Dynamics of terrestrial locomotion

Periplaneta americana

video courtesy of Poly-PEDAL Lab, UC Berkeley
Empirically, animals use few degrees-of-freedom

Cockroach dynamics $\sim 7$ dimensional \hspace{1cm} (Revzen & Guckenheimer 2011)
Empirically, animals use few degrees-of-freedom

Cockroach dynamics $\sim 7$ dimensional

Mechanisms:

- Neural synchronization
  - Cohen et al. 1982
- Physiological symmetry
  - Golubitsky et al. 1999
- Muscle activation synergy
  - Ting & Macpherson 2005
- Granular media solidification
  - Li et al. 2009

(Revzen & Guckenheimer 2011)
Reduced-order model describes dynamic locomotion

physical system

animal, robot

↓ ↓

detailed model

10–100 DOF

↓ ↓

reduced-order model

< 10 DOF

Full & Koditschek 1999
Mechanical self-stabilization in animals

physical system

*Blaberus discoidalis*
Jindrich & Full 2002

↓

reduced model

Lateral Leg-Spring
Schmitt & Holmes 2000

video courtesy of Poly-PEDAL Lab
Fast & maneuverable dynamic robots

Saranli et al. 2001

physical system
RHex, DynaRoACH

Hoover, Burden et al. 2010

↓

reduced-order model
SLIP, LLS

Saranli et al. 2001

Ghigliazza et al. 2003

Proctor & Holmes 2008
Obstacles to using reduced-order models

- Physical system: animal, robot
- Detailed model: 10–100 DOF
- Reduced-order model: < 10 DOF

Reduction and ID for Hybrid Models

Sam Burden

March 22, 2013 7
Obstacles to using reduced-order models

physical system
animal, robot

detailed model
10–100 DOF

reduced-order model
< 10 DOF

Mazor et al. 1998
Ferrari-Trecate et al. 2003
Vidal 2008
Grizzle et al. 2002
Ames et al. 2006
Proctor et al. 2010
Obstacles to using reduced-order models

- Physical system: animal, robot
- Detailed model: 10–100 DOF
- Reduced-order model: < 10 DOF

Reduction:
- Grizzle et al. 2002
- Ames et al. 2006
- Proctor et al. 2010
Obstacles to using reduced-order models

physical system
animal, robot

identification

detailed model
10–100 DOF

reduction

reduced-order model
< 10 DOF

identification
Mazor et al. 1998
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reduction
Grizzle et al. 2002
Ames et al. 2006
Proctor et al. 2010
Overview

Motivation

reduced-order models describe dynamic locomotion

Reduction

hybrid dynamics reduce dimensionality near periodic orbits

Identification

reduction enables scalable algorithm for parameter estimation

Conclusion

novel quantitative predictions for biomechanics
model-based design and control of dynamic robots
Reduction

physical system
animal, robot

detailed model
10–100 DOF

reduction

reduced-order model
< 10 DOF
Example (vertical hopper)
Hybrid dynamical system

\[ \dot{x} = F_1(x) \]
\[ \dot{x} = F_2(x) \]
Trajectory for a hybrid dynamical system
Trajectory for a hybrid dynamical system
Periodic orbit $\gamma$ for a hybrid dynamical system

\[ \dot{x} = F_1(x) \]

\[ \dot{x} = F_2(x) \]
Assumptions on hybrid periodic orbit $\gamma$
Assumptions on hybrid periodic orbit $\gamma$

Assumption (transversality)

*periodic orbit* $\gamma$ *passes transversely through each guard* $G_j$
Assumptions on hybrid periodic orbit $\gamma$

**Assumption (transversality)**

*periodic orbit* $\gamma$ *passes transversely through each guard* $G_j$

**Assumption (dwell time)**

$\exists \varepsilon > 0$ : *periodic orbit* $\gamma$ *spends at least* $\varepsilon$ *time units in each domain* $D_j$
Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

The Poincaré map $P$ is smooth in a neighborhood of $\xi$. 

Poincaré map for periodic orbit $\gamma$
Poincaré map for periodic orbit $\gamma$

**smooth dynamical system**

Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

The Poincaré map $P$ is smooth in a neighborhood of $\xi$.

**hybrid dynamical system**

The Poincaré map $P^i$ is smooth in a neighborhood of $\xi$. 

$D$, $\Sigma$, $x = F(x)$

$G_1$, $D_1$, $x = F_1(x)$

$G_2$, $D_2$, $x = F_2(x)$

$R_1$, $R_2$, $\gamma$
Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

The Poincaré map \( P \) is smooth in a neighborhood of \( \xi \).
Poincaré map for periodic orbit $\gamma$

**Smooth Dynamical System**

$$\dot{x} = F(x)$$

**Hybrid Dynamical System**

$$\dot{x} = F_1(x)$$

Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

The Poincaré map $P$ is smooth in a neighborhood of $\xi$. 

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Reduction and ID for Hybrid Models

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Poincaré map for periodic orbit $\gamma$

**smooth dynamical system**

$$D \sum \xi P(x_0) \dot{x} = F(x)$$

**hybrid dynamical system**

$$\dot{x} = F_1(x) \quad \dot{x} = F_2(x)$$
Poincaré map for periodic orbit $\gamma$

Smooth dynamical system

$$\sum \xi \quad P(x_0) \quad \dot{x} = F(x)$$

Hybrid dynamical system

$$\sum x_0 \quad P(x_0) \quad \dot{x} = F_1(x)$$

Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

The Poincaré map $P$ is smooth in a neighborhood of $\xi$. 
Rank of Poincaré map $P$ with fixed point $P(\xi) = \xi$

**smooth dynamical system**

$D \sum \xi \dot{x} = F(x)$

**hybrid dynamical system**

$\dot{x} = F_1(x)$

$\dot{x} = F_2(x)$

Hirsch and Smale 1974

$\text{rank } D_P(\xi) \leq \min_j \text{dim } D_j - 1$

Wendel and Ames 2010
Rank of Poincaré map $P$ with fixed point $P(\xi) = \xi$

**smooth dynamical system**

\[ \dot{x} = F(x) \]

\[ \text{rank } DP(\xi) = \dim D - 1 \]

Hirsch and Smale 1974

**hybrid dynamical system**

\[ \dot{x} = F_1(x) \]

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\[
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**hybrid dynamical system**

\[
\text{rank } DP(\xi) \leq \min_j \dim D_j - 1
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Wendel and Ames 2010
Example (rank-deficient Poincaré map)

\[A \in \mathbb{R}^{n \times n}\]

If \(A\) is nilpotent (i.e. \(A^n = 0\)), then \(\text{rank}(D) = 0\).

\[y \in \mathbb{R}^n\]
\[\dot{y} = 0\]
\[x \in [0, 1]\]
\[\dot{x} = 1\]

\[(1, y) \mapsto (0, Ay)\]
Example (rank-deficient Poincaré map)

If $A \in \mathbb{R}^{n \times n}$ is nilpotent (i.e. $A^n = 0_{n \times n}$), then $\text{rank } DP^n = 0$. 
Theorem (Burden, Revzen, Sastry CDC 2011)

Let $n = \min_j \dim D_j - 1$. If $\text{rank} \, D_P n = r$ near $\xi$, then trajectories starting near $\gamma$ contract to a collection of hybrid-invariant $(r + 1)$-dimensional submanifolds $M_j \subset D_j$ in finite time.
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Model reduction near hybrid periodic orbit $\gamma$

Corollary (Burden, Revzen, Sastry CDC 2011)

The submanifolds $M_j$ determine a hybrid system with periodic orbit $\gamma$. 

Model reduction near hybrid periodic orbit $\gamma$

Corollary (Burden, Revzen, Sastry CDC 2011)

The submanifolds $M_j$ determine a hybrid system with periodic orbit $\gamma$.

$\gamma$ is asymptotically stable in the original hybrid system $\iff \gamma$ is asymptotically stable in the reduced hybrid system.
Example (exact model reduction in vertical hopper)

Numerically linearizing Poincaré map, we find \( DP(\xi) \) has eigenvalues \( \approx -0.25 \pm 0.70 j \), therefore \( DP^2 \) is constant rank near \( \xi \).

Theorem \( \Rightarrow \) dynamics collapse to 1-DOF hopper.

Interpretation: unilateral (Lagrangian) constraint appears after one “hop”.

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Reduction and ID for Hybrid Models
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Interpretation: unilateral (Lagrangian) constraint appears after one “hop”
Approximate model reduction near hybrid periodic orbit $\gamma$

Theorem (Burden, Revzen, Sastry (in preparation))

If $\xi$ is exponentially stable and $\text{rank} \, DP_n(\xi) = r$, then trajectories starting near $\gamma$ contract super-exponentially to a collection of hybrid-invariant $(r + 1)$-dimensional submanifolds $M_j \subset D_j$. 
Theorem (Burden, Revzen, Sastry (in preparation))

*If* $\xi$ *is exponentially stable and* $\operatorname{rank} DP^n(\xi) = r$, *then trajectories starting near* $\gamma$ *contract super-exponentially to a collection of hybrid-invariant* $(r + 1)$-*dimensional submanifolds* $M_j \subset D_j$. 
Example (structural stability in vertical hopper)

There exists a deadbeat control for vertical hopper, i.e. smooth actuator feedback law $a(x,y,\dot{x},\dot{y})$ such that the hopper exactly tracks periodic orbit after one “hop.”

Carver, Cowen, & Guckenheimer 2009

However, this is sensitive to parameter values: perturbing parameters $k, \ell_0, m, \mu, b$ increases rank $\text{DP}$.

Theorem $= \Rightarrow$ hopper contracts to periodic orbit at rate bounded by magnitude of perturbation.
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Theorem $\implies$ hopper contracts to periodic orbit at rate bounded by magnitude of perturbation
Identification

physical system
animal, robot

identification

detailed model
10–100 DOF

reduced-order model
< 10 DOF
Identification of initial conditions

\[ y(t) = y(t) \]

\[ \eta_i = y(\phi(t^*, z^*)) + w_i, \quad w_i \text{ iid random variables} \]

Identification problem

\[ \text{Solve } \arg \min_{z \in D_j} \varepsilon(z, \{\eta_i\}), \text{ where } \varepsilon(z, \{\eta_i\}) := \sum_i \|y(\phi(t, z)) - \eta_i\|^2_2. \]
Identification of initial conditions

$Y(\phi(t, z)) = y(t)$
Identification of initial conditions

\[ Y(\phi(t, z)) = y(t) \]

\[ \eta_i = Y(\phi(iT, z^*)) + w_i, \]

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Identification of initial conditions

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**Identification problem**

Solve \( \arg \min_{z \in D_j} \varepsilon(z, \{\eta_i\}), \) where \( \varepsilon(z, \{\eta_i\}) := \sum_i \|Y(\phi(iT, z)) - \eta_i\|^2. \)
Assumption (smooth observations)

\( Y \) is smooth along trajectories, i.e. \( Y(\phi(t, z)) \) is a smooth function of \( t \).
Identification on reduced hybrid model

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Identification on \( \bigcup_j D_j \)

\[
\arg\min_{\substack{z \in D_j \in M_j}} \varepsilon(z, \{\eta_i\})
\]

- \( \nabla \varepsilon \) undefined on \( G_j \subset D_j \)
- \( R_j \) not generally invertible
Identification on reduced hybrid model

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**global optimization** needed
Identification on reduced hybrid model

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**Identification on reduced hybrid model**

**Assumption (smooth observations)**

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- \( R_j \) not generally invertible

**global optimization** needed

- \( \nabla \varepsilon \) well-defined on \( G_j \cap M_j \)
- \( R_j \mid_{M_j} \) invertible

**first-order algorithms** applicable
Example (initial condition for vertical hopper)

Observe position of upper mass at 20Hz, additive noise with variance 0.2.

\[
\begin{align*}
\sigma_0, y_0, \dot{y}_0 &\approx (8.0, 1.5, 1.1) : \text{initial} \\
\sigma, y, \dot{y} &\approx (4.7, 1.6, 1.0) : \text{actual} \\
\sigma^*, y^*, \dot{y}^* &\approx (4.6, 1.6, 1.1) : \text{estimated}
\end{align*}
\]
Reduction & Identification

physical system
animal, robot

identification

detailed model
10–100 DOF

reduction

reduced-order model
< 10 DOF
Novel quantitative predictions for biomechanics

- **Observation**: Neural feedback appears at a delay
  - Revzen, Burden et al. 2013

- **Identification**

- **Prediction**: Passive mechanics sensitive to inertia
  - Full et al. 2002

Burden, Revzen, Moore, Sastry, & Full SICB 2013
Model-based design and control of dynamic robots

**design**

minimal use of actuators

Hoover et al. 2010

**identification**

**control**

asymmetric leg stiffness change

Proctor & Holmes 2008

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Hoover, Burden, Fu, Sastry, & Fearing IEEE BIOROB 2010
Model enables translation across morphology, scale

Physical system: animal, robot

Identification

Detailed model: 10–100 DOF

Reduction

Reduced-order model: < 10 DOF
Model enables translation across morphology, scale

**physical system**
- animal, robot

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**reduction**

**reduced-order model**
- < 10 DOF
Discussion & Questions — Thanks for your time!

**Reduction**
Hybrid dynamics reduce dimensionality near periodic orbits.

**Identification**
Reduction enables scalable algorithm for parameter estimation.

**Collaborators**
- Prof. Shankar Sastry
- Prof. Henrik Ohlsson
- Prof. Robert Full
- Prof. Aaron Hoover
- Prof. Shai Revzen
- Talia Moore
Locomotion is self-manipulation

Johnson, Haynes, & Koditschek, IROS 2012
Example (rank $DP^n$ generically non-constant)

$P(x, y) = (x^2, x)$

$P^2(\mathbb{R}^2)$

$P(\mathbb{R}^2)$
Example \((\text{rank } DP^n \text{ generically non-constant})\)

\[
P(x, y) = (x^2, x)
\]

\[
DP(x, y) = \begin{pmatrix}
2x & 0 \\
1 & 0
\end{pmatrix}
\]

\[
\rightarrow \quad \text{rank } DP = 1
\]

\[
DP^2(x, y) = \begin{pmatrix}
4x^3 & 0 \\
2x & 0
\end{pmatrix}
\]

\[
\rightarrow \quad \text{rank } DP^2(x, y) = \begin{cases}
0, & x = y = 0 \\
1, & \text{else}
\end{cases}
\]

Note that \(P\) contracts superexponentially since \(DP(0, 0)\) is nilpotent: for all \(\epsilon > 0\) there exists \(\delta > 0\) and \(|\cdot|_\epsilon\) such that

\[
\|P(x, y)\| < \delta \Rightarrow \|P(x, y)\| < \epsilon \|P(x, y)\|_\epsilon
\]
Example \((\text{rank } DP^n \text{ generically non-constant})\)

\[
P(x, y) = (x^2, x)
\]

\[
DP(x, y) = \begin{pmatrix} 2x & 0 \\ 1 & 0 \end{pmatrix}
\]

\[
\Rightarrow \text{rank } DP = 1
\]

\[
DP^2(x, y) = \begin{pmatrix} 4x^3 & 0 \\ 2x & 0 \end{pmatrix}
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Note that \(P\) contracts superexponentially since \(DP(0, 0)\) is nilpotent:

for all \(\varepsilon > 0\) there exists \(\delta > 0\) and \(\|\cdot\|_\varepsilon\) such that

\[
\| (x, y) \| < \delta \implies \| P(x, y) \| < \varepsilon \| (x, y) \|_\varepsilon
\]
Gluing smooth dynamical systems

Let $F_j$ be a smooth vector field on $n$-dimensional manifold $M_j$, $j \in \{1, 2\}$. If $R: \partial M_1 \to \partial M_2$ is a diffeomorphism, $F_1$ points outward on $\partial M_1$, and $F_2$ points inward on $\partial M_2$, then the quotient $\tilde{M} = M_1 \cup M_2 \partial M_1 \cong \partial M_2$ is a smooth manifold, $M_j \subset \tilde{M}$ is a smooth submanifold, and the vector field $\tilde{F}(x) = \begin{cases} F_1(x), & x \in M_1; \\ F_2(x), & x \in M_2; \end{cases}$ is smooth on $\tilde{M}$. 

\[ \dot{x} = F_1(x) \]

\[ \dot{x} = F_2(x) \]
**Lemma (Hirsch 1976)**

Let $F_j$ be a smooth vector field on $n$-dimensional manifold $M_j$, $j \in \{1, 2\}$.

If $R: \partial M_1 \to \partial M_2$ is a diffeomorphism, $F_1$ points outward on $\partial M_1$, and $F_2$ points inward on $\partial M_2$, then the quotient $\tilde{M} = M_1 \cup M_2 \bigcup \partial M_1 \simeq \partial M_2$ is a smooth manifold, $M_j \subset \tilde{M}$ is a smooth submanifold, and the vector field $\tilde{F}(x) = \{F_1(x), x \in M_1; F_2(x), x \in M_2; \}$ is smooth on $\tilde{M}$. 

**Gluing smooth dynamical systems**
Gluing smooth dynamical systems

Let $F_j$ be a smooth vector field on $n$-dimensional manifold $M_j$, $j \in \{1, 2\}$. If $R: \partial M_1 \to \partial M_2$ is a diffeomorphism, $F_1$ points outward on $\partial M_1$, and $F_2$ points inward on $\partial M_2$, then the quotient $\tilde{M} = M_1 \cup M_2 / \partial M_1 \simeq \partial M_2$ is a smooth manifold, $M_j \subset \tilde{M}$ is a smooth submanifold, and the vector field $\tilde{F}(x) = \begin{cases} F_1(x), & x \in M_1; \\ F_2(x), & x \in M_2; \end{cases}$ is smooth on $\tilde{M}$. 

[Diagram showing gluing process]
Lemma (Hirsch 1976)

Let $F_j$ be a smooth vector field on $n$-dimensional manifold $M_j$, $j \in \{1, 2\}$. If $R : \partial M_1 \to \partial M_2$ is a diffeomorphism, $F_1$ points outward on $\partial M_1$, and $F_2$ points inward on $\partial M_2$, then the quotient $\tilde{M} = \frac{M_1 \cup M_2}{\partial M_1 \simeq \partial M_2}$ is a smooth manifold, $M_j \subset \tilde{M}$ is a smooth submanifold, and the vector field

$$\tilde{F}(x) = \begin{cases} F_1(x), & x \in M_1; \\ F_2(x), & x \in M_2; \end{cases}$$

is smooth on $\tilde{M}$. 

Sam Burden
Reduction and ID for Hybrid Models
March 22, 2013
Smoothing reduced-order hybrid system

\[ \dot{x} = F_1(x) \]

\[ \dot{x} = F_2(x) \]

Corollary (Burden, Revzen, Sastry CDC 2011)

The topological quotient

\[ \tilde{M} = \bigcup_j M_j \cap (G_j \cap M_j) \]

\[ \cong \]

is a smooth manifold, \( M_j \subset \tilde{M} \) is a smooth submanifold, and the vector field

\[ \tilde{F}(x) = \begin{cases} F_1(x), & x \in M_1; \\ \vdots \\ F_j(x), & x \in M_j; \\ \vdots \end{cases} \]

is smooth on \( \tilde{M} \).
Smoothing reduced-order hybrid system

\[ \hat{x} = F_1(x) \]

\[ M_1 \]

\[ R_2|_{G_2 \cap M_2} \]

\[ M_2 \cap G_2 \]

\[ \gamma \]

\[ \hat{x} = F_2(x) \]

\[ G_1 \cap M_1 \]

\[ R_1|_{G_1 \cap M_1} \]

\[ \bigcup_j M_j \]

\[ (G_j \cap M_j) \simeq R_j(G_j \cap M_j) \]
Smoothing reduced-order hybrid system

\[ \dot{x} = F_1(x) \]

\[ \dot{x} = F_2(x) \]

\[ R_2|_{G_2 \cap M_2} \]

\[ R_1|_{G_1 \cap M_1} \]

\[ G_1 \cap M_1 \]

\[ M_1 \]

\[ M_2 \cap G_2 \]

\[ M_2 \]

\[ \bigcup_j M_j \]

\[ (G_j \cap M_j) \simeq R_j(G_j \cap M_j) \]

\[ \dot{x} = \tilde{F}(x) \]

\[ \tilde{M} \]
**Corollary (Burden, Revzen, Sastry CDC 2011)**

The topological quotient \( \widetilde{M} = \frac{\bigcup_j M_j}{(G_j \cap M_j) \simeq R_j(G_j \cap M_j)} \) is a smooth manifold, \( M_j \subset \widetilde{M} \) is a smooth submanifold, and the vector field

\[
\widetilde{F}(x) = \begin{cases} 
F_1(x), & x \in M_1; \\
\vdots & \vdots \\
F_j(x), & x \in M_j; \\
\vdots & \vdots 
\end{cases}
\]

is smooth on \( \widetilde{M} \).