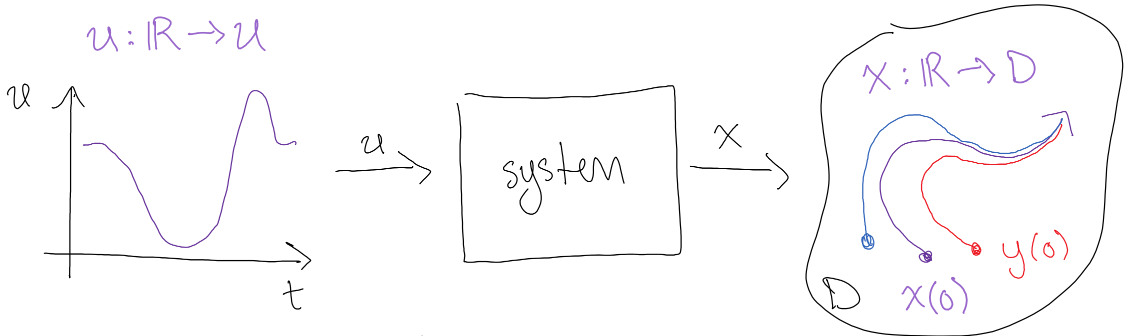
conceptual framework

- suppose: feedforward input yields stable behavior:



- then trajectories forget initial conditions: $\lim_{t\to\infty} d(x(t), y(t)) = 0$
- if forgetting happens at an exponential rate,
  $d(x(t), y(t)) \leq e^{-ct} \cdot d(x(0), y(0))$, call the system <u>contractive</u>

– Lohmiller, Slotine Automatica 1998

– Aminzare, Sontag CDC 2014

classical contractivity

- classical (smooth) differential / difference equations:

(globally) contractive if (and only if*) locally contractive

$d(x(t), y(t))$

$\leq e^{ct} \cdot d(x(0), y(0))$ $\quad \dot{x} = F(t, x)$ $\quad \lambda_{max} \frac{1}{2}(D_x F + D_x F^T) < c$

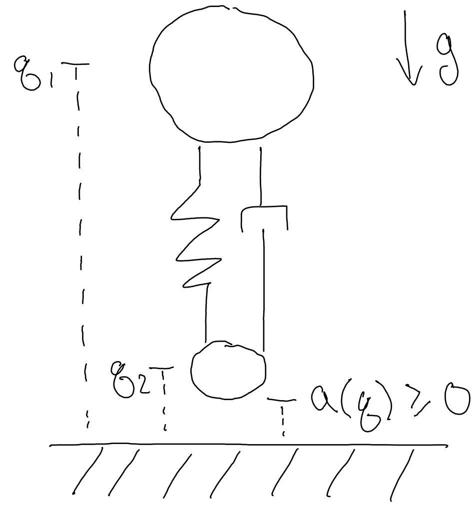
$\leq K^t \cdot d(x(0), y(0))$ $\quad x^+ = R(t, x)$ $\quad \sigma_{max}(D_x R) < K$

<u>takeaway</u>: global property (feedforward inputs yield stable behaviors)
is intimately connected to local design properties
(does your system behave locally like a spring-mass-damper?)

* in the right coordinates / with the right distance metric

mechanical systems subject to unilateral constraints

ex:



- unfortunately, classical contraction doesn't apply to legs, since dynamics are <u>hybrid</u> (piecewise-defined):

$$M(q)\,\ddot{q} = f(t, q, \dot{q}, u) + \lambda \cdot Da(q)$$

$$\dot{q}^+ = \Delta(q)\,\dot{q}^-, \quad a(q) = 0$$

$q, \dot{q}$: positions, velocities

$M(q)$: mass matrix

f : forces (applied & Coriolis)

$a(q) \geq 0$: nonpenetration constraints

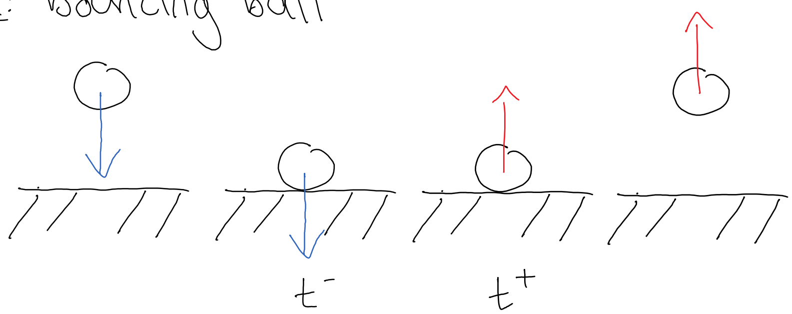
λ : contact forces

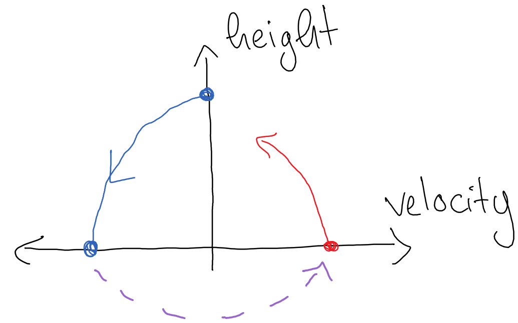
$\Delta(q)$: impact law

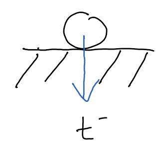
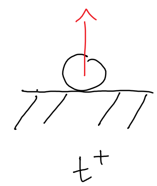
Ballard Arch. Rat. Mech. Anal. 2000

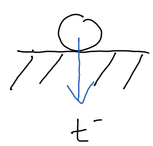
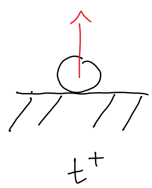
aside: distance in mechanical systems

ex: bouncing ball





Question: what is the distance between  &  ?

(my) Answer: d( , ) = 0

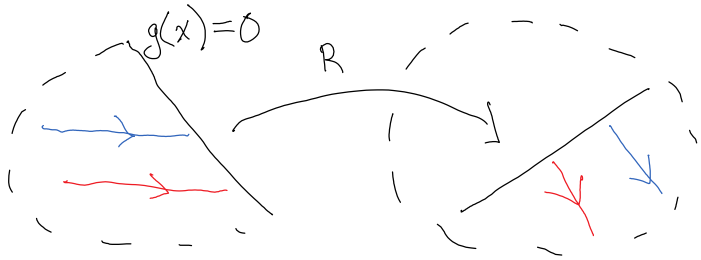
- Schatzman Math. Comput. Modelling 1998
- Burden, Gonzalez, Vasudevan, Bajcsy, Sastry IEEE TAC 2015

generalizing contractivity to hybrid systems

- mechanical systems mix continuous-time (CT) $\dot{x} = F(t, x)$, $g(x) > 0$
  and discrete-time (DT) $x^+ = R(x)$, $g(x) \leq 0$
- so it makes sense that these <u>hybrid</u> systems are contractive if they are locally contractive in both:

$$(CT)\ \lambda_{max}\, \tfrac{1}{2}(D_x F^T + D_x F) < 0 \quad \& \quad (DT)\ \sigma_{max}(\cancel{D_x R}) < 1$$

* DT condition must be modified to account for discontinuity in F across g:



$$S = D_x R + (F^+ - D_x R \cdot F^-) \cdot \frac{Dg}{Dg \cdot F^-}$$

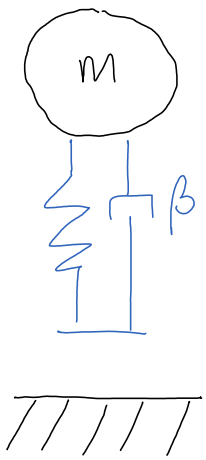
(saltation matrix)

– DT $\sigma_{max}(S) < 1$

– Aizerman, Gantmacher Quart. Journ. Mech. and Applied Math. 1958
– Burden, Coogan arXiv: 1804.04122

limb impact is NOT contractive (!)

ex:



elastic limb

massless foot

$$S = \begin{bmatrix} 1 & 0 \\ \pm\frac{\beta}{m} & 1 \end{bmatrix}$$

$$\sigma_{max}(S) > 1$$

ex:

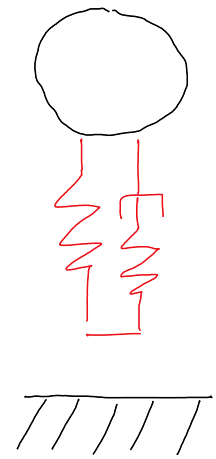
elastic limb

massed foot

$$S_{TD} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta/m_1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_{max}(S_{TD}) > 1$$

ex:



viscoelastic limb

massless foot

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1+\delta\cdot\alpha \end{bmatrix}$$

$$\sigma_{max}(S) > 1$$

- based on the vignettes, I did not expect this...