Extensible Normalization for Multi-Species Plasma Models

U. Shumlak

July 8, 2016

Normalizing equation systems provide convenient means to evaluate the relative importance of each term or effect for a given set of physical parameters. The general normalization procedure involves defining characteristic values for the system variables and then deriving nondimensional parameters. Nondimensional numbers are useful for determining dynamic similitude – guaranteeing self-similar solutions among differing scale problems.

Since the derivation of normalized equation systems requires choosing characteristic values, normalizations are not unique. However, a consistent normalization permits comparisons among different problems, and some choices during the normalization procedure results in nondimensional parameters that are well-defined among a variety of models. This technical note presents a normalization for multi-species plasma models that is consistent for the Boltzmann-Maxwell equation system and for reduced plasma kinetic and fluid models, including the Vlasov-Poisson [1], multi-fluid [2–6], and magnetohydrodynamic (MHD) [7] plasma models. The normalization and variable definitions establishes **the standard** that should be used by the University of Washington Computational Plasma Dynamics Group. The normalization selected in this derivation results in physically relevant nondimensional parameters, which indicate the relative importance of the electromagnetic effects.

In an effort to simplify the presentation of the normalized multi-species plasma models, collisions and atomic reactions are ignored and the pressure is assumed to be isotropic. Relaxing these assumptions does not change the steps of the derivation, but it does introduce additional terms. For example, a three-fluid plasma-neutral model is presented in Ref. [8].

The set of plasma models presented in this technical note provides a hierarchy of model fidelity with a consistent normalization with nondimensional numbers that can be compared based on the plasma parameters. The highest fidelity model considered is the Boltzmann-Maxwell system, and it forms the basis from which all reduced models stem. However, the organization of this document follows the derivation of the normalization, which begins with a plasma model of intermediate fidelity and then extends the normalization to lower and higher fidelity plasma models.

1 Normalized Multi-Fluid Plasma Model

A normalization is sought that is preserved through the asymptotic approximations of the center-of-mass (COM) single-fluid MHD model. A primary result of the approximations is the pre-Maxwellian form of Ampere's law and suggests that Ampere's law should inform the appropriate normalization.

The dimensional form of Ampere's law in SI units is expressed as

$$-\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{j}, \tag{1}$$

where $\mathbf{j} = \sum_\alpha q_\alpha n_\alpha \mathbf{v}_\alpha$. To eliminate the vacuum permittivity and permeability, the reference proton plasma frequency and the reference Alfvén speed in a proton plasma are introduced through

$$\epsilon_0 = \frac{e^2 n_0}{m_p \omega_p^2}, \tag{2}$$

and

$$\mu_0 = \frac{B_0^2}{m_p n_0 V_A^2}. \tag{3}$$

Reference values are denoted with a 0 subscript. Substituting these expressions and reference values into Eq. (1) gives

$$-\frac{e^2 n_0}{m_p \omega_p^2} \frac{E_0}{\tau} \frac{\partial \tilde{\mathbf{E}}}{\partial \tilde{t}} + \frac{m_p n_0 V_A^2}{B_0^2} \frac{B_0}{L} \tilde{\nabla} \times \tilde{\mathbf{B}} = e n_0 v_0 \tilde{\mathbf{j}} = e n_0 v_0 \sum_\alpha Z_\alpha \tilde{n}_\alpha \tilde{\mathbf{v}}_\alpha,$$

where all variables in the differential equation with a tilde are normalized by the reference values, e.g. $\mathbf{E} = E_0 \tilde{\mathbf{E}}$. Relating the reference values defines a set of normalizations,

$$E_0 = v_0 B_0, \tag{4}$$

$$v_0 = \frac{\tau}{L} = V_A. \tag{5}$$

Note that the normalization of Eq. (5) specifies the reference time to be the characteristic Alfvén transit time. Introducing the proton cyclotron frequency

$$\omega_c = \frac{e B_0}{m_p}, \tag{6}$$

reduces Ampere's law to a normalized form,

$$-\frac{(\omega_c \tau)^2}{(\omega_p \tau)^2} \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = (\omega_c \tau)\, \mathbf{j}, \tag{7}$$

where $\mathbf{j} = \sum_\alpha Z_\alpha n_\alpha \mathbf{v}_\alpha$ and tildes have been dropped for clarity. The normalization process has introduced nondimensional parameters for the plasma frequency $\omega_p \tau$ and the cyclotron frequency $\omega_c \tau$.

Faraday's law is normalized using the same definitions used for Ampere's law. The dimensional form of Faraday's law in SI units is

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{8}$$

which when expressed using the reference values gives

$$\frac{B_0}{\tau}\frac{\partial\tilde{\mathbf{B}}}{\partial t} + \frac{E_0}{L}\tilde{\nabla}\times\tilde{\mathbf{E}} = 0.$$

Replacing the reference value for E_0 from Eq. (4) simplifies Faraday's law to normalized form,

$$\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0, \tag{9}$$

with the tildes removed.

Gauss's law is normalized in a similar fashion. The dimensional form in SI units is

$$\epsilon_0\nabla\cdot\mathbf{E} = \rho_c = \sum_\alpha q_\alpha n_\alpha, \tag{10}$$

where $\rho_c = \sum_\alpha q_\alpha n_\alpha$. Using the reference values and normalizations gives

$$\frac{e^2 n_0}{m_p\omega_p^2}\frac{E_0}{L}\tilde{\nabla}\cdot\tilde{\mathbf{E}} = en_0\tilde{\rho}_c = en_0\sum_\alpha Z_\alpha\tilde{n}_\alpha,$$

which reduces to

$$\frac{(\omega_c\tau)}{(\omega_p\tau)^2}\nabla\cdot\mathbf{E} = \rho_c, \tag{11}$$

where $\rho_c = \sum_\alpha Z_\alpha n_\alpha$.

The fluid equations for the multi-fluid plasma model provide governing equations for the number density, momentum, and total energy for each species. The continuity equation is

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot(n_\alpha\mathbf{v}_\alpha) = 0, \tag{12}$$

for each species α. Substituting the reference values into Eq. (12) gives

$$\frac{n_0}{\tau}\frac{\partial\tilde{n}_\alpha}{\partial\tilde{t}} + \frac{n_0 v_0}{L}\tilde{\nabla}\cdot(\tilde{n}_\alpha\tilde{\mathbf{v}}_\alpha) = 0.$$

The coefficients cancel from the definitions of the normalizations, and the normalized continuity equation simplifies to

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot(n_\alpha\mathbf{v}_\alpha) = 0, \tag{13}$$

without tildes.

The momentum equation is

$$m_\alpha\frac{\partial\left(n_\alpha\mathbf{v}_\alpha\right)}{\partial t} + m_\alpha\nabla\cdot(n_\alpha\mathbf{v}_\alpha\mathbf{v}_\alpha) + \nabla p_\alpha = q_\alpha n_\alpha\mathbf{E} + q_\alpha n_\alpha\mathbf{v}_\alpha\times\mathbf{B} \tag{14}$$

for each species α. The momentum equation is rewritten with the same reference values used in deriving Eqs. (7,9,13) with an additional reference value for pressure, p_0. The species mass and charge are normalized by introducing nondimensional parameters for the mass ratio $A_\alpha \equiv m_\alpha/m_p$, which is normalized to the proton mass – effectively the atomic mass, and for the ionization state $Z_\alpha \equiv q_\alpha/e$, which is normalized to the elementary charge. Substituting these reference values and nondimensional parameters into Eq. (14) gives

$$\frac{m_p n_0 v_0}{\tau} A_\alpha \frac{\partial \left(\tilde{n}_\alpha \tilde{\mathbf{v}}_\alpha\right)}{\partial \tilde{t}} + \frac{m_p n_0 v_0^2}{L} A_\alpha \tilde{\nabla} \cdot \left(\tilde{n}_\alpha \tilde{\mathbf{v}}_\alpha \tilde{\mathbf{v}}_\alpha\right) + \frac{p_0}{L} \tilde{\nabla} \tilde{p}_\alpha = \\ e n_0 E_0 Z_\alpha \tilde{n}_\alpha \tilde{\mathbf{E}} + e n_0 v_0 B_0 Z_\alpha \tilde{n}_\alpha \tilde{\mathbf{v}}_\alpha \times \tilde{\mathbf{B}}.$$

Dividing by the leading coefficient, substituting the normalizations gives

$$A_\alpha \frac{\partial \left(\tilde{n}_\alpha \tilde{\mathbf{v}}_\alpha\right)}{\partial \tilde{t}} + A_\alpha \tilde{\nabla} \cdot \left(\tilde{n}_\alpha \tilde{\mathbf{v}}_\alpha \tilde{\mathbf{v}}_\alpha\right) + \frac{p_0}{m_p n_0 v_0^2} \tilde{\nabla} \tilde{p}_\alpha = \left(\frac{e B_0}{m_p} \tau\right) Z_\alpha \tilde{n}_\alpha \left(\tilde{\mathbf{E}} + \tilde{\mathbf{v}}_\alpha \times \tilde{\mathbf{B}}\right). \tag{15}$$

Relating the reference pressure to the reference magnetic field defines an additional normalization

$$p_0 = m_p n_0 v_0^2 = \frac{B_0^2}{\mu_0}. \tag{16}$$

Inserting the nondimensional cyclotron frequency gives the normalized momentum equation

$$A_\alpha \frac{\partial \left(n_\alpha \mathbf{v}_\alpha\right)}{\partial t} + A_\alpha \nabla \cdot \left(n_\alpha \mathbf{v}_\alpha \mathbf{v}_\alpha\right) + \nabla p_\alpha = \left(\omega_c \tau\right) Z_\alpha n_\alpha \left(\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}\right). \tag{17}$$

The tildes have again been dropped for clarity. Since the Lorentz force couples the charged fluids through the electromagnetic fields, the momentum balance law does not reduce to a conservation law.

The fluid energy equation is

$$\frac{\partial \varepsilon_\alpha}{\partial t} + \nabla \cdot \left(\left(\varepsilon_\alpha + p_\alpha\right) \mathbf{v}_\alpha\right) = q_\alpha n_\alpha \mathbf{v}_\alpha \cdot \mathbf{E} \tag{18}$$

for each species α, where

$$\varepsilon_\alpha = \frac{1}{\gamma - 1} p_\alpha + \frac{1}{2} m_\alpha n_\alpha v_\alpha^2. \tag{19}$$

Using the same normalized variables and reference values and setting the reference energy value such that $\varepsilon_0 = p_0$, Eq. (18) is transformed into

$$\frac{p_0}{\tau} \frac{\partial \tilde{\varepsilon}_\alpha}{\partial \tilde{t}} + \frac{p_0 v_0}{L} \tilde{\nabla} \cdot \left(\left(\tilde{\varepsilon}_\alpha + \tilde{p}_\alpha\right) \tilde{\mathbf{v}}_\alpha\right) = e n_0 v_0 E_0 Z_\alpha \tilde{n}_\alpha \tilde{\mathbf{v}}_\alpha \cdot \tilde{\mathbf{E}},$$

which reduces to the normalized energy equation,

$$\frac{\partial \varepsilon_\alpha}{\partial t} + \nabla \cdot \left(\left(\varepsilon_\alpha + p_\alpha\right) \mathbf{v}_\alpha\right) = \left(\omega_c \tau\right) Z_\alpha n_\alpha \mathbf{v}_\alpha \cdot \mathbf{E}, \tag{20}$$

through cancellation and the definition of the nondimensional cyclotron frequency. The normalized total energy is now given by

$$\varepsilon_\alpha = \frac{1}{\gamma - 1} p_\alpha + \frac{1}{2} A_\alpha n_\alpha v_\alpha^2. \tag{21}$$

The tildes have been dropped from Eqs. (20,21) for clarity.

1.1 Alternative Nondimensional Parameter

While the normalized equations given in the previous section are complete, the presence of the nondimensional cyclotron frequency implies a magnetized plasma. Alternative expressions can be derived by replacing this nondimensional parameter with an equivalent one.

$$\omega_c\tau = \frac{eB_0}{m_p}\frac{L}{V_A} = \frac{eB_0}{m_p}\frac{L\sqrt{\mu_0 m_p n_0}}{B_0} = \sqrt{\frac{e^2 n_0}{\epsilon_0 m_p}}\frac{L}{c} = \frac{\omega_p L}{c} = \frac{L}{\delta_p}, \tag{22}$$

where δ_p is the proton skin depth.

Replacing the nondimensional cyclotron frequency with the nondimensional skin depth, which is defined even in an unmagnetized plasma, results in the Maxwell's equations of

$$-\frac{1}{(\omega_p\tau)^2}\left(\frac{L}{\delta_p}\right)^2\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B} = \left(\frac{L}{\delta_p}\right)\mathbf{j}, \tag{23}$$

$$\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0, \tag{9}$$

$$\frac{1}{(\omega_p\tau)^2}\frac{L}{\delta_p}\nabla\cdot\mathbf{E} = \rho_c. \tag{24}$$

Note that the nondimensional parameters that multiply the displacement current in the normalized Ampere's law, Eq. (23), can be reduced as

$$\frac{1}{(\omega_p\tau)^2}\left(\frac{L}{\delta_p}\right)^2 = \left(\frac{V_A}{c}\right)^2.$$

Using the nondimensional skin depth, the fluid equations become

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot(n_\alpha\mathbf{v}_\alpha) = 0, \tag{13}$$

$$\frac{\partial(n_\alpha\mathbf{v}_\alpha)}{\partial t} + \nabla\cdot(n_\alpha\mathbf{v}_\alpha\mathbf{v}_\alpha) + \frac{1}{A_\alpha}\nabla p_\alpha = \left(\frac{L}{\delta_p}\right)\frac{Z_\alpha}{A_\alpha}n_\alpha(\mathbf{E} + \mathbf{v}_\alpha\times\mathbf{B}), \tag{25}$$

$$\frac{\partial\varepsilon_\alpha}{\partial t} + \nabla\cdot((\varepsilon_\alpha + p_\alpha)\mathbf{v}_\alpha) = \left(\frac{L}{\delta_p}\right)Z_\alpha n_\alpha\mathbf{v}_\alpha\cdot\mathbf{E}. \tag{26}$$

These equations constitute the normalization from which the other plasma models are extended.

2 Center-of-Mass Multi-Fluid Plasma Model

Extending the multi-fluid plasma model of Sec. 1 to the single-fluid MHD model begins by expressing the equation system for a center-of-mass (COM) fluid. The COM mass density and velocity are defined as

$$\rho = \sum_\alpha A_\alpha \tilde{n}_\alpha = \frac{1}{m_p n_0} \sum_\alpha m_\alpha n_\alpha, \tag{27}$$

$$\mathbf{v} = \frac{\sum_\alpha A_\alpha \tilde{n}_\alpha \tilde{\mathbf{v}}_\alpha}{\sum_\alpha A_\alpha \tilde{n}_\alpha} = \frac{1}{v_0} \frac{\sum_\alpha m_\alpha n_\alpha \mathbf{v}_\alpha}{\sum_\alpha m_\alpha n_\alpha}, \tag{28}$$

in both normalized and dimensional values. The tildes are dropped for the remainder of the derivation.

The continuity equation, Eq. (13), is rewritten as a mass conservation law for each species,

$$\frac{\partial \left(A_\alpha n_\alpha\right)}{\partial t} + \nabla \cdot \left(A_\alpha n_\alpha \mathbf{v}_\alpha\right) = 0, \tag{29}$$

which is summed over all species to give the COM continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{30}$$

A drift velocity is defined for each species that relates the species velocity to the COM velocity,

$$\mathbf{w}_\alpha = \mathbf{v}_\alpha - \mathbf{v}, \tag{31}$$

which can be used to describe the evolution of the species densities,

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot \left(n_\alpha \left(\mathbf{v} + \mathbf{w}_\alpha\right)\right) = 0. \tag{32}$$

Note that from Eqs. (27,28,31) the relative drift velocities have the property of $\sum_\alpha A_\alpha n_\alpha \mathbf{w}_\alpha = 0$.

The COM momentum equation is derived by summing the species momentum equations over all species,

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla \cdot \left(\sum_\alpha A_\alpha n_\alpha \left(\mathbf{w}_\alpha \mathbf{v} + (\mathbf{v} + \mathbf{w}_\alpha) \mathbf{w}_\alpha \right) \right) + \nabla p = \left(\frac{L}{\delta_p}\right) \left(\sum_\alpha Z_\alpha n_\alpha \mathbf{E} + \sum_\alpha Z_\alpha n_\alpha \mathbf{v}_\alpha \times \mathbf{B} \right)$$

or

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla \cdot \left(\sum_\alpha A_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha \right) + \nabla p = \left(\frac{L}{\delta_p}\right) \left(\rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B}\right), \tag{33}$$

where the total pressure is defined as the sum of the partial pressures, $p = \sum_\alpha p_\alpha$.

The evolution of the species momenta is described by the separate species momentum equations, Eq. (25), and eliminating one species momentum equation since it is redundant. Alternatively, an evolution equation can be derived for the relative drift velocities for each species.

Using Eq. (30), Eq. (33) can be expressed as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + \frac{1}{\rho}\nabla\cdot\left(\sum_\alpha A_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha\right) + \frac{1}{\rho}\nabla p = \left(\frac{L}{\delta_p}\right)\left(\frac{\rho_c}{\rho}\mathbf{E} + \frac{\mathbf{j}}{\rho}\times\mathbf{B}\right). \tag{34}$$

In a similar manner the species momentum equation can be manipulated to give

$$\frac{\partial \mathbf{v}_\alpha}{\partial t} + \mathbf{v}_\alpha\cdot\nabla\mathbf{v}_\alpha + \frac{1}{A_\alpha n_\alpha}\nabla p_\alpha = \left(\frac{L}{\delta_p}\right)\frac{Z_\alpha}{A_\alpha}\left(\mathbf{E} + \mathbf{v}_\alpha\times\mathbf{B}\right). \tag{35}$$

Subtracting the two velocity equations yields an evolution equation for the drift velocity of each species,

$$\frac{\partial \mathbf{w}_\alpha}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{w}_\alpha + \mathbf{w}_\alpha\cdot\nabla\left(\mathbf{v}+\mathbf{w}_\alpha\right) - \frac{1}{\rho}\nabla\cdot\left(\sum_\alpha A_\alpha n_\alpha \mathbf{w}_\alpha\mathbf{w}_\alpha\right) + \frac{1}{A_\alpha n_\alpha}\nabla p_\alpha - \frac{1}{\rho}\nabla p = \left(\frac{L}{\delta_p}\right)\left(\left(\frac{Z_\alpha}{A_\alpha} - \frac{\rho_c}{\rho}\right)\mathbf{E} + \left(\frac{Z_\alpha \mathbf{v}_\alpha}{A_\alpha} - \frac{\mathbf{j}}{\rho}\right)\times\mathbf{B}\right). \tag{36}$$

While a relative drift velocity equation exists for each species, it only needs to be solved for all but one species. The remaining drift velocity can be computed from $\sum_\alpha A_\alpha n_\alpha \mathbf{w}_\alpha = 0$.

The COM energy equation is defined by summing over all species,

$$\frac{\partial \varepsilon}{\partial t} + \nabla\cdot\left(\sum_\alpha\left(\varepsilon_\alpha + p_\alpha\right)\left(\mathbf{v}+\mathbf{w}_\alpha\right)\right) = \left(\frac{L}{\delta_p}\right)\sum_\alpha Z_\alpha n_\alpha \mathbf{v}_\alpha\cdot\mathbf{E},$$

or

$$\frac{\partial \varepsilon}{\partial t} + \nabla\cdot\left(\left(\varepsilon + p\right)\mathbf{v}\right) + \nabla\cdot\left(\sum_\alpha\left(\varepsilon_\alpha + p_\alpha\right)\mathbf{w}_\alpha\right) = \left(\frac{L}{\delta_p}\right)\mathbf{j}\cdot\mathbf{E}, \tag{37}$$

where the COM total energy is defined by

$$\varepsilon = \sum_\alpha \varepsilon_\alpha \tag{38}$$

$$= \sum_\alpha \frac{1}{\gamma-1}p_\alpha + \frac{1}{2}A_\alpha n_\alpha v_\alpha^2 \tag{39}$$

$$= \frac{1}{\gamma-1}p + \frac{1}{2}\rho v^2 + \frac{1}{2}\sum_\alpha A_\alpha n_\alpha w_\alpha^2. \tag{40}$$

The species energy equation is expressed as

$$\frac{\partial \varepsilon_\alpha}{\partial t} + \nabla\cdot\left(\left(\varepsilon_\alpha + p_\alpha\right)\left(\mathbf{v}+\mathbf{w}_\alpha\right)\right) = \left(\frac{L}{\delta_p}\right)Z_\alpha n_\alpha\left(\mathbf{v}+\mathbf{w}_\alpha\right)\cdot\mathbf{E}. \tag{41}$$

The current density in COM variables is

$$\mathbf{j} = \mathbf{v}\sum_\alpha Z_\alpha n_\alpha + \sum_\alpha Z_\alpha n_\alpha \mathbf{w}_\alpha$$

$$= \rho_c\mathbf{v} + \sum_\alpha Z_\alpha n_\alpha \mathbf{w}_\alpha. \tag{42}$$

The fluid equations are coupled to the complete Maxwell's equations which are expressed using the COM variables

$$-\frac{1}{(\omega_p \tau)^2}\left(\frac{L}{\delta_p}\right)^2 \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \left(\frac{L}{\delta_p}\right)\mathbf{j}, \tag{23}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0. \tag{9}$$

The derivation of the COM equation system for the multi-fluid plasma model has introduced no approximations and the system is mathematically equivalent to the system from Sec. 1.

2.1 Infinite Speed of Light - Charge Neutral Multi-Fluid Plasma Model

Applying the asymptotic approximation of infinite speed of light is achieved through the limit of $\omega_p \tau \to \infty$. Ampere's law, Eq. (23), becomes

$$\mathbf{j} = \left(\frac{\delta_p}{L}\right)\nabla \times \mathbf{B}, \tag{43}$$

and Gauss's law, Eq. (24), reduces to $\rho_c = 0$, charge neutrality. Faraday's law, Eq. (9), remains unaffected.

Propagating these reductions through the fluid equations results in a total charge continuity equation of

$$\frac{\partial\left(\sum_\alpha Z_\alpha n_\alpha\right)}{\partial t} + \nabla \cdot \left(\left(\sum_\alpha Z_\alpha n_\alpha\right)\mathbf{v}\right) + \nabla \cdot \left(\sum_\alpha Z_\alpha n_\alpha \mathbf{w}_\alpha\right) = \nabla \cdot \left(\sum_\alpha Z_\alpha n_\alpha \mathbf{w}_\alpha\right) = 0, \tag{44}$$

due to charge neutrality.

The COM momentum equation, Eq. (33), becomes

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla \cdot \left(\sum_\alpha A_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha\right) + \nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{45}$$

and the relative drift velocity is given by

$$\frac{\partial \mathbf{w}_\alpha}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{w}_\alpha + \mathbf{w}_\alpha \cdot \nabla(\mathbf{v} + \mathbf{w}_\alpha) - \frac{1}{\rho}\nabla\cdot\left(\sum_\alpha A_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha\right) + \frac{1}{A_\alpha n_\alpha}\nabla p_\alpha - \frac{1}{\rho}\nabla p = \left(\frac{L}{\delta_p}\right)\frac{Z_\alpha}{A_\alpha}(\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}) - \frac{1}{\rho}(\nabla \times \mathbf{B}) \times \mathbf{B}. \tag{46}$$

The COM energy equation becomes

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot ((\varepsilon + p)\mathbf{v}) + \nabla \cdot \left(\sum_\alpha (\varepsilon_\alpha + p_\alpha)\mathbf{w}_\alpha\right) = (\nabla \times \mathbf{B}) \cdot \mathbf{E}, \tag{47}$$

and the species energy equation remains unchanged. The COM and species continuity equations also remain unchanged.

2.2 Negligible Electron Inertia - Massless Electron Multi-Fluid Plasma Model

Applying the asymptotic approximation of negligible electron inertia to the complete COM equation system is accomplished through the limit of $A_e \to 0$, which does not alter Maxwell's equations.

The COM continuity equation is unchanged except that the mass density and COM velocity are reinterpreted as

$$\rho = \sum_{\substack{\alpha \\ \alpha \neq e}} A_\alpha \tilde{n}_\alpha,$$

$$\mathbf{v} = \frac{\sum_{\substack{\alpha \\ \alpha \neq e}} A_\alpha \tilde{n}_\alpha \mathbf{v}_\alpha}{\sum_{\substack{\alpha \\ \alpha \neq e}} A_\alpha \tilde{n}_\alpha}.$$

The summation of the relative advection term in the COM momentum equation, Eq. (33), now excludes the electron species.

$$\frac{\partial\left(\rho\mathbf{v}\right)}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\mathbf{v}\right) + \nabla\cdot\left(\sum_{\substack{\alpha \\ \alpha \neq e}} A_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha\right) + \nabla p = \left(\frac{L}{\delta_p}\right)\left(\rho_c\mathbf{E} + \mathbf{j}\times\mathbf{B}\right) \tag{48}$$

The relative drift velocity equation, Eq. (36), is unchanged except the summation again excludes the electron species.

$$\frac{\partial\mathbf{w}_\alpha}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{w}_\alpha + \mathbf{w}_\alpha\cdot\nabla\left(\mathbf{v}+\mathbf{w}_\alpha\right) - \frac{1}{\rho}\nabla\cdot\left(\sum_{\substack{\alpha \\ \alpha \neq e}} A_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha\right) + \frac{1}{A_\alpha n_\alpha}\nabla p_\alpha - \frac{1}{\rho}\nabla p = \left(\frac{L}{\delta_p}\right)\left(\left(\frac{Z_\alpha}{A_\alpha} - \frac{\rho_c}{\rho}\right)\mathbf{E} + \left(\frac{Z_\alpha\mathbf{v}_\alpha}{A_\alpha} - \frac{\mathbf{j}}{\rho}\right)\times\mathbf{B}\right) \tag{49}$$

These seemingly minor modifications are particularly consequential in a two-species (electron, ion) plasma, since $\mathbf{v} = \mathbf{v}_i$ and $\mathbf{w}_i = 0$. In this case, the COM momentum equation becomes

$$\frac{\partial\left(\rho\mathbf{v}\right)}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\mathbf{v}\right) + \nabla p = \left(\frac{L}{\delta_p}\right)\left(\rho_c\mathbf{E} + \mathbf{j}\times\mathbf{B}\right). \tag{50}$$

Instead of using the relative drift velocity equation, an appropriate electron momentum equation is most easily derived from the multi-fluid form by setting $Z_e = -1$ and applying the limit $A_e \to 0$.

$$-\frac{1}{n_e}\nabla p_e = \left(\frac{L}{\delta_p}\right)\left(\mathbf{E} + \mathbf{v}_e\times\mathbf{B}\right),$$

or

$$-\frac{1}{n_e}\nabla p_e = \left(\frac{L}{\delta_p}\right)\left(\mathbf{E} + \left(\mathbf{v}+\mathbf{w}_e\right)\times\mathbf{B}\right) \tag{51}$$

to provide a state equation that can be solved for the electron relative drift velocity.

The COM energy equation, Eq. (37), is unchanged except that the electron energy only contains the internal energy component, $\varepsilon_e = p_e/(\gamma - 1)$. For the two-species plasma, the COM energy equation becomes

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \left((\varepsilon + p)\, \mathbf{v} + \frac{\gamma}{\gamma - 1} p_e \mathbf{w}_e \right) = \left(\frac{L}{\delta_p} \right) \mathbf{j} \cdot \mathbf{E}. \tag{52}$$

2.3 Hall-MHD Model

Applying both asymptotic approximations from Secs. 2.1 and 2.2 yields the Hall-MHD model. The evolution of the electromagnetic fields is described by Eq. (9)

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{9}$$

and Eq. (43)

$$\mathbf{j} = \left(\frac{\delta_p}{L} \right) \nabla \times \mathbf{B}, \tag{43}$$

where they have been repeated for convenience.

Charge neutrality results from the first asymptotic approximation and results in $\rho_c = 0$ and $\nabla \cdot \mathbf{j} = 0$. While a general set of fluid equations with arbitrary number of species can be derived for the Hall-MHD model, the fluid equations for the two-species plasma are most common. The fluid equations for the two-species are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{30}$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B} \tag{53}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \left(\frac{\delta_p}{L} \right) \frac{1}{n_e} \left((\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p_e \right) \tag{54}$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \left((\varepsilon + p)\, \mathbf{v} + \frac{\gamma}{\gamma - 1} p_e \mathbf{w}_e \right) = (\nabla \times \mathbf{B}) \cdot \mathbf{E} \tag{55}$$

where the electron drift velocity is given by

$$\mathbf{w}_e = -\frac{\mathbf{j}}{n_e} \tag{56}$$

$$= -\left(\frac{\delta_p}{L} \right) \frac{\nabla \times \mathbf{B}}{n_e}, \tag{57}$$

as a result of charge neutrality.

The MHD energy often represents the sum of the plasma energy ε and the magnetic field energy $B^2/(2\mu_0)$. Performing the dot product of Eq. (9) with $\mathbf{B}$ yields

$$\frac{\partial}{\partial t} \left(\frac{B^2}{2} \right) = -\nabla \times \mathbf{E} \cdot \mathbf{B} = -\nabla \times \mathbf{B} \cdot \mathbf{E} - \nabla \cdot (\mathbf{E} \times \mathbf{B}). \tag{58}$$

Performing the cross product of Eq. (51) with $\mathbf{B}$ and rearranging the terms gives

$$\begin{aligned}\mathbf{E}\times\mathbf{B} &= -\left(\frac{\delta_p}{L}\right)\frac{1}{n_e}\nabla p_e\times\mathbf{B} - ((\mathbf{v}+\mathbf{w}_e)\times\mathbf{B})\times\mathbf{B}\\ &= -\left(\frac{\delta_p}{L}\right)\frac{1}{n_e}\nabla p_e\times\mathbf{B} + B^2\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})\,\mathbf{B} + B^2\mathbf{w}_e - (\mathbf{w}_e\cdot\mathbf{B})\,\mathbf{B}.\end{aligned} \tag{59}$$

Substituting Eq. (59) into Eq. (58) gives an evolution equation for the magnetic field energy, which is $B^2/2$ in normalized form. The MHD total energy equation is found by summing this equation with Eq. (55).

$$\frac{\partial}{\partial t}\left(\varepsilon+\frac{B^2}{2}\right)+\nabla\cdot\left(\left(\varepsilon+p+B^2\right)\mathbf{v}+(\mathbf{B}\cdot\mathbf{v})\,\mathbf{B}\right)+\nabla\cdot\left(\left(\varepsilon_e+p_e+B^2\right)\mathbf{w}_e+(\mathbf{B}\cdot\mathbf{w}_e)\,\mathbf{B}\right) = \left(\frac{\delta_p}{L}\right)\nabla\cdot\left(\frac{1}{n_e}\nabla p_e\times\mathbf{B}\right)$$

The total energy is defined as $e=\varepsilon+B^2/2$ and similarly for the electron total energy, $e_e=\varepsilon_e+B^2/2$. Using these definitions and Eq. (57) gives the Hall MHD total energy equation

$$\frac{\partial e}{\partial t}+\nabla\cdot\left(\left(e+p+\frac{B^2}{2}\right)\mathbf{v}+(\mathbf{B}\cdot\mathbf{v})\,\mathbf{B}\right) = \left(\frac{\delta_p}{L}\right)\nabla\cdot\left(\left(e_e+p_e+\frac{B^2}{2}\right)\frac{\nabla\times\mathbf{B}}{n_e}-\frac{\mathbf{B}\cdot\nabla\times\mathbf{B}}{n_e}\mathbf{B}+\frac{1}{n_e}\nabla p_e\times\mathbf{B}\right), \tag{60}$$

where the total energy has its usual definition of

$$e=\frac{1}{\gamma-1}p+\frac{1}{2}A_i n_i v^2+\frac{B^2}{2}. \tag{61}$$

Note that in the ideal MHD limit, $\delta_p/L\to 0$, and the right-hand side of Eq. (60) vanishes.

3 Continuum Kinetic Plasma Model

The normalization presented in the previous sections leads to consistent plasma models that extend from the multi-fluid plasma model to the Hall-MHD plasma model. The same normalization can be extended to the continuum kinetic plasma model, namely the Boltzmann-Maxwell equation system. For simplicity the collision term is expressed as a BGK operator [9, 10], though more complete collision operators can be used. The Boltzmann equation describes the evolution of continuum distribution functions for each species α

$$\frac{\partial f_\alpha}{\partial t}+\mathbf{v}\cdot\nabla f_\alpha+\frac{q_\alpha}{m_\alpha}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f_\alpha=\nu_\alpha\left(f_\alpha^M-f_\alpha\right), \tag{62}$$

where ν_α is the spatially-dependent self-relaxation rate and f_α^M is a Maxwellian distribution function defined taking the first three moments of f_α. Additional collisional effects can be included and do not alter the normalization procedure presented here. Maxwell's equations govern the evolution of the electric and magnetic fields through Eqs. (1) and (8) and couple to the evolution of the plasma state.

The normalized variables have been defined for the lower fidelity plasma models using reference values

$$t = \tilde{t}\tau \qquad \mathbf{x} = \tilde{\mathbf{x}}L \qquad v_0 = \frac{L}{\tau} = V_A \qquad E_0 = v_0 B_0$$
$$f_\alpha = \tilde{f}_\alpha n_0 \qquad q_\alpha = Z_\alpha e \qquad m_\alpha = A_\alpha m_p \qquad \nu_\alpha = \tilde{\nu}_\alpha \nu_p$$
$$V_A = \left(\frac{B_0^2}{\mu_0 m_p n_0}\right)^{1/2} \qquad \omega_p = \left(\frac{e^2 n_0}{\epsilon_0 m_p}\right)^{1/2} \qquad \omega_c = \frac{eB_0}{m_p} \qquad \delta_p = \frac{c}{\omega_p}$$

where the reference self-relaxation rate is defined for a proton species. Substituting the normalizations into Eq. (62) gives

$$\frac{n_0}{\tau}\frac{\partial \tilde{f}_\alpha}{\partial \tilde{t}} + \frac{n_0 v_0}{L}\tilde{\mathbf{v}} \cdot \tilde{\nabla} \tilde{f}_\alpha + \frac{n_0 e B_0}{m_p}\frac{Z_\alpha}{A_\alpha}\left(\tilde{\mathbf{E}} + \tilde{\mathbf{v}} \times \tilde{\mathbf{B}}\right) \cdot \tilde{\nabla}_v \tilde{f}_\alpha = n_0 \nu_p \tilde{\nu}_\alpha \left(\tilde{f}_\alpha^M - \tilde{f}_\alpha\right),$$

which simplifies to

$$\frac{\partial \tilde{f}_\alpha}{\partial \tilde{t}} + \tilde{\mathbf{v}} \cdot \tilde{\nabla} \tilde{f}_\alpha + (\omega_c \tau)\frac{Z_\alpha}{A_\alpha}\left(\tilde{\mathbf{E}} + \tilde{\mathbf{v}} \times \tilde{\mathbf{B}}\right) \cdot \tilde{\nabla}_v \tilde{f}_\alpha = (\nu_p \tau)\, \tilde{\nu}_\alpha \left(\tilde{f}_\alpha^M - \tilde{f}_\alpha\right).$$

Using the relationship between the nondimensional parameters, the normalized Boltzmann equation becomes

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \left(\frac{L}{\delta_p}\right)\frac{Z_\alpha}{A_\alpha}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \nabla_v f_\alpha = (\nu_p \tau)\, \nu_\alpha \left(f_\alpha^M - f_\alpha\right). \tag{63}$$

Note that $L/\delta_p = \omega_p L/c$, which commonly appears as the nondimensional parameter for the normalized Boltzmann equation with a reference time of L/c. In the continuum kinetic model, the current density is given by

$$\mathbf{j} = \sum_\alpha Z_\alpha \int d\mathbf{v}' \mathbf{v}' f_\alpha(\mathbf{v}'). \tag{64}$$

The normalized forms of Maxwell's equations

$$-\frac{1}{(\omega_p \tau)^2}\left(\frac{L}{\delta_p}\right)^2 \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \left(\frac{L}{\delta_p}\right)\mathbf{j}, \tag{23}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{9}$$

complete the Boltzmann-Maxwell continuum kinetic plasma model.

3.1 Vlasov-Poisson Plasma Model

A common simplification of the Boltzmann-Maxwell continuum kinetic plasma model is to assume a collisionless, electrostatic plasma. The resulting equation system constitutes the Vlasov-Poisson continuum kinetic plasma model. The collisionless form of the normalized Boltzmann equation, Eq. (63), is expressed as

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha + \left(\frac{L}{\delta_p}\right)\frac{Z_\alpha}{A_\alpha}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f_\alpha = 0, \tag{65}$$

where the magnetic field is independent of plasma dynamics. The electric field is described by an electrostatic potential ϕ, such that $\mathbf{E} = -\nabla\phi$. Poisson's equation is derived by combining the electrostatic potential with Gauss's law, Eq. (10). The normalization follows that of Eqs. (11) and (24) to yield

$$-\frac{1}{(\omega_p\tau)^2}\frac{L}{\delta_p}\nabla^2\phi = \rho_c. \tag{66}$$

The charge density is given by

$$\rho_c = \sum_\alpha Z_\alpha \int d\mathbf{v}' f_\alpha(\mathbf{v}') \tag{67}$$

in the continuum kinetic model.

4 Summary of Normalized Plasma Models

The equations systems are summarized and ordered from highest fidelity to lowest fidelity plasma models.

Boltzmann-Maxwell Plasma Model

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha + \left(\frac{L}{\delta_p}\right)\frac{Z_\alpha}{A_\alpha}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f_\alpha = \left(\nu_p\tau\right)\nu_\alpha\left(f_\alpha^M - f_\alpha\right) \tag{63}$$

$$\mathbf{j} = \sum_\alpha Z_\alpha \int d\mathbf{v}'\mathbf{v}' f_\alpha(\mathbf{v}') \tag{64}$$

$$-\frac{1}{(\omega_p\tau)^2}\left(\frac{L}{\delta_p}\right)^2\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B} = \left(\frac{L}{\delta_p}\right)\mathbf{j} \tag{23}$$

$$\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0 \tag{9}$$

Vlasov-Poisson Plasma Model

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha + \left(\frac{L}{\delta_p}\right)\frac{Z_\alpha}{A_\alpha}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f_\alpha = 0 \tag{65}$$

$$\rho_c = \sum_\alpha Z_\alpha \int d\mathbf{v}' f_\alpha(\mathbf{v}') \tag{67}$$

$$\mathbf{E} = -\nabla\phi \tag{68}$$

$$-\frac{1}{(\omega_p\tau)^2}\frac{L}{\delta_p}\nabla^2\phi = \rho_c. \tag{66}$$

Multi-Fluid Plasma Model

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot\left(n_\alpha\mathbf{v}_\alpha\right) = 0 \tag{13}$$

$$\frac{\partial\left(n_\alpha\mathbf{v}_\alpha\right)}{\partial t} + \nabla\cdot\left(n_\alpha\mathbf{v}_\alpha\mathbf{v}_\alpha\right) + \frac{1}{A_\alpha}\nabla p_\alpha = \left(\frac{L}{\delta_p}\right)\frac{Z_\alpha}{A_\alpha}n_\alpha\left(\mathbf{E} + \mathbf{v}_\alpha\times\mathbf{B}\right) \tag{25}$$

$$\frac{\partial\varepsilon_\alpha}{\partial t} + \nabla\cdot\left(\left(\varepsilon_\alpha + p_\alpha\right)\mathbf{v}_\alpha\right) = \left(\frac{L}{\delta_p}\right)Z_\alpha n_\alpha\mathbf{v}_\alpha\cdot\mathbf{E} \tag{26}$$

$$\mathbf{j} = \sum_\alpha Z_\alpha n_\alpha\mathbf{v}_\alpha \tag{69}$$

$$-\frac{1}{(\omega_p\tau)^2}\left(\frac{L}{\delta_p}\right)^2\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B} = \left(\frac{L}{\delta_p}\right)\mathbf{j} \tag{23}$$

$$\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0 \tag{9}$$

Center-of-Mass Multi-Fluid Plasma Model Note the COM multi-fluid plasma model is equivalent to the multi-fluid plasma model, and therefore shares the same fidelity.

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\right) = 0 \tag{30}$$

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot\left(n_\alpha\left(\mathbf{v} + \mathbf{w}_\alpha\right)\right) = 0 \tag{32}$$

$$\frac{\partial\left(\rho\mathbf{v}\right)}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\mathbf{v}\right) + \nabla\cdot\left(\sum_\alpha A_\alpha n_\alpha\mathbf{w}_\alpha\mathbf{w}_\alpha\right) + \nabla p = \left(\frac{L}{\delta_p}\right)\left(\rho_c\mathbf{E} + \mathbf{j}\times\mathbf{B}\right) \tag{33}$$

$$\begin{aligned}\frac{\partial\mathbf{w}_\alpha}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{w}_\alpha + \mathbf{w}_\alpha\cdot\nabla\left(\mathbf{v} + \mathbf{w}_\alpha\right) - \frac{1}{\rho}\nabla\cdot\left(\sum_\alpha A_\alpha n_\alpha\mathbf{w}_\alpha\mathbf{w}_\alpha\right) + \\ \frac{1}{A_\alpha n_\alpha}\nabla p_\alpha - \frac{1}{\rho}\nabla p = \left(\frac{L}{\delta_p}\right)\left(\left(\frac{Z_\alpha}{A_\alpha} - \frac{\rho_c}{\rho}\right)\mathbf{E} + \left(\frac{Z_\alpha\mathbf{v}_\alpha}{A_\alpha} - \frac{\mathbf{j}}{\rho}\right)\times\mathbf{B}\right)\end{aligned} \tag{36}$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \left(\left(\varepsilon + p \right) \mathbf{v} \right) + \nabla \cdot \left(\sum_\alpha \left(\varepsilon_\alpha + p_\alpha \right) \mathbf{w}_\alpha \right) = \left(\frac{L}{\delta_p} \right) \mathbf{j} \cdot \mathbf{E} \tag{37}$$

$$\frac{\partial \varepsilon_\alpha}{\partial t} + \nabla \cdot \left(\left(\varepsilon_\alpha + p_\alpha \right) \left(\mathbf{v} + \mathbf{w}_\alpha \right) \right) = \left(\frac{L}{\delta_p} \right) Z_\alpha n_\alpha \left(\mathbf{v} + \mathbf{w}_\alpha \right) \cdot \mathbf{E} \tag{41}$$

$$\rho_c = \sum_\alpha Z_\alpha n_\alpha \tag{70}$$

$$\mathbf{j} = \rho_c \mathbf{v} + \sum_\alpha Z_\alpha n_\alpha \mathbf{w}_\alpha \tag{42}$$

$$-\frac{1}{\left(\omega_p \tau \right)^2} \left(\frac{L}{\delta_p} \right)^2 \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \left(\frac{L}{\delta_p} \right) \mathbf{j} \tag{23}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{9}$$

Charge Neutral Multi-Fluid Plasma Model

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) = 0 \tag{30}$$

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot \left(n_\alpha \left(\mathbf{v} + \mathbf{w}_\alpha \right) \right) = 0 \tag{32}$$

$$\frac{\partial \left(\rho \mathbf{v} \right)}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \mathbf{v} \right) + \nabla \cdot \left(\sum_\alpha A_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha \right) + \nabla p = \left(\nabla \times \mathbf{B} \right) \times \mathbf{B} \tag{45}$$

$$\frac{\partial \mathbf{w}_\alpha}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{w}_\alpha + \mathbf{w}_\alpha \cdot \nabla \left(\mathbf{v} + \mathbf{w}_\alpha \right) - \frac{1}{\rho} \nabla \cdot \left(\sum_\alpha A_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha \right) + \frac{1}{A_\alpha n_\alpha} \nabla p_\alpha - \frac{1}{\rho} \nabla p = \left(\frac{L}{\delta_p} \right) \frac{Z_\alpha}{A_\alpha} \left(\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B} \right) - \frac{1}{\rho} \left(\nabla \times \mathbf{B} \right) \times \mathbf{B} \tag{46}$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \left(\left(\varepsilon + p \right) \mathbf{v} \right) + \nabla \cdot \left(\sum_\alpha \left(\varepsilon_\alpha + p_\alpha \right) \mathbf{w}_\alpha \right) = \left(\nabla \times \mathbf{B} \right) \cdot \mathbf{E} \tag{47}$$

$$\frac{\partial \varepsilon_\alpha}{\partial t} + \nabla \cdot \left(\left(\varepsilon_\alpha + p_\alpha \right) \left(\mathbf{v} + \mathbf{w}_\alpha \right) \right) = \left(\frac{L}{\delta_p} \right) Z_\alpha n_\alpha \left(\mathbf{v} + \mathbf{w}_\alpha \right) \cdot \mathbf{E} \tag{41}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{9}$$

Massless Electron Multi-Fluid Plasma Model

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{30}$$

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha (\mathbf{v} + \mathbf{w}_\alpha)) = 0 \tag{32}$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla \cdot \left(\sum_{\substack{\alpha \\ \alpha \neq e}} A_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha \right) + \nabla p = \left(\frac{L}{\delta_p}\right) (\rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B}) \tag{48}$$

$$\frac{\partial \mathbf{w}_\alpha}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{w}_\alpha + \mathbf{w}_\alpha \cdot \nabla (\mathbf{v} + \mathbf{w}_\alpha) - \frac{1}{\rho} \nabla \cdot \left(\sum_{\substack{\alpha \\ \alpha \neq e}} A_\alpha n_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha \right) + \frac{1}{A_\alpha n_\alpha} \nabla p_\alpha - \frac{1}{\rho} \nabla p = \left(\frac{L}{\delta_p}\right) \left(\left(\frac{Z_\alpha}{A_\alpha} - \frac{\rho_c}{\rho} \right) \mathbf{E} + \left(\frac{Z_\alpha \mathbf{v}_\alpha}{A_\alpha} - \frac{\mathbf{j}}{\rho} \right) \times \mathbf{B} \right) \tag{49}$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot ((\varepsilon + p)\mathbf{v}) + \nabla \cdot \left(\sum_\alpha (\varepsilon_\alpha + p_\alpha) \mathbf{w}_\alpha \right) = \left(\frac{L}{\delta_p}\right) \mathbf{j} \cdot \mathbf{E} \tag{37}$$

$$\frac{\partial \varepsilon_\alpha}{\partial t} + \nabla \cdot ((\varepsilon_\alpha + p_\alpha)(\mathbf{v} + \mathbf{w}_\alpha)) = \left(\frac{L}{\delta_p}\right) Z_\alpha n_\alpha (\mathbf{v} + \mathbf{w}_\alpha) \cdot \mathbf{E} \tag{41}$$

$$\mathbf{j} = \rho_c \mathbf{v} + \sum_\alpha Z_\alpha n_\alpha \mathbf{w}_\alpha \tag{42}$$

$$-\frac{1}{(\omega_p \tau)^2} \left(\frac{L}{\delta_p}\right)^2 \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \left(\frac{L}{\delta_p}\right) \mathbf{j} \tag{23}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{9}$$

Massless Electron Two-Fluid Plasma Model (Special Case)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{30}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e (\mathbf{v} + \mathbf{w}_e)) = 0 \tag{71}$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = \left(\frac{L}{\delta_p}\right) (\rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B}) \tag{50}$$

$$-\frac{1}{n_e} \nabla p_e = \left(\frac{L}{\delta_p}\right) (\mathbf{E} + (\mathbf{v} + \mathbf{w}_e) \times \mathbf{B}) \tag{51}$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \left((\varepsilon + p)\,\mathbf{v} + \frac{\gamma}{\gamma - 1} p_e \mathbf{w}_e \right) = \left(\frac{L}{\delta_p} \right) \mathbf{j} \cdot \mathbf{E} \tag{52}$$

$$\frac{\partial p_e}{\partial t} + \nabla \cdot \left(\gamma p_e \left(\mathbf{v} + \mathbf{w}_e \right) \right) = \left(\frac{L}{\delta_p} \right) (\gamma - 1)\, Z_\alpha n_\alpha \left(\mathbf{v} + \mathbf{w}_e \right) \cdot \mathbf{E} \tag{72}$$

$$\mathbf{j} = \rho_c \mathbf{v} - e n_e \mathbf{w}_e \tag{73}$$

$$-\frac{1}{(\omega_p \tau)^2} \left(\frac{L}{\delta_p} \right)^2 \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \left(\frac{L}{\delta_p} \right) \mathbf{j} \tag{23}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{9}$$

Hall-MHD Plasma Model

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{30}$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B} \tag{53}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \left(\frac{\delta_p}{L} \right) \frac{1}{n_e} \left((\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p_e \right) \tag{54}$$

$$\frac{\partial e}{\partial t} + \nabla \cdot \left(\left(e + p + \frac{B^2}{2} \right) \mathbf{v} + (\mathbf{B} \cdot \mathbf{v})\,\mathbf{B} \right) = \left(\frac{\delta_p}{L} \right) \nabla \cdot \left(\left(e_e + p_e + \frac{B^2}{2} \right) \frac{\nabla \times \mathbf{B}}{n_e} - \frac{\mathbf{B} \cdot \nabla \times \mathbf{B}}{n_e} \mathbf{B} + \frac{1}{n_e} \nabla p_e \times \mathbf{B} \right) \tag{60}$$

$$e = \frac{1}{\gamma - 1} p + \frac{1}{2} A_i n_i v^2 + \frac{B^2}{2} \tag{61}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{9}$$

References

[1] G. V. Vogman, P. Colella, and U. Shumlak. Dory–Guest–Harris instability as a benchmark for continuum kinetic Vlasov–Poisson simulations of magnetized plasmas. *Journal of Computational Physics*, 277(0):101 – 120, 2014.

[2] U. Shumlak and J. Loverich. Approximate Riemann solver for the two-fluid plasma model. *Journal of Computational Physics*, 187(2):620–638, 2003.

[3] A. Hakim, J. Loverich, and U. Shumlak. A high resolution wave propagation scheme for ideal two-fluid plasma equations. *Journal of Computational Physics*, 219(1):418 – 442, 2006.

[4] A. Hakim and U. Shumlak. Two-fluid physics and field-reversed configurations. *Physics of Plasmas*, 14(5):055911, 2007.

[5] U. Shumlak, R. Lilly, N. Reddell, E. Sousa, and B. Srinivasan. Advanced physics calculations using a multi-fluid plasma model. *Computer Physics Communications*, 182(9):1767–1770, 2011.

[6] B. Srinivasan and U. Shumlak. Analytical and computational study of the ideal full two-fluid plasma model and asymptotic approximations for Hall-magnetohydrodynamics. *Physics of Plasmas*, 18(9):092113, 2011.

[7] J. P. Freidberg. Ideal magnetohydrodynamic theory of magnetic fusion systems. *Reviews of Modern Physics*, 54(3):801–902, July 1982.

[8] E. T. Meier and U. Shumlak. A general nonlinear fluid model for reacting plasma-neutral mixtures. *Physics of Plasmas*, 19(7):072508, 2012.

[9] P. L. Bhatnagar, E. P. Gross, and M. Krook. A model for collision processes in gases. i. small amplitude processes in charged and neutral one-component systems. *Physical Review*, 94:511–525, May 1954.

[10] S. Livi and E. Marsch. Comparison of the Bhatnagar-Gross-Krook approximation with the exact Coulomb collision operator. *Physical Review A*, 34:533–540, Jul 1986.