Characterization and Computation of Local Nash Equilibria in Continuous Games

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Abstract— We present derivative–based necessary and sufficient conditions ensuring player strategies constitute local Nash equilibria in non–cooperative continuous games. Our results can be interpreted as generalizations of analogous second–order conditions for local optimality from nonlinear programming and optimal control theory. Drawing on this analogy, we propose an iterative steepest descent algorithm for numerical approximation of local Nash equilibria and provide a sufficient condition ensuring local convergence of the algorithm. We demonstrate our analytical and computational techniques by computing local Nash equilibria in games played on a finite–dimensional differentiable manifold or an infinite–dimensional Hilbert space.

I. Introduction

When resources are scarce, competition develops between self–interested agents. Game theory is an established technique for modeling this interaction, and it has emerged as an engineering tool for analysis and synthesis of systems comprised of dynamically–coupled decision–making agents possessing competing interests [1], [2]. We focus on games with a finite number of agents where the strategy space is continuous, either a finite–dimensional differentiable manifold or an infinite–dimensional Hilbert space.

Previous work on continuous games with convex strategy spaces and player costs led to global characterization and computation of Nash equilibria [3]–[7]. Imposing a differentiable structure on the strategy spaces yielded other global conditions ensuring existence and uniqueness of Nash equilibria and Pareto optima [8]–[10]. In contrast, we seek to analytically characterize and numerically compute *local* Nash equilibria in continuous games. Bounding the rationality of agents can result in *myopic* behavior [11], meaning that agents seek strategies that that are optimal locally but not necessarily globally. Further, it is common in engineering applications for strategy spaces or player costs to be non–convex, for example when an agent's configuration space is a constrained set or a differentiable manifold [12], [13]. These observations suggest that techniques for characterization and computation of local Nash equilibria may have important practical applications.

Motivated by systems with myopic agents and non–convex strategy spaces, we seek a characterization for local Nash

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equilibria that is amenable to computation. By generalizing derivative–based conditions for local optimality in nonlinear programming [14] and optimal control [15], we provide necessary first– and second–order conditions that local Nash equilibria must satisfy, and further develop a second–order sufficient condition ensuring player strategies constitute a local Nash equilibrium. We term points satisfying this sufficient condition *differential* Nash equilibria. In contrast to a pure optimization problem, this second–order condition is insufficient to guarantee a differential Nash equilibrium is isolated; in fact, we show in an example that games may possess a continuum of differential Nash equilibria. Consequently, we provide an additional second–order condition ensuring a differential Nash equilibria is isolated. Further exploiting the analogy with nonlinear programming, we propose a *steepest descent* algorithm for iterative numerical approximation of differential Nash equilibria. Adopting a dynamical systems view of the algorithm, we derive a sufficient condition ensuring local convergence of the iteration.

Our results are applicable to strategy spaces that are finite–dimensional differentiable manifolds or infinite–dimensional Hilbert spaces. Verifying that a strategy constitutes a Nash equilibrium in such spaces requires testing that a non–convex inequality is satisfied on an open set, a task we regard as generally intractable. In contrast, our sufficient conditions for local Nash equilibria require only the evaluation of player costs and their derivatives at a single point. Further, our framework allows for numerical computations to be carried out when the strategy spaces of the players, as well as the cost functions, are non–convex. Hence, we provide tractable tools for characterization and computation of differential Nash equilibria in continuous games.

The paper is organized as follows. In Section II, we discuss the necessary mathematical preliminaries. We provide the game formulation and characterization of differential Nash equilibria in Section III. Subsequently, in Section IV we propose a steepest descent algorithm for numerical computation of differential Nash equilibria and study convergence properties of the algorithm. In Section V, we numerically compute differential Nash equilibria in games with nonlinear and infinite–dimensional strategy spaces. Finally, we summarize the contributions of the paper and discuss future work in Section VI.

II. Mathematical Preliminaries

We begin by introducing the standard mathematical objects used throughout this paper (see [16] for a more detailed introduction). A *topological m–dimensional manifold M is*

a topological space which is Hausdorff, second-countable, and is locally Euclidean of dimension m, i.e. every point $p \in M$ has a neighborhood $U \subset M$ containing p that is homeomorphic to $\mathbb{R}^m$ via the map $\varphi : U \to \mathbb{R}^m$. The pair (U, φ) is called a *coordinate chart* and the map φ is called the *coordinate map*. We define the component functions $(u^1, \ldots, u^m)$ of φ by $\varphi(p) = (u^1(p), \ldots, u^m(p))$ and we call $(u^1, \ldots, u^m)$ the *local coordinates*. We say two charts (U, φ) and (V, ψ) are *smoothly compatible* if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a smooth bijective map with a smooth inverse, i.e. it is a diffeomorphism. A family of smoothly compatible charts whose domain covers M is called a *smooth atlas* for M. A *smooth m–dimensional manifold M* is a topological manifold with a smooth atlas. A smooth manifold *without boundary* is a topological manifold with empty boundary.

A function $J : U \to \mathbb{R}^n$ where $U \subset \mathbb{R}^m$ is said to be of class C^k if all the partial derivatives of J of order less than or equal to k exist and are continuous functions on U. A function that is of class C^k for all $k \geq 0$ is said to be *smooth*. A function $J : M \to \mathbb{R}$ is smooth if for every $p \in M$, there exits a smooth chart (U, φ) on M with $p \in U$ and such that $J \circ \varphi^{-1}$ is smooth on $\varphi(U)$.

A linear map $v : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ is a derivation at p if for all $J_1, J_2 \in C^\infty(M, \mathbb{R})$ it has the property that

$$v(J_1 \cdot J_2) = J_1(p) \cdot v(J_2) + J_2(p) \cdot v(J_2). \qquad (1)$$

The *tangent space to M* at $p \in M$ is the set of all derivations of $C^\infty(M, \mathbb{R}) = \{ J : M \to \mathbb{R} | \ J \text{ is } C^\infty \}$ at p and is denoted by $T_p M$. Elements of $T_p M$ are called *tangent vectors*. The disjoint union of the tangent spaces is the *tangent bundle* $TM = \coprod_{p \in M} T_p M$. The *co-tangent space to M* at $p \in M$, denoted $T_p^* M$, is the set of all real-valued linear functionals on the tangent space $T_p M$, and the disjoint union of the co–tangent spaces is the *co–tangent bundle* $T^* M = \coprod_{p \in M} T_p^* M$. Both TM and $T^* M$ are naturally smooth manifolds. There is a natural projection $\pi : T^* M \to M$ mapping elements in $T_p^* M$ to p. A *1–form* of $T^* M$ is a continuous map $\omega : M \to T^* M$ satisfying $\pi \circ \omega = \mathrm{Id}_M$. For a smooth function $J : M \to \mathbb{R}$, the exterior derivative $dJ : M \to T^* M$ is a 1–form and $dJ(p) : T_p M \to \mathbb{R}$ is a linear functional on the tangent space $T_p M$.

Consider topological manifolds M_1 and M_2 of dimension m_1 and m_2 respectively. The product space $M_1 \times M_2$ is a topological manifold of dimension $m_1 + m_2$ in the following sense. Suppose we have charts (U_1, φ_1) on M_1 and (U_2, φ_2) on M_2. Then, the product map

$$\varphi_1 \times \varphi_2 : U_1 \times U_2 \to \mathbb{R}^{m_1 + m_2} \qquad (2)$$

is a homeomorphism onto its image. The product of the coordinate domains in atlases for M_1 and M_2 cover $M_1 \times M_2$. Thus, $M_1 \times M_2$ has an atlas with charts of the form $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ and we refer to charts of this form as *product charts*. $M_1 \times M_2$ is a smooth manifold if the charts in the atlas $\{(U_1 \times U_2, \varphi_1 \times \varphi_2)_i\}$ are smoothly compatible. We will exploit the product structure of this atlas on $M_1 \times M_2$ in the sections that follow.

There is a canonical isomorphism at each point such that the cotangent bundle of the product manifold splits

$$T_{(p,q)}^*(M_1 \times M_2) \cong T_p^* M_1 \oplus T_q^* M_2 \qquad (3)$$

(see [16] Problem 3-3 for a similar result). Equation (3) says that elements in the co-tangent space at (p, q) on the product manifold $M_1 \times M_2$ can be identified with the sum of an element in the co-tangent space of M_1 at p and an element in the co-tangent space of M_2 at q. For example, this means that $(du_1^1, \ldots, du_1^{m_1}, du_2^1, \ldots, du_2^{m_2})$ can be identified with $\sum_{i=1}^{m_1} du_1^i + \sum_{i=1}^{m_2} du_2^i$. There are natural bundle maps $\pi_{M_1}, \pi_{M_2} : T^*(M_1 \times M_2) \to T^*(M_1 \times M_2)$ annihilating the last m_2 components and the first m_1 components respectively.

Consider a function $J \in C^\infty(M_1 \times M_2, \mathbb{R})$ and a product chart $(U_1 \times U_2, \varphi)$ on $M_1 \times M_2$ where $p \in U_1$, $q \in U_2$ and $\varphi = \varphi_1 \times \varphi_2$. Let the local coordinates be denoted by $(u_1^1, \ldots, u_1^{m_1}, u_2^1, \ldots, u_2^{m_2})$ where $\varphi_1(p) = (u_1^1(p), \ldots, u_1^{m_1}(p))$ and $\varphi_2(q) = (u_2^1(q), \ldots, u_2^{m_2}(q))$. We refer to the first m_1 coordinates by $u_1 = (u_1^1, \ldots, u_1^{m_1})$ and the last m_2 coordinates by $u_2 = (u_2^1, \ldots, u_2^{m_2})$. Then, we define

$$D_{u_1}^\varphi J(p, q) = \left[\left. \frac{\partial(J \circ \varphi^{-1})}{\partial u_1^1} \right|_{\varphi(p,q)} \cdots \left. \frac{\partial(J \circ \varphi^{-1})}{\partial u_1^{m_1}} \right|_{\varphi(p,q)} \right] \qquad (4)$$

and we define $D_{u_2}^\varphi J(p, q)$ similarly. The superscript notation indicates that the derivatives are taken with respect to the chart φ. A *critical point* (p, q) of J is such that $D_{u_1}^\varphi J(p, q)$ and $D_{u_2}^\varphi J(p, q)$ are zero covectors of the appropriate dimension. Since the transition map between two sets of coordinates is a diffeomorphism, stationary of critical points is coordinate invariant.

The Hessian of a real-valued function $J : \mathbb{R}^m \to \mathbb{R}$ is a bilinear form defined by

$$D_z^2 J(y) = D^2 J(z)(y, y) = \sum_{i, j} \frac{\partial^2 J}{\partial x^i x^j}(p) y^i y^j \qquad (5)$$

where $y, z \in \mathbb{R}^m$. The following proposition is a result from Morse theory [17]. We will use it to prove coordinate invariance of the results in Section III.

Proposition 1: Let $U \subset \mathbb{R}^m$ be open and let $J : U \to \mathbb{R}$ be a smooth map. Consider a critical point $z \in U$. The bilinear form $D_z^2 J(y)$ is invariant under diffeomorphism.

The proof is by direct calculation, and hence, we exclude it. Proposition 1 implies that the following diagram commutes:

We introduce the following notation for defining the Hessian of a real-valued function on the manifold $M_1 \times M_2$.

The Hessian of $J \in C^\infty(M_1 \times M_2, \mathbb{R})$ is a quadratic form $D^2_{(p,q)}J : T_{(p,q)}(M_1 \times M_2) \to \mathbb{R}$ defined by the composition

$$D^2_{\varphi(p,q)}(J \circ \varphi^{-1}) \circ D\varphi(p,q) \tag{6}$$

where $D^2_{\varphi(p,q)}(J \circ \varphi^{-1}) : \mathbb{R}^m \to \mathbb{R}$ is the usual Hessian of a real-valued function on $\mathbb{R}^m$ and $D\varphi : T_{(p,q)}(M_1 \times M_2) \to T\mathbb{R}^m$. We partition the Hessian of J as follows

$$D^2_{(p,q)}J = \begin{bmatrix} H_{11}J(p,q) & H_{12}J(p,q) \\ H_{21}J(p,q) & H_{22}J(p,q) \end{bmatrix} \tag{7}$$

where we define in local coordinates

$$D^2_{12}J(p,q) = \begin{bmatrix} \frac{\partial^2 J}{\partial u_2^1 \partial u_1^1}(p,q) & \cdots & \frac{\partial^2 J}{\partial u_2^{m_2} \partial u_1^1}(p,q) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 J}{\partial u_2^1 \partial u_1^{m_1}}(p,q) & \cdots & \frac{\partial^2 J}{\partial u_2^{m_2} \partial u_1^{m_1}}(p,q) \end{bmatrix}. \tag{8}$$

$D^2_{11}J(p,q)$, $D^2_{21}J(p,q)$, and $D^2_{22}J(p,q)$ are defined similarly. Note that $D^2_{(p,q)}J$ is symmetric since J is C^∞. We use the same notation for a partition of $D^2_z J$ where $J : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ is a real-valued function on $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$. Note that the Hessian of J is well defined.

Consider a critical point (p,q). We use the notation $D^2_{11}J(p,q) > 0$ to indicate that in local coordinates this matrix is positive definite. The definiteness of $D^2_{11}J(p,q)$ is invariant with respect to choice of coordinates in the following sense. Suppose that $D^2_{11}J(p,q) > 0$ with respect to the product chart $(U_1 \times U_2, \varphi)$ whose local coordinates are (u_1, u_2). Consider another product chart $(V_1 \times V_2, \psi)$ with $\psi = \psi_1 \times \psi_2$ whose local coordinates are (v_1, v_2) and that is smoothly compatible with $(U_1 \times U_2, \varphi)$. By the commutative diagram following Proposition 1 and direct calculation using the fact that (p,q) is a critical point, we have that for fixed q with $x_2 = \varphi_2(q)$ and $y_2 = \psi_2(q)$

$$D^2_{11}(J \circ \varphi^{-1})(x_1, x_2) = D^2_{11}(J \circ \psi^{-1})(D(\psi_1^{-1} \circ \varphi_1)(y_1),$$
$$D(\psi_1^{-1} \circ \varphi_1)(y_1)) \tag{9}$$

Hence, the definiteness of $D^2_{11}J(p,q)$ is invariant with respect to choice of chart. The definiteness of the Hessian will be used in Section III to show that the second–order results we derive do not depend on the choice of coordinate chart.

III. CHARACTERIZATION OF DIFFERENTIAL NASH EQUILIBRIA

The theory of games we consider concerns situations in which several rational agents, generally having different interests and objectives, interact within their environment. We refer to the rational agents as *players*. Competition arises due to the fact that the players have opposing interests. We note that the game formulation as presented in this section and the results that follow easily extend to games with any finite number of players. We choose to present the results for two player games in an effort to be clear and concise.

Let us begin by considering a game in which we have two selfish players with competing interests. In homage to von Neumann, Morgenstern, and Ekeland, we call the first player Urbain and the second player Victor [9], [18]. The strategy spaces of Urbain and Victor are topological spaces M_1 and M_2 respectively. Urbain and Victor are interested in minimizing a cost function representing their interests by choosing an element from their strategy space. We define Urbain's cost function to be $J_1 : M_1 \times M_2 \to \mathbb{R}$ and Victor's cost function to be $J_2 : M_1 \times M_2 \to \mathbb{R}$.

Definition 1: A strategy $(p,q) \in M_1 \times M_2$ is a **local Nash equilibrium** if there exist open sets $W_1 \subset M_1$, $W_2 \subset M_2$ such that $p \in W_1$, $q \in W_2$,

$$J_1(p,q) \le J_1(p',q), \quad \forall \, p' \in W_1 \backslash \{p\} \tag{10}$$

and

$$J_2(p,q) \le J_2(p,q') \quad \forall \, q' \in W_2 \backslash \{q\}. \tag{11}$$

If $W_1 = M_1$ and $W_2 = M_2$, then (p,q) is a **global Nash equilibrium**. Further, if the above inequalities are strict, then we say (p,q) is a **strict local Nash equilibrium**.

We consider continuous games with finite–dimensional strategy spaces as well as a class of continuous games with infinite–dimensional strategy spaces.

A. Finite–Dimensional Strategy Spaces

In this subsection, the strategy spaces of Urbain and Victor are smooth manifolds (without boundary) M_1 and M_2 respectively. Let the dimensions of M_1 and M_2 be m_1 and m_2 respectively. Further, we define $m = m_1 + m_2$. Urbain and Victor's cost functions, J_1 and J_2 respectively, are assumed to be a real-valued, smooth functions on the product manifold $M_1 \times M_2$, i.e. $J_1, J_2 \in C^\infty(M_1 \times M_2, \mathbb{R})$.

We now define the coordinate representation of J_i for each $i \in \{1, 2\}$. Let $\varphi_1 \times \varphi_2 : U_1 \times U_2 \to \mathbb{R}^m$ be a coordinate map on $M_1 \times M_2$. Then, the coordinate representation of J_i with respect to the coordinate map $\varphi_1 \times \varphi_2$ is the map $\tilde{J}_i : \mathbb{R}^m \to \mathbb{R}$ defined by

$$\tilde{J}_i = J_i \circ (\varphi_1 \times \varphi_2)^{-1}. \tag{12}$$

The following definition of a differential game form is due to N. Stein [19].

Definition 2: A **differential game form** is a differential 1–form $\omega : M_1 \times M_2 \to T^*(M_1 \times M_2)$ defined by

$$\omega = \pi_{M_1}(dJ_1) + \pi_{M_2}(dJ_2) \tag{13}$$

and, in coordinates, is defined by

$$\omega = \sum_{i=1}^{m_1} \frac{\partial \tilde{J}_1}{\partial u_1^i} du_1^i + \sum_{j=1}^{m_2} \frac{\partial \tilde{J}_2}{\partial u_2^j} du_2^j \tag{14}$$

where $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ is a coordinate chart on $M_1 \times M_2$ with local coordinates $(u_1^1, \ldots, u_1^{m_1}, u_2^1, \ldots, u_2^{m_2})$.
The above definition of a differential game form captures a differential view of the strategic interaction between the players. Indeed, ω indicates the direction in which Urbain and Victor can change their strategies to decrease their individual cost functions most rapidly. Note that Urbain's cost function is dependent on Urbain's strategy choice as well as Victor's, but Urbain can only affect his payoff by adjusting his strategy (and similarly for Victor).
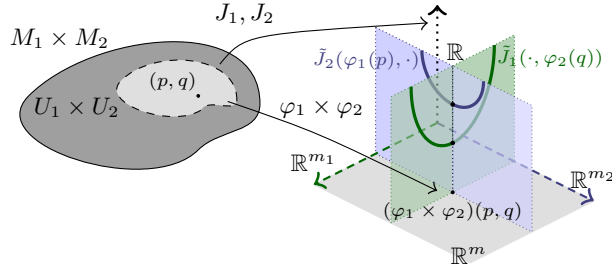
Fig. 1. The map $\tilde{J}_i : \mathbb{R}^m \to \mathbb{R}$ is the coordinate representation of J_i and it is defined by $\tilde{J}_i = J_i \circ (\varphi_1 \times \varphi_2)^{-1}$ where $\varphi_1 \times \varphi_2 : U_1 \times U_2 \to \mathbb{R}^m$ is the coordinate map. Player i, whose cost function is J_i, can only adjust his payoff by changing is strategy in directions corresponding to $\mathbb{R}^{m_i}$. $\tilde{J}_1(\cdot, \varphi_2(q))$ and $\tilde{J}_2(\varphi_1(p), \cdot)$ are slices of the coordinate representation of J_1 and J_2, respectively.

Definition 3: In the case of finite–dimensional strategy spaces, a strategy $(p, q) \in M_1 \times M_2$ is a **differential Nash equilibrium** if $\omega(p, q) = 0$, $D^2_{11} J_1(p, q) > 0$ and $D^2_{22} J_2(p, q) > 0$.

The second–order conditions used to define differential Nash equilibria are motivated by results in nonlinear programming that use first– and second–order conditions to define when a critical point is a local optima [15], [14]. Further parallels to nonlinear programming in the development of an algorithm for computation of differential Nash equilibria are made in Section IV.

The following proposition provides first– and second–order necessary conditions for local Nash equilibria. We remark that these conditions are reminiscent of those seen in nonlinear programming for optimality of critical points.

Proposition 2: A local Nash equilibrium (p, q) satisfies $\omega(p, q) = 0$ and $D^2_{ii} J_i(p, q) \geq 0$ for each $i \in \{1, 2\}$.

Proof: Suppose that $(p, q) \in M_1 \times M_2$ is a local Nash equilibrium. Then,

$$J_1(p, q) \leq J_1(p', q), \quad \forall\, p' \in W_1 \backslash \{p\} \tag{15}$$

and

$$J_2(p, q) \leq J_2(p, q') \quad \forall\, q' \in W_2 \backslash \{q\}. \tag{16}$$

for open $W_i \subset M_i$, $i = 1, 2$. Suppose that we have a chart $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ such that $(p, q) \in U_1 \times U_2$. Let $\varphi_1(p) = x_1$ and $\varphi_2(q) = x_2$. Then, since $\varphi_1 \times \varphi_2$ is a homeomorphism, we have that for all $x'_1 \in \varphi_1(W_1 \cap U_1) \backslash \{x_1\}$,

$$\tilde{J}_1(x_1, x_2) \leq \tilde{J}_1(x'_1, x_2), \tag{17}$$

and for all $x'_2 \in \varphi_2(W_2 \cap U_2) \backslash \{x_2\}$

$$\tilde{J}_2(x_1, x_2) \leq \tilde{J}_2(x_1, x'_2). \tag{18}$$

Now, we can apply Proposition 1.1.1 from [14] to $\tilde{J}_i$ for each $i \in \{1, 2\}$ to get that $D^{\varphi_1 \times \varphi_2}_{u_i} J_i(x_1, x_2) = 0$ and $D^2_{ii} \tilde{J}_i(x_1, x_2) \geq 0$ for each $i \in \{1, 2\}$. Invariance of the stationarity of critical points and sign of the Hessian with respect to coordinate change gives us $\omega(p, q) = 0$ and $D^2_{ii} J_1(p, q) \geq 0$ for each $i \in \{1, 2\}$. ∎

We now show that the conditions defining a differential Nash equilibrium are sufficient to guarantee a strict local Nash equilibrium.

Theorem 1: A differential Nash equilibrium is a strict local Nash equilibrium.

Proof: Suppose that $(p, q) \in M_1 \times M_2$ is a differential Nash equilibrium. Then, by the definition of differential Nash equilibrium, $\omega(p, q) = 0$, $D^2_{11} J_1(p, q) > 0$ and $D^2_{22} J_2(p, q) > 0$. The latter two inequalities imply that for a coordinate chart $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ with $p \in U_1$, $q \in U_2$ and local coordinates $(u^1_1, \ldots, u^{m_1}_1, u^1_2, \ldots, u^{m_2}_2)$, $D^2_{11} \tilde{J}_1(x_1, x_2) > 0$ and $D^2_{22} \tilde{J}_2(x_1, x_2) > 0$ where we have defined $\varphi_1(p) = x_1$ and $\varphi_2(q) = x_2$. Note that we have argued that positive definiteness of the Hessian at a critical point is invariant with respect to the choice of coordinate chart.

Using the isomorphism introduced in Equation (3), we have that $\omega(p, q) = 0$ implies that for each $i \in \{1, 2\}$

$$D^{\varphi_1 \times \varphi_2}_{u_i} J_i(x_1, x_2) = 0 \tag{19}$$

where the right-hand side of the above equation is a zero co-vector of the appropriate dimension.

Now, we can apply Proposition 1.1.3 from [14] to $\tilde{J}_1$ with x_2 fixed. The result of which gives us the existence of a neighborhood $\tilde{W}_1 \in \mathbb{R}^{m_1}$ such that for all $x' \in \tilde{W}_1$

$$\tilde{J}_1(x', x_2) > \tilde{J}_1(x_1, x_2). \tag{20}$$

Since $\tilde{J}_1 = J_1 \circ (\varphi_1 \times \varphi_2)^{-1}$ and $\varphi_1 \times \varphi_2$ is a homeomorphism, there exists a neighborhood $W_1 \subset M_1$ of p such that for all $p' \in W_1 \backslash \{p\}$

$$J_1(p, q) < J_1(p', q) \tag{21}$$

where $W_1 = \varphi^{-1}_1(\tilde{W}_1)$.

A symmetric argument applied to $\tilde{J}_2$ with x_1 fixed shows that there exists a neighborhood $W_2 \subset M_2$ of q such that for all $q' \in W_2 \backslash \{q\}$

$$J_2(p, q) < J_2(p, q'). \tag{22}$$

Therefore, differential Nash equilibria are strict local Nash equilibria independent of the choice of coordinates where coordinate invariance is due to the fact that stationarity of critical points is coordinate invariant and definiteness of the Hessian is coordinate invariant. ∎

We remark that the conditions for differential Nash equilibria are not sufficient to guarantee that the equilibria is isolated. The following example shows games may possess a continuum of differential Nash equilibria.

Example 1: Let Urbain's strategy space be $M_1 = \mathbb{R}$ and his cost function $J_1(x, y) = \frac{x^2}{2} - xy$. Similarly, let Victor's strategy space be $M_2 = \mathbb{R}$ and his cost function $J_2(x, y) = \frac{y^2}{2} - xy$. Fix $y = q$, and calculate

$$\frac{\partial J_1}{\partial x} = x - q \tag{23}$$

Then, Urbain's optimal response to Victor playing $y = q$ is $x = q$. Similarly, if we fix $x = p$, then Victor's optimal

response to Urbain playing $x = p$ is $y = p$. For all $x \in \mathbb{R}\backslash\{q\}$

$$-\frac{q^2}{2} < \frac{x^2}{2} - xq \qquad (24)$$

so that $J_1(q, q) < J_1(x, q)$ for all $x \in \mathbb{R}\backslash\{q\}$. Again, similarly, for all $y \in \mathbb{R}\backslash\{p\}$

$$-\frac{p^2}{2} < \frac{y^2}{2} - yp \qquad (25)$$

so that $J_2(p, p) < J_2(p, y)$ for all $y \in \mathbb{R}\backslash\{p\}$. Hence, all the points on the line $x = y$ in $M_1 \times M_2 = \mathbb{R}^2$ are strict local Nash equilibria (in fact, they are strict global Nash equilibria).

As the above example shows, it can be the case that equilibria in continuous games are not isolated even locally. In the context of convex games in finite dimensions where the strategy spaces are convex subsets of $\mathbb{R}^n$ and the players' costs are convex, these ideas were studied at the global level by Rosen [3]. In particular, Rosen defined the notion of diagonal strict convexity of a weighted sum of the players' cost functions for the purpose of constructing a sufficient condition for global uniqueness of Nash equilibria in these type of convex games.

We propose a sufficient condition to guarantee that differential Nash equilibria are isolated and hence, strict local Nash equilibria are isolated. We do so by combining ideas introduced by Rosen for convex games with concepts from Morse theory, in particular second–order conditions on non-degenerate critical points of real-valued functions on manifolds. By an abuse of notation, we define the following:

$$D\omega(p, q) = \begin{bmatrix} D^2_{11} J_1(p, q) & D^2_{21} J_1(p, q) \\ D^2_{12} J_2(p, q) & D^2_{22} J_2(p, q) \end{bmatrix}. \qquad (26)$$

Theorem 2: If (p, q) is a differential Nash equilibrium and $D\omega(p, q)$ is invertible, then (p, q) is an isolated strict local Nash equilibrium.

Proof: Since (p, q) is a differential Nash equilibrium, Theorem 1 gives us that it is a strict local Nash equilibrium. The following argument shows that it is isolated.

Let us define a function $g : \mathbb{R}^m \to \mathbb{R}^m$ as follows

$$g(x_1, x_2) = \begin{bmatrix} D^{\varphi_1 \times \varphi_2}_{u_1} J_1(x_1, x_2) \\ D^{\varphi_1 \times \varphi_2}_{u_2} J_2(x_1, x_2) \end{bmatrix}. \qquad (27)$$

Note that g is just the coordinate representation of the game form ω. Zeros of the function g define critical Nash points of the game. The derivative of g is $D\omega$. Since $D\omega(p, q)$ is invertible, the inverse function theorem (Theorem 4.5 [16]) implies that g is a local diffeomorphism. Thus, only $(\varphi_1(p), \varphi_2(q))$ could be mapped to zero near $(\varphi_1(p), \varphi_2(q))$. Invertibility of $D\omega(p, q)$ is invariant under coordinate change. Therefore, (p, q) is a isolated independent of choice of coordinates. ∎

In an effort to parallel Morse theory, we term differential Nash equilibria such that $D\omega(p, q)$ is invertible *non–degenerate*.

B. Infinite–Dimensional Strategy Spaces

We now consider the class of continuous games with infinite–dimensional strategy spaces regarded as open–loop differential games.

We will use the notation and optimal control framework developed by Polak [15]. Let $L_2[0, T]$ denote the space of square integrable functions from $[0, T]$ into $\mathbb{R}^m$. Let $L_{\infty, 2}[0, T]$ denote the space of bounded functions from $[0, T]$ into $\mathbb{R}^m$ endowed with the $L_2[0, T]$ inner product and norm. Consider a two player continuous game in which Urbain and Victor's strategy spaces are $M_1 = M_2 = L_2[0, T] \cap L_{\infty, 2}[0, T]$. For each $t \in [0, T]$, let $x(t) \in \mathbb{R}^n$ denote the state of the game. The state evolves according to the dynamics

$$\dot{x}(t) = f(x(t), \mu_1(t), \mu_2(t)) \quad \forall\, t \in [0, T] \qquad (28)$$

where $\mu_1 \in M_1$ is Urbain's strategy choice and $\mu_2 \in M_2$ is Victor's strategy choice. We assume that $f(x, \mu_1, \mu_2)$ is continuously differentiable, Lipschitz continuous and all the derivatives in all its arguments are Lipschitz continuous. We denote by $J_1(x^{(x(0), \mu_1, \mu_2)}(T))$ Urbain's cost function and by $J_2(x^{(x(0), \mu_1, \mu_2)}(T))$ Victor's. Suppose that J_1 and J_2 are C^2–Fréchet–differentiable. The superscript notation on the state x indicates the dependence of the state on the initial state and the strategies of the players. We pose each player's optimization problem as

$$\min_{\mu_i} J_i(x^{(x(0), \mu_i, \mu_{-i})}(T)) \qquad (29)$$

where we use the notation $-i$ to denote the set of players excluding the i–th player. Then, the costate for player i evolves according to

$$\dot{p}_i(t) = -p_i(t) \frac{\partial f}{\partial x}(x(t), \mu_i(t), \mu_{-i}(t)) \qquad (30)$$

with final time condition $p_i(T) = D_x J_i(x^{(x(0), \mu_i, \mu_{-i})}(T))$. Then, the derivative of the i–th player's cost function is given by

$$(D_i J_i)(t) = p_i(t) \frac{\partial f}{\partial \mu_i}(x(t), \mu_i(t), \mu_{-i}(t)). \qquad (31)$$

Analogously to the previous subsection, the directions in which player i can adjust his payoff at time t are captured in Equation (31).

Definition 4: We say in the infinite dimensional case, that a point (μ_1^*, μ_2^*) is a **differential Nash equilibrium** if for each $i \in \{1, 2\}$ we have that $D_i J_i(\mu_1^*, \mu_2^*) \equiv 0$ and $D_{ii}^2 J_i(\mu_1^*, \mu_2^*)$ is a positive definite bilinear form, i.e. for each $i \in \{1, 2\}$

$$(D_{ii}^2 J_i)(\mu_1^*, \mu_2^*)(\nu, \nu) \geq \alpha \|\nu\|^2, \ \forall \nu \in M_i. \qquad (32)$$

Similar to the previous subsection, we state two propositions. The first of which provides necessary conditions for local Nash equilibria of open–loop differential games. The second shows that the conditions for differential Nash equilibria in games with infinite–dimensional strategy spaces are sufficient for strict local Nash equilibria. They are reminiscent of the necessary and sufficient conditions for optimality in nonlinear programming.

Proposition 3: A local Nash equilibrium (μ_1^*, μ_2^*) satisfies $D_i J_i(\mu_1^*, \mu_2^*) \equiv 0$ and $D_{ii}^2 J_i(\mu_1^*, \mu_2^*)$ is a positive semi–definite bilinear form for each $i \in \{1, 2\}$.

Proof: Suppose that (μ_1^*, μ_2^*) is a local Nash equilibrium. Then, by definition, for each $i \in \{1, 2\}$ we have that for all $\nu \in U_i \backslash \{\mu_i\}$

$$J_i(\mu_i^*, \mu_{-i}^*) \leq J_i(\nu, \mu_{-i}^*) \tag{33}$$

where U_i is an open subset of M_i. Then, with μ_{-i}^* fixed, we can apply Theorem 4.2.3 (a) and Theorem 4.2.4 (a) from [15] to get that $D_i J_i(\mu_i^*, \mu_{-i}^*) \equiv 0$ and $D_{ii}^2 J_i(\mu_i^*, \mu_{-i}^*) \geq 0$ respectively. ∎

Theorem 3: In an open-loop differential game, a differential Nash equilibrium is a strict local Nash equilibrium.

Proof: Suppose that (μ_1^*, μ_2^*) is a differential Nash equilibrium of an open–loop differential game with two players. Thus, for each $i \in \{1, 2\}$, $D_i J_i(\mu_1^*, \mu_2^*) \equiv 0$ and $D_{ii}^2 J_i(\mu_1^*, \mu_2^*)$ is a positive definite bilinear form obeying Equation (32). We may apply Theorem 4.2.6 (a) from [15] to each J_i with μ_{-i}^* fixed for each $i \in \{1, 2\}$. This gives us that μ_i^* is a local minimizer of $J_i(\cdot, \mu_{-i}^*)$ for each $i \in \{1, 2\}$. Thus, (μ_1^*, μ_2^*) is a strict local Nash equilibrium. ∎

Remark 1: If the state of the game decomposes into pieces that are associated to each player, then it is possible to extend the above results to the case in which each player chooses their initial state in addition to their strategy. For instance, suppose $x_i \in \mathbb{R}^{n_i}$ is player i's state and that the full state of the game $x(t) = [x_i^T(t) \ x_{-i}^T(t)]^T \in \mathbb{R}^n$ where $n = \sum_i n_i$. Then, player i's optimization problem maybe reformulated as

$$\min_{\xi_i} J_i(x^{(\xi_i, \xi_{-i})}(T)) \tag{34}$$

where $\xi_i = (x_i(0), \mu_i) \in \mathbb{R}^{n_i} \times M_i$. The above results may be easily modified to accommodate this scenario.

IV. COMPUTATION OF DIFFERENTIAL NASH EQUILIBRIA

Our sufficient conditions for local Nash equilibria based on first– and second–order properties of player costs closely parallel theoretical developments in nonlinear programming [14] and optimal control [15]. In this section we further exploit this analogy by proposing an iterative *steepest descent* algorithm for computation of differential Nash equilibria.

Consider a two–player game over the finite–dimensional strategy space $U_1 \times U_2$ with player costs $J_1, J_2 : U_1 \times U_2 \to \mathbb{R}$. For each $i \in \{1, 2\}$ let $D_i J_i$ denote the derivative of player i's cost with respect to u_i. We study the continuous–time dynamical system generated by the negative of these gradients; with $u = (u_1, u_2) \in U_1 \times U_2$, we let

$$\dot{u} = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -D_1 J_1(u_1, u_2) \\ -D_2 J_2(u_1, u_2) \end{bmatrix} = -\omega(u). \tag{35}$$

If $\mu \in U_1 \times U_2$ is a differential Nash equilibrium, then $\omega(\mu) = 0$. Linearizing ω around μ, we obtain the following sufficient condition ensuring μ attracts nearby strategies under the gradient flow (35).

Proposition 4: If μ is a differential Nash equilibrium and all eigenvalues of $-D\omega(\mu)$ are in the open left–half plane, then μ is an exponentially stable fixed point of the continuous–time dynamical system (35).

Toward developing a numerical algorithm that approximates Nash equilibria, we study the forward–Euler approximation to (35). Fixing a step size $h > 0$, we obtain the discrete–time dynamical system

$$u^{k+1} = u^k - h\omega(u^k). \tag{36}$$

Note that a differential Nash equilibrium is a fixed point of (36). Linearizing around such an equilibrium, we obtain the following sufficient condition ensuring nearby strategies converge to the Nash equilibrium under iteration of (36).

Proposition 5: If μ is a differential Nash equilibrium and all eigenvalues of $-D\omega(\mu)$ are in the open left–half plane, then there exists $\eta > 0$ such that for all $h \in (0, \eta)$, μ is an exponentially stable fixed point of the discrete–time dynamical system (36).

We interpret iteration of (36) as a *steepest–descent* algorithm analogous to techniques in nonlinear programming [14], and terminate the iteration when $\|\omega(u^k)\|$ becomes sufficiently small. In fact, if the players are identical so that $J_1 = J_2 = J$, the algorithm exactly reduces to gradient descent on J with constant stepsize. A less trivial case where (36) reduces to gradient descent arises when $J_1 \neq J_2$ yet ω is an *exact* 1–form [16]. In this case, there exists a smooth function J such that $\omega = dJ$, and hence (36) is again equivalent to gradient descent on J. The case when ω is exact is referred to as a *potential game* [20].

The analogy between gradient descent algorithms for nonlinear programming and the formula in (36) suggests a technique to numerically approximate differential Nash equilibria in the class of open–loop differential games described in III-B. In particular, the derivative (31) can be approximated using techniques from numerical optimal control [15], and hence the formula in (36) may be iterated to approximate differential Nash equilibria in the game.

Note that Proposition 5 only ensures *local* convergence of iterates of (36) to differential Nash equilibria. However, we have observed empirically in the examples described in the next section that our proposed algorithm converges to a stationary point of (36) when initialized from almost every randomly–sampled initial condition. We are actively pursuing generalizations of *sufficient descent* techniques from nonlinear programming [14], [15] to develop algorithms which provably converge to differential Nash equilibria over larger regions of attraction.

Existing methods for iterative approximation of Nash equilibria generally employ the *relaxation technique*, where players alternately update their strategies by averaging their current strategy with the best response to the other player's current strategy,

$$u_i^{k+1} = \alpha u_i^k + (1 - \alpha) \arg\min_{\mu_i \in U_i} J_i(\mu_i, u_{-i}^k), \tag{37}$$

where $\alpha \in (0, 1)$ is a parameter and we again use the notation u_{-i} to denote the strategies of all players other than i. Assuming convexity in the strategy space and cost

functions to ensure there exists a unique Nash equilibrium, it is known that iterating (37) converges to the Nash equilibrium [5]–[7]. Each iteration of (37) requires the solution of a (generally non–convex) optimization problem at every iteration; on contrast, our scheme requires only the evaluation of derivatives of the player costs at a single point. In future work, we intend to compare the performance of our steepest descent algorithm with algorithms based on the relaxation technique.

V. EXAMPLES

In this section we demonstrate the preceding theoretical and algorithmic developments in examples with (i) nonlinear and (ii) infinite–dimensional strategy spaces. Our proposed method applies broadly, but we present examples where Nash equilibria are known explicitly so that we may evaluate the scalability and accuracy of our algorithm. The sourcecode for these examples will be made available at `http://purl.org/sburden/allerton2013`.

A. Location Game

Here we consider two–player game on the unit circle, S^1. The player costs $J_i : S^1 \times S^1 \to \mathbb{R}$ are given by

$$J_1(\theta_1, \theta_2) = -\cos\theta_1 + \alpha_1 \cos(\theta_1 - \theta_2)$$
$$J_2(\theta_1, \theta_2) = -\cos\theta_2 + \alpha_2 \cos(\theta_2 - \theta_1)$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ are parameters. An interpretation of these costs is that both players wish to be near zero but far from each other. This game is a location game that is an abstraction of a game that has many applications. In coordinates, the game form $\omega(\theta_1, \theta_2)$ is

$$\left[\ \sin\theta_1 - \alpha_1 \cos(\theta_1 - \theta_2)\quad \sin\theta_2 - \alpha_2 \cos(\theta_2 - \theta_1)\ \right]$$

and the Hessian $D\omega(\theta_1, \theta_2)$ is

$$\left[\begin{array}{cc} \cos\theta_1 - \alpha_1 \cos(\theta_1 - \theta_2) & \alpha_1 \cos(\theta_1 - \theta_2) \\ \alpha_2 \cos(\theta_2 - \theta_1) & \cos\theta_2 - \alpha_2 \cos(\theta_2 - \theta_1) \end{array}\right].$$

Theorem 2 implies that any point (θ_1, θ_2) for which $\omega(\theta_1, \theta_2) = 0$ and $D\omega(\theta_1, \theta_2) > 0$ is an isolated local Nash equilibrium. Numerically, with $\alpha_1 = 1, \alpha_2 = 1.05$ we find two such equilibria situated symmetrically around the zero angle: one near $(1, -1.1)$ and the other near $(-1, 1.1)$. Points where $\omega = 0$ but $D\omega$ is not positive–definite are located at $(0, \pi)$ and (π, π). Applying the steepest descent algorithm of Section IV with constant step–size 0.1 and termination tolerance 1×10^{-3}, we find empirically that most initial conditions converge to one of the two stable Nash equilibria within a few hundred iterations. See Figure 2 for a visualization of this example.

B. Open–loop Linear Quadratic Differential Game

As an illustration of the generality and scalability of our proposed algorithm, we numerically determine open–loop Nash equilibrium inputs in a linear–quadratic (LQ) game played between m players. We consider the linear time–invariant differential equation
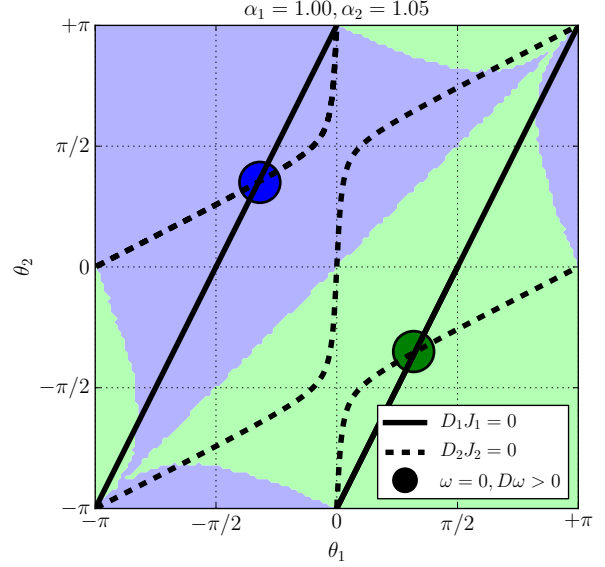
$$\dot{x} = Ax + \sum_{j=1}^{m} B_j u_j, \ x(0) = x_0$$

Fig. 2. Visualization of a two–player game played on a nonlinear strategy space (the torus, $S^1 \times S^1$) as described in Section V-A. Curves indicate the set of points where $D_i J_i = 0$ for each $i \in \{1, 2\}$; we have $\omega = 0$ wherever the curves intersect. There are two differential Nash equilibria, indicated by a dark circle; most initial conditions converge to one of these via the steepest descent algorithm of Section IV. The empirical basin of attraction for each Nash equilibria is illustrated by the filled region containing the point.

with player costs

$$J_i = \int_0^T x^T(t) Q_i x(t) + \sum_{j=1}^{m} u_j^T(t) R_{ij} u_j(t) dt$$

for each $i \in \{1, \dots, m\}$. It is known that, under non–degeneracy conditions on the game parameters [21], the unique open–loop Nash equilibria strategy is given by the linear state feedback $u_i(t) = -R_{ii}^{-1} B_i^T P_i(t) x(t)$ for each $i \in \{1, \dots, m\}$ where $P_i \in \mathbb{R}^{n \times n}$ satisfies $P_i(T) = 0$ and

$$-\dot{P}_i = P_i A + A^T P_i + Q_i - P_i \sum_{j=1}^{m} B_j R_{jj}^{-1} B_j^T P_j.$$

Using the discretization scheme for optimal control problems described in Chapter 4 of [15], we numerically approximate differential Nash equilibria for this game using the steepest descent algorithm of Section IV and compare the result with the corresponding time–discretized approximation of this closed–form expression. By considering randomly–generated examples where the entries of A, B_i, Q_i, and R_{ij} are chosen from a standard normal distribution (and subsequently positive–semidefiniteness is enforced for Q_i and R_{ij}) for each $i, j \in \{1, \dots, m\}$, we find empirically that the limit point of our algorithm is insensitive to initialization and yields strategies that are quantitatively similar to the time–discretized analytical formula. Figure 3 shows how the relative error between the two solutions decreases as the number N of time samples increases in a typical randomly–generated example; specifically,

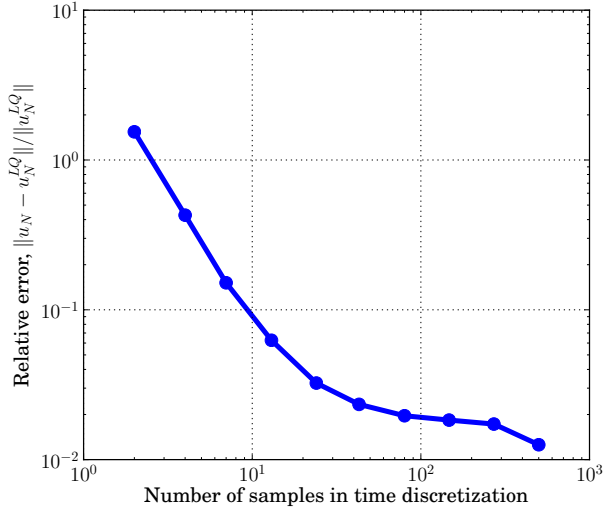$$A = \left[\begin{array}{cc} -2.28 & 0.96 \\ 0.69 & 0.23 \end{array}\right],$$

Fig. 3. Relative error between open–loop Nash equilibria obtained from discretized analytical formula, u_N^{LQ}, and from steepest descent algorithm of Section IV, u_N, in the infinite–dimensional (linear–quadratic) game described in Section V-B, with N samples in the time discretization. The relative error between the two solutions decreases as N increases.

$$B_1 = \begin{bmatrix} -0.53 \\ 0.39 \end{bmatrix}, B_2 = \begin{bmatrix} -0.04 \\ -0.49 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1.69 & -0.64 \\ -0.64 & 3.02 \end{bmatrix}, Q_2 = \begin{bmatrix} 1.36 & -0.08 \\ -0.08 & 4.39 \end{bmatrix},$$

$$R_{11} = 1.85, \ R_{12} = 0.06, \ R_{21} = 1.1, \ R_{22} = 1.38.$$

We use a stepsize of 1 and a termination tolerance of 10^{-4}.

VI. CONCLUSION

Motivated by engineering applications comprised of my-opic agents operating in non–convex strategy spaces, we studied local Nash equilibria in continuous games. General-izing derivative–based conditions for local optimality from nonlinear programming and optimal control, we derived necessary first– and second–order conditions that local Nash equilibria must satisfy, and further developed a second–order sufficient condition ensuring player strategies constitute a local Nash equilibrium. Further exploiting the analogy with nonlinear programming, we proposed a *steepest descent* algorithm for numerical computation of local Nash equilibria and provided a condition ensuring local convergence.

We believe this work provides a foundation enabling fur-ther development of analysis and synthesis tools for multi–agent engineered systems, and are actively pursuing exten-sions in several directions. First, the local convergence result for the steepest descent algorithm of Section IV is weak when compared with the global convergence obtained for steepest descent algorithms in nonlinear programming [14], [15]; we are actively pursuing generalizations of *sufficient de-scent* techniques. Second, examples demonstrate that global Nash equilibria may fail to persist under arbitrarily small

changes in player costs [9]; we conjecture that the isolated differential Nash equilibria studied in this paper may be less sensitive to such perturbations. Finally, by combining these analytical and computational advancements, we anticipate developing novel schemes for decentralized control in engi-neered systems as well as on–line identification techniques for human agents.

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