Parameter Identification Near Periodic Orbits of Hybrid Dynamical Systems

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Hybrid periodic orbits model interesting physical systems

**biochemistry:** reaction networks  
Elowitz and Leibler 2000, Alur et al. 2001

**biomechanics:** terrestrial locomotion  
Holmes et al. 2006, Revzen 2009
Example: vertical hopper
Hybrid dynamical system

\[ \dot{x} = F_1(x) \]

\[ \dot{x} = F_2(x) \]
Trajectory for a hybrid dynamical system

$\dot{x} = F_1(x)$

$\dot{x} = F_2(x)$
Trajectory for a hybrid dynamical system

\[ \dot{x} = F_1(x) \]

\[ \phi(t, x_0) \]

\[ \dot{x} = F_2(x) \]
Periodic orbit $\gamma$ for a hybrid dynamical system

\[ \dot{x} = F_1(x) \]

\[ \dot{x} = F_2(x) \]
Identification of initial conditions

\[ Y(\varphi(t, z)) = y(t) \]

\[ \eta_i = Y(\varphi(iT, z^*)) + w_i \]

\[ w_i \text{ iid random variables} \]

Identification problem

\[ \text{arg min}_{z \in D_j} \varepsilon(z, \{\eta_i\}) \]

where

\[ \varepsilon(z, \{\eta_i\}) := \sum_i \| Y(\varphi(iT, z)) - \eta_i \|^2 \]
Identification of initial conditions

\[ Y(\phi(t, z)) = y(t) \]

\[ \eta_i = Y(\phi(iT, z^*)) + w_i, \quad w_i \text{ iid random variables} \]

Identification problem

\[ \arg\min_{z \in D} \varepsilon(z, \{\eta_i\}) \]

\[ \varepsilon(z, \{\eta_i\}) := \sum_i \|Y(\phi(iT, z)) - \eta_i\|_2 \]
Identification of initial conditions

\[ Y(\phi(t, z)) = y(t) \]

\[ \eta_i = Y(\phi(iT, z^*)) + w_i, \quad w_i \text{ iid random variables} \]
Identification of initial conditions

$$Y(\phi(t, z)) = y(t)$$

$$\eta_i = Y(\phi(iT, z^*)) + w_i,$$

$$w_i \text{ iid random variables}$$

Identification problem

Solve $$\arg \min_{z \in D_j} \mathcal{E}(z, \{ \eta_i \}),$$ where

$$\mathcal{E}(z, \{ \eta_i \}) := \sum_i \| Y(\phi(iT, z)) - \eta_i \|^2.$$
Identification on hybrid model

Assumption (smooth observations)

\( Y \) is smooth along trajectories, i.e. \( Y(\phi(t,z)) \) is a smooth function of \( t \).
Assumption (smooth observations)

$Y$ is smooth along trajectories, i.e. $Y(\phi(t, z))$ is a smooth function of $t$.

Identification on $\bigcup_j D_j$

$$\arg\min_{z \in D_j} \varepsilon(z, \{\eta_i\})$$
Identification on hybrid model

Assumption (smooth observations)

\( Y \) is smooth along trajectories, i.e. \( Y(\phi(t, z)) \) is a smooth function of \( t \).

Identification on \( \bigcup_j D_j \)

\[
\arg \min_{z \in D_j} \varepsilon(z, \{ \eta_i \})
\]

- \( \nabla \varepsilon \) undefined on \( G_j \subset D_j \)
- \( R_j \) not generally invertible
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**global optimization** needed
Theorem (Hirsch and Smale 1974, Grizzle et al. 2002) The Poincaré map $P$ is smooth in a neighborhood of $\xi$. 

smooth dynamical system 

\[ \dot{x} = F(x) \]

hybrid dynamical system 

\[ \dot{x} = F_1(x) \]

\[ \dot{x} = F_2(x) \]
The Poincaré map for periodic orbit $\gamma$

**Smooth Dynamical System**
- $D$
- $\Sigma$
- $\xi$
- $\dot{x} = F(x)$

**Hybrid Dynamical System**
- $\dot{x} = F_1(x)$
- $\dot{x} = F_2(x)$
- $\Sigma$
- $\gamma$
- $D_1$
- $D_2$
- $G_1$
- $G_2$
- $R_1$
- $R_2$
Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

The Poincaré map $\mathbf{P}$ is smooth in a neighborhood of $\xi$. 

smooth dynamical system

$\mathbf{D} \quad \Sigma \quad x_0 \quad \mathbf{\dot{x}} = F(x)$

hybrid dynamical system

$\Sigma \quad x_0 \quad \mathbf{\dot{x}} = F_1(x) \quad \mathbf{\dot{x}} = F_2(x)$
The Poincaré map for periodic orbit $\gamma$

Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

The Poincaré map $P$ is smooth in a neighborhood of $\xi$.

smooth dynamical system

hybrid dynamical system
The Poincaré map for periodic orbit $\gamma$

**smooth dynamical system**

$\dot{x} = F(x)$

**hybrid dynamical system**

$\dot{x} = F_1(x)$ and $\dot{x} = F_2(x)$
Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

The Poincaré map $P$ is smooth in a neighborhood of $\xi$. 

smooth dynamical system

$D \ni x_0 \rightarrow \sum \rightarrow P(x_0) \rightarrow \gamma$

$\dot{x} = F(x)$

hybrid dynamical system

$D_1 \ni x_0 \rightarrow \sum \rightarrow P(x_0) \rightarrow G_1$

$\dot{x} = F_1(x)$

$D_2 \ni x_0 \rightarrow \sum \rightarrow P(x_0) \rightarrow G_2$

$\dot{x} = F_2(x)$

$R_1 \rightarrow R_2$
Rank of the Poincaré map $P$ with fixed point $P(\xi) = \xi$

- **Smooth dynamical system**
  \[ \dot{x} = F(x) \]

- **Hybrid dynamical system**
  \[ \dot{x} = F_1(x) \]

Hirsch and Smale 1974

Wendel and Ames 2010
Rank of the Poincaré map $P$ with fixed point $P(\xi) = \xi$

**smooth dynamical system**

\[
\begin{align*}
D & \quad \Sigma \quad \xi \\
\nearrow & \quad \searrow \\
P(x_0) & \quad x_0
\end{align*}
\]

\[\dot{x} = F(x)\]

\[\text{rank } DP(\xi) = \dim D - 1\]

Hirsch and Smale 1974

**hybrid dynamical system**

\[
\begin{align*}
R_2 & \quad D_1 \quad P(x_0) \\
G_2 & \quad \nearrow \\
R_1 & \quad D_2
\end{align*}
\]

\[\dot{x} = F_1(x)\]

\[\dot{x} = F_2(x)\]
Rank of the Poincaré map $P$ with fixed point $P(\xi) = \xi$

**smooth dynamical system**

$$\begin{align*}
\text{rank } DP(\xi) &= \dim D - 1 \\
\text{Hirsch and Smale 1974}
\end{align*}$$

**hybrid dynamical system**

$$\begin{align*}
\text{rank } DP(\xi) &\leq \min_j \dim D_j - 1 \\
\text{Wendel and Ames 2010}
\end{align*}$$
Model reduction near a periodic orbit

Theorem (Burden, Revzen, Sastry CDC 2011)

Let $n = \min_j \dim D_j$. If $DP_n$ has constant rank $r$ near $\xi$, then after a finite amount of time all trajectories starting near $\gamma$ collapse to a collection of hybrid invariant $(r+1)$-dimensional submanifolds $M_j \subset D_j$. 

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Model reduction near a periodic orbit

\[ \dot{x} = F_1(x) \]

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Corollary (Burden, Revzen, Sastry CDC 2011)

The submanifolds \( M_j \) determine a hybrid system with periodic orbit \( \gamma \).
Corollary (Burden, Revzen, Sastry CDC 2011)

The submanifolds $M_j$ determine a hybrid system with periodic orbit $\gamma$. Each $R_j|_{G_j \cap M_j}$ is a diffeomorphism.
Reduction in the vertical hopper

With parameters

\[ m = 1, \mu = 2, k = 10, b = 5, \ell_0 = 2, a = 20, \dot{\sigma} = 2\pi, g = 2 \]
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Numerically linearizing Poincaré map \( P \) on ground we find \( DP(\xi) \) has eigenvalues \( \simeq -0.25 \pm 0.70j \)

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Theorem \( \implies \) after one cycle, dynamics collapse to 1-DOF hopper
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Solve \( \arg \min_{z \in D_j} \varepsilon(z, \{\eta_i\}) \), where \( \varepsilon(z, \{\eta_i\}) := \sum_i \|Y(\phi(iT, z)) - \eta_i\|^2 \).
Identification on reduced hybrid model

Assumption (smooth observations)

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Identification on \( \bigcup_j D_j \)

\[ \arg \min_{z \in D_j} \varepsilon(z, \{\eta_i\}) \]

- \( \nabla \varepsilon \) undefined on \( G_j \subset D_j \)
- \( R_j \) not generally invertible

**global optimization** needed
Identification on reduced hybrid model

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<table>
<thead>
<tr>
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<th>Identification on ( \bigcup_j M_j )</th>
</tr>
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Identification on \( \bigcup_j M_j \)

\[
\arg \min_{z \in M_j} \varepsilon(z, \{\eta_i\})
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- \( \nabla \varepsilon \) well-defined on \( G_j \cap M_j \)
- \( R_j \mid_{M_j} \) invertible
### Assumption (smooth observations)

$Y$ is smooth along trajectories, i.e. $Y(\phi(t, z))$ is a smooth function of $t$.

### Identification on $\bigcup_j D_j$

$$\arg \min_{z \in D_j} \varepsilon(z, \{\eta_i\})$$

- $\nabla \varepsilon$ undefined on $G_j \subset D_j$
- $R_j$ not generally invertible

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### Identification on $\bigcup_j M_j$

$$\arg \min_{z \in M_j} \varepsilon(z, \{\eta_i\})$$

- $\nabla \varepsilon$ well-defined on $G_j \cap M_j$
- $R_j|_{M_j}$ invertible

**first-order algorithms** applicable
Identifying initial condition for vertical hopper

Observe position of upper mass at 20Hz, additive noise with variance 0.2.

$$\sigma_0, y_0, \dot{y}_0 \approx (8.0, 1.5, 1.1)$$: initial
$$\sigma, y, \dot{y} \approx (4.7, 1.6, 1.0)$$: actual
$$\sigma^*, y^*, \dot{y}^* \approx (4.6, 1.6, 1.1)$$: estimated
Identifying initial condition for vertical hopper

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\[(\sigma^*, y^*, \dot{y}^*) \approx (4.6, 1.6, 1.1) : \text{estimated}\]
Open Problems

Determine identifiability of parameters on reduced model.

Numerically approximate coordinates for reduced model.

Accommodate aperiodic, open-loop, or stochastic dynamics in reduction.
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Our contribution

We provide a first-order algorithm for parameter identification in hybrid dynamical models of biomechanics & biochemistry.
Appendix
Assumptions on hybrid periodic orbit $\gamma$

**Assumption (transversality)**

Periodic orbit $\gamma$ passes transversely through each guard $G_j$

**Assumption (dwell time)**

$\exists \varepsilon > 0 : \text{periodic orbit } \gamma \text{ spends at least } \varepsilon \text{ time units in each domain } D_j$