On the Characterization of Local Nash Equilibria in Continuous Games

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***Abstract*—We present a unified framework for characterizing local Nash equilibria in continuous games on either infinite-dimensional or finite-dimensional non-convex strategy spaces. We provide intrinsic necessary and sufficient first- and second-order conditions ensuring strategies constitute local Nash equilibria. We term points satisfying the sufficient conditions *differential Nash equilibria*. Further, we provide a sufficient condition (non-degeneracy) guaranteeing differential Nash equilibria are isolated and show that such equilibria are structurally stable. We present tutorial examples to illustrate our results and highlight degeneracies that can arise in continuous games.**

***Index Terms*—Game theory, optimization.**

I. Introduction

Many engineering systems are complex networks in which intelligent actors make decisions regarding usage of shared, yet scarce, resources. Game theory provides established techniques for modeling competitive interactions that have emerged as tools for analysis and synthesis of systems comprised of dynamically-coupled decision-making agents possessing diverse or opposing interests (see, e.g. [1], [2]). We focus on games with a finite number of agents where their strategy spaces are continuous, either a finite-dimensional differentiable manifold or an infinite-dimensional Banach manifold.

Previous work on continuous games with convex strategy spaces and player costs led to global characterization and computation of Nash equilibria [3]–[5]. Adding constraints led to extensions of nonlinear programming concepts, such as constraint qualification conditions, to games with generalized Nash equilibria [6]–[8]. Imposing a differentiable structure on the strategy spaces yielded other global conditions ensuring existence and uniqueness of Nash equilibria and Pareto optima [9]–[11]. In contrast to these prior efforts, we aim to analytically characterize and numerically compute *local* Nash equilibria in continuous games where the strategy space is convex but the player costs are non-convex, or where the strategy space itself is non-convex.

Bounding the rationality of agents can result in *myopic* behavior [12], meaning that agents seek strategies that are optimal locally but not necessarily globally. Further, it is common in engineering applications for strategy spaces or player costs to be non-convex, for example when an agent's configuration space is a constrained set or a differentiable manifold [13]–[19]. These observations suggest that techniques for characterization and computation of local Nash equilibria have important practical applications.

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Motivated by systems with myopic agents and non-convex strategy spaces, we seek an intrinsic characterization for local Nash equilibria that is structurally stable and amenable to computation. By generalizing derivative-based conditions for local optimality in nonlinear programming [20] and optimal control [21], we provide necessary first- and second-order conditions that local Nash equilibria must satisfy, and further develop a second-order sufficient condition ensuring player strategies constitute a local Nash equilibrium. We term points satisfying this sufficient condition *differential* Nash equilibria. In contrast to a pure optimization problem, this second-order condition is insufficient to guarantee a differential Nash equilibrium is isolated; in fact, games may possess a continuum of differential Nash equilibria. Hence, we introduce an additional second-order condition ensuring a differential Nash equilibrium is isolated.

Verifying that a strategy constitutes a Nash equilibrium in general requires testing that a non-convex inequality constraint is satisfied on an open set, a task we regard as generally intractable. In contrast, our sufficient conditions for local Nash equilibria require only the evaluation of player costs and their derivatives at a single point. Further, our framework allows for numerical computations to be carried out when players' strategy spaces and cost functions are non-convex. Hence, we provide tractable tools for characterization of differential Nash equilibria in continuous games.

We show that non-degenerate differential Nash equilibria are structurally stable. Consequently, model uncertainty or error that gives rise to a nearby game does not result in drastically different equilibrium behavior. This *structural stability* property is desirable in both the design of games as well as inverse modeling of agent behavior in competitive environments. Finally, we provide sufficient conditions ensuring that the flow generated by the gradient of each player's cost (*gradient play* [2] or *myopic tâtonnement* [22]) converges locally to a differential Nash equilibrium.

The results in this paper extend our earlier work [23], [24] by generalizing the second-order non-degeneracy condition [23, Thm. 2] to infinite-dimensions in Theorem 2, extending the structural stability result [24, Thm. 2] beyond parametric perturbations in Theorem 3, and presenting tutorial examples with practical relevance (see Example 1 throughout the text and Example 2 in Section VII).

The rest of the paper is organized as follows. In Section II we present the game formulation. We follow with the characterization of local Nash equilibria in Section III. In Section IV, we show that the characterization we provide—non-degenerate differential Nash equilibrium—is structurally stable. We take a dynamical systems point of view in Section V and show that approximate myopic best-response (*gradient play*) converges to stable equilibria. Throughout the paper we carry an example that provides insight into the importance of the results and in Section VI we return to the example and highlight the utility of the results for incentive design. In Section VII, we introduce an example of coupled oscillators that illustrates the significance of the results for engineering applications. Finally, we conclude with discussion in Section VIII. The necessary mathematical background and notation is contained in the Appendix.

II. Game Formulation

The theory of games we consider concerns interaction between a finite number of rational agents that we refer to as *players*.

Consider a game of *complete information* in which we have n players. The strategy spaces are topological spaces M_i for each $i \in \{1, \ldots, n\}$. Note these can be finite-dimensional smooth manifolds or infinite-dimensional Banach manifolds. We denote the joint strategy space by $M = \prod_{i=1}^{n} M_i$. The players are each interested in minimizing a cost function representing their interests by choosing an element from their strategy space. We define player i's cost to be a twice-differentiable function $f_i \in C^2(M, \mathbb{R})$. The following definition describes the equilibrium behavior we are interested in:

Definition 1: A strategy $(u_1, \ldots, u_n) \in M$ is a **local Nash equilibrium** if for each $i \in \{1, \ldots, n\}$ there exists an open set $W_i \subset M_i$ such that $u_i \in W_i$ and

$$f_i(u_1, \ldots, u_i, \ldots, u_n) \leqslant f_i(u_1, \ldots, u_i', \ldots, u_n), \ \ \forall\, u_i' \in W_i \backslash \{u_i\}. \tag{1}$$

If the above inequalities are strict, then we say $(u_1, \ldots, u_n)$ is a **strict local Nash equilibrium**. If $W_i = M_i$ for each i, then $(u_1, \ldots, u_n)$ is a **global Nash equilibrium**. Simply put, the above definition states that no player can unilaterally deviate from the Nash strategy and decrease their cost.

Prior to moving on to the characterization of local Nash equilibria, we describe the types of games the results apply to and why they are important in engineering applications.

Continuous games with finite-dimensional strategy spaces are described by the player strategy spaces $M_1, \ldots, M_n$ and their cost functions $(f_1, \ldots, f_n)$. They arise in a number of engineering and economic applications, for instance, in modeling one-shot decision making problems arising in transportation, communication and power networks [25]–[27], or mixed strategies over discrete strategy spaces [2].

On the other hand, the consideration of mixed strategies in games with continuous finite-dimensional strategy spaces lead to games on infinite-dimensional strategy spaces. In particular, the mixed strategies are probability measures over the pure strategies [28].

Continuous games with infinite-dimensional strategy spaces, regarded as open-loop differential games, are used in engineering applications in which there are agents coupled through dynamics [29]. They arise in problems such as building energy management [30], pricing of network security [31], travel-time optimization in transportation networks [32], and integration of renewables into energy systems [33].

Let us consider a simple, but motivating, example that exhibits very interesting behavior.

Example 1 (Betty-Sue: Thermodynamic Coupling): Consider a two player game between Betty and Sue. Let Betty's strategy space be $M_1 = \mathbb{R}$ and her cost function $f_1(u_1, u_2) = (1/2)u_1^2 - u_1 u_2$. Similarly, let Sue's strategy space be $M_2 = \mathbb{R}$ and her cost function $f_2(u_1, u_2) = (1/2)u_2^2 - u_1 u_2$. This game can be thought of as an abstraction of two agents in a building occupying adjoining rooms where cross terms model the effect of heat transfer. The first term in each of their costs represents an energy cost and the second term is a cost from thermodynamic coupling. The agents try to maintain the temperature at a desired set-point in thermodynamic equilibrium.

Definition 1 specifies that a point (μ_1, μ_2) is a Nash equilibrium if no player can unilaterally deviate and decrease their cost, i.e., $f_1(\mu_1, \mu_2) < f_1(u_1, \mu_2)$ for all $u_1 \in \mathbb{R}$ and $f_2(\mu_1, \mu_2) < f_2(\mu_1, u_2)$ for all $u_2 \in \mathbb{R}$.

Fix Sue's strategy $u_2 = \mu_2$, and calculate $D_1 f_1 = \partial f_1 / \partial u_1 = u_1 - \mu_2$. Then, Betty's optimal response to Sue playing $u_2 = \mu_2$ is $u_1 = \mu_2$. Similarly, if we fix $u_1 = \mu_1$, then Sue's optimal response to Betty playing $u_1 = \mu_1$ is $u_2 = \mu_1$. For all $u_1 \in \mathbb{R} \setminus \{\mu_2\}$

$$-\frac{1}{2}\mu_2^2 < \frac{1}{2}u_1^2 - u_1\mu_2 \tag{2}$$

so that $f_1(\mu_2, \mu_2) < f_1(u_1, \mu_2)$ for all $u_1 \in \mathbb{R} \setminus \{\mu_2\}$. Again, similarly, for all $u_2 \in \mathbb{R} \setminus \{\mu_1\}$

$$-\frac{1}{2}\mu_1^2 < \frac{1}{2}u_2^2 - u_2\mu_1 \tag{3}$$

so that $f_2(\mu_1, \mu_1) < f_2(\mu_1, u_2)$ for all $u_2 \in \mathbb{R} \setminus \{\mu_1\}$. Hence, all the points on the line $u_1 = u_2$ in $M_1 \times M_2 = \mathbb{R}^2$ are strict local Nash equilibria—in fact, they are strict global Nash equilibria. ■

Of course, the example is simple. Nonetheless, we can use it to illustrate some important concepts. As the example shows, continuous games can exhibit a continuum of equilibria. Throughout the text we will return to this example.

III. Characterization of Local Nash Equilibria

In this section, we characterize local Nash equilibria by paralleling results in nonlinear programming and optimal control that provide first- and second-order necessary and sufficient conditions for local optima.

The following definition of a differential game form is due to Stein [34].

Definition 2: A **differential game form** is a differential 1-form $\omega : M_1 \times \cdots \times M_n \to T^*(M_1 \times \cdots \times M_n)$ defined by

$$\omega = \sum_{i=1}^{n} \psi_{M_i} \circ df_i \tag{4}$$

where ψ_{M_i} are the natural bundle maps defined in (20) that annihilate those components of the covector field df_i not corresponding to M_i.

Remark 1: If each M_i is a finite-dimensional manifold of dimension m_i, then the differential game form has the following coordinate representation:

$$\omega_\varphi = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\partial (f_i \circ \varphi^{-1})}{\partial y_i^j} dy_i^j \tag{5}$$

where (U, φ) is a product chart on M at $u = (u_1, \ldots, u_n)$ with local coordinates $(y_1^1, \ldots, y_1^{m_1}, \ldots, y_n^1, \ldots, Y_n^{m_n})$ and where $U = \prod_{i=1}^{n} U_i$ and $\varphi = X_{i=1}^{n} \varphi_i = \varphi_1 \times \cdots \times \varphi_n$. In addition, $f_i \circ \varphi^{-1}$ is the coordinate representation of f_i for $i \in \{1, \ldots, n\}$. In particular, $\varphi_i(u_i) = (y_i^1, \ldots, y_i^{m_i})$ where each $y_i^j : U_i \to \mathbb{R}$ is a coordinate function so that dy_i^j is its derivative. ■

The differential game form captures a differential view of the strategic interaction between the players. Note that each player's cost function depends on its own choice variable as well as all the other players' choice variables. However, each player can only affect their payoff by adjusting their own strategy.

Definition 3: A strategy $u = (u_1, \ldots, u_n) \in M_1 \times \cdots \times M_n$ is a **differential Nash equilibrium** if $\omega(u) = 0$ and $D_{ii}^2 f_i(u)$ is positive-definite for each $i \in \{1, \ldots, n\}$.

The second-order conditions used to define differential Nash equilibria are motivated by results in nonlinear programming that use first- and second-order conditions to assess whether a critical point is a local optimum [20], [21].

The following proposition provides first- and second-order necessary conditions for local Nash equilibria.

Proposition 1: If $u = (u_1, \ldots, u_n)$ is a local Nash equilibrium, then $\omega(u) = 0$ and $D_{ii}^2 f_i(u)$ is positive semi-definite for each $i \in \{1, \ldots, n\}$.

Proof: Suppose that $u = (u_1, \ldots, u_n) \in M$ is a local Nash equilibrium. Then

$$f_i(u) \leqslant f_i(u_1, \ldots, u_i', \ldots, u_n), \ \forall u_i' \in W_i \setminus \{u_i\} \tag{6}$$

for open $W_i \subset M_i$, $i \in \{1, \ldots, n\}$. Suppose that we have a product chart (U, φ), where $U = \prod_{i=1}^n U_i$ and $\varphi = X_{i=1}^n \varphi_i$, such that $u \in U$. Let $\varphi_i(u_i) = v_i$ for each i. Then, since φ is continuous, for each $i \in \{1, \ldots, n\}$, we have that for all $v_i' \in \varphi_i(W_i \cap U_i) \setminus \{\varphi_i(u_i)\}$

$$f_i \circ \varphi^{-1}(v_1, \ldots, v_i, \ldots, v_n) \leqslant f_i \circ \varphi^{-1}(v_1, \ldots, v_i', \ldots, v_n). \tag{7}$$

Now, we apply Proposition 1.1.1 from [20], if M_i is finite-dimensional, or Theorem 4.2.3(1) and Theorem 4.2.4(a) from [21], if M_i is infinite-dimensional, to $f_i \circ \varphi^{-1}$. We conclude that for each $i \in \{1, \ldots, n\}$, $D_i(f_i \circ \varphi^{-1})(v_1, \ldots, v_n) = 0$ and for all $\nu \in \varphi_i(U_i \cap W_i)$

$$D_{ii}^2(f_i \circ \varphi^{-1})(v_1, \ldots, v_n)(\nu, \nu) \geqslant \alpha \|\nu\|^2 \tag{8}$$

i.e., it is a positive semi-definite bilinear form on $\varphi_i(U_i \cap W_i)$.

Invariance of the stationarity of critical points and the index of the Hessian with respect to coordinate change gives us that $\omega(u) = 0$ and $D_{ii}^2 f_i(u)$ is a positive semi-definite for each $i \in \{1, \ldots, n\}$. ■

We now show that the conditions defining a differential Nash equilibrium are sufficient to guarantee a strict local Nash equilibrium.

Theorem 1: A differential Nash equilibrium is a strict local Nash equilibrium.

Proof: Suppose that $u = (u_1, \ldots, u_n) \in M$ is a differential Nash equilibrium. Then, by the definition of differential Nash equilibrium, $\omega(u) = 0$ and $D_{ii}^2 f_i(u)$ is positive definite for each $i \in \{1, \ldots, n\}$. The second-derivative conditions imply that $D_{ii}^2(f_i \circ \varphi^{-1})(v_1, \ldots, v_n)$ is a positive-definite bilinear form where $v_i = \varphi_i(u_i)$ for any coordinate chart (U, φ), with $\varphi = X_i \varphi_i$, $U = \prod_i U_i$, and $u_i \in U_i$ for each $i \in \{1, \ldots, n\}$.

Using the isomorphism introduced in the Appendix in (19), $\omega(u) = 0$ implies that for each $i \in \{1, \ldots, n\}$, $D_i(f_i \circ \varphi^{-1})(v_1, \ldots, v_n) = 0$. Let E_i be the model space, i.e., the underlying Banach space, in either the finite-dimensional or infinite-dimensional case. Applying either Proposition 1.1.3 from [20] or Theorem 4.2.6 (a) from [21] to to each $f_i \circ \varphi^{-1}$ with $(\varphi_1(u_1), \ldots, \varphi_{i-1}(u_{i-1}), \varphi_{i+1}(u_{i+1}), \ldots, \varphi_n(u_n))$ fixed yields a neighborhood $W_i \subset E_i$ such that for all $v' \in W_i$

$$f_i \circ \varphi^{-1}(v_1, \ldots, v_i, \ldots, v_n) < f_i \circ \varphi^{-1}(v_1, \ldots, v', \ldots, v_n). \tag{9}$$

Since φ is continuous, there exists a neighborhood $V_i \subset M_i$ of u_i such that for $V_i = \varphi_i^{-1}(W_i)$ and all $u_i' \in V_i \setminus \{u_i\}$

$$f_i(u_1, \ldots, u_i, \ldots, u_n) < f_i(u_1, \ldots, u_i', \ldots, u_n). \tag{10}$$

Therefore, differential Nash equilibria are strict local Nash equilibria. Since both $\omega(u) = 0$ and definiteness of the Hessian are coordinate invariant, this is independent of choice of coordinate chart. ■

We remark that the conditions for differential Nash equilibria are not sufficient to guarantee that an equilibrium is isolated.

Example 1 (Betty-Sue: Continuum of Differential Nash): Returning to the Betty-Sue example, at all the points such that $u_1 = u_2$, $\omega(u_1, u_2) = 0$ and $D_{ii}^2 f_i(u_1, u_2) = 1 > 0$ for each $i \in \{1, 2\}$. Hence, there is a continuum of differential Nash equilibria. ■

We propose a sufficient condition to guarantee that differential Nash equilibria are isolated. We do so by combining ideas introduced by Rosen for convex games with concepts from Morse theory; in particular, second-order conditions on non-degenerate critical points of real-valued functions on manifolds.

At a differential Nash equilibrium $u = (u_1, \ldots, u_n)$, consider the derivative of the differential game form

$$d\omega = \sum_{i=1}^n d(\psi_{M_i} \circ df_i). \tag{11}$$

Intrinsically, this derivative is a tensor field $d\omega \in T_2^0(M)$; at a point $u \in M$ where $\omega(u) = 0$ it determines a bilinear form constructed from the uniquely determined continuous, symmetric, bilinear forms $\{d^2 f_i(u)\}_{i=1}^n$. We will refer to its local representation as the *Hessian* of the differential game form.

Theorem 2: If $u = (u_1, \ldots, u_n)$ is a differential Nash equilibrium and $d\omega(u)$ is non-degenerate, then u is an isolated strict local Nash equilibrium.

Proof: Since u is a differential Nash equilibrium, Theorem 1 gives us that it is a strict local Nash equilibrium. The following argument shows that it is isolated. Non-degeneracy of $d\omega(u)$ is invariant with respect to choice of coordinates. It suffices to choose a coordinate chart (U, φ) with $\varphi = X_{i=1}^n \varphi_i$ and $U = \prod_{i=1}^n U_i$ and show the result with respect to φ. Let E denote the underlying model space of the manifold $M_1 \times \cdots \times M_n$. Define the map $g : E \to E$ by

$$g(\varphi(u)) = \sum_{i=1}^n D_i(f_i \circ \varphi^{-1})(\varphi(u)) \tag{12}$$

where g is the coordinate representation of the differential game form ω. Zeros of the function g define critical points of the game and its derivative at critical points is $d\omega_\varphi$. Since u is a differential Nash equilibrium, $\omega(u) = 0$. Further, since $d\omega_\varphi(u)$ is non-degenerate—the map $A(v)(w) = d\omega_\varphi(u)(v, w)$ is a linear isomorphism—we can apply the Inverse Function Theorem [35, Thm. 2.5.2] to get that g is a local diffeomorphism at u, i.e., there exists an open neighborhood V of u such that the restriction of g to V establishes a diffeomorphism between V and an open subset of E. Thus, only $\varphi(u)$ could be mapped to zero near $\varphi(u)$. Therefore, independent of the choice of φ, u is isolated. ■

Definition 4: Differential Nash equilibria $u = (u_1, \ldots, u_n)$ such that $d\omega(u)$ is non-degenerate are termed *non-degenerate*.

Example 1 (Betty-Sue: Degeneracy and Breaking Symmetry): Return again to the Betty-Sue example in which we showed that there is a continuum of differential Nash equilibria. At each of these points, it is straightforward to check that $\det(d\omega(u_1, u_2)) = 0$. Hence, all of the equilibria are *degenerate*. By breaking the symmetry in the game, we can make (0,0) a non-degenerate differential Nash equilibrium; i.e., we can remove all but one of the equilibria. Indeed, let Betty's cost be given by $\tilde{f}_1(u_1, u_2) = (1/2)u_1^2 - a u_1 u_2$ and let Sue's cost remain unchanged. Then the local representation of the derivative of the differential game form $\tilde{\omega}$ of the game $(\tilde{f}_1, f_2)$ is

$$d\tilde{\omega}(u_1, u_2) = \begin{bmatrix} 1 & -a \\ -1 & 1 \end{bmatrix}. \tag{13}$$

Thus for any value of $a \neq 1$, $d\tilde{\omega}$ is invertible and hence (0,0) is a non-degenerate differential Nash equilibrium. This shows that small modeling errors can remove degenerate differential Nash equilibria. ■

In a neighborhood of a non-degenerate differential Nash equilibrium there are no other Nash equilibria. This property is desirable particularly in applications where a central planner is designing incentives to induce a socially optimal or otherwise desirable equilibrium that optimizes the central planner's cost; if the desired equilibrium resides on a continuum of equilibria, then due to measurement noise or myopic play, agents may be induced to play a nearby equilibrium that is suboptimal for the central planner. In Section VI, we extend

Example 1 by introducing a central planner. But first, we show that non-degenerate differential Nash equilibria are structurally stable.

IV. Structural Stability

Examples demonstrate that global Nash equilibria may fail to persist under arbitrarily small changes in player costs [10]. A natural question arises: do local Nash equilibria persist under perturbations? Applying structural stability analysis from dynamical systems theory, we answer this question affirmatively for non-degenerate differential Nash equilibria subject to smooth perturbations in player costs.

Let $M = M_1 \times \cdots \times M_n$ and $f_1, \ldots, f_n : M \to \mathbb{R}$ be C^2 player cost functions, $\omega : M \to T^*M$ the associated differential game form (4), and suppose $u \in M$ is a non-degenerate differential Nash equilibrium, i.e., $\omega(u) = 0$ and $d\omega(u)$ is non-degenerate. We show that for all $\tilde{f}_i \in C^\infty(M, \mathbb{R})$ sufficiently close to f_i there exists a unique non-degenerate differential Nash equilibrium $\tilde{u} \in M$ for $(\tilde{f}_1, \ldots, \tilde{f}_n)$ near u.

In our previous work, we showed that non-degenerate differential Nash are parametrically structurally stable [24, Theorem 2]. We remark that this results extends directly to any finitely-parameterized perturbation. For an arbitrary perturbation, we have the following.

Theorem 3 (Structural Stability): Non-degenerate differential Nash equilibria are structurally stable: let $u \in M$ be a non-degenerate differential Nash equilibrium for $(f_1, \ldots, f_n) \in C^2(M, \mathbb{R}^n)$. Then there exist neighborhoods $U \subset C^2(M, \mathbb{R}^n)$ of $(f_1, \ldots, f_n)$ and $W \subset M$ of u and a C^2 Fréchet-differentiable function $\sigma \in C^2(U, W)$ such that for all $(\tilde{f}_1, \ldots, \tilde{f}_n) \in U$ the point $\sigma(\tilde{f}_1, \ldots, \tilde{f}_n)$ is the unique non-degenerate differential Nash equilibrium for $(\tilde{f}_1, \ldots, \tilde{f}_2)$ in W.

Proof: Consider the operator $\Omega \in C^1(C^1(M, \mathbb{R}^n) \times M, \mathbb{R}^n)$ defined by

$$\Omega\left((\tilde{f}_1, \ldots, \tilde{f}_n), (u_1, \ldots, u_n)\right) = \sum_{i=1}^{n} \psi_{M_i} \circ d\tilde{f}_i(u_1, \ldots, u_n). \tag{14}$$

Note that the right-hand side is the differential game form $\tilde{\omega}(u_1, \ldots, u_n)$ for the game $(\tilde{f}_1, \ldots, \tilde{f}_n)$. Suppose that $u = (u_1, \ldots, u_n)$ is a non-degenerate differential Nash equilibrium. A straightforward application of Proposition 2.4.20 [35] implies that the operator Ω is C^1 Fréchet-differentiable. In addition,

$$D_2\Omega\left((f_1, \ldots, f_n), (u_1, \ldots, u_n)\right) = d\omega(u_1, \ldots, u_n). \tag{15}$$

Since $d\omega(u)$ is an isomorphism by assumption, we can apply the Implicit Function Theorem [35, Prop. 3.3.13 (iii)] to Ω to get an open neighborhood $W \subset M$ of u and $V \subset C^2(M, \mathbb{R}^n)$ of $(f_1, \ldots, f_n)$ and a smooth function $\sigma \in C^2(V, W)$ such that

$$\forall\, \tilde{f} \in V,\ v \in W :\ \Omega(\tilde{f}, v) = 0 \iff v = \sigma(\tilde{f})$$

where $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)$. Furthermore, since Ω is continuously differentiable, there exists a neighborhood $U \subset V$ of $(f_1, \ldots, f_n)$ such that $d\Omega(\tilde{f}, \sigma(\tilde{f}))$ is an isomorphism for all $\tilde{f} \in U$. Thus, for all $\tilde{f} \in U$, $\sigma(\tilde{f}) \in M$ is the unique non-degenerate differential Nash equilibrium on W. ■

Let us return to Example 1 and examine what can happen in the degenerate case.

Example 1 (Betty-Sue: Structural Instability): Recall the Betty-Sue example admitting a continuum of differential Nash equilibria. An arbitrarily small perturbation can make *all* the equilibria disappear. Indeed, let $\varepsilon \neq 0$ be arbitrarily small and consider Betty's perturbed cost function $\tilde{f}_1(u_1, u_2) = (1/2)u_1^2 - u_1u_2 + \varepsilon u_1$. Let Sue's cost function remain unchanged. A necessary condition for (u_1, u_2) to be a Nash equilibrium is $\omega(u_1, u_2) = 0$ (see Proposition 1) thereby implying $D_1\tilde{f}_1(u_1, u_2) = u_1 - u_2 + \varepsilon = 0$ and $D_2 f_2(u_1, u_2) = u_2 - u_1 = 0$. This can only happen for $\varepsilon = 0$. Hence, *any* perturbation εu_1 with $\varepsilon \neq 0$ will remove all the Nash equilibria. ■

We remark that in the finite-dimensional case we can show that non-degenerate differential Nash equilibria are generic among local Nash equilibria [24]. Genericity implies that local Nash equilibria in an open-dense set of continuous games (in the C^r topology on player costs) are non-degenerate differential Nash equilibria. Furthermore, structural stability implies that these equilibria persist under smooth perturbations to player costs. As a consequence, small modeling errors or environmental disturbances generally do not result in games with drastically different equilibrium behavior.

V. Convergence of Gradient Play

We adopt a dynamical systems perspective of a two-player game over the strategy space $U = \prod_{i=1}^{n} U_i$ with player costs $f_i : U \to \mathbb{R}$. Specifically, we consider the continuous-time dynamical system generated by the negative of the player's individual gradients:

$$\dot{u} = -\omega(u). \tag{16}$$

Gradient play may be viewed as a *better response* strategy instead of a *best response* strategy; in particular, this is a myopic tâtonnement process in which players adjust their current strategy in a gradient direction [2]. If $(\mu_1, \mu_2) \in U_1 \times U_2$ is a differential Nash equilibrium, then $\omega(\mu_1, \mu_2) = 0$. These dynamics are *uncoupled* in the sense the dynamics $\dot{u}_i$ for each player do not depend on the cost function of the other player. It is known that such uncoupled dynamics need not converge to local Nash equilibria [36]. However, we provide the following result on convergence of these dynamics.

Proposition 2: If μ is a differential Nash equilibrium and the spectrum of $d\omega$ is strictly in the right-half plane, then μ is an exponentially stable stationary point of the continuous-time dynamical system (16).

The above result was stated for the finite-dimensional case in [23, Prop. 4] and the proof of the stated result is an application of [35, Thm. 4.3.4].

We say a non-degenerate differential Nash equilibrium is *stable* if the spectrum of $d\omega$ is strictly in the right-half plane. Equilibria that are stable—thereby attracting using decoupled myopic approximate best-response—persist under small perturbations [23, Section IV]. Furthermore, Theorem 3 shows that convergence of uncoupled gradient play to such *stable* non-degenerate differential Nash equilibria persists under small smooth perturbations to player costs since the spectrum varies continuously [37, Lemma 6.3].

VI. Inducing a Nash Equilibrium

The problem of inducing Nash equilibria through incentive mechanisms appears in engineering applications including energy management [30] and network security [38]. The central planner aims to shift the Nash equilibrium of the agents' game to one that is desirable from its perspective. Thus the central planner optimizes its cost subject to constraints given by the inequalities that define a Nash equilibrium. This requires verifying satisfaction of non-convex inequalities on an open set—a generally intractable task. A natural solution is to replace these inequalities with first- and second-order sufficient conditions on each agent's optimization problem. As the Betty-Sue example shows (Example 1), these necessary conditions are not enough to guarantee the desired Nash is isolated; the additional constraint that $d\omega$ be non-degenerate must be enforced.

Example 1 (Betty-Sue: Inducing Nash): A central planner desires to optimize the cost of deviating from the temperature τ

$$f_p(u_1,u_2) = (u_1-\tau)^2 + (u_2-\tau)^2 \tag{17}$$

by induce the agents to play $(u_1,u_2)=(\tau,\tau)$. The planner does so by selecting $a\in\mathbb{R}$ and augmenting Betty's and Sue's costs:

$$\tilde{f}_1^a(u_1,u_2) = f_1(u_2,u_2) + \frac{a}{2}(u_1-\tau)^2$$
$$\tilde{f}_2^a(u_1,u_2) = f_2(u_1,u_2) + \frac{a}{2}(u_2-\tau)^2.$$

The differential game form of the augmented game $(\tilde{f}_1^a,\tilde{f}_2^a)$ is

$$\tilde{\omega}(u_1,u_2)=(u_1-u_2+a(u_1-\tau))\,du_1+(u_2-u_1+a(u_2-\tau))\,du_2$$

and the Hessian of the differential game form at the differential Nash equilibrium is

$$d\tilde{\omega}(u_1,u_2) = \begin{bmatrix} 1+a & -1 \\ -1 & 1+a \end{bmatrix}.$$

For any $a\in(-1,\infty)$, (τ,τ) is a differential Nash equilibrium of $(\tilde{f}_1^a,\tilde{f}_2^a)$ since $\tilde{\omega}(\tau,\tau)=0$ and $d_{ii}^2\tilde{f}_i^a(\tau,\tau)>0$. For any $a\in(-1,0]$, the game $(\tilde{f}_1^a,\tilde{f}_2^a)$ exhibits undesirable behavior. Indeed, recall (16) in which we consider the gradient dynamics for a two player game. For values of $a\in(-1,0)$, $d\tilde{\omega}$ is indefinite so that the equilibrium of the gradient system is a saddle point. Hence, if agents perform gradient play and happen to initialize on the unstable manifold, then they will not converge to any equilibrium. Further, while $a=0$ seems like a natural choice since it means not augmenting the players costs at all, it in fact gives rise to a continuum of equilibria. However, for $a>0$, $d\tilde{\omega}$ is positive definite so that, as Proposition 2 points out, the gradient dynamics will converge and the value of a determines the contraction rate. ■

This example indicates how undesirable behavior can arise when the operator $d\omega$ is degenerate. Further, if the goal is to induce a particular Nash equilibrium amongst competitive agents, then it is not enough to consider only necessary and sufficient conditions for Nash equilibria; inducing stable non-degenerate differential Nash equilibria leads to desirable and structurally stable behavior.

VII. A Game of Coupled Oscillators

In this section we present an illustrative example of coupled oscillators on a non-convex strategy space, the torus. Coupled oscillator models are used widely for applications including power networks [15], traffic networks [16], robotics [17], biological networks [18], and in coordinated motion control [19]. Often coupled oscillators are viewed in a game-theoretic context in order to gain further insight into the system properties [39], [40].

Let us first define the notion of a potential game, consistent with the definition introduced in the seminal work by Monderer and Shapley [41], for games on non-convex strategy space.

Definition 5: An n-player game on a smooth, connected manifold $M=M_1\times\cdots\times M_n$ is a *potential game* if its differential game form is exact, i.e. there exists $\phi\in C^\infty(M,\mathbb{R})$ such that $\omega=d\phi$.

Example 2 (Coupled Oscillators): Consider n-coupled oscillators with an interaction structure specified by a undirected, complete graph where the nodes represent the oscillators and the edges indicate coupling between oscillators. Let the phase of oscillator i be denoted by $\theta_i\in\mathbb{S}^1$ and let its cost be $f_i=-(1/n)\sum_{i\in N_i}\cos(\theta_i-\theta_j)$ where N_i is the index set of oscillators that are coupled to oscillator i. The form of the cost is derived from the Laplacian potential function [19]. It is straightforward to check that the differential game form for the oscillator game satisfies $\omega=d\phi$ where

$$\phi(\theta_1,\ldots,\theta_n) = -\frac{1}{2n}\sum_{i=1}^n\left(\sum_{j\in N_i}\cos(\theta_i-\theta_j)\right).$$

We claim that all points in the set

$$\left\{(\theta_1,\ldots,\theta_n)\in\mathbb{S}^1\times\cdots\times\mathbb{S}^1\,\middle|\,\theta_i=\theta_j,\forall\, i,j\in\{1,\ldots,n\}\right\}$$

are global Nash equilibria of the game. Indeed, fix $\theta_j=\beta$ for all $j\neq i$. Then $\theta_i=\beta$ is a best-response—i.e., in the set of optimizers—by oscillator i to all other oscillators playing $\theta_j=\beta$. In particular,

$$\phi(\beta_1,\ldots,\beta_n) = -\frac{|N_i|}{n} < -\frac{1}{n}\sum_{j\in N_i}\cos(\theta_i'-\beta),\ \forall\,\theta_i'\in\mathbb{S}^1\setminus\{\beta\}$$

where $|N_i|$ denotes the cardinality of the set N_i. Thus there is a continuum of Nash equilibria for which the oscillators are *synchronized*. In fact there is a continuum of differential Nash equilibria; this is easily seen by checking that $D_{ii}^2 f_i(\theta,\ldots,\theta)>0$. ■

It is interesting to note that if we considered the same game with the modification that $(f_1,\ldots,f_n)$ are now utility functions and the oscillators are utility maximizers, then there is a continuum of Nash equilibria now at all $(\theta_1,\ldots,\theta_n)$ such that the oscillators are *balanced* [13].

While one may notice the symmetry in the game described in Example 2, breaking that symmetry may still result in multiple Nash equilibria.

Example 3: Consider now a simple game with $n=2$ oscillators managed by Jean and Paul respectively. Let Jean's cost be $f_1=-(1/2)\cos(\gamma\theta_1-\theta_2)$ and Paul's cost be $f_2=-(1/2)\cos(\theta_2-\theta_1)$ where in this example Jean and Paul have different preferences for their phase. Allowing γ to take values in $\mathbb{N}\setminus\{1\}$, there are at least $\gamma-1$ non-degenerate differential Nash equilibria:

$$\left\{(\theta_1,\theta_2)\in\mathbb{S}^1\times\mathbb{S}^1|\,\theta_1=\theta_2=\frac{2(n-1)\pi}{\gamma-1}, n\in\{1,\ldots,\gamma-1\}\right\}.$$

The above set contains only stable, non-degenerate differential Nash equilibria of the game (f_1,f_2) since points in this set satisfy $\omega(\theta_1,\theta_2)=0$, $D_{ii}^2 f_i(\theta_1,\theta_2)>0$, and $\det(d\omega(\theta_1,\theta_2))\neq 0$. In fact, they are (non-strict) global Nash equilibria. ■

In our framework, stable equilibria attract nearby trajectories under the gradient flow of the game, a fact which can be leveraged by a central planner. Indeed, the n-coupled oscillator game can be regarded as an abstraction of generators or inverters—perhaps even microgrids—coupling to the grid [15], [42] where each oscillator is individually managed. Due to the existence of a continuum of Nash equilibria, it is possible that the players will equilibrate on a socially undesirable outcome. A central planner vying to coordinate the individuals would therefore benefit from considering these second-order conditions when designing incentives.

VIII. Discussion

By paralleling results in non-linear programming and optimal control, we developed first- and second-order necessary and sufficient conditions that characterize local Nash equilibria in continuous games on both finite- and infinite-dimensional strategy spaces. We provided a second-order sufficient condition guaranteeing differential Nash equilibria are non-degenerate and, hence, isolated. We showed that non-degenerate differential Nash equilibria are structurally stable and thus small modeling errors or environmental disturbances will not result in games with drastically different equilibrium behavior. As a result of structural stability, our characterization of non-degenerate differential Nash equilibria is amenable to computation. We illustrate through examples that such a characterization has value for the design of incentives to induce a desired equilibrium. By enforcing not only non-degeneracy but also stability of a differential Nash equilibrium, the central planner can ensure that the desired equilibrium is isolated and that gradient play (or myopic tâtonnement) will converge locally.

APPENDIX
MATHEMATICAL PRELIMIARIES

This appendix contains standard mathematical objects used throughout this paper (see [35], [43] for a more detailed introduction). A *smooth manifold* is a topological manifold with a smooth atlas. We use the term *manifold* generally; we specify whether it is a finite- or infinite-dimensional manifold only when necessary and if it is not clear from context. If a covering by charts takes their values in a Banach space E, then E is called the *model space* and we say that M is a C^r*-Banach manifold*. For a vector space E we define the vector space of continuous $(r+s)$-multilinear maps $T^r_s(E) = L^{r+s}(E^*, \dots, E^*, E, \dots, E; \mathbb{R})$ with s copies of E and r copes of E^* and where E^* denotes the dual. We say elements of $T^r_s(E)$ are *tensors* on E, and we use the notation $T^r_s(M)$ to denote the vector bundle of such tensors [35, Def. 5.2.9].

Suppose $f : M \to N$ is a mapping of one manifold into another, and $u \in M$, then by means of charts we can interpret the derivative of f on each chart at u as a linear mapping $df(u) : T_uM \to T_{f(u)}N$. When $N = \mathbb{R}$, the collection of such maps defines a 1-*form* $df : M \to T^*M$. A point $u \in M$ is said to be a *critical point* of a map $f \in C^r(M, \mathbb{R})$, $r \geq 2$ if $df(u) = 0$. At a critical point $u \in M$, there is a uniquely determined continuous, symmetric, bilinear form $d^2f(u) \in T^0_2(M)$ such that $d^2f(u)$ is defined for all $v, w \in T_uM$ by $d^2(f \circ \varphi^{-1})(\varphi(u))(v_\varphi, w_\varphi)$ where φ is any product chart at u and v_φ, w_φ are the local representations of v, w respectively [44, Prop. in §7]. We say $d^2f(u)$ is *positive semi-definite* if there exists $\alpha \geq 0$ such that for any chart φ

$$d^2(f \circ \varphi^{-1})\,(\varphi(u))\,(v, v) \geqslant \alpha \|v\|^2, \;\; \forall\, v \in T_{\varphi(u)}E. \tag{18}$$

If $\alpha > 0$, then we say $d^2f(u)$ is *positive-definite*. Both critical points and positive definiteness are invariant with respect to the choice of coordinate chart.

Consider smooth manifolds $M_1, \dots, M_n$. The product space $\prod_{i=1}^n M_i = M_1 \times \cdots \times M_n$ is naturally a smooth manifold [35, Def. 3.2.4]. There is a canonical isomorphism at each point such that the cotangent bundle of the product manifold splits:

$$T^*_{(u_1,\dots,u_n)}(M_1 \times \cdots \times M_n) \cong T^*_{u_1}M_1 \oplus \cdots \oplus T^*_{u_n}M_n \tag{19}$$

where $\oplus$ denotes the direct sum of vector spaces. There are natural bundle maps annihilating the all the components other than those corresponding to M_i of an element in the cotangent bundle:

$$\psi_{M_i} : T^*(M_1 \times \cdots \times M_n) \to T^*(M_1 \times \cdots \times M_n). \tag{20}$$

In particular, $\psi_{M_i}(\omega_1, \dots, \omega_n) = (0_1, \dots, 0_{i-1}, \omega_i, 0_{i+1}, \dots, 0)$ where $\omega = (\omega_1, \dots, \omega_n) \in T^*_u(M_1 \times \cdots \times M_n)$ and $0_j \in T^*_{u_j}M_j$ for each $j \neq i$ is the zero functional. Let $M = M_1 \times \cdots \times M_n$. For a function $f : M \to \mathbb{R}$, we use the notation $D_i f(u)$ for the *partial derivatives* of f at $u \in M$ [35, Prop. 2.4.12]. For second-order partial derivatives, we use the notation $D^2_{ij}f(u) = D_i(D_j f)(u)$.

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