Reduction and Robustness via Intermittent Contact

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Motivation
Dynamic interaction involves intermittent contact

Nao humanoid robot

Contact yields a *hybrid* dynamical system

\[
\dot{x} = F_1(x)
\]

\[
\dot{x} = F_2(x)
\]
Contact yields a \textit{hybrid} dynamical system
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Combinatorial \# of discrete modes, each generally possessing nonlinear dynamics.
Focus on rhythmic behaviors

Animals utilize rhythmic behaviors for locomotion & manipulation

Grillner, Science 1985
Focus on rhythmic behaviors

Animals utilize rhythmic behaviors for locomotion & manipulation
Grillner, Science 1985

Represented by periodic orbits in hybrid dynamical system
Focus on locomotion

Note that locomotion is self-manipulation

Johnson, Haynes, & Koditschek, IROS 2012
Overview of this talk

Motivation
interaction with environment involves intermittent contact

Reduction
low-dimension subsystem appears near hybrid periodic orbit

Robustness
simultaneous hybrid transitions yield robust stability

Applications
identification of neuromechanical control architecture in animals
design and optimization of gaits and maneuvers for robots
Reduction
Example (vertical hopper)
Hybrid dynamical system

\[ \dot{x} = F_1(x) \]

\[ \dot{x} = F_2(x) \]
Trajectory for a hybrid dynamical system

\[ \dot{x} = F_1(x) \]

\[ \dot{x} = F_2(x) \]
Periodic orbit $\gamma$ for a hybrid dynamical system

$\dot{x} = F_1(x)$

$\dot{x} = F_2(x)$
Assumptions on hybrid periodic orbit $\gamma$
Assumptions on hybrid periodic orbit $\gamma$

**Assumption (transversality)**

*periodic orbit* $\gamma$ *passes transversely through each guard* $G_j$.
Assumptions on hybrid periodic orbit $\gamma$

Assumption (transversality)

*periodic orbit $\gamma$ passes transversely through each guard $G_j$*

Assumption (dwell time)

$\exists \varepsilon > 0: \text{periodic orbit } \gamma \text{ spends at least } \varepsilon \text{ time units in each domain } D_j$
Poincaré map for periodic orbit $\gamma$

**Smooth Dynamical System**

$D$

$\dot{x} = F(x)$

**Hybrid Dynamical System**

$\dot{x} = F_1(x)$

$\dot{x} = F_2(x)$
Poincaré map for periodic orbit $\gamma$

**smooth dynamical system**

$D \; \Sigma \; \xi \; \dot{x} = F(x)$

**hybrid dynamical system**

$G_2 \; D_2 \; R_1 \; \dot{x} = F_2(x)$

$G_1 \; D_1 \; R_2 \; \dot{x} = F_1(x)$
Poincaré map for periodic orbit $\gamma$
Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

The Poincaré map $P$ is smooth in a neighborhood of $\xi$. 

### smooth dynamical system

$$D \xrightarrow{\Sigma} x_0 \xrightarrow{\xi} \gamma \quad \dot{x} = F(x)$$

### hybrid dynamical system

$$\begin{align*}
G_1 & \xrightarrow{\Sigma} \xi \xrightarrow{x_0} \gamma \quad \dot{x} = F_1(x) \\
R_2 & \xrightarrow{\Sigma} D_2 \xrightarrow{\gamma} \quad \dot{x} = F_2(x)
\end{align*}$$
Poincaré map for periodic orbit $\gamma$

**smooth dynamical system**

$$\dot{x} = F(x)$$

**hybrid dynamical system**

$$\dot{x} = F_1(x)$$

Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

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Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

*The Poincaré map* $P$ *is smooth in a neighborhood of* $\xi$. 

The Poincaré map for periodic orbit $\gamma$
Rank of Poincaré map $P$ with fixed point $P(\xi) = \xi$

Smooth dynamical system:

$$\dot{x} = F(x)$$

Hybrid dynamical system:

$$\dot{x} = F_1(x)$$

$D$ and $\Sigma$
Rank of Poincaré map $P$ with fixed point $P(\xi) = \xi$

**smooth dynamical system**

\[ \begin{align*}
D \quad & \sum \quad \xi \\
\dot{x} = F(x) \\
P(x_0) \quad & x_0
\end{align*} \]

\[ \text{rank } DP(\xi) = \dim D - 1 \]

Hirsch and Smale 1974

**hybrid dynamical system**

\[ \begin{align*}
\dot{x} = F_1(x) \\
\sum \quad \xi \\
x = F_2(x)
\end{align*} \]
Rank of Poincaré map \( P \) with fixed point \( P(\xi) = \xi \)

\[
\text{rank } DP(\xi) = \dim D - 1
\]
Hirsch and Smale 1974

\[
\text{rank } DP(\xi) \leq \min_j \dim D_j - 1
\]
Wendel and Ames 2010
Example (rank-deficient Poincaré map)

\[ A \in \mathbb{R}^{n \times n} \text{ is nilpotent} \Rightarrow \text{rank} \, DP_n = 0. \]
Example (rank-deficient Poincaré map)

If $A \in \mathbb{R}^{n \times n}$ is nilpotent (i.e. $A^n = 0_{n \times n}$), then $\text{rank } DP^n = 0$. 
Exact model reduction near hybrid periodic orbit $\gamma$

Let $\min_j \dim D_j$. If $\text{rank} \, \text{D}_n = r$ near $\xi$, then trajectories starting near $\gamma$ contract to a collection of hybrid-invariant $(r + 1)$-dimensional submanifolds $M_j \subset D_j$ in finite time.
Theorem (Burden, Revzen, Sastry CDC 2011)

Let \( n = \min_j \dim D_j \). If \( \text{rank}\, D_P^n = r \) near \( \xi \), then trajectories starting near \( \gamma \) contract to a collection of hybrid-invariant \((r + 1)\)-dimensional submanifolds \( M_j \subset D_j \) in finite time.
Exact model reduction near hybrid periodic orbit $\gamma$

Theorem (Burden, Revzen, Sastry CDC 2011)

Let $n = \min_j \dim D_j$. If $\text{rank } DP^n = r$ near $\xi$, then trajectories starting near $\gamma$ contract to a collection of hybrid-invariant $(r + 1)$–dimensional submanifolds $M_j \subset D_j$ in finite time.
Example (exact model reduction in vertical hopper)

Numerically linearizing Poincaré map
we find
we find $\mathbf{D}P(\xi)$ has eigenvalues $\lambda_1 \approx -0.25 \pm 0.70j$,
therefore $\mathbf{D}^2P_2$ is constant rank near $\xi$.

Theorem $\Rightarrow$ dynamics collapse to 1-DOF hopper
Interpretation: unilateral (Lagrangian) constraint appears after one "hop"
Example (exact model reduction in vertical hopper)

Numerically linearizing Poincaré map $P$ on ground, we find $DP(\xi)$ has eigenvalues $\simeq -0.25 \pm 0.70j$, therefore $DP^2$ is constant rank near $\xi$. 

![Diagram of vertical hopper model](image)
Example (exact model reduction in vertical hopper)

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**Theorem $\implies$** dynamics collapse to 1-DOF hopper

**Interpretation:** unilateral (Lagrangian) constraint appears after one “hop”
Example (rank $DP^n$ generically non-constant)

$P(x, y) = (x^2, x)$

$P^2(\mathbb{R}^2)$

$P(\mathbb{R}^2)$
Example \((\text{rank } DP^n \text{ generically non-constant})\)

\[ P(x, y) = (x^2, x) \]

\[ DP(x, y) = \begin{pmatrix} 2x & 0 \\ 1 & 0 \end{pmatrix} \]

\[\implies \text{rank } DP = 1 \]

\[ DP^2(x, y) = \begin{pmatrix} 4x^3 & 0 \\ 2x & 0 \end{pmatrix} \]

\[\implies \text{rank } DP^2(x, y) = \begin{cases} 0, & x = y = 0 \\ 1, & \text{else} \end{cases} \]
Example \((\text{rank } DP^n \text{ generically non-constant})\)

\[
P(x, y) = (x^2, x)
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DP(x, y) = \begin{pmatrix} 2x & 0 \\ 1 & 0 \end{pmatrix}
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\Rightarrow \text{rank } DP = 1
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\[
DP^2(x, y) = \begin{pmatrix} 4x^3 & 0 \\ 2x & 0 \end{pmatrix}
\]

\[
\Rightarrow \text{rank } DP^2(x, y) = \begin{cases} 0, & x = y = 0 \\ 1, & \text{else} \end{cases}
\]

Note that \(P\) contracts arbitrarily rapidly since \(DP(0, 0)\) is nilpotent: for all \(\varepsilon > 0\) there exists \(\delta > 0\) and \(\|\cdot\|_\varepsilon\) such that

\[
\|(x, y)\| < \delta \quad \Rightarrow \quad \|P(x, y)\| < \varepsilon \|(x, y)\|_\varepsilon
\]
If \( \text{rank} \, \mathbf{D}_P(\xi) = r \) and \( \text{spec} \, \mathbf{D}_P(\xi) \subset B_1(0) \subset \mathbb{C} \), then for any \( \epsilon > 0 \), trajectories starting sufficiently near \( \gamma \) contract exponentially fast with rate \( \epsilon \) to a collection of \((r+1) - \text{dimensional submanifolds} \, M_j \subset D_j \).
Theorem (Burden, Revzen, Sastry (in preparation))

If $\text{rank } DP^n(\xi) = r$ and $\text{spec } DP(\xi) \subset B_1(0) \subset \mathbb{C}$, then for any $\varepsilon > 0$ trajectories starting sufficiently near $\gamma$ contract exponentially fast with rate $\varepsilon$ to a collection of $(r + 1)$-dimensional submanifolds $M_j \subset D_j$. 

Approximate model reduction near hybrid periodic orbit $\gamma$
Example (deadbeat control of vertical hopper)

There exists a deadbeat control for vertical hopper, i.e., smooth actuator feedback law \( a(x, y, \dot{x}, \dot{y}) \) such that the hopper exactly tracks periodic orbit after one "hop". (Carver, Cowen, & Guckenheimer, Chaos 2009)

However, this is sensitive to parameter values: perturbing parameters \( k, \ell_0, m, \mu, b \) yields rank of \( DP \) = 2

Theorem => hopper contracts to orbit at rate bounded by size of parameter perturbation.
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However, this is sensitive to parameter values: perturbing parameters \( k, \ell_0, m, \mu, b \) yields rank \( DP = 2 \)

Theorem \( \implies \) hopper contracts to orbit at rate bounded by size of parameter perturbation
Robustness
Empirically, simultaneous limb touchdown typical for animal gaits

Golubitsky et al. Nature 1999
Motivation Reduction Robustness Applications

Simultaneous Transitions Normal Form Robust Stability

Simultaneous hybrid transitions

Empirically, simultaneous limb touchdown typical for animal gaits
Golubitsky et al. Nature 1999

(Consequently) also typical for polyped robot gaits
Saranli et al. IJRR 2001; Kim et al. IJRR 2006; Hoover et al. IROS 2008
Assumptions on simultaneous hybrid transitions

\[ \dot{x} = F(x) \]
Assumptions on simultaneous hybrid transitions

Assumption (transversality)

\[ n = \dim D \text{ transition surfaces } \{ S_j \}_{1}^{n} \text{ intersect transversely at } \sigma \in D. \]
Assumptions on simultaneous hybrid transitions

Assumption (transversality)

\[ n = \text{dim} \ D \text{ transition surfaces } \{S_j\}_{1}^{n} \text{ intersect transversely at } \sigma \in D. \]

Assumption (piecewise smooth vector field)

All points where \( F \) is discontinuous or nonsmooth are contained in \( \bigcup_{j} S_j \).
Assumptions on simultaneous hybrid transitions

\[ \dot{x} = F(x), \quad \dot{y} = Dh(F)(y) \]

\[ h(\phi(t, x)) \]

\( h(U) \subset \mathbb{R}^2 \)

\( h(\sigma) \)

\( h(U) \subset \mathbb{R}^2 \)

\( h(\sigma) \)

\( h(\sigma) \)

Assumption (no sliding modes)

For \( q \in \{-1, +1\} \), let

\[ F_q = \lim_{y \to 0} y \in D_q \quad Dh(F)(y) \]

and assume \( F_q \in \text{Int} D_{+1} \).

Theorem (Filippov 1988)

The flow \( \phi \) is well-defined and continuous in a neighborhood of \( \sigma \).

Assumptions on simultaneous hybrid transitions

Assumption (no sliding modes)

For \( q \in \{-1, +1\}^n \), let \( F_q = \lim_{y \to 0} \frac{Dh(F)(y)}{y \in D_q} \) and assume \( F_q \in \text{Int } D_+ \).
Assumptions on simultaneous hybrid transitions

**Assumption (no sliding modes)**

For \( q \in \{-1, +1\}^n \), let \( F_q = \lim_{y \to 0} \frac{Dh(F)(y)}{y} \) and assume \( F_q \in \text{Int } D_{+1} \).

**Theorem (Filippov 1988)**

The flow \( \phi \) is well-defined and continuous in a neighborhood of \( \sigma \).
Normal form for simultaneous hybrid transitions

\[ \phi(t, x) \]

\[ F_{-1} \]

\[ D_{-1} \]

\[ D_{+1} \]

\[ T \]

\[ R \]

\[ G \]

With \( \omega := \frac{1}{\sqrt{n}} \), let \( \Pi := I - \omega \omega^T \) be orthogonal projection onto \( \ker \omega^T \).
Normal form for simultaneous hybrid transitions

\[ \phi(t, x) \]

\[ D 
\]

\[ D^{-1,+1} \]

\[ D^{+1} \]

\[ \psi(x) \]

\[ \omega \]

\[ F(-1,+1) \]

\[ F^{+1} \]

\[ F_{-1} \]

\[ F^{(1,-1)} \]

\[ G \]

\[ T \]

\[ R \]

\[ \omega \]

\[ x \]

\[ -\omega \]

\[ R : G \to T \text{ continuous, } \psi : T \to G \text{ obtained by integrating flow } \phi. \]
Normal form for simultaneous hybrid transitions

\[ R : G \to T \text{ continuous, } \psi : T \to G \text{ obtained by integrating flow } \phi. \]
With \( \omega := \frac{1}{\sqrt{n}} \mathbb{1} \), let \( \Pi := I - \omega \omega^T \) be orthogonal projection onto \( \ker \omega^T \).
Sufficient condition for stability

\[ \gamma \phi(t,x) \]

\[ F(-1,+1) \]

\[ F(+1,-1) \]

\[ D \]

\[ D(-1,+1) \]

\[ D_{+1} \]

\[ D_{-1} \]

\[ D_{(+1,-1)} \]

\[ F_{+1} \]

\[ F_{-1} \]

\[ \omega \]

\[ \psi(x) \]

\[ \omega \]

\[ \phi(t,x) \]

\[ x \]

\[ -\omega \]

\[ R \]

\[ G \]

\[ T \]

Theorem (Burden, Revzen, Koditschek, Sastry (in preparation))

\[ \Pi \]

\[ F \]

\[ q \in \text{Int} \]

\[ D_{-q} \]

\[ \text{for all } q \neq \pm 1 \Rightarrow \exists c \in (0,1) : \|\Pi \psi(x)\| < c \|\Pi x\| \]

If \( R \) Lipschitz with constant \( 1/c \) and \( R(\omega) = -\omega \) then \( \gamma \) exp. stable.
Sufficient condition for stability

Theorem (Burden, Revzen, Koditschek, Sastry (in preparation))

\[ \Pi F_q \in \text{Int} \; D_{-q} \; \text{for all} \; q \neq \pm 1 \implies \exists c \in (0, 1) : \| \Pi \psi(x) \| < c \| \Pi x \|. \]
Sufficient condition for stability

Theorem (Burden, Revzen, Koditschek, Sastry \textit{(in preparation)})

\[ \Pi F_q \in \text{Int } D_{-q} \text{ for all } q \neq \pm 1 \implies \exists c \in (0, 1) : \|\Pi \psi(x)\| < c \|\Pi x\|. \]

\textit{If } R \text{ Lipschitz with constant } 1/c \text{ and } R(\omega) = -\omega \text{ then } \gamma \text{ exp. stable.}
Applications
Identify neuromechanical control architecture in animals

Revzen et al. (in review) 2012
Mechanical self-stabilization vs. neural feedback
Mechanical self-stabilization vs. neural feedback
Mechanical self-stabilization vs. neural feedback

![Graph showing phase change, lateral velocity, and cart acceleration over time.](image-url)
Identify neuromechanical control architecture in animals

Burden et al. SysID 2012; Burden et al. SICB 2013
Identify neuromechanical control architecture in animals

Identification problem

$$\arg \min_{z \in D_j} \varepsilon(z, \{\eta_i\})$$

Burden et al. SysID 2012; Burden et al. SICB 2013
Identify neuromechanical control architecture in animals

**Identification problem**

$$\arg \min_{z \in D_j} \varepsilon (z, \{\eta_i\})$$

- $\nabla \varepsilon$ undefined on $G_j \subset D_j$

**global optimization** needed

Burden *et al.* SysID 2012; Burden *et al.* SICB 2013
Identify neuromechanical control architecture in animals

Identification problem
\[
\arg \min_{z \in D_j} \varepsilon (z, \{\eta_i\})
\]

- \(\nabla \varepsilon\) undefined on \(G_j \subset D_j\)

**global optimization** needed

Identification on \(\bigcup_j M_j\)
\[
\arg \min_{z \in M_j} \varepsilon (z, \{\eta_i\})
\]

- \(\nabla \varepsilon\) defined on \(G_j \cap M_j\)

**first-order methods** apply

*Burden et al.* SysID 2012; *Burden et al.* SICB 2013
Design and optimize gaits and maneuvers for robots

RHex robot

video courtesy of KodLab, http://kodlab.seas.upenn.edu/
Exploit hybrid transitions for robust stability of gaits

smooth leg coordination

\[ T^2 = S^1 \times S^1 \]

Burden et al. (in preparation)
Exploit hybrid transitions for robust stability of gaits

smooth leg coordination

\[ T^2 = S^1 \times S^1 \]

hybrid leg coordination

\[ T^2 = S^1 \times S^1 \]

Burden et al. (in preparation)
Motivation

Reduction

Hybrid dynamics generically reduce dimensionality near a periodic orbit.

Robustness

Simultaneous hybrid transitions can lend robust stability to a periodic orbit.

Collaborators

- Prof. Shankar Sastry
- Prof. Dan Koditschek
- Prof. Shai Revzen
- Prof. Robert Full
- Prof. Henrik Ohlsson
- Prof. Aaron Hoover
- Talia Moore
- Justin Starr
- Mike Choi

Discussion & Questions — Thanks for your time!