# Genericity and Structural Stability of Non–Degenerate Differential Nash Equilibria

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*Abstract*— **We show that non–degenerate differential Nash equilibria are generic among local Nash equilibria in games with smooth costs and continuous strategy spaces, and demonstrate that such equilibria are structurally stable with respect to smooth perturbations in player costs. This implies that second–order conditions suffice to characterize local Nash equilibria in an open–dense set of games where player costs are smooth functions. Furthermore, equilibria that are computable using decoupled myopic approximate best–response persist under perturbations to the cost functions of individual players.**

## I. Introduction

Significant interest has developed around multi–agent distributed control in biological systems [1], the smart grid [2], and cyber–physical systems [3]. In these applications, competition develops between self–interested agents when resources are scarce. Game theory is an established technique for modeling this interaction, and it has emerged as an engineering tool for analysis and synthesis of systems comprised of dynamically–coupled decision–making agents possessing competing interests [4]–[6]. We focus on games with a finite number of agents where the strategy space is a finite–dimensional differentiable manifold. We emphasize that this setting is general, encompassing in particular mixed strategies in finite games [7].

In applications, player behavior is subject to disturbances from the environment and perturbations due to imperfect modeling or sensing. This implies that the player costs cannot be known with arbitrary precision, and hence that techniques developed to analyze or synthesize game behavior must be robust to such imperfections. In this setting, we focus on *generic* and *structurally stable* game phenomena that manifest in all or almost–all games and persist under perturbations or disturbances.

Previous work on continuous games led to global characterization and computation of Nash equilibria and Pareto optima [8]–[15] by imposing convexity on player costs or strategy spaces. However, it is common in applications for strategy spaces to be non–convex, for example a constrained set or a differentiable manifold [16], [17]. Further, bounding the rationality of agents can result in *myopic* behavior [18], meaning that agents seek strategies that that are optimal locally but not necessarily globally.

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Motivated by systems comprised of myopic agents acting in non–convex strategy spaces, in previous work [19] we developed a second–order characterization of local Nash equilibria that is amenable to computation. Such *non–degenerate differential Nash equilibria* are always strict local Nash equilibria. In this paper we use techniques from differential topology [20], [21] to show the two equilibrium concepts are generically equivalent.

Examples demonstrate that global Nash equilibria may fail to persist under arbitrarily small changes in player costs [14]. A natural question arises: do local Nash equilibria persist under perturbations? Applying structural stability analysis from dynamical systems theory [22], [23], we answer this question affirmatively for non–degenerate differential Nash equilibria subject to smooth perturbations in player costs.

Genericity and structural stability of non–degenerate differential Nash equilibria implies that local Nash equilibria in an open–dense set of continuous games are non–degenerate differential Nash equilibria, and furthermore these equilibria persist under perturbations to player costs. As a consequence, small modeling errors or environmental disturbances generally do not result in games with drastically different equilibrium behavior. For instance, equilibria that are computable using decoupled myopic approximate best–response persist under small perturbations.

The paper is organized as follows. In Sections II and III, we discuss the necessary mathematical preliminaries and game formulation, respectively. We show that non–degenerate differential Nash equilibria are generic among local Nash equilibria in continuous games in Section IV. Subsequently, in Section V we show that non–degenerate differential Nash equilibria are structurally stable. Finally, we summarize the contributions of the paper and discuss future work in Section VI.

## II. Mathematical Preliminaries

We rely on tools from differential geometry [23] to provide an *intrinsic* characterization of equilibrium play in continuous games, and techniques from differential topology [21] to establish the genericity and structural stability results.

### A. Differential Geometry

A *topological $m$–dimensional manifold $M$* is a topological space which is Hausdorff, second-countable, and is locally Euclidean of dimension $m$, i.e. every point $p \in M$ has a neighborhood $U \subset M$ containing $p$ that is homeomorphic to $\mathbb{R}^m$ via a map $\varphi : U \to \mathbb{R}^m$. The pair $(U, \varphi)$ is called a *coordinate chart* and the component functions

$(u^1, \ldots, u^m) = \varphi$ are referred to as *local coordinates*. We say two charts $(U, \varphi)$ and $(V, \psi)$ are *smoothly compatible* if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a smooth bijective map with a smooth inverse, i.e. it is a diffeomorphism. A family of smoothly compatible charts whose domain covers $M$ is called a *smooth atlas* for $M$. A *smooth $m$–dimensional manifold $M$* is a topological manifold with a smooth atlas. A smooth manifold *without boundary* is a topological manifold with empty boundary. If $S$ is an $n$–dimensional submanifold of $M$, then we say $S$ has *co–dimension* $m - n$ in $M$.

A function $f : U \to \mathbb{R}^n$ defined on an open set $U \subset \mathbb{R}^m$ is said to be $C^k$ if all the partial derivatives of $f$ of order less than or equal to $k$ exist and are continuous functions on $U$. A function that is of class $C^k$ for all $k \geq 0$ is said to be *smooth*. A function $f : M \to N$ is $C^k$ if for every $p \in M$ there exist smooth charts $(U, \varphi)$ on $M$ and $(V, \psi)$ on $N$ with $p \in U$ and $f(U) \subset V$ such that $\psi \circ f \circ \varphi^{-1}$ is $C^k$ on $\varphi(U)$. In this case we will write $f \in C^k(M, N)$.

Each $p \in M$ has an associated *tangent space* $T_pM$, and the disjoint union of the tangent spaces is the *tangent bundle* $TM = \coprod_{p \in M} T_pM$. The *co-tangent space to* $M$ at $p \in M$, denoted $T_p^*M$, is the set of all real-valued linear functionals on the tangent space $T_pM$, and the disjoint union of the co–tangent spaces is the *co–tangent bundle* $T^*M = \coprod_{p \in M} T_p^*M$. Both $TM$ and $T^*M$ are naturally smooth manifolds. At each $p \in M$ there is an associated linear map $(f_*)_p : T_pM \to T_{f(p)}N$ called the *pushforward*. A 1*–form* on $M$ is a continuous map $\omega : M \to T^*M$ satisfying $\pi \circ \omega = \mathrm{Id}_M$ where $\pi : T^*M \to M$ is the natural projection. Each $f \in C^k(M, \mathbb{R})$ determines a 1–form $df : M \to T^*M$ that is $C^{k-1}$.

Consider topological manifolds $M_1$ and $M_2$ of dimension $m_1$ and $m_2$ respectively. The product space $M_1 \times M_2$ is naturally a smooth manifold of dimension $m_1 + m_2$. In particular, there is an atlas on $M_1 \times M_2$ composed of *product charts* $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ where $(U_i, \varphi_i)$ is a chart on $M_i$ for $i \in \{1, 2\}$. There is a canonical isomorphism at each point such that the cotangent bundle of the product manifold splits

$$T^*_{(p,q)}(M_1 \times M_2) \cong T_p^*M_1 \oplus T_q^*M_2. \quad (1)$$

In addition, there are natural bundle maps $\pi_{M_1}, \pi_{M_2} : T^*(M_1 \times M_2) \to T^*(M_1 \times M_2)$ annihilating the last $m_2$ components and the first $m_1$ components respectively.

Consider a function $f \in C^\infty(M_1 \times M_2, \mathbb{R})$ and a product chart $(U, \varphi)$ on $M_1 \times M_2$. Let the local coordinates be denoted by $(u_1^1, \ldots, u_1^{m_1}, u_2^1, \ldots, u_2^{m_2})$. Then we define

$$D^\varphi f = \begin{bmatrix} D_1^\varphi f_1 & D_2^\varphi f_2 \end{bmatrix} \quad (2)$$

with

$$D_1^\varphi f = \left[ \frac{\partial (f \circ \varphi^{-1})}{\partial u_1^1} \cdots \frac{\partial (f \circ \varphi^{-1})}{\partial u_1^{m_1}} \right] \quad (3)$$

and we define $D_2^\varphi f$ similarly. The superscript notion indicates the dependence on chart and we suppress it when its clear from context. A *critical point* $(p, q)$ of $f$ is such that $D_1 f(p, q)$ and $D_2 f(p, q)$ are zero covectors in the appropriate co–tangent spaces. We use the notation $D^2 f$ to denote the Hessian of $f : M \to \mathbb{R}$ and we partition it as

$$D^2 f = \begin{bmatrix} D_{11}^2 f & D_{12}^2 f \\ D_{21}^2 f & D_{22}^2 f \end{bmatrix} \quad (4)$$

where $D_{12}^2 f = \left[ \frac{\partial^2 (f \circ \varphi^{-1})}{\partial u_1^i \partial u_2^j} \right]_{i,j}$ and similarly for the other blocks. The Hessian is well–defined (see [21] for a more detailed exposition).

### B. Differential Topology

Consider smooth manifolds $M$ and $N$ of dimension $m$ and $n$ respectively. An *$k$–jet* from $M$ to $N$ is an equivalence class $[x, f, U]_k$ of triples $(x, f, U)$ where $U \subset M$ is an open set, $x \in U$, and $f : U \to N$ is a $C^k$ map. The equivalence relation satisfies $[x, f, U]_k = [y, g, V]_k$ if $x = y$ and in some (and hence any) pair of charts adapted to $f$ at $x$, $f$ and $g$ have the same derivatives up to order $k$. We use the notation $[x, f, U]_k = j^k f(x)$ to denote the $k$–jet of $f$ at $x$. The set of all $k$–jets from $M$ to $N$ is denoted by $J^k(M, N)$. The jet bundle $J^k(M, N)$ is a smooth manifold (see [21] Chapter 2 for the construction). For each $C^k$ map $f : M \to N$ we define a map $j^k f : M \to J^k(M, N)$ by $x \mapsto j^k f(x)$ and refer to it as the *$k$–jet extension*.

*Definition 1:* Let $M$, $N$ be smooth manifolds and $f : M \to N$ be a smooth mapping. Let $Z$ be a smooth submanifold of $N$ and $p$ a point in $M$. Then $f$ *intersects* $Z$ *transversally at* $p$ (denoted $f \pitchfork Z$ at $p$) if either $f(p) \notin Z$ or $f(p) \in Z$ and $T_{f(p)}N = T_{f(p)}Z + (f_*)_p(T_pM)$.

For $1 \leq k < s \leq \infty$ consider the jet map

$$j^k : C^s(M, N) \to C^{s-k}(M, J^k(M, N)) \quad (5)$$

and let $Z \subset J^k(M, N)$ be a submanifold. Define

$$\pitchfork^s(M, N; j^k, Z) = \{h \in C^s(M, N) |\ j^k h \pitchfork Z\}. \quad (6)$$

A subset of a topological space $X$ is *residual* if it contains the intersection of countably many open–dense sets. We say a property is *generic* if the set of all points of $X$ which possess this property is residual [24].

The following results will be used to prove genericity of non–degenerate differential Nash equilibria.

*Theorem 2.8 in [21]* (Jet Transversality) Let $M$, $N$ be $C^\infty$ manifolds without boundary, and let $Z \subset J^k(M, N)$ be a $C^\infty$ submanifold. Suppose that $1 \leq k < s \leq \infty$. Then, $\pitchfork^s(M, N; j^k, Z)$ is residual and thus dense in $C^s(M, N)$ endowed with the strong topology, and open if $Z$ is closed.

*Proposition 1:* (See [25], Chapter II.4, Proposition 4.2) Let $M, N$ be smooth manifolds and $Z \subset N$ a submanifold. Suppose that $\dim M < \mathrm{codim} Z$. Let $f : M \to N$ be smooth and suppose that $f \pitchfork Z$. Then, $f(M) \cap Z = \emptyset$.

The definition of residual set, and hence genericity, implies a notion of *almost all* which will be necessary for showing that local Nash equilibria are generically non-degenerate differential Nash equilibria. The latter have nice properties; they are isolated, structurally stable as we will see in Section V, and hence computable.

The Jet Transversality Theorem and Proposition 1 can be used to show a subset of a jet bundle having a particular set of desired properties is generic. Indeed, consider the jet bundle $J^k(M,N)$ and recall that it is a manifold that contains jets $j^k f : M \to J^k(M,N)$ as its elements where $f \in C^k(M,N)$. Let $Z \subset J^k(M,N)$ be the submanifold of the jet bundle that *does not* possess the desired properties. If $\dim M < \text{codim } Z$, then for a generic function $f \in C^k(M,N)$ the image of the $k$–jet extension is disjoint from $Z$ implying that there is an open–dense set of functions having the desired properties. We will use this argument in Section IV to show that non–degenerate differential Nash equilibria are generic among local Nash equilibria.

## III. Game Formulation

The theory of games we consider concerns situations in which several rational agents, generally having different interests and objectives, interact within their environment. We refer to the rational agents as *players*. Competition arises due to the fact that the players have opposing interests. We note that the game formulation presented in this section and the results that follow easily extend to games with any finite number of players. We choose to present the results for two player games in an effort to be clear and concise.

Let us begin by considering a game in which we have two selfish players, Urbain and Victor, with competing interests. The strategy spaces of Urbain and Victor are smooth manifolds without boundary $M_1$ and $M_2$ respectively. The dimension of $M_i$ is $m_i$ for each $i \in \{1,2\}$, and we let $m = m_1 + m_2$. Urbain and Victor are interested in minimizing a cost function representing their interests by choosing elements from their (respective) strategy spaces. We define Urbain's cost function to be $f_1 : M_1 \times M_2 \to \mathbb{R}$ and Victor's cost function to be $f_2 : M_1 \times M_2 \to \mathbb{R}$ such that $f_1, f_2 \in C^\infty(M_1 \times M_2, \mathbb{R})$.

*Remark 1:* We note that it is only necessary to consider functions $f_1, f_2 \in C^2(M_1 \times M_2, \mathbb{R})$. We consider $C^\infty$ cost functions only to simplify the notation in the proofs in the sections that follow.

*Definition 2:* A strategy $(p,q) \in M_1 \times M_2$ is a ***local Nash equilibrium*** if there exist open sets $W_1 \subset M_1$, $W_2 \subset M_2$ such that $p \in W_1$, $q \in W_2$,

$$f_1(p,q) \le f_1(p',q) \quad \forall\ p' \in W_1 \backslash \{p\}, \tag{7}$$

and

$$f_2(p,q) \le f_2(p,q') \quad \forall\ q' \in W_2 \backslash \{q\}. \tag{8}$$

If $W_1 = M_1$ and $W_2 = M_2$, then $(p,q)$ is a ***global Nash equilibrium***. Further, if the above inequalities are strict, then we say $(p,q)$ is a ***strict local Nash equilibrium***.

*Definition 3:* A ***differential game form*** is a differential 1–form $\omega : M_1 \times M_2 \to T^*(M_1 \times M_2)$ defined by

$$\omega = \pi_{M_1}(df_1) + \pi_{M_2}(df_2) \tag{9}$$

and, in coordinates, is defined by

$$\omega = \sum_{i=1}^{m_1} \frac{\partial (f_1 \circ \varphi^{-1})}{\partial u_1^i} du_1^i + \sum_{j=1}^{m_2} \frac{\partial (f_2 \circ \varphi^{-1})}{\partial u_2^j} du_2^j \tag{10}$$

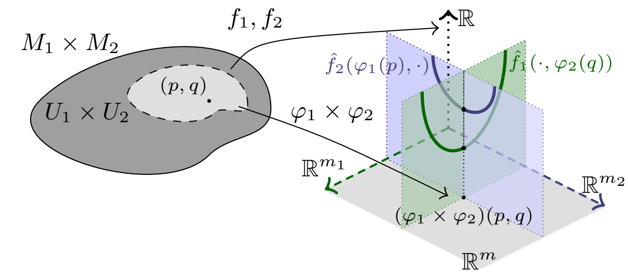


Fig. 1. The map $\hat{f}_i : \mathbb{R}^m \to \mathbb{R}$ is the coordinate representation of $f_i$ and it is defined by $\hat{f}_i = f_i \circ (\varphi_1 \times \varphi_2)^{-1}$ where $\varphi_1 \times \varphi_2 : U_1 \times U_2 \to \mathbb{R}^m$ is the coordinate map. Player $i$, whose cost function is $f_i$, can only adjust his payoff by changing is strategy in directions corresponding to $\mathbb{R}^{m_i}$. $\hat{f}_1(\cdot, \varphi_2(q))$ and $\hat{f}_2(\varphi_1(p), \cdot)$ are slices of the coordinate representation of $f_1$ and $f_2$, respectively.

where $(U_1 \times U_2, \varphi)$ is a product chart on $M_1 \times M_2$ with the form $\varphi = \varphi_1 \times \varphi_2$ and with local coordinates $(u_1^1, \ldots, u_1^{m_1}, u_2^1, \ldots, u_2^{m_2})$.

The above definition of a differential game form captures a differential view of the strategic interaction between the players. Indeed, $\omega$ indicates the direction in which Urbain and Victor can change their strategies to decrease their individual cost functions most rapidly. Note that Urbain's cost function is dependent on Urbain's strategy choice as well as Victor's, but Urbain can only affect his payoff by adjusting his strategy (and similarly for Victor).

*Definition 4:* A strategy $(p,q) \in M_1 \times M_2$ is a ***differential Nash equilibrium*** if $\omega(p,q) = 0$ and $D_{ii}^2 f_i(p,q) > 0$ for $i \in \{1,2\}$.

The above definition was introduced in [19] and is motivated by results in nonlinear programming that use first– and second–order conditions to characterize when a critical point is a local optimum [26], [27]. The conditions of the definition of differential Nash equilibrium guarantee the cost functions are locally convex as is illustrated in Figure 1.

Consider the matrix $D\omega(p,q)$ given by

$$D\omega(p,q) = \begin{bmatrix} D_{11}^2 f_1(p,q) & D_{21}^2 f_1(p,q) \\ D_{12}^2 f_2(p,q) & D_{22}^2 f_2(p,q) \end{bmatrix}. \tag{11}$$

By Theorem 2 of [19], $\det(D\omega(p,q)) \neq 0$ is a sufficient condition for a differential Nash equilibrium $(p,q)$ to be isolated. We say a differential Nash equilibria is ***degenerate*** if $\det(D\omega(p,q)) = 0$ and ***non–degenerate*** otherwise. Both Definition 4 and the non–degeneracy of a differential Nash equilibrium $(p,q)$ are coordinate–invariant.

*Remark 2:* The condition $\det(D\omega(p,q)) \neq 0$ is sufficient but not necessary for $(p,q)$ to be an isolated differential Nash equilibrium. There can be isolated differential Nash equilibria such that $\det(D\omega(p,q)) = 0$; we will see in the subsequent section that such equilibria are non–generic.

We now introduce an example we will return to in the sections that follow. The example illustrates some of the degeneracies that occur in non–generic games.

*Example 1:* Let Urbain's strategy space be $M_1 = \mathbb{R}$ and his cost function $f_1(x,y) = \frac{x^2}{2} - xy$. Similarly, let Victor's strategy space be $M_2 = \mathbb{R}$ and his cost function $f_2(x,y) = \frac{y^2}{2} - xy$. Fix $y = q$, and calculate

$$\frac{\partial f_1}{\partial x} = x - q \tag{12}$$

Then, Urbain's optimal response to Victor playing $y = q$ is $x = q$. Similarly, Victor's optimal response to Urbain playing $x = p$ is $y = p$. For all $x \in \mathbb{R}\backslash\{q\}$

$$-\frac{q^2}{2} < \frac{x^2}{2} - xq \tag{13}$$

so that $f_1(q,q) < f_1(x,q)$ for all $x \in \mathbb{R}\backslash\{q\}$. Similarly, for all $y \in \mathbb{R}\backslash\{p\}$

$$-\frac{p^2}{2} < \frac{y^2}{2} - yp \tag{14}$$

so that $f_2(p,p) < f_2(p,y)$ for all $y \in \mathbb{R}\backslash\{p\}$. Hence, all the points on the line $x = y$ in $M_1 \times M_2 = \mathbb{R}^2$ are strict local Nash equilibria (in fact, they are strict global Nash equilibria).

In the above example, any slice of Urbain's cost function is convex in his choice variable. Similarly, Victor's cost function at a fixed value for Urbain is convex. Yet, there is a continuum of local Nash equilibria. As we will show when we return to this example in Section IV and V, this example is neither generic or structurally stable.

## IV. Genericity

In this section, we show local Nash equilibria are generically non–degenerate differential Nash equilibria; there is an open–dense set of games whose local Nash equilibria are non–degenerate differential Nash equilibria. Non–degenerate differential Nash equilibria can be amenable to computation since they satisfy first– and second–order conditions reminiscent of those from nonlinear programming [19]. The following result is analogous to the fact in dynamical systems theory that non–degenerate singularities are generic [24].

*Theorem 1 (Genericity):* Non–degenerate differential Nash equilibria are generic among local Nash equilibria: for any smooth boundaryless manifolds $M_1, M_2$ there exists an open–dense subset $G \subset C^\infty(M_1 \times M_2, \mathbb{R}^2)$ such that for all $(f_1, f_2) \in G$, if $(p,q) \in M_1 \times M_2$ is a local Nash equilibrium for $(f_1, f_2)$, then $(p,q)$ is a non–degenerate differential Nash equilibrium for $(f_1, f_2)$.

*Proof:* Consider a two player game where player $i$'s cost function is $f_i \in C^\infty(M_1 \times M_2, \mathbb{R})$. Let $J^2(M_1 \times M_2, \mathbb{R}^2)$ denote the second order jet bundle containing 2–jets $j^2 f$ such that $f = (f_1, f_2) : M_2 \times M_2 \to \mathbb{R}^2$. Let $(U, \varphi)$ be a product chart on $M_1 \times M_2$ that contains $(p,q)$. The dimensions of $M_1$ and $M_2$ are $m_1$ and $m_2$ respectively and we define $m = m_1 + m_2$. We define $S(m)$ to be the symmetric $m \times m$ matrices as follows

$$S(m) = \{A \in \mathbb{R}^{m\times m} |\ A = A^T\}. \tag{15}$$

For $(A_1, A_2) \in S(m)^2$, we can partition each $A_i$ as follows:

$$A_i = \begin{bmatrix} A_i^{11} & A_i^{12} \\ A_i^{21} & A_i^{22} \end{bmatrix} \tag{16}$$

where $A_i^{kj} \in \mathbb{R}^{m_k \times m_j}$ for $j,k \in \{1,2\}$. The non–degeneracy of a differential Nash equilibrium is determined by the determinant of $D\omega$. Recall that $D\omega$ is constructed from components of the symmetric matrices $D^2 f_1$ and $D^2 f_2$, i.e. the Hessians of $f_1$ and $f_2$. Hence, we partition the space $S(m)^2$ into two subsets $S_1(m)$ and $S_2(m)$ defined as follows:

$$S_1(m) = \left\{ \begin{bmatrix} A_1^{11} & A_1^{21} \\ A_2^{12} & A_2^{22} \end{bmatrix} \in \mathbb{R}^{m \times m} \middle|\ A_1, A_2 \in S(m) \right\} \tag{17}$$

and

$$S_2(m) = \left\{ \begin{bmatrix} A_2^{11} & A_2^{21} \\ A_1^{12} & A_1^{22} \end{bmatrix} \in \mathbb{R}^{m \times m} \middle|\ A_1, A_2 \in S(m) \right\} \tag{18}$$

where $S_1(m)$ is the space corresponding to $D\omega$ and $S_2(m)$ is the space in which matrices constructed from the other pieces of the player Hessians that were excluded in the construction of $D\omega$. Then $J^2(M_1 \times M_2, \mathbb{R}^2)$ is locally diffeomorphic to

$$\mathbb{R}^m \times \mathbb{R}^2 \times \mathbb{R}^{m_1+m_2} \times \mathbb{R}^{m_1+m_2} \times \mathbb{R}^{\frac{m(m+1)}{2}} \times \mathbb{R}^{\frac{m(m+1)}{2}} \tag{19}$$

and the 2–jet extension of $f = (f_1, f_2)$ at a point $(p,q) \in M_1 \times M_2$, namely $j^2 f(p,q)$, in coordinates is given by

$$\begin{aligned}(\varphi(p,q), ((f_1 \circ \varphi^{-1})(\varphi(p,q)), (f_2 \circ \varphi^{-1})(\varphi(p,q))),\\ Df_1(p,q), Df_2(p,q), D^2 f_1(p,q), D^2 f_2(p,q)).\end{aligned} \tag{20}$$

Define

$$Z(m) = \{A \in S_1(m) |\ \det(A) = 0\}. \tag{21}$$

$Z(m)$ is an algebraic set and hence, admits a canonical Whitney stratification having finitely many algebraic strata (see Chapter 1, Theorem 2.7 of [28]), i.e. it is the finite union of submanifolds. By its construction, $Z(m)$ has no interior points. Hence, it has co–dimension at least 1. Now, we consider the subset of the jet bundle $J^2(M_1 \times M_2, \mathbb{R}^2)$ defined by

$$\begin{aligned} G_1 = &\mathbb{R}^m \times \mathbb{R}^2 \times \{0_{\mathbb{R}^{m_1}}\} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_1} \times \{0_{\mathbb{R}^{m_2}}\} \\ &\times Z(m) \times S_2(m) \end{aligned} \tag{22}$$

where $0_{\mathbb{R}^{m_i}}$ is the zero vector in $\mathbb{R}^{m_i}$. Note that $\{0_{\mathbb{R}^{m_i}}\}$ has co–dimension $m_i$. Hence, $G_1$ is the union of submanifolds of co–dimension at least $m_1 + m_2 + 1$. By the Jet Transversality Theorem (see Section II-B or Theorem 2.8 in [21]) and Proposition 1, since $m_1 + m_2 + 1 > m_1 + m_2$, for generic $f = (f_1, f_2)$, the image of the 2–jet extension $j^2 f$ is disjoint from $G_1$. Hence, there is an open–dense set of functions $f = (f_1, f_2)$ such that for each $(p,q) \in M_1 \times M_2$, whenever $D_1 f_1(p,q) = 0$ and $D_2 f_2(p,q) = 0$ (i.e. $\omega(p,q) = 0$), the derivative of the differential game form has non–zero determinant (i.e. $\det D\omega(p,q) \neq 0$). Note that the conditions $\omega(p,q) = 0$ and $\det(D\omega(p,q)) \neq 0$ are coordinate–invariant. Hence, this result is independent of the choice of chart.

Similarly, consider another subset of $J^2(M_1 \times M_2, \mathbb{R}^2)$ defined by

$$\begin{aligned} G_2 = &\mathbb{R}^m \times \mathbb{R}^2 \times \{0_{\mathbb{R}^{m_1}}\} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_1} \times \{0_{\mathbb{R}^{m_2}}\} \\ &\times Z(m_1) \times \mathbb{R}^{m_1 \times m_2} \times S(m_2) \\ &\times S(m_1) \times \mathbb{R}^{m_1 \times m_2} \times Z(m_2) \end{aligned} \tag{23}$$

where $Z(m_i)$ is the subset of symmetric matrices $S(m_i)$ such that for $A \in Z(m_i)$, $\det(A) = 0$. Since $Z(m_i)$ are algebraic and have no interior points, we may again use the Whitney stratification theorem to get that each $Z(m_i)$ is the union of submanifolds of co–dimension at least 1. Hence, $G_2$ is the union of submanifolds and has co–dimension at least $m_1 + m_2 + 2$. Application of the Jet Transversality Theorem and Proposition 1 yields an open–dense set of functions $f = (f_1, f_2)$ such that when $\omega(p, q) = 0$ we have $\det(D_{ii}^2 f_i(p, q)) \neq 0$ for each $i \in \{1, 2\}$.

Since the intersection of two open–dense sets is open–dense, we have an open–dense set of functions $f = (f_1, f_2)$ such that for each $(p, q) \in M_1 \times M_2$ whenever $\omega(p, q) = 0$, $\det(D_{ii}^2 f_i(p, q)) \neq 0$ for each $i \in \{1, 2\}$ and $\det(D\omega(p, q)) \neq 0$ independent of the choice of chart.

Thus, there exists an open–dense set $G \subset C^\infty(M_1 \times M_2, \mathbb{R}^2)$ such that for all $f = (f_1, f_1) \in G$, if $(p, q) \in M_1 \times M_2$ is a local Nash equilibrium, then $(p, q)$ is a non–degenerate differential Nash equilibrium. Indeed, suppose $(f_1, f_2) \in G$ and $(p, q) \in M_1 \times M_2$ is a local Nash equilibrium. Then, by Proposition 2 of [19], $(p, q)$ necessarily satisfies $\omega(p, q) = 0$ and $D_{ii}^2 f_i(p, q) \geq 0$ for each $i \in \{1, 2\}$. However, since $(f_1, f_2) \in G$, $\det(D_{ii}^2 f_i(p, q)) \neq 0$ so that $D_{ii}^2 f_i(p, q) > 0$. Hence, $(p, q)$ is a differential Nash equilibrium. Further, $(f_1, f_2) \in G$ implies that $\det(D\omega(p, q)) \neq 0$; hence, $(p, q)$ is non–degenerate. ■

Given $f_1, f_2 \in C^\infty(M_1 \times M_2, \mathbb{R})$, we define the set of local Nash equilibria

$$\begin{aligned} \mathrm{LN}(f_1, f_2) = \{ & (p, q) \in M_1 \times M_2 | W_1 \subset M_1, W_2 \subset M_2 \\ & f_1(p, q) \leq f_1(p', q)\ \forall p' \in W_1 \backslash \{p\} \\ & f_2(p, q) \leq f_2(p, q')\ \forall q' \in W_2 \backslash \{q\} \} \end{aligned} \tag{24}$$

and the set of non–degenerate differential Nash equilibria

$$\begin{aligned} & \mathrm{DN}(f_1, f_2) = \{(p, q) \in M_1 \times M_2 | \omega(p, q) = 0, \\ & D_{ii}^2 f_i(p, q) > 0 \text{ for each } i \in \{1, 2\}, \det(D\omega(p, q)) \neq 0\}. \end{aligned} \tag{25}$$

In [19], we showed that $\mathrm{DN}(f_1, f_2) \subset \mathrm{LN}(f_1, f_2)$ for all $f_1, f_2 \in C^\infty(M_1 \times M_2, \mathbb{R})$. Theorem 1 shows that $\mathrm{LN}(f_1, f_2) = \mathrm{DN}(f_1, f_2)$ for all $(f_1, f_2)$ in an open–dense subset $G \subset C^\infty(M_1 \times M_2, \mathbb{R})$. In other words, the set of local Nash equilibria is generically equivalent to the set of non–degenerate differential Nash equilibria.

*Example 1 (continued):* Continuing with Example 1, we can see that there is a continuum of differential Nash equilibria at which $\det(D\omega(p, q)) = 0$. Indeed, all points on the line $x = y$ are differential Nash equilibria and

$$D\omega(p, q) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{26}$$

The presence of such *degenerate* differential Nash equilibria indicate that the game in this example is not generic.

Another interpretation of Theorem 1 is that degenerate local Nash equilibria can become non–degenerate differential Nash equilibria under an arbitrarily small perturbation to the game. We will show in the following section that the converse is not true: non–degenerate differential Nash equilibria persist under small smooth perturbations to player costs.

## V. Structural Stability

Let $f_1, f_2 : M_1 \times M_2 \to \mathbb{R}$ be smooth player cost functions, $\omega : M_1 \times M_2 \to T^*(M_1 \times M_2)$ the associated differential game form (10), and suppose $(p, q) \in M_1 \times M_2$ is a non–degenerate differential Nash equilibrium, i.e. $\omega(p, q) = 0$ and $D\omega(p, q)$ is invertible. We show that for all $\widetilde{f}_1, \widetilde{f}_2 \in C^\infty(M_1 \times M_2, \mathbb{R})$ sufficiently close to $f_1, f_2$ there exists a unique non–degenerate differential Nash equilibrium $(\widetilde{p}, \widetilde{q}) \in M_1 \times M_2$ for $(\widetilde{f}_1, \widetilde{f}_2)$ near $(p, q)$.

*Theorem 2 (Structural Stability):* Non–degenerate differential Nash equilibria are structurally stable: given $f_1, f_2 \in C^\infty(M_1 \times M_2, \mathbb{R})$, $\zeta_1, \zeta_2 \in C^\infty(M_1 \times M_2, \mathbb{R})$, and a non–degenerate differential Nash equilibrium $(p, q) \in M_1 \times M_2$ for $(f_1, f_2)$, there exist neighborhoods $U \subset \mathbb{R}$ of 0 and $W \subset M_1 \times M_2$ of $(p, q)$ such that for all $s \in U$ there exists a unique non–degenerate differential Nash equilibrium $(\widetilde{p}(s), \widetilde{q}(s)) \in W$ for $(f_1 + s\zeta_1, f_2 + s\zeta_2)$.

*Proof:* Define $\widetilde{f}_j : M_1 \times M_2 \times \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}_j(x, y, s) = f_j(x, y) + s\zeta_j(x, y)$$

and $\widetilde{\omega} : M_1 \times M_2 \times \mathbb{R} \to T^*(M_1 \times M_2)$ by

$$\widetilde{\omega}(x, y, s) = (D_1 \widetilde{f}_1(x, y, s),\ D_2 \widetilde{f}_2(x, y, s))$$

for all $s \in \mathbb{R}$ and $(x, y) \in M_1 \times M_2$. Observe that $D_{(1,2)}\widetilde{\omega}(p, q, 0)$ is invertible since $(p, q)$ is a non–degenerate differential Nash equilibrium for $(f_1, f_2)$. Therefore by the Implicit Function Theorem (see Theorem 7.8 in [23]), there exist neighborhoods $V \subset \mathbb{R}$ of 0 and $W \subset M_1 \times M_2$ of $(p, q)$ and a smooth function $\sigma \in C^\infty(V, W)$ such that

$$\forall s \in V, (x, y) \in W : \widetilde{\omega}(x, y, s) = 0 \iff (x, y) = \sigma(s).$$

Furthermore, since $\widetilde{\omega}$ is continuously differentiable, there exists a neighborhood $U \subset V$ of 0 such that $D\widetilde{\omega}(\sigma(s), s)$ is invertible for all $s \in U$. We conclude for all $s \in U$ that $\sigma(s) \in M_1 \times M_2$ is the unique Nash equilibrium for $((f_1 + s\zeta_1)|_W, (f_2 + s\zeta_2)|_W)$, and furthermore that $\sigma(s)$ is a non–degenerate differential Nash equilibrium. ■

We remark that the preceding analysis extends directly to any finitely–parameterized perturbation.

*Example 1 (continued):* Again we return to Example 1 and recall that it is of a game admitting a continuum of differential Nash equilibria. We can show that an arbitrarily small perturbation will make *all* the equilibria disappear. Let $\varepsilon > 0$ be arbitrarily small and consider Urbain's perturbed cost function

$$\tilde{f}_1(x, y) = \frac{x^2}{2} - xy + \varepsilon x. \tag{27}$$

Let Victor's cost function remain the same. Then, *all* Nash equilibria disappear. Indeed, a necessary condition that a Nash equilibrium $(x, y) \in M_1 \times M_2$ must satisfy is $\omega(x, y) = 0$. Then, $D_1 \tilde{f}_1 = x - y + \varepsilon = 0$ and $D_2 f_2(x, y) = y - x = 0$ must both hold. This can only happen for $\varepsilon = 0$. Hence, *any* perturbation of the form $\varepsilon x$ with $\varepsilon > 0$ will remove all the

Nash equilibria. As we noted in the previous section, this unperturbed game is not generic since $\det(D\omega(p,q)) = 0$ for every $(p,q)$. We see here that degeneracy leads to structural instability.

*Example 2 (Convergence of Gradient Play):* We adopt a dynamical systems perspective of a two–player game over the finite–dimensional strategy space $U_1 \times U_2$ with player costs $f_1, f_2 : U_1 \times U_2 \to \mathbb{R}$. Specifically, we consider the continuous–time dynamical system generated by the negative of the player's individual gradients:

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -D_1 f_1(u_1, u_2) \\ -D_2 f_2(u_1, u_2) \end{bmatrix} = -\omega(u). \qquad (28)$$

If $(\mu_1, \mu_2) \in U_1 \times U_2$ is a differential Nash equilibrium, then $\omega(\mu_1, \mu_2) = 0$. These dynamics are *uncoupled* in the sense the dynamics for each player $\dot{u}_i$ do not depend on the cost function of the other player. It is known that such uncoupled dynamics need not converge to local Nash equilibria [29]. However, Proposition 4 in [19] shows that the subset of non–degenerate differential Nash equilibria where all eigenvalues of $D\omega$ have strictly positive real parts are exponentially stable stationary points of (28). Theorem 2 shows that convergence of uncoupled gradient play to such *stable* non–degenerate differential Nash equilibria persists under small smooth perturbations to player costs.

## VI. Conclusion

Given a second–order characterization of local Nash equilibria, namely the differential Nash equilibrium concept, we used techniques from differential topology to show that non–degenerate differential Nash equilibria are generic among local Nash equilibria. Applying structural stability analysis from dynamical systems theory, we showed that non–degenerate differential Nash equilibria persist under smooth perturbations in player costs. As a consequence of the genericity and structural stability results, small modeling errors or environmental disturbances generally do not result in games with drastically different equilibrium behavior.

We believe that this work is essential for decentralized control in engineered systems as well as on–line identification techniques for human agents. Decentralized control involves the design of control strategies for cooperative or non–cooperative agents. Genericity and structural stability of the equilibrium concept allows for modeling errors to have minimal impact on synthesis of control strategies. Both from an analysis and synthesis point of view, it is particularly useful for a designer to be able to accurately estimate agent cost functions. Again, the notions of genericity and structural stability of the equilibrium concept result in minimal effects of uncertainties, whether they are computational or measurement errors, on the structure of the game, and thereby the estimation of agents' cost functions.

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