find the frequency of small oscillations for particle moving in the potential
 $U(x) = V(ax^2 - \sin^2 bx)$.

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6. Evaluate a Poisson brackets $[\mathbf{p}, (\mathbf{a}\mathbf{p})]$

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$[\mathbf{p}, (\mathbf{a}\mathbf{p})] = 0$

7. Find a differential cross-section for the scattering of fast particles by the potential

$U = V \exp(-\kappa r^2)$

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