Dimension Reduction Near Periodic Orbits of Hybrid Systems

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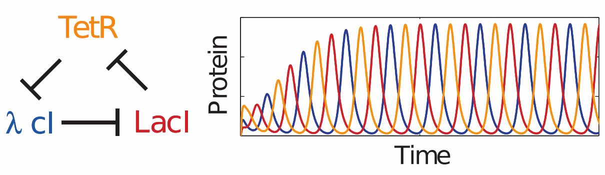
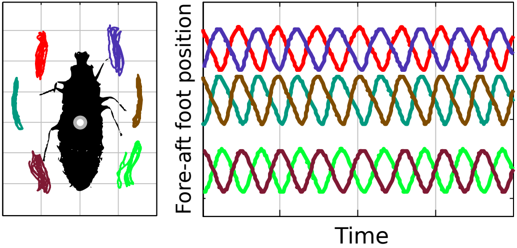
Hybrid periodic orbits model interesting physical systems



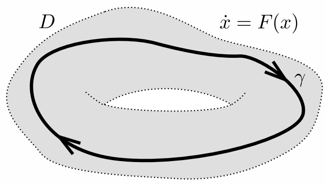
biochemistry: chemical reaction networks Elowitz and Leibler 2000, Alur et al. 2001



biomechanics: terrestrial locomotion Holmes et al. 2006, Revzen 2009

Tools for hybrid periodic orbits generalize smooth theory

smooth dynamical system



Feedback linearization

Sastry 1999

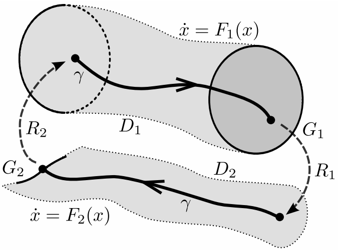
Symmetric reduction

Marsden and Ratiu 1999

Averaging theory

Guckenheimer and Holmes 1983

hybrid dynamical system



Hybrid zero dynamics

Westervelt et al. 2003

Reduction in mechanical systems

Ames et al. 2006

Averaging for neuromechanical systems

Proctor et al. 2010

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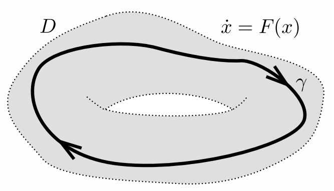

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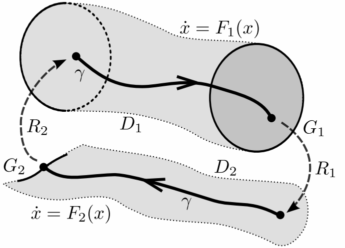

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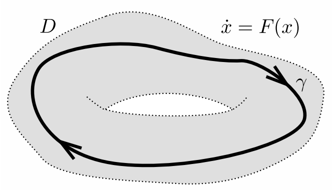

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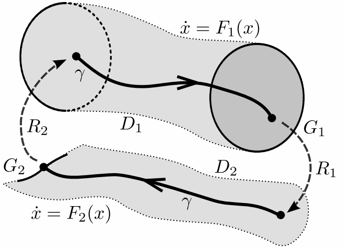

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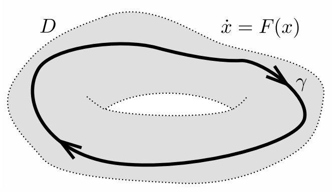

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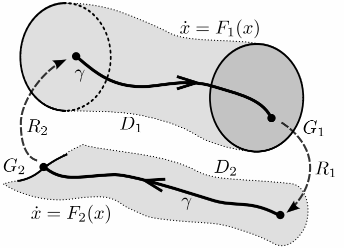

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Generalizing tools from smooth dynamical systems theory

Open problem

Can arbitrary tools be generalized from smooth systems to hybrid systems?

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Our contribution

We provide a sufficient condition under which hybrid dynamics **exactly** reduce to a smooth dynamical system near a periodic orbit.

Overview of this talk

Motivation
hybrid periodic orbits model physical phenomena

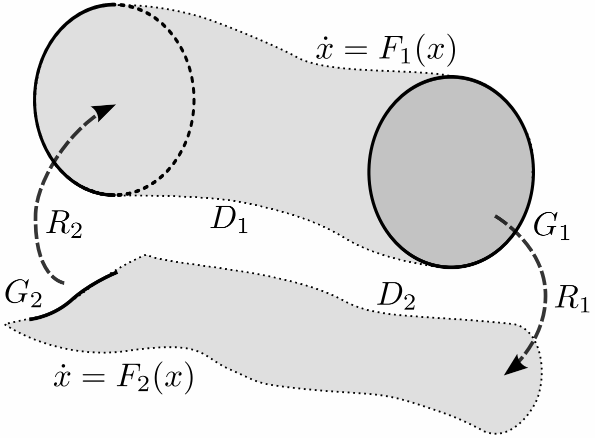
Reduction
model reduction near a hybrid periodic orbit

Smoothing
smoothing reduced-order hybrid system

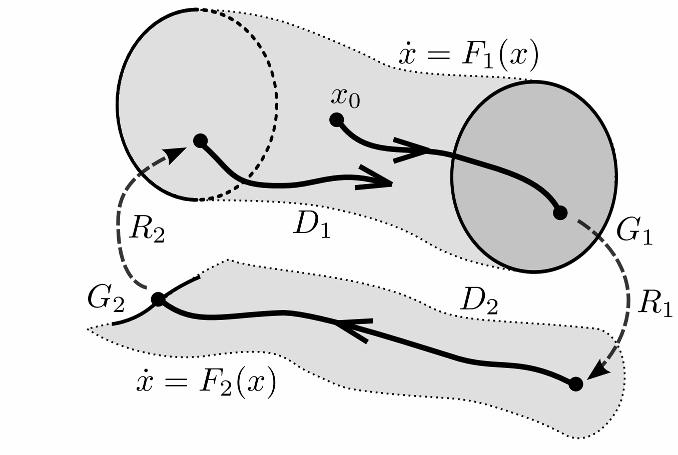
Example
model reduction in a mechanical hybrid system

Conclusion
generalizing tools from smooth systems theory

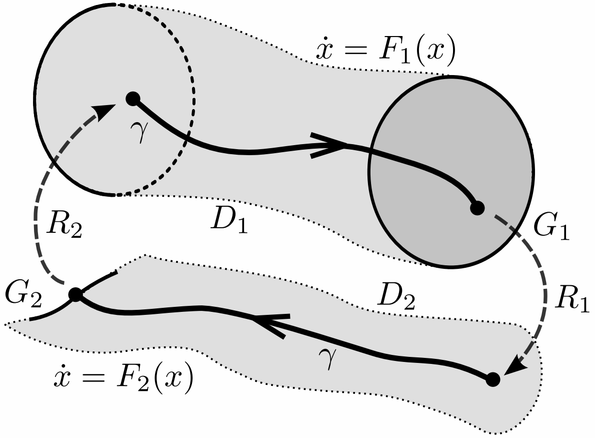
Hybrid dynamical system



Trajectory for a hybrid dynamical system



Periodic orbit γ for a hybrid dynamical system



Assumptions on hybrid periodic orbit γ

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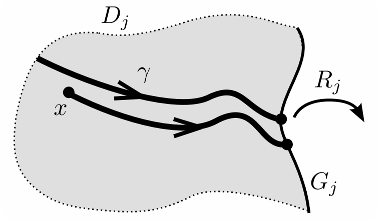
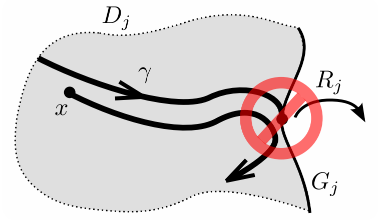
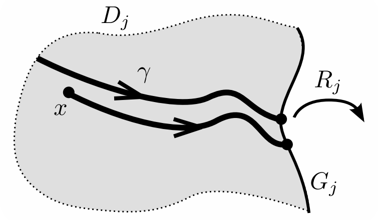


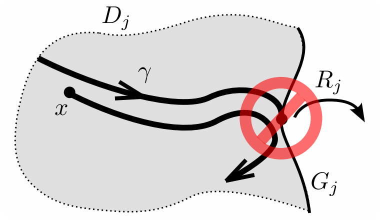


Assumption (transversality)

periodic orbit γ *passes transversely through each guard* G_j

Assumptions on hybrid periodic orbit γ





Assumption (transversality)

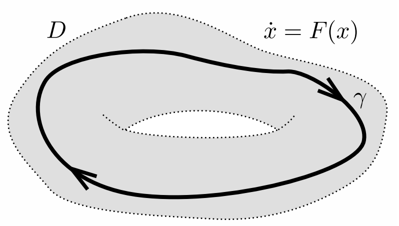
periodic orbit γ *passes transversely through each guard* G_j

Assumption (dwell time)

$\exists \varepsilon > 0$: *periodic orbit* γ *spends at least* ε *time units in each domain* D_j

Poincaré map for periodic orbit γ

smooth dynamical system



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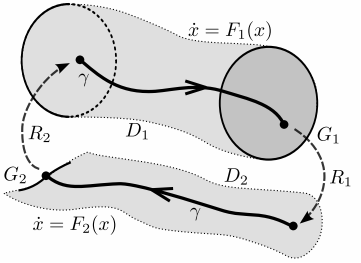

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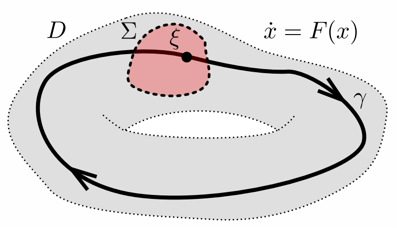

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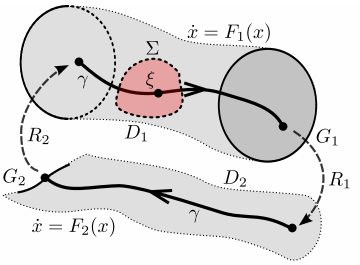

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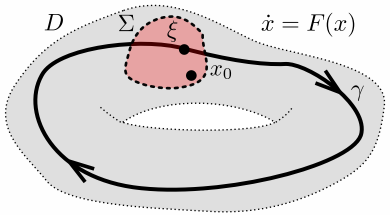

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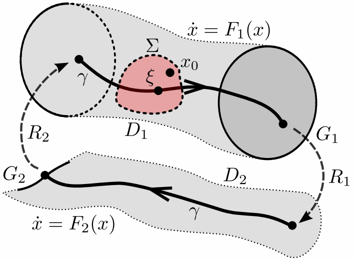

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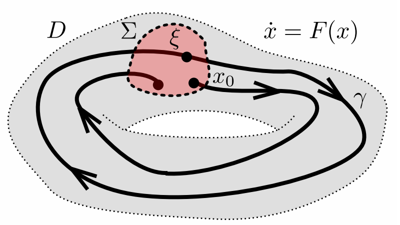

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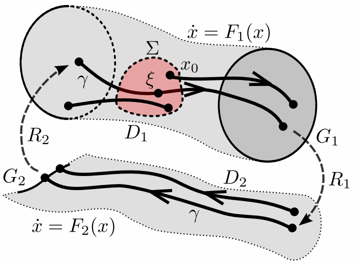

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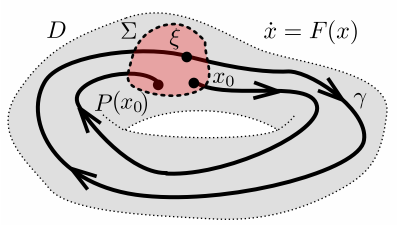

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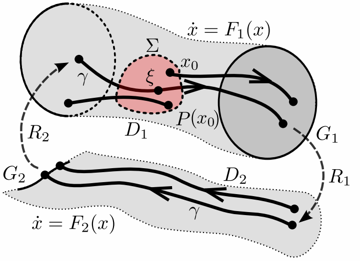

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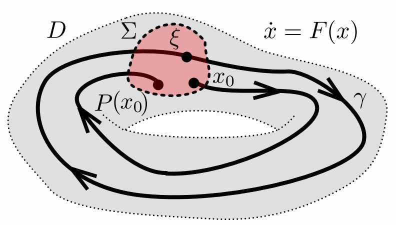

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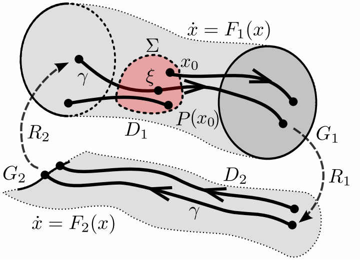
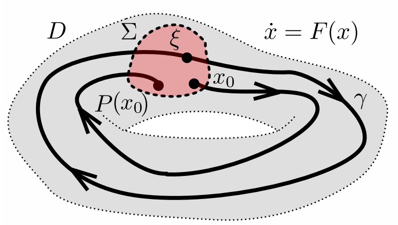


Theorem (Hirsch and Smale 1974, Grizzle et al. 2002)

The Poincaré map P is smooth in a neighborhood of ξ.

Rank of the Poincaré map P with fixed point $P(\xi) = \xi$

smooth dynamical system



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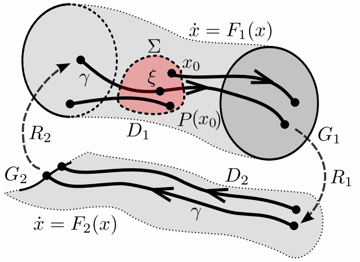

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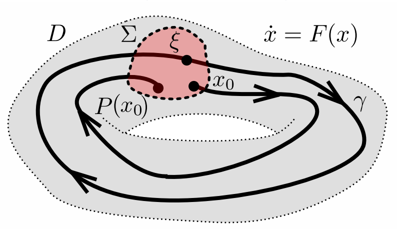
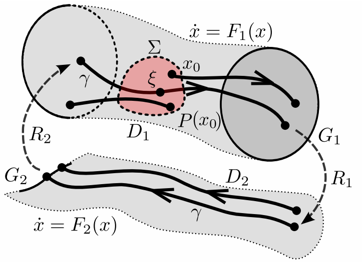


$\operatorname{rank} DP(\xi) = \dim D - 1$

Hirsch and Smale 1974

hybrid dynamical system



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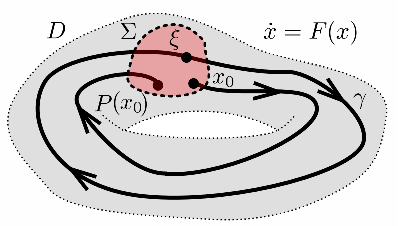

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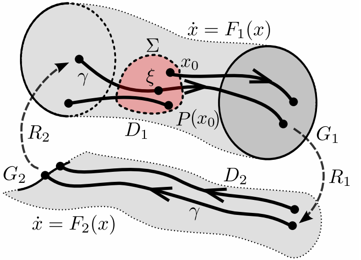

$$\operatorname{rank} DP(\xi) \leq \min_j \dim D_j - 1$$

Wendel and Ames 2010

Example (rank-deficient Poincaré map)

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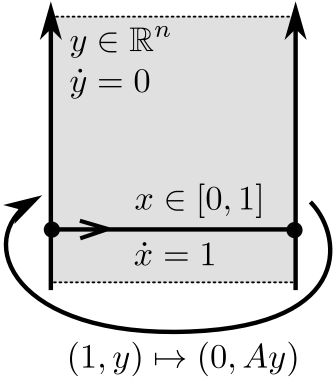
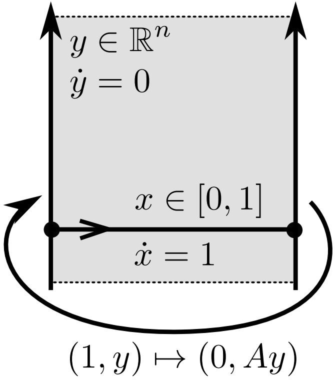
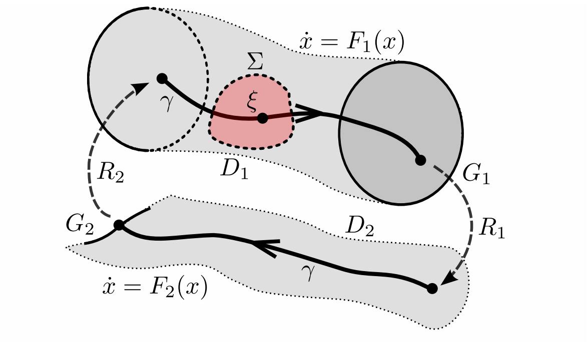


Example (rank-deficient Poincaré map)



If $A \in \mathbb{R}^{n \times n}$ nilpotent (i.e. $A^n = 0_{n \times n}$) then $\operatorname{rank} DP^n = 0$.

Invariant submanifold of Poincaré map



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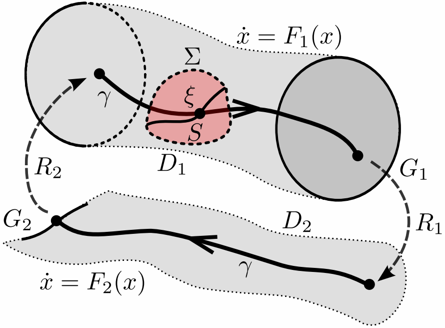

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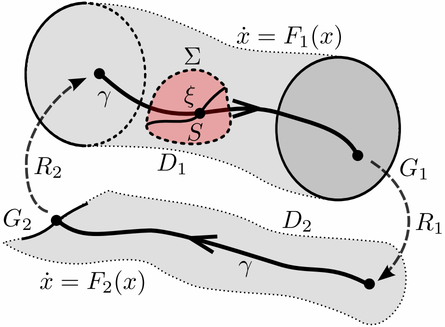
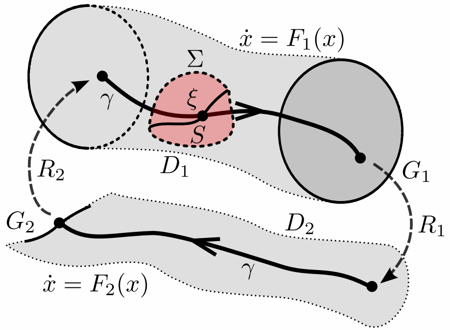


Lemma

Let $n = \min_j \dim D_j$.
If DP^n *has constant rank* r *near* ξ, *then* $S = P^n(\Sigma)$ *is an* r*-dimensional submanifold near* ξ *and* $P|_S$ *is a diffeomorphism of* S *near* ξ.

Model reduction near a periodic orbit



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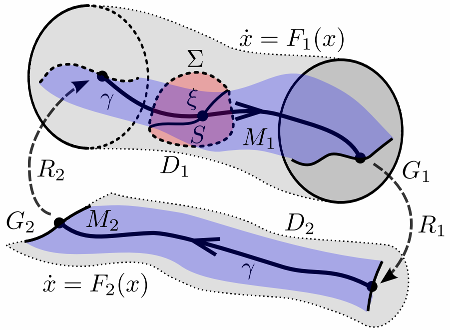

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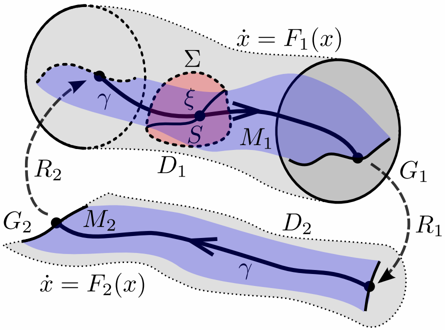
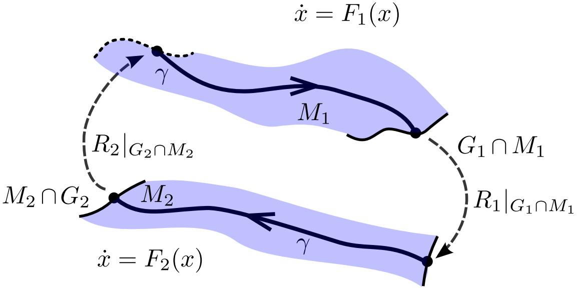


Theorem

Let $n = \min_j \dim D_j$.
If DP^n *has constant rank* r *near* ξ, *then after a finite amount of time all trajectories starting near* γ *collapse to a collection of hybrid invariant* $(r+1)$ *dimensional submanifolds* $M_j \subset D_j$.

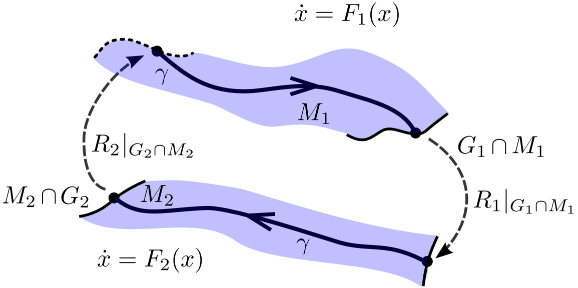
Model reduction near a periodic orbit



Corollary

The submanifolds M_j determine a hybrid system with periodic orbit γ.

Model reduction near a periodic orbit



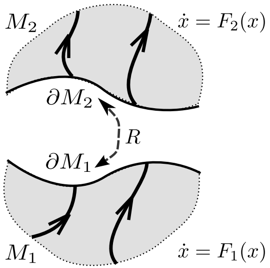
Corollary

The submanifolds M_j determine a hybrid system with periodic orbit γ.

γ is asymptotically stable in the original hybrid system

$\iff$ γ is asymptotically stable in the reduced hybrid system.

Gluing smooth dynamical systems



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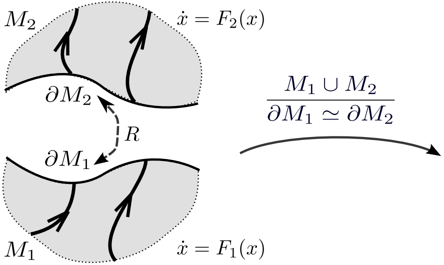

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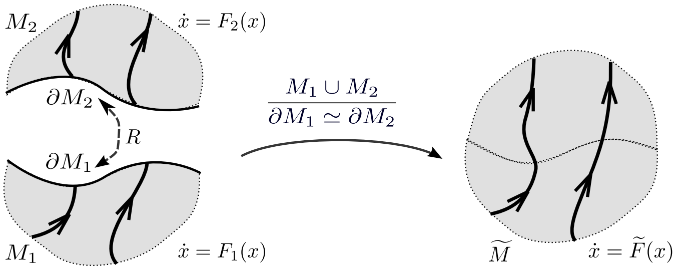

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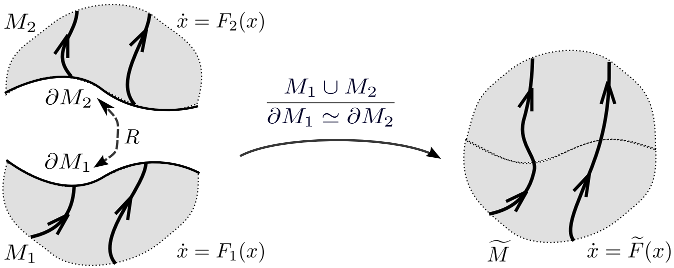


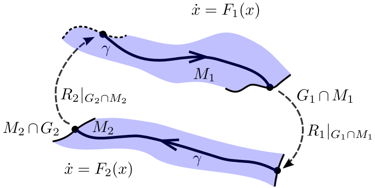
Lemma (Hirsch 1976)

Let F_j be a smooth vector field on n-dimensional manifold M_j, $j \in \{1, 2\}$. If $R : \partial M_1 \to \partial M_2$ is a diffeomorphism, F_1 points outward on ∂M_1, and F_2 points inward on ∂M_2, then the quotient $\widetilde{M} = \frac{M_1 \cup M_2}{\partial M_1 \simeq \partial M_2}$ is a smooth manifold, $M_j \subset \widetilde{M}$ is a smooth submanifold, and the vector field

$$\widetilde{F}(x) = \begin{cases} F_1(x), & x \in M_1; \\ F_2(x), & x \in M_2; \end{cases}$$

is smooth on $\widetilde{M}$.

Smoothing reduced-order hybrid system



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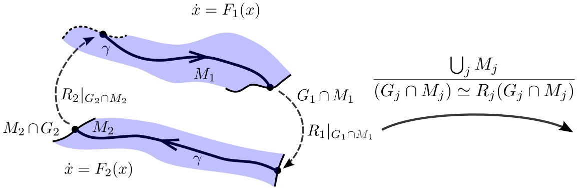

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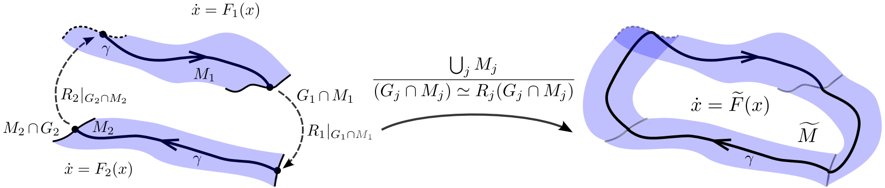

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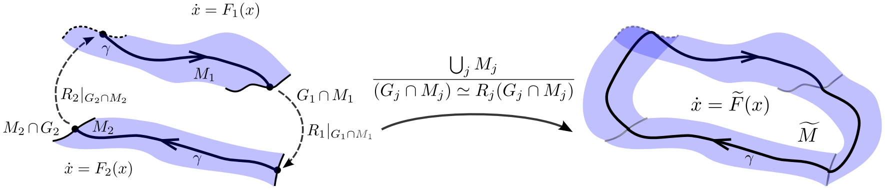


Corollary

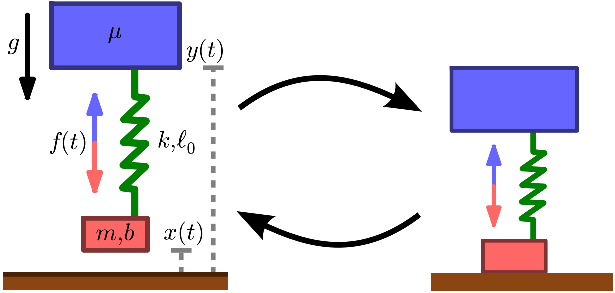
The topological quotient $\widetilde{M} = \frac{\bigcup_j M_j}{(G_j \cap M_j) \simeq R_j(G_j \cap M_j)}$ *is a smooth manifold,* $M_j \subset \widetilde{M}$ *is a smooth submanifold, and the vector field*

$$\widetilde{F}(x) = \begin{cases} F_1(x), & x \in M_1; \\ \vdots & \vdots \\ F_j(x), & x \in M_j; \\ \vdots & \vdots \end{cases}$$

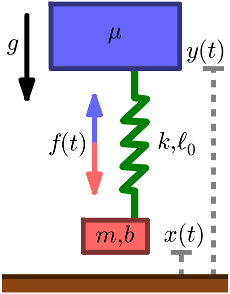
is smooth on $\widetilde{M}$.

Example (vertical hopper)

vertical hopper



Reduction in the vertical hopper



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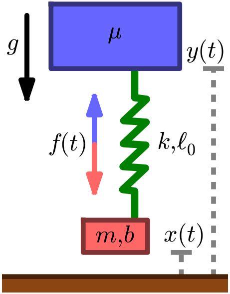
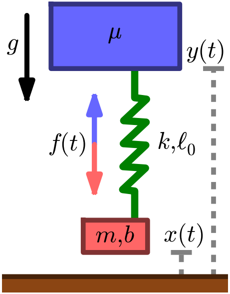


With parameters

$m = 1, \mu = 2, k = 10, b = 5, \ell_0 = 2, a = 20, \omega = 2\pi, g = 2$

Reduction in the vertical hopper

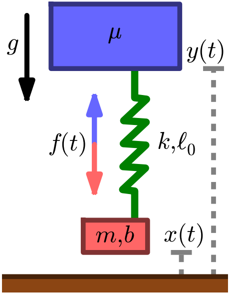


With parameters

$m = 1, \mu = 2, k = 10, b = 5, \ell_0 = 2, a = 20, \omega = 2\pi, g = 2$

Numerically linearizing Poincaré map P on ground

Reduction in the vertical hopper



With parameters

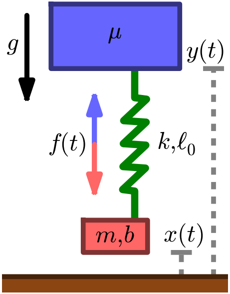
$m = 1, \mu = 2, k = 10, b = 5, \ell_0 = 2, a = 20, \omega = 2\pi, g = 2$

Numerically linearizing Poincaré map P on ground

we find $DP(\xi)$ has eigenvalues $\simeq -0.25 \pm 0.70j$

therefore DP^2 is constant rank

Reduction in the vertical hopper



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Theorem $\Longrightarrow$ after one cycle, dynamics collapse to 1-DOF hopper

Discussion & Questions — Thanks for your time!

Our contribution

We provide a sufficient condition under which hybrid dynamics **exactly** reduce to a smooth dynamical system near a periodic orbit.

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Enables generalization of first-order tools for parameter identification

Burden, Ohlsson, Sastry, Submitted to IFAC SysID 2012

Partially explains empirically-observed dimension loss in animals

Full and Koditschek 1999, Revzen 2009