Piecewise–Differentiable Trajectory Outcomes in Mechanical Systems Subject to Unilateral Constraints

Andrew M. Pace
Univeristy of Washington
Seattle, WA, USA
apace2@uw.edu

Samuel A. Burden*
Univeristy of Washington
Seattle, WA, USA
sburden@uw.edu

ABSTRACT

We provide conditions under which trajectory outcomes in mechanical systems subject to unilateral constraints depend piecewise–differentiably on initial conditions, even as the sequence of constraint activations and deactivations varies. This builds on prior work that provided conditions ensuring existence, uniqueness, and continuity of trajectory outcomes, and extends previous differentiability results that applied only to fixed constraint (de)activation sequences. We discuss extensions of our result and implications for assessing stability and controllability.

Keywords

mechanical systems; stability; controllability

1. INTRODUCTION

To move through and interact with the world, terrestrial agents intermittently contact terrain and objects. The dynamics of this interaction are, to a first approximation, *hybrid*, with transitions between contact modes summarized by abrupt changes in system velocities [16]. Such phenomenological models are known in general to exhibit a range of pathologies that plague hybrid systems, including non–existence or non–uniqueness of trajectories [15, 34] [2, Sec. 5], or discontinuous dependence of trajectory outcomes on initial conditions (i.e. states and parameters) [28] [2, Sec. 7]; see Fig. 1 (*left*). Although instances of these pathologies can occur in physical systems [13], these occurrences are rare in everyday experience involving locomotion and manipulation with limbs. Our view is that these pathologies lie chiefly in the modeling formalism, and can be effectively removed by appropriately restricting the models under consideration without loss of relevance for many physical systems of interest.

*This material is based upon work supported by the U. S. Army Research Laboratory and the U. S. Army Research Office under contract/grant number W911NF-16-1-0158.

HSCC'17, April 18-20, 2017, Pittsburgh, PA, USA
© 2017 ACM. ISBN 978-1-4503-4590-3/17/04...$15.00
DOI: http://dx.doi.org/10.1145/3049797.3049807

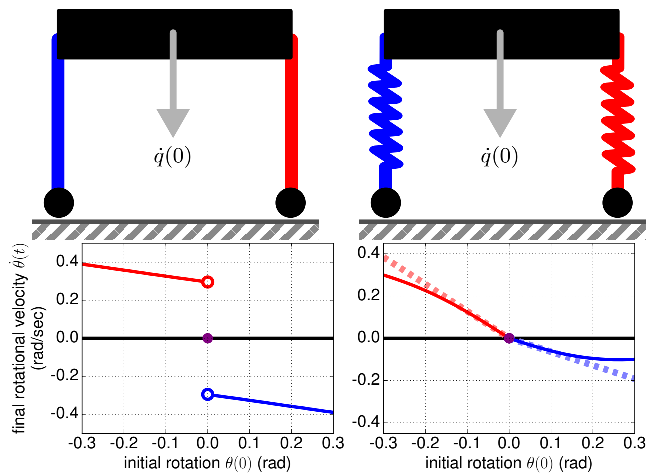

Figure 1: Trajectory outcomes in mechanical systems subject to unilateral constraints. (*left*) In general, trajectory outcomes depend discontinuously on initial conditions. In the pictured model for rigid–leg trotting (adapted from [28]), discontinuities arise when two legs touch down: if the legs impact simultaneously (corresponding to rotation $\theta(0) = 0$), then the post–impact rotational velocity is zero; if the left leg impacts before the right leg ($\theta(0) > 0$, blue) or vice–versa ($\theta(0) < 0$, red), then the post–impact rotational velocities are bounded away from zero. (*right*) In the pictured model for soft–leg trotting (adapted from [5] with the addition of a nonlinear damper coupling the body and limbs), trajectory outcomes (solid lines) are continuous and piecewise–differentiable at $\theta(0) = 0$ (dashed lines). These figures were generated using simulations of the depicted models; the sourcecode is available at https://bitbucket.org/apace2/2017hscc_figure1.

Specifically, this paper provides mathematical conditions on *mechanical* systems subject to *unilateral* constraints that ensure trajectory outcomes vary continuously and piecewise–differentiably with respect to initial conditions. Conditions that ensure continuity are known; see for instance Schatzman's work on the one–dimensional impact problem [32] or Ballard's seminal result [2, Thm. 20]. Furthermore, when the sequence of constraint activations and deactivations is held fixed, it has been known for some time that outcomes depend differentiably on initial conditions; see [1] for the earliest instance of this result we found in the English literature and [14, 11, 36, 6] for modern treatments. Our contribution is a proof that imposing an additional *admissibility* condition ensures continuous trajectory outcomes are *piecewise*–differentiable with respect to initial conditions, even as the sequence of constraint activations and deactivations varies; see Fig. 1 (*right*). The operative notion of piecewise–differentiability was originally developed by the nonsmooth analysis community to study structural stability of nonlinear programs [30], and has enabled a generalization of Calculus based on non–linear first–order approximations [33]. In the terminology of that community, we provide conditions that ensure the flow of a mechanical system subject to unilateral constraints is PC^r, and therefore possesses a piecewise–linear *Bouligand* (or *B*–)derivative.

As discussed in more detail in Sec. 6, we envision the existence and straightforward computability of the B–derivative of the flow to be useful in practice because it supports generalization of familiar control techniques to a class of hybrid systems with physical significance. In particular, building on related work that dealt with differential equations with discontinuous right–hand–sides [7, 4], the B–derivative can be used to assess stability, controllability, or optimality of trajectories in mechanical systems subject to unilateral constraints. As control of dynamic and dexterous robots increasingly relies on scalable algorithms for optimization and learning that presume the existence of first–order approximations (i.e. gradients or gradient–like objects) [26, 18, 22, 19], it is important to place application of such algorithms on a firm theoretical foundation. From a theoretical perspective, the results in this paper dovetail with recent advances in simulation of hybrid systems [5] in that one of the conditions necessary for the B–derivative to exist (namely, continuity of trajectory outcomes) is also requisite for convergence of numerical simulations. Taken together, these observations suggest that a unified analytical and computational framework for modeling and control of mechanical systems subject to unilateral constraints may be within reach.

1.1 Organization

We begin in Sec. 2 by specifying the class of dynamical systems under consideration, namely, *mechanical* systems subject to *unilateral* constraints. Sec. 3 summarizes the well–known fact that, when the contact mode sequence is fixed, trajectories vary differentiably with respect to initial conditions. In Sec. 4, we observe (as others have) that trajectories generally vary discontinuously with respect to initial conditions as the contact mode sequence varies, but provide a sufficient condition that is known to restore continuity. Sec. 5 leverages continuity to provide conditions under which trajectories vary piecewise–differentiably with respect to initial conditions across contact mode sequences, and Sec. 6 discusses extensions and implications for a systems theory for mechanical systems subject to unilateral constraints.

1.2 Relation to prior work

The technical content in Sec. 2, Sec. 3, and Sec. 4 appeared previously in the literature and is (more–or–less) well–known; we collate the results here in a sequence of technical Lemmas[1] to contextualize our contributions in Sec. 5.

2. MECHANICAL SYSTEMS SUBJECT TO UNILATERAL CONSTRAINTS

In this paper, we study the dynamics of a mechanical system with configuration coordinates $q \in Q = \mathbb{R}^d$ subject to (perfect, holonomic, scleronomic)[2] unilateral constraints $a(q) \geq 0$ specified by a differentiable function $a\colon Q \to \mathbb{R}^n$ where $d, n \in \mathbb{N}$ are finite. We are primarily interested in systems with $n > 1$ constraints, whence we regard the inequality $a(q) \geq 0$ as being enforced componentwise. Given any $J \subset \{1, \ldots, n\}$, and letting $|J|$ denote the number of elements in the set J, we let $a_J\colon Q \to \mathbb{R}^{|J|}$ denote the function obtained by selecting the component functions of a indexed by J, and we regard the equality $a_J(q) = 0$ as being enforced componentwise. It is well–known (see e.g. [2, Sec. 3] or [16, Sec. 2.4, 2.5]) that with $J = \{j \in \{1, \ldots, n\} : a_j(q) = 0\}$ the system's dynamics take the form

$$M(q)\ddot{q} = f(q, \dot{q}) + c(q, \dot{q})\dot{q} + Da_J(q)^\top \lambda_J(q, \dot{q}), \tag{1a}$$

$$\dot{q}^+ = \Delta_J(q, \dot{q}^-)\dot{q}^-, \tag{1b}$$

where $M\colon Q \to \mathbb{R}^{d\times d}$ specifies the mass matrix (or *inertia tensor*) for the mechanical system in the q coordinates, $f\colon TQ \to \mathbb{R}^d$ is termed the *effort map* [2] and specifies[3] the internal and applied forces, $c\colon TQ \to \mathbb{R}^{d\times d}$ denotes the *Coriolis matrix* determined[4] by M, $Da_J\colon Q \to \mathbb{R}^{|J|\times d}$ denotes the (Jacobian) derivative of the constraint function a_J with respect to the coordinates, $\lambda_J\colon TQ \to \mathbb{R}^{|J|}$ denotes the reaction forces generated in contact mode J to enforce the constraint $a_J(q) \geq 0$, $\Delta_J\colon TQ \to \mathbb{R}^{d\times d}$ specifies the collision restitution law that instantaneously resets velocities to ensure compatibility with the constraint $a_J(q) = 0$,

$$\Delta_J(q, \dot{q}) = I_d - (1 + \gamma(q, \dot{q}))M(q)^{-1} Da_J(q)^\top \Lambda_J(q) Da_J(q), \tag{2}$$

where I_d is the d–dimensional identity matrix, $\gamma\colon TQ \to [0, \infty)$ specifies the *coefficient of restitution*, $\dot{q}^+$ (resp. $\dot{q}^-$) denotes the right– (resp. left–)handed limits of the velocity vector with respect to time, and $\Lambda_J\colon Q \to R^{d\times d}$ is given by

$$\Lambda_J(q) = \left(Da_J(q) M(q)^{-1} Da_J(q)^\top\right)^{-1}. \tag{3}$$

Definition 1 (contact modes). *With* $A = \{q \in Q : a(q) \geq 0\}$ *denoting the set of* admissible *configurations, the constraint functions* $\{a_j\}_{j=1}^n$ *partition* A *into a finite collection*[5] $\{A_J\}_{J\in 2^n}$ *of* contact modes*:*

$$\forall J \in 2^n : A_J = \{q \in Q \mid a_J(q) = 0, \\ \forall i \notin J : a_i(q) > 0\}. \tag{4}$$

[1] For uniformity and clarity of exposition, we present previous results here as Lemmas regardless of the form in which they originally appeared.

[2] A constraint is: *perfect* if it only generates force in the direction normal to the constraint surface; *holonomic* if it varies with configuration but not velocity; *scleronomic* if it does not vary with time. We will discuss the inclusion of *imperfect*, *nonholonomic*, or *nonscleronomic* constraints in Sec. 6.

[3] We let $TQ = \mathbb{R}^d \times \mathbb{R}^d$ denote the *tangent bundle* of the configuration space Q; an element $(q, \dot{q}) \in TQ$ can be regarded as a pair containing a vector of generalized configurations $q \in \mathbb{R}^d$ and velocities $\dot{q} \in \mathbb{R}^d$; we write $\dot{q} \in T_qQ$.

[4] For each $\ell, m \in \{1, \ldots, d\}$ the (ℓ, m) entry $c_{\ell m}$ is determined from the entries of M via $c_{\ell m} = -\frac{1}{2}\sum_{k=1}^d (D_k M_{\ell m} + D_m M_{\ell k} - D_\ell M_{km})$.

[5] We let $2^n = \{J \subset \{1, \ldots, n\}\}$ denote the *power set* (i.e. the set containing all subsets) of $\{1, \ldots, n\}$.

We let $TA = \{(q, \dot{q}) \in TQ : q \in A\}$ *and* $TA_J = \{(q, \dot{q}) \in TQ : q \in A_J\}$ *for each* $J \in 2^n$.

Remark 1. *In Def. 1 (contact modes),* $J = \{1, \ldots, n\}$ *indexes the maximally constrained contact mode and* $J = \emptyset$ *indexes the unconstrained contact mode. Since any velocity is allowable in the unconstrained mode, we adopt the convention* $\Delta_\emptyset(q, \dot{q}) = I_d$.

In the present paper, we will assume that appropriate conditions have been imposed to ensure trajectories of (1) exist on a region of interest in time and state.

Assumption 1 (existence and uniqueness [2, Thm. 10]). *There exists a* flow *for (1), that is, a function* $\phi : \mathcal{F} \to TA$ *where* $\mathcal{F} \subset [0, \infty) \times TA$ *is an open subset containing* $\{0\} \times TA$ *and for each* $(t, (q, \dot{q})) \in \mathcal{F}$ *the restriction* $\phi|_{[0,t] \times \{(q,\dot{q})\}} : [0, t] \to TQ$ *is the unique left–continuous trajectory for (1) initialized at* $(q, \dot{q})$.

Remark 2. *The problem of ensuring trajectories of (1) exist and are unique has been studied extensively; we refer the reader to [2, Thm. 10] for a specific result and [16] for a general discussion of this problem.*

Since we are concerned with differentiability properties of the flow, we assume the elements in (1) are differentiable.

Assumption 2 (C^r vector field and reset map [2, §3.4]). *The vector field (1a) and reset map (1b) are continuously differentiable to order* $r \in \mathbb{N}$.

Remark 3. *If we restricted our attention to the continuous–time dynamics in (1), then Assump. 2 would suffice to provide the local existence and uniqueness of trajectories imposed by Assump. 1; as illustrated by [2, Ex. 2], Assump. 2 does not suffice when the vector field (1a) is coupled to the reset map (1b).*

3. DIFFERENTIABILITY WITHIN CONTACT MODE SEQUENCES

It is possible to satisfy Assump. 1 (existence and uniqueness of flow) under mild conditions that allow trajectories to exhibit phenomena such as *grazing* (wherein the trajectory activates a new constraint without undergoing impact) or *Zeno* (wherein the trajectory undergoes an infinite number of impacts in a finite time interval). In this and subsequent sections, where we seek to study differentiability properties of the flow, we will not be able to accommodate grazing or Zeno phenomena. Therefore we proceed to restrict the trajectories under consideration.

Definition 2 (constraint activation/deactivation [8, Chpt. 2]). *The trajectory* $\phi^{(q,\dot{q})}$ *initialized at* $(q, \dot{q}) \in TA_J \subset TQ$ activates constraints $I \in 2^n$ at time $t > 0$ *if (i) no constraint in* I *was active immediately before time* t *and (ii) all constraints in* I *become active at time* t*. Formally,*[6]

$$\exists \varepsilon > 0 : \; I \cap J = \emptyset, \; (i)\ \phi((t - \varepsilon, t), (q, \dot{q})) \subset TA_J, \quad (ii)\ \phi(t, (q, \dot{q})) \in TA_{I \cup J}. \tag{5}$$

We refer to t *as a* constraint activation time *for* $\phi^{(q,\dot{q})}$. *Similarly, the trajectory* $\phi^{(q,\dot{q})}$ deactivates constraints $I \in 2^n$ at time $t > 0$ *if (i) all constraints in* I *were active at time* t *and (ii) no constraint in* I *remains active immediately after time* t*. Formally,*

$$\exists \varepsilon > 0 : \; I \subset J, \; (i)\ \phi(t, (q, \dot{q})) \in TA_J, \quad (ii)\ \phi((t, t + \varepsilon), (q, \dot{q})) \subset TA_{J \setminus I}. \tag{6}$$

We refer to t *as a* constraint deactivation time *for* $\phi^{(q,\dot{q})}$.

Definition 3 (admissible activation/deactivation). *A constraint activation time* $t > 0$ *for* $\phi^{(q,\dot{q})}$ *is* admissible *if the constraint velocity*[7] *for all activated constraints* $I \in 2^n$ *is negative. Formally, with* $(\rho, \dot{\rho}^-) = \lim_{s \to t^-} \phi(s, (q, \dot{q}))$ *denoting the left–handed limit of the trajectory at time* t,

$$\forall i \in I : D_t \left[a_i \circ \phi \right] (0, (\rho, \dot{\rho}^-)) = Da_i(\rho) \dot{\rho}^- < 0. \tag{7}$$

A constraint deactivation time $t > 0$ *for* $\phi^{(q,\dot{q})}$ *is* admissible *if, for all deactivated constraints* $I \in 2^n$*: (i) the constraint velocity or constraint acceleration*[8] *is positive, or (ii) the time derivative of the contact force is negative. Formally, with* $(\rho, \dot{\rho}^+) = \lim_{s \to t^+} \phi(s, (q, \dot{q}))$ *denoting the right–handed limit of the trajectory at time* t*, for all* $i \in I$ *:*

$$\begin{aligned} (i)\ & D_t \left[a_i \circ \phi \right] (0, (\rho, \dot{\rho}^+)) > 0 \text{ or } \\ & D_t^2 \left[a_i \circ \phi \right] (0, (\rho, \dot{\rho}^+)) > 0, \\ \text{or } (ii)\ & D_t \left[\lambda_i \circ \phi \right] (0, (\rho, \dot{\rho}^+)) < 0. \end{aligned} \tag{8}$$

Remark 4. *The conditions for admissible constraint deactivation in case (i) of (8) can only arise at admissible constraint activation times; otherwise the trajectory is continuous, whence active constraint velocities and accelerations are zero.*

Definition 4 (admissible trajectory). *A trajectory* $\phi^{(q,\dot{q})}$ *is* admissible on $[0, t] \subset \mathbb{R}$ *if (i) it has a finite number of constraint activation (hence, deactivation) times on* $[0, t]$*, and (ii) every constraint activation and deactivation is admissible; otherwise the trajectory is* inadmissible.

Remark 5 (admissible trajectories). *The key property admissible trajectories possess that will be leveraged in what follows is: time–to–activation and time–to–deactivation are differentiable with respect to initial conditions; the same is not generally true of inadmissible trajectories.*

Remark 6 (grazing is not admissible). *The restriction in Def. 4 (admissible trajectory) that all constraint activation/deactivation times are admissible precludes admissibility of grazing.*

Remark 7 (Zeno is not admissible). *The restriction in Def. 4 (admissible trajectory) that a finite number of constraint activations occur on a compact time interval precludes admissibility of Zeno.*

[6] $\phi((t_1, t_2), (q, \dot{q})) = \{\phi(t, (q, \dot{q})) : t \in (t_1, t_2)\} \subset TQ$ denotes the *image* of $\phi^{(q,\dot{q})}$ over the interval $(t_1, t_2) \subset [0, \infty)$.

[7] Formally, the *Lie derivative* [20, Prop. 12.32] of the constraint along the vector field specified by (1a). Although constraint functions are technically only functions of configuration $q \in Q$ and not the full state $(q, \dot{q}) \in TQ$, by a mild abuse of notation we allow ourselves to consider compositions $a \circ \phi$ rather than the formally correct $a \circ \pi_Q \circ \phi$ where $\pi_Q : TQ \to Q$ is the canonical projection.

[8] Formally, the second Lie derivative of the constraint along the vector field specified by (1a).

DEFINITION 5 (CONTACT MODE SEQUENCE [16, DEF. 4]). *The* contact mode sequence *associated with a trajectory* $\phi^{(q,\dot{q})}$ *that is admissible on* $[0,t] \subset \mathbb{R}$ *is the unique function*

$$\omega\colon \{0,\ldots,m\} \to 2^n \tag{9}$$

such that there exists a finite sequence of times $\{t_\ell\}_{\ell=0}^{m+1} \subset [0,t]$ *for which* $0 = t_0 < t_1 < \cdots < t_{m+1} = t$ *and*

$$\forall \ell \in \{0,\ldots,m\} : \phi((t_\ell, t_{\ell+1}), (q,\dot{q})) \subset TA_{\omega(\ell)}. \tag{10}$$

REMARK 8. *In Def. 8 (contact mode sequence), the sequence* ω *is easily seen to be unique by the admissibility of the trajectory; indeed, the associated time sequence consists of start, stop, and constraint activation/deactivation times. Note that successive modes in the sequence need not be related by set containment (i.e.* $\omega(\ell) \subset \omega(\ell+1)$ *or* $\omega(\ell) \supset \omega(\ell+1)$*) since, e.g., one constraint could activate and another deactivate at the same time instant as in Fig. 2. Thus,* ω *is not simply a discrete "counter" as in* hybrid time domains *[10, §3.2].*

ASSUMPTION 3 (INDEPENDENT CONSTRAINTS [2, §3.4]). *The constraints are independent:*

$$\begin{aligned}\forall J \in 2^n,\ q \in a_J^{-1}(0) : \{Da_j(q)\}_{j\in J} \subset T_q^* Q \\ \text{is linearly independent.}\end{aligned} \tag{11}$$

REMARK 9. *Algebraically, Assump. 3 (independent constraints) implies that the constraint forces* λ_J *are well–defined, and that there are no more constraints than degrees–of–freedom,* $n \leq d$. *Geometrically, it implies for each* $J \in 2^n$ *that* $a_J^{-1}(0) \subset Q$ *is an embedded codimension–*$|J|$ *submanifold, and that the codimension–1 submanifolds* $\left\{a_j^{-1}(0)\right\}_{j\in J}$ *intersect transversally; this follows from [20, Thm. 5.12] since each* $a_J\colon Q \to \mathbb{R}$ *must be constant–rank on its zero section.*

We now state the well–known fact[9] that, if the contact mode sequence is fixed, then admissible trajectory outcomes are differentiable with respect to initial conditions.

LEMMA 1 (DIFFERENTIABILITY WITHIN MODE SEQ. [1]). *Under Assump. 1 (existence and uniqueness of flow), Assump. 2 (*C^r *vector field and reset map), and Assump. 3 (independent constraints), with* $\phi\colon [0,\infty) \times TA \to TA$ *denoting the flow, if* $\Sigma \subset TQ$ *is a* C^r *embedded submanifold such that all trajectories initialized in* $\Sigma \cap TA$

(i) are admissible on $[0,t] \subset \mathbb{R}$ *and*

(ii) have the same contact mode sequence,

then the restriction $\phi|_{\{t\}\times\Sigma}$ *is* C^r.

4. (DIS)CONTINUITY ACROSS CONTACT MODE SEQUENCES

As stated in Sec. 1, the point of this paper is to provide sufficient conditions that ensure trajectories of (1) vary differentiably as the contact mode sequence varies. A precondition for differentiability is continuity, whence in this section we consider what condition must be imposed to give rise to continuity in general. We begin in Sec. 4.1 by demonstrating that the transversality of constraints imposed by Assump. 3 (independent constraints) generally gives rise to discontinuity, then introduce an orthogonality condition in Sec. 4.2 that suffices to restore continuity.

4.1 Discontinuity across contact mode sequences

Consider an unconstrained initial condition $(q,\dot{q}) \in TA_\emptyset \subset TQ$ that impacts (i.e. admissibly activates) exactly two constraints $i,j \in \{1,\ldots,n\}$ at time $t > 0$; with $(\rho,\dot{\rho}^-) = \phi(t,(q,\dot{q}))$ we have

$$a_{\{i,j\}}(\rho) = 0,\ Da_i(\rho)\dot{\rho}^- < 0,\ Da_j(\rho)\dot{\rho}^- < 0. \tag{12}$$

The pre–impact velocity $\dot{\rho}^-$ abruptly resets via (1b):

$$\dot{\rho}^+ = \Delta_{\{i,j\}}(\rho)\dot{\rho}^-. \tag{13}$$

As noted in Remark 9 (independent constraints), the constraint surfaces $a_i^{-1}(0)$, $a_j^{-1}(0)$ intersect transversally. Therefore given any $\varepsilon > 0$ it is possible to find $(q_i,\dot{q}_i)$ and $(q_j,\dot{q}_j)$ in the open ball of radius ε centered at $(q,\dot{q})$ such that the trajectory $\phi^{(q_i,\dot{q}_i)}$ impacts constraint i before constraint j and $\phi^{(q_j,\dot{q}_j)}$ impacts j before i. As $\varepsilon > 0$ tends toward zero, the time spent flowing according to (1a) tends toward zero, hence the post–impact velocities tend toward the twofold iteration of (1b):

$$\begin{aligned}\dot{\rho}_i^+ &= \Delta_{\{i,j\}}(\rho)\Delta_i(\rho)\dot{\rho}^-,\\ \dot{\rho}_j^+ &= \Delta_{\{i,j\}}(\rho)\Delta_j(\rho)\dot{\rho}^-.\end{aligned} \tag{14}$$

Recalling for all $J \in 2^n$ that $\Delta_J \in \mathbb{R}^{d\times d}$ is an orthogonal projection[10] onto the tangent plane of the codimension–$|J|$ submanifold $a_J^{-1}(0)$, observe that $\dot{\rho}_i^+ = \dot{\rho}^+ = \dot{\rho}_j^+$ if and only if $Da_i(\rho)$ is orthogonal to $Da_j(\rho)$. Therefore if constraints intersect transversally but non–orthogonally, outcomes from the dynamics in (1) vary discontinuously as the contact mode sequence varies.

REMARK 10 (DISCONTINUOUS LOCOMOTION OUTCOMES). *The analysis of a saggital–plane quadruped in [28] provides an instructive example of the behavioral consequences of transverse but non–orthogonal constraints in a model of legged locomotion. As summarized in [28, Table 2], the model possesses three distinct but nearby trot (or trot–like) gaits, corresponding to whether two legs impact simultaneously (as in (13)) or at different time instants (as in (14)); the trot that undergoes simultaneous impact is unstable due to discontinuous dependence of trajectory outcomes on initial conditions.*

4.2 Continuity across contact mode sequences

To preclude the discontinuous dependence on initial conditions exhibited in Sec. 4.1, we strengthen the *transversality* of constraints implied by Assump. 3 (independent constraints) by imposing *orthogonality* of constraints.

ASSUMPTION 4 (ORTHOGONAL CONSTRAINTS [2, THM. 20]). *Constraint surfaces intersect orthogonally:*

$$\begin{aligned}\forall i,j \in \{1,\ldots,n\},\ i \neq j,\ q \in a_i^{-1}(0) \cap a_j^{-1}(0) : \\ \langle Da_i(q), Da_j(q)\rangle_{M^{-1}} = 0.\end{aligned} \tag{15}$$

REMARK 11. *Note that Assump. 4 (orthogonal constraints) is strictly stronger than Assump. 3 (independent constraints). Physically, the assumption can be interpreted as asserting that any two independent limbs that can undergo impact simultaneously must be inertially decoupled. This can be*

[9] The result follows via a straightforward composition of smooth flows with smooth time–to–impact maps; we refer the interested reader to [6, App. A1] for details.

[10] relative to the inner product $\langle\cdot,\cdot\rangle_M$. For further discussion of orthogonal projection in constraint activation of mechanical systems, see [16, §1.3.4].

achieved in artifacts by introducing series compliance in a sufficient number of degrees–of–freedom.

Sec. 4.1 demonstrated that Assump. 4 (orthogonal constraints) is necessary in general to preclude discontinuous dependence on initial conditions. The following result demonstrates that this assumption is sufficient to ensure continuous dependence on initial conditions, even as the contact mode sequence varies.[11]

LEMMA 2 (CONTINUITY ACROSS MODE SEQ. [2, THM. 20]). *Under Assump. 1 (existence and uniqueness of flow), Assump. 2 (C^r vector field and reset map), and Assump. 4 (orthogonal constraints), with $\phi\colon [0,\infty) \times TA \to TA$ denoting the flow, if $t \in \mathbb{R}$ and $(p,\dot p) \in TA \subset TQ$ are such that t is not a constraint activation time for $(p,\dot p)$, then ϕ is continuous at $(t,(p,\dot p))$.*

REMARK 12 (CONTINUITY ACROSS MODE SEQ.). *The preceding result implies that the flow ϕ is continuous almost everywhere in both time and state, without needing to restrict to admissible trajectories. Thus orthogonal constraints ensure the flow ϕ depends continuously on initial conditions, even along trajectories that exhibit grazing and Zeno phenomena.[12] For the reason described in Remark 5 (admissible trajectories), we will not be able to accommodate these phenomena when we study differentiability properties of trajectories in the next section.*

5. DIFFERENTIABILITY ACROSS CONTACT MODE SEQUENCES

We now provide conditions that ensure trajectories depend *differentiably* on initial conditions, even as the contact mode sequence varies. In general, the flow does not possess a classical Jacobian (alternately called *Fréchet* or *F*–)derivative, i.e. there does not exist a single linear map that provides a first–order approximation for the flow. Instead, under the admissibility conditions introduced in Sec. 3, we show that the flow admits a *piecewise*–linear first–order approximation termed[13] a *Bouligand* (or *B*–)derivative [33, Ch. 3.1]. Though perhaps unfamiliar, this derivative is nevertheless quite useful. Significantly, unlike functions that are merely directionally differentiable, B–differentiable functions admit generalizations of many techniques familiar from calculus, including the Chain Rule [33, Thm 3.1.1] (and hence Product and Quotient Rules [33, Cor. 3.1.1]), Fundamental Theorem of Calculus [33, Prop. 3.1.1], and Implicit Function Theorem [33, Thm. 4.2.3], and the B–derivative can be employed to implement scalable algorithms [17] for optimization or learning.

We proceed by showing that the flow is piecewise–differentiable in the sense defined in [33, Ch. 4.1] and recapitulated here; functions that are piecewise–differentiable in this sense are always B–differentiable [33, Prop. 4.1.3]. Let $r \in \mathbb{N} \cup \{\infty\}$ denote an order of differentiability[14] and $D \subset \mathbb{R}^m$ be open. A continuous function $\psi\colon D \to \mathbb{R}^\ell$ is called *piecewise–C^r* if the graph of ψ is everywhere locally covered by the graphs of a finite collection of functions that are r times continuously differentiable (C^r–functions).[15] Formally, for every $x \in D$ there must exist an open set $U \subset D$ containing x and a finite collection $\left\{\psi_\omega\colon U \to \mathbb{R}^\ell\right\}_{\omega\in\Omega}$ of C^r–functions such that for all $x \in U$ we have $\psi(x) \in \left\{\psi_\omega(x)\right\}_{\omega\in\Omega}$.

We now state and prove the main result of this paper: whenever the flow of a mechanical system subject to unilateral constraints is continuous and admissible, it is piecewise–C^r; see Fig. 2 for an illustration.

THEOREM 1 (PIECEWISE–DIFFERENTIABLE FLOW). *Under Assump. 1 (existence and uniqueness of flow), Assump. 2 (C^r vector field and reset map), and Assump. 4 (orthogonal constraints), with $\phi\colon [0,\infty) \times TA \to TA$ denoting the flow, if $t \in [0,\infty)$, $(p,\dot p) \in TA \subset TQ$, and $\Sigma \subset TQ$ is a C^r embedded submanifold containing $(p,\dot p)$ such that*

(i) the trajectory $\phi^{(p,\dot p)}$ activates and/or deactivates constraints at time $s \in (0,t)$,

(ii) $\phi^{(p,\dot p)}$ has no other activation or deactivation times in $[0,t]$,

(iii) trajectories initialized in $\Sigma \cap TA$ are admissible on $[0,t]$, and

(iv) the set Ω of contact mode sequences for trajectories initialized in $\Sigma \cap TA$ is finite,

then the restriction $\phi|_{[0,\infty)\times\Sigma}$ is piecewise–C^r at $(t,(p,\dot p))$.

PROOF. We seek to show that the restriction $\phi|_{[0,\infty)\times\Sigma}$ is piecewise–C^r at $(t,(p,\dot p))$. We will proceed by constructing a finite set of r times continuously differentiable selection functions for ϕ on $[0,t]\times\Sigma$. In the example given in Fig. 2, there are two selection functions, one corresponding to a perturbation along $(v_r,\dot v_r)$, colored red, and the other along $(v_b,\dot v_b)$, colored blue. These selection functions will be indexed by a pair of functions (ω,η) where: $\omega\colon \{0,\dots,m\} \to 2^n$ is a contact mode sequence, i.e. $\omega \in \Omega$; $\eta\colon \{0,\dots,m-1\} \to \{1,\dots,n\}$ indexes constraints that undergo admissible activation or deactivation[16] at the contact mode transition indexed by $\ell \in \{0,\dots,m-1\}$. For instance, in Fig. 2 the index functions for the (de)activation sequence starting from $(v_r,\dot v_r)$, in red, are $\omega_r(0) = \{1\}, \omega_r(1) = \emptyset, \omega_r(2) = \{2\}$, $\eta_r(0) = 1, \eta_r(1) = 2$, and the index functions for the (de)activation sequence starting from $(v_b,\dot v_b)$, in blue, are

[11] We note for the interested reader that the result on continuity with respect to initial conditions in [25] is inapplicable along trajectories that simultaneously activate and/or deactivate more than one constraint; such trajectories do not satisfy hypotheses 4 and 5 of [25, Thm. III.2].

[12] We remark that [2, Thm. 20] implies the function ϕ is continuous everywhere with respect to the quotient metric defined in [5, Sec. III], whence the numerical simulation algorithm in [5, Sec. IV] is provably–convergent for all trajectories (even those that graze) up to the first occurrence of Zeno.

[13] This terminology was introduced, to the best of our knowledge, by Robinson [30].

[14] We let context specify whether $r = \infty$ indicates "mere" smoothness or the more stringent condition of analyticity.

[15] The definition of piecewise–C^r may at first appear unrelated to the intuition that a function ought to be piecewise–differentiable precisely if its "domain can be partitioned locally into a finite number of regions relative to which smoothness holds" [31, Section 1]. However, as shown in [31, Theorem 2], piecewise–C^r functions are always piecewise–differentiable in this intuitive sense.

[16] In light of Remark 4, we only consider deactivations of type (ii) in Def. 3 (admissible constraint activation/deactivation). In some systems, a deactivation of type (ii) may only arise following a (simultaneous) activation; it suffices to restrict to functions η that do not index such deactivations.

$\omega_b(0) = \{1\}, \omega_b(1) = \{1,2\}, \omega_b(2) = \{2\}$, $\eta_b(0) = 2, \eta_b(1) = 1$. Note that for each $\omega \in \Omega$ the set $H(\omega)$ of possible η's is finite; since the set Ω is finite by assumption, the set of pairs (ω, η) is finite.

Let $(\omega\colon \{0,\ldots,m\} \to 2^n) \in \Omega$ and $(\eta\colon \{0,\ldots,m-1\} \to \{1,\ldots,n\}) \in H(\omega)$ be as described above. Let $\mu\colon \{0,\ldots,m\} \to 2^n$ be defined as $\mu(k) = \bigcup_{i=0}^{k-1} \{\eta(i)\}$, where we adopt the convention that $\bigcup_{i=0}^{-1}\{i\} = \emptyset$; note that μ is uniquely determined by η.[17] For the sake of readability, we suppress dependence on η and ω until (22). Let $(\rho, \dot\rho^-) = \lim_{u\uparrow s} \phi(u,(p,\dot p))$. For all $k \in \{0,\ldots,m\}$ define $\dot\rho_k = \Delta_{\mu(k)}(\rho)\dot\rho^-$. There exists an open neighborhood $U_k \subset TQ$ containing $(\rho, \dot\rho_k)$ such that the vector field determined by (1a) at $\omega(k)$ admits a C^r extension to $F_k\colon U_k \to \mathbb{R}^{2d}$. (Note that for $k = m$ (resp. $k = 0$) the neighborhood U_k can be taken to additionally include $\phi((s,t],(p,\dot p))$ (resp. $\phi([0,s),(p,\dot p))$).)

By the Fundamental Theorem on Flows [20, Thm. 9.12], F_k determines a unique maximal flow $\phi_k\colon \mathcal{F}_k \to U_k$ over a maximal flow domain $\mathcal{F}_k \subset \mathbb{R}\times U_k$, which is an open set that contains $\{0\}\times U_k$, and the flow ϕ_ℓ is C^r. (Note that $(t-s,(\rho,\dot\rho_m)) \in \mathcal{F}_m$ and $(s,(p,\dot p)) \in \mathcal{F}_0$.)

If $\eta(\ell)$ indexes an admissible constraint activation (recall that $\ell \in \{0,\ldots,m-1\}$), then there exists a time–to–activation $\tau_\ell\colon U_\ell \to \mathbb{R}$ defined over an open set $U_\ell \subset TQ$ containing $(\rho,\dot\rho_\ell)$ such that

$$\forall (q,\dot q) \in U_\ell : a_{\eta(\ell)} \circ \phi_\ell(\tau_\ell(q,\dot q),(q,\dot q)) = 0. \tag{16}$$

If instead $\eta(\ell)$ indexes an admissible constraint deactivation, then there exists a time–to–deactivation $\tau_\ell\colon U_\ell \to \mathbb{R}$ defined over an open set $U_\ell \subset TQ$ containing $(\rho,\dot\rho_\ell)$ such that

$$\forall (q,\dot q) \in U_\ell : \lambda_{\eta(\ell)} \circ \phi_\ell(\tau_\ell(q,\dot q),(q,\dot q)) = 0. \tag{17}$$

In either case, τ_ℓ exists and is C^r by the Implicit Function Theorem [20, Thm. C.40] due to admissibility of trajectories initialized in Σ. (Note for $\ell = 0$ the neighborhood U_ℓ can be extended to include $\phi([0,s),(p,\dot p))$ using the semi–group property[18] of the flow ϕ_ℓ.) See Fig. 2 for an illustration of constraint activations and deactivations.

Let $\varphi_\ell\colon \mathbb{R}\times U_\ell \to \mathbb{R}\times U_\ell$ be defined for all $(u,(q,\dot q)) \in \mathbb{R}\times U_\ell$ by

$$\varphi_\ell(u,(q,\dot q)) = (u - \tau_\ell(q,\dot q), \phi_\ell(\tau_\ell(q,\dot q),(q,\dot q))). \tag{18}$$

The map φ_ℓ flows a state $(q,\dot q)$ using the vector field from contact mode $\omega(\ell)$ until constraint $\eta(\ell)$ undergoes admissible activation/deactivation, and deducts the time required from the given budget u. The total derivative of φ_ℓ at $(0,(\rho,\dot\rho_\ell))$ (see also [7, § 7.1.4]) is

$$D\varphi_\ell(0,(\rho,\dot\rho_\ell)) = \begin{bmatrix} 1 & \frac{1}{gf} g \\ 0 & I_{2d} - \frac{1}{gf} fg \end{bmatrix}, \tag{19}$$

where $f = F(\rho,\dot\rho_\ell)$ and $g = Dh_{\eta(\ell)}(\rho,\dot\rho_\ell)$ where $h_\ell\colon TQ \to \mathbb{R}$ is defined for all $(q,\dot q) \in TQ$ by $h_\ell(q,\dot q) = a_{\eta(\ell)}(q)$.

Let $\Gamma_\ell\colon \mathbb{R}\times TQ \to \mathbb{R}\times TQ$ be defined for all $(u,(q,\dot q)) \in \mathbb{R}\times TQ$ by

$$\Gamma_\ell(u,(q,\dot q)) = (u,(q,\Delta_{\mu(\ell)}(q)\dot q)). \tag{20}$$

[17] η is not uniquely determined by ω due to the possibility of instantaneous activation/deactivation for the same constraint; consider for instance the bounce of an elastic ball [12, Ch. 2.4].

[18] $\phi_\ell(u+v,x) = \phi_\ell(u,\phi_\ell(v,x))$ whenever $(v,x),(u+v,x),(u,\phi_\ell(v,x)) \in \mathcal{F}_\ell$.

The map Γ_ℓ resets velocities to be compatible with contact mode $\omega(\ell)$ while leaving positions and times unaffected. The total derivative of Γ_ℓ at $(u,(q,\dot q))$ is given by

$$D\Gamma_\ell(u,(q,\dot q)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_d & 0 \\ 0 & D_q(\Delta_{\mu(\ell)}(q)\dot q) & \Delta_{\mu(\ell)}(q) \end{bmatrix}. \tag{21}$$

For each $\omega \in \Omega$ and $\eta \in H(\omega)$ define ϕ^η_ω by the formal composition

$$\phi^\eta_\omega = \phi_m \circ \prod_{\ell=0}^{m-1} (\Gamma_{\ell+1} \circ \varphi_\ell). \tag{22}$$

We take as the domain of ϕ^η_ω the set

$$\mathcal{F}^\eta_\omega = (\phi^\eta_\omega)^{-1}(TQ) \subset \mathbb{R}\times TQ, \tag{23}$$

noting that $\mathcal{F}^\eta_\omega$ is (i) open since each function in the composition is continuous, and (ii) nonempty since $(t,(p,\dot p)) \in \mathcal{F}^\eta_\omega$. The map ϕ^η_ω flows states via a given contact mode sequence for a specified amount of time; note that some of the resulting "trajectories" are not physically realizable, as they may evaluate the flows $\{\phi_k\}_{k=0}^m$ in backward time. An example of such a physically unrealizable "trajectory" is illustrated in Fig. 2 by $\phi^{\omega_r}_{\eta_r}(t,(v_b,\dot v_b))$, which first flows forward in time via the extended vector field $F_{\{1\}}$ past the constraint surface $\{a_2(q) = 0\}$ until constraint 1 deactivates and then flows backwards in time until constraint 2 activates, ultimately terminating in $TA_{\{2\}}$.

With $\mathcal{F} = \bigcap\{\mathcal{F}^\eta_\omega : \omega\in\Omega, \eta\in H(\omega)\} \subset [0,\infty)\times TQ$, for any $(u,(q,\dot q)) \in \mathcal{F}\cap([0,\infty)\times TA)$ with contact mode sequence $\omega\in\Omega$ and constraint sequence $\eta \in H(\omega)$, the trajectory outcome is obtained by applying ϕ^η_ω to $(u,(q,\dot q))$, i.e. $\phi(u,(q,\dot q)) = \phi^\eta_\omega(u,(q,\dot q))$. See Fig. 2 for an illustration of trajectories with different contact mode sequences.

The maps φ_ℓ, Γ_ℓ, and ϕ^η_ω are C^r on their domains since they are each obtained from a finite composition of C^r functions. Therefore the restriction[19] $\phi|_{[0,\infty)\times\Sigma}$ is a continuous selection of the finite collection of C^r functions

$$\{\phi^\eta_\omega : \omega\in\Omega, \eta\in H(\omega)\}$$

on the open neighborhood $\mathcal{F} \subset TQ$ containing $(t,(p,\dot p))$, i.e. $\phi|_{[0,\infty)\times\Sigma}$ is piecewise–C^r at $(t,(p,\dot p))$. See Fig. 2 for an illustration the piecewise–differentiability of trajectory outcomes arising from a transition between contact mode sequences.

Remark 13 (satisfying Theorem hypotheses). *Models of animal or robot behaviors involving intermittent contact with terrain—walking, running, climbing, leaping, dancing, juggling, grasping—generally satisfy our hypotheses, so long as they possess sufficient compliance as in Fig. 1 (right).*

Remark 14 (relaxing Theorem hypotheses). *Since the class of piecewise–differentiable functions is closed under finite composition, conditions (i) and (ii) in the preceding*

[19] As a technical aside, we remark that the domain of ϕ is confined to $[0,\infty)\times TA$, whence invoking the definition of piecewise–differentiability requires a continuous extension $\widetilde\phi$ of ϕ defined on a neighborhood of $(t,(p,\dot p))$ that is open relative to $[0,\infty)\times TQ$. One such extension is obtained by composing ϕ with a sufficiently differentiable retraction [20, Ch. 6] of TQ onto TA (such a retraction is guaranteed to exist locally by transversality of constraint surfaces).

Theorem can be readily relaxed to accommodate a finite number of constraint activation/deactivation times in the interval $(0, t)$. Conditions (iii) and (iv) are more difficult to relax since there are systems wherein trajectories initialized arbitrarily close to an admissible trajectory fail to be admissible themselves. As a familiar example, consider a 1 degree–of–freedom elastic impact oscillator [12, Ch. 2.4] (i.e. a bouncing ball): the stationary trajectory (initialized with $q, \dot{q} = 0$) is admissible for all time, but all nearby trajectories (initialized with $q \neq 0$ or $\dot{q} \neq 0$) exhibit the Zeno phenomenon. We will discuss further possible extensions in Sec. 6.1.1.

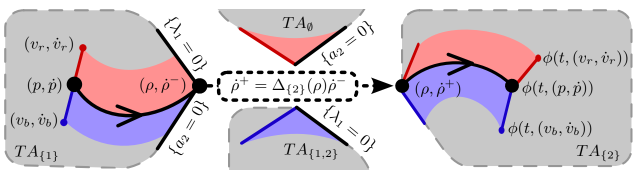

Figure 2: Illustration of trajectory undergoing simultaneous constraint activation and deactivation: the trajectory initialized at $(p, \dot{p}) \in TA_{\{1\}} \subset TQ$ flows via (1a) to a point $(\rho, \dot{\rho}^-) \in TA_{\{1\}}$ where both the constraint force λ_1 and constraint function a_2 are zero, instantaneously resets velocity via (1b) to $\dot{\rho}^+ = \Delta_{\{2\}}(\rho)\dot{\rho}^-$, then flows via (1a) to $\phi(t, (p, \dot{p})) \in TA_{\{2\}} \subset TQ$. Nearby trajectories undergo activation and deactivation at distinct times: trajectories initialized in the red region, e.g. $(v_r, \dot{v}_r)$, deactivate constraint 1 and flow through contact mode $TA_\emptyset$ before activating constraint 2—their contact mode sequence is $(\{1\}, \emptyset, \{2\})$—while trajectories initialized in the blue region, e.g. $(v_b, \dot{v}_b)$, activate 2 and flow through $TA_{\{1,2\}}$ before deactivating 1—their contact mode sequence is $(\{1\}, \{1, 2\}, \{2\})$. Piecewise–differentiability of the trajectory outcome is illustrated by the fact that red outcomes lie along a different subspace than blue.

Under the hypotheses of the preceding Theorem, the continuous flow ϕ is piecewise–differentiable at a point $(t, (p, \dot{p})) \in [0, \infty) \times TA$, that is, near $(t, (p, \dot{p}))$ the graph of ϕ is has an open covering by the graphs of a finite collection $\{\phi^\eta_\omega : \omega \in \Omega, \eta \in H(\omega)\}$ of differentiable functions (termed selection functions). This implies in particular that there exists a continuous and piecewise–linear function

$$D\phi(t, (p, \dot{p})) : T_{(t,(p,\dot{p}))} ([0, \infty) \times TA) \to T_{\phi(t,(p,\dot{p}))} A \quad (24)$$

(termed the Bouligand or B–derivative) that provides a first–order approximation for how trajectory outcomes vary with respect to initial conditions. Formally, for all $(u, (v, \dot{v})) \in T_{(t,(p,\dot{p}))} ([0, \infty) \times TA)$, the vector $D\phi(t, (p, \dot{p}); u, (v, \dot{v})) \in \mathbb{R}^{2d}$ is the directional derivative of $\phi(t, (p, \dot{p}))$ in the $(u, (v, \dot{v}))$ direction:

$$\begin{aligned}\lim_{\alpha \downarrow 0} \frac{1}{\alpha} &[(\phi(t + \alpha u, (p + \alpha v, \dot{p} + \alpha \dot{v})) - \phi(t, (p, \dot{p}))) - \\ &D\phi(t, (p, \dot{p}); u, (v, \dot{v}))] = 0.\end{aligned} \quad (25)$$

Furthermore, this directional derivative is contained within the collection of directional derivatives of the selection functions. Formally, for all $(u, (v, \dot{v})) \in T_{(t,(p,\dot{p}))} ([0, \infty) \times TA)$,

$$\begin{aligned}&D\phi(t, (p, \dot{p}); u, (v, \dot{v})) \in \\ &\quad \{D\phi^\eta_\omega(t, (p, \dot{p}); u, (v, \dot{v})) : \omega \in \Omega, \eta \in H(\omega)\}.\end{aligned} \quad (26)$$

The selection functions are classically differentiable, whence their directional derivatives can be computed via matrix–vector multiplication between a classical (Jacobian/Fréchet) derivative matrix and the perturbation vector. Formally, for all $(u, (v, \dot{v})) \in T_{(t,(p,\dot{p}))} ([0, \infty) \times TA)$, $\omega \in \Omega$, $\eta \in H(\omega)$,

$$D\phi^\eta_\omega(t, (p, \dot{p}); u, (v, \dot{v})) = D\phi^\eta_\omega(t, (p, \dot{p})) \begin{bmatrix} u \\ v \\ \dot{v} \end{bmatrix}, \quad (27)$$

where $D\phi^\eta_\omega(t, (p, \dot{p})) \in \mathbb{R}^{(2d)\times(1+2d)}$ is the classical derivative of the selection function ϕ^η_ω. The matrix $D\phi^\eta_\omega(t, (p, \dot{p}))$ can be obtained by applying the (classical) chain rule to the definition of ϕ^η_ω from (22).

6. DISCUSSION

We conclude by discussing possible routes (or obstacles) to extend our result, and implications for assessing stability and controllability.

6.1 Extending our result

6.1.1 Relaxing hypotheses

The hypotheses used to state Thm. 1 (piecewise differentiability across contact mode sequences) restrict either the systems or system trajectories under consideration; we will discuss the latter before addressing the former.

Trajectories we termed admissible exhibit neither *grazing* nor *Zeno* phenomena. Since grazing generally entails constraint activation times that are not even Lipschitz continuous with respect to initial conditions, the flow is not piecewise–C^r along grazing trajectories. This fact has been shown by others [8, Ex. 2.7], and is straightforward to see in an example. Indeed, consider the trajectory of a point mass moving vertically in a uniform gravitational field subject to a maximum height (i.e. ceiling) constraint. The grazing trajectory is a parabola, whence the time–to–activation function involves a square root of the initial position. Zeno trajectories, on the other hand, can exhibit differentiable trajectory outcomes following an accumulation of constraint activations (and, hence, deactivations); consider, for instance, the (stationary) outcome that follows the accumulation of impacts in a model of a bouncing ball [12, Ch. 2.4]. Thus we cannot at present draw any general conclusions regarding differentiability of the flow along Zeno trajectories, and speculate that it might be possible to recover piecewise–differentiability along such trajectories in the *completion* of the mechanical system [27, Sec. IV] after establishing continuity with respect to initial conditions in the intrinsic state–space metric [5, Sec. III].

The (so–called [2]) effort map f was not allowed to vary with the contact mode, while the dynamics in (1) vary with the contact mode $J \subset \{1, \ldots, n\}$ due to intermittent activation of unilateral constraints $a_J(q) \geq 0$. Contact–dependent effort can easily introduce nonexistence or nonuniqueness. Indeed, this phenomenon was investigated thoroughly by Carathéodory and, later, Filippov [9, Ch. 1]. For a specific example of the potential challenges in allowing contact-dependent forcing, note that the introduction of simple friction models into mechanical systems subject to unilateral constraints is known to produce pathologies including nonexistence and nonuniqueness of trajectories [34]. To generalize the preceding results to allow the above phenomena, one would need to provide conditions ensuring that trajectories (i) exist uniquely, (ii) depend continuously on initial conditions, and (iii) admit differentiable selection functions along trajectories of interest.

6.1.2 *Including control inputs*

We focused on autonomous dynamics in (1); however, parameterized control inputs can be incorporated through a standard state augmentation technique in such a way that Theorem 1 implies trajectory outcomes depend piecewise–differentiably on initial states and input parameters, even as the contact mode sequence varies.

Specifically, suppose (1a) is replaced with

$$M(q)\ddot{q} = \tilde{f}((q,\dot{q}),u) + c(q,\dot{q})\dot{q} + Da_J(q)^\top \tilde{\lambda}_J((q,\dot{q}),u), \tag{28a}$$

$$\dot{q}^+ = \widetilde{\Delta}_J((q,\dot{q}^-),u)\dot{q}^-, \tag{28b}$$

where $\tilde{f}\colon TQ \times U \to \mathbb{R}^d$ is an effort map that accepts a constant input parameter $u \in U = \mathbb{R}^m$, $\tilde{\lambda}_J\colon TQ \times U \to \mathbb{R}^{|J|}$ is the reaction force that results from applying effort $\tilde{f}(q,\dot{q},u)$ in contact mode J, and $\widetilde{\Delta}_J\colon TQ \times U \to \mathbb{R}^d$ is a reset map that accepts input parameter u as well. We interpret the vector u as parameterizing an open– or closed–loop input to the system; once initialized, u remains constant.[20] It is possible to generalize the proof of Thm. 1 (piecewise differentiability across contact mode sequences) to provide conditions under which there exists a continuous flow $\widetilde{\phi} : \widetilde{\mathcal{F}} \to TA$ for (28) that is piecewise–differentiable with respect to initial conditions $(q,\dot{q}) \in TA$ and input parameters $u \in U$ over an open subset $\widetilde{\mathcal{F}} \subset [0,\infty) \times TA \times U$ containing $\{0\} \times TA \times U$.

6.2 Assessing (in)stability of periodic orbits

In this section we consider the problem of assessing stability (or instability) of a periodic orbit in a mechanical system subject to unilateral constraints. Suppose $(\rho,\dot{\rho}) \in TA_\emptyset$ is an initial condition that lies on a *periodic orbit*, i.e. there exists $T > 0$ so that $\phi(T,(\rho,\dot{\rho})) = (\rho,\dot{\rho})$ and $\phi(t,(\rho,\dot{\rho})) \neq (\rho,\dot{\rho})$ for all $t \in (0,T)$. If the trajectory $\phi^{(\rho,\dot{\rho})}$ undergoes constraint activations and deactivations at isolated instants in time, then prior work has shown that ϕ is C^1 at $(T,(\rho,\dot{\rho}))$, and the classical derivative $D\phi(T,(\rho,\dot{\rho}))$ can be used to assess stability of the periodic orbit [1]. If instead the trajectory activates and/or deactivates some constraints simultaneously as in Fig. 3, then (so long as constraint activations/deactivations are admissible on and near $\phi^{(\rho,\dot{\rho})}$) the results of Sec. 5 ensure that ϕ is PC^1 at $(T,(\rho,\dot{\rho}))$ and the B–derivative $D\phi(T,(\rho,\dot{\rho}))$ is not generally given by a single linear map, whence classical tests for stability are not applicable. In what follows we generalize the classical techniques to use this B–derivative to assess stability (or instability) of the periodic orbit $\phi^{(\rho,\dot{\rho})}$.

[20] A control policy represented using a universal function approximator such as an artificial neural network [22, 19] provides an example of a parameterized closed–loop input, while a control signal represented using a finite truncation of an expansion in a chosen basis [26, 18] provides an example of a parameterized open-loop input.

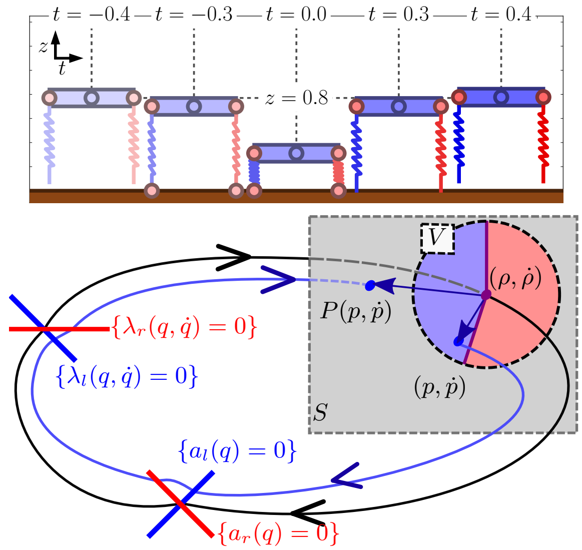

Figure 3: Illustration of a periodic orbit in the system depicted in Fig. 1(*right*) undergoing simultaneous activation (and, subsequently, simultaneous deactivation) of unilateral constraints. (*top*) Snapshots of the trajectory at apex ($t = \pm 0.4$), touchdown/liftoff ($t = \pm 0.3$), and nadir ($t = 0.0$). (*bottom*) Illustration of Poincaré map $P : V \to S$ over the apex section S, defined in Sec. 6.2; trajectories initialized in the open subset $V \subset S$ return to S. The set V is partitioned into regions where selection functions for the piecewise–C^r map P are active: initial conditions with $\theta > 0$, where the left leg constraint activates before the right, are colored blue; initial conditions with $\theta < 0$ are colored red. Along the trajectory generated by the fixed point $P(\rho,\dot{\rho}) = (\rho,\dot{\rho})$ (colored black), simultaneous constraint activation is indicated by the trajectory passing through the intersection of the constraint surfaces for the right ($\{a_r = 0\}$) and left ($\{a_l = 0\}$) legs; similarly for simultaneous deactivation through the intersection $\{\lambda_l = 0\} \cap \{\lambda_r = 0\}$. A nearby trajectory initialized at $(p,\dot{p}) \in V$ (colored blue) undergoes constraint activation and deactivation at distinct instants in time.

We start by constructing a Poincaré map for the periodic orbit $\phi^{(\rho,\dot{\rho})}$. Let $S \subset TQ$ be a Poincaré section for $\phi^{(\rho,\dot{\rho})}$ at $(\rho,\dot{\rho})$, i.e. a C^r embedded codimension–1 submanifold containing $(\rho,\dot{\rho})$ that is transverse to the vector field in (1a). For a concrete example we refer to the model in Fig. 3 where S is a Poincaré section about the apex and ρ the position vector with body height $z = 0.8$, rotation $\theta = 0$, and the legs oriented perpendicular to the body orientation. Given zero initial velocity, the orbit's period is $T \simeq 0.8$.

Since ϕ is continuous by Lem. 2 (continuity across contact mode sequences), there exists a *first–return time* $\tau\colon V \to (0,\infty)$ defined over an open neighborhood $V \subset S$ containing $(\rho,\dot{\rho})$ such that $\phi(\tau(q,\dot{q}),(q,\dot{q})) \in S$ for all $(q,\dot{q}) \in V$ and $\tau(\rho,\dot{\rho}) = T$; we let $P : V \to S$ be the Poincaré (or *first–return*) map defined by

$$\forall (q,\dot{q}) \in V\colon P(q,\dot{q}) = \phi(\tau(q,\dot{q}),(q,\dot{q})) \in S. \tag{29}$$

As an illustration, $(p, \dot{p}) \in V$ in Fig. 3(*bottom*) generates a trajectory initialized near $(\rho, \dot{\rho})$ that undergoes admissible constraint activations and deactivations at distinct instants in time, activating the left leg constraint before activating the right leg constraint, then deactivating both constraints in the same order. Since ϕ is PC^r and S is a C^r manifold we conclude that τ is PC^r [7, Thm. 10], whence P is PC^r. To assess exponential stability of $\phi^{(\rho,\dot{\rho})}$, it suffices to determine conditions under which the piecewise–linear map $DP(\rho, \dot{\rho})$ is exponentially contractive or expansive. This task is nontrivial since, as is well–known [3, Ex. 2.1], a piecewise–linear system constructed from stable subsystems may be unstable; similarly, a system constructed from unstable subsystems may be stable. We refer to [23, Sec. II-A] for a review of state–of–the–art methods for assessing stability of piecewise–linear systems, and provide an example test below.

Since P is PC^r, there exists a finite collection $\{P_\omega\}_{\omega\in\Omega}$ of C^r selection functions for P, and we assume the neighborhood V was chosen sufficiently small that $P_\omega \colon V \to S$ for each $\omega \in \Omega$. Let $R_\omega \subset V$ denote the region where the selection function P_ω is *active* (i.e. where $P|_{R_\omega} = P_\omega|_{R_\omega}$). The first order approximation for P_ω is given by the classical (Jacobian/Fréchet) derivative $DP_\omega \colon TV \to TS$, which can be calculated using the (classical) chain rule. If there is a norm $\|\cdot\| : \mathbb{R}^{2d-1} \to \mathbb{R}$ with respect to which $DP_\omega(\rho, \dot{\rho})$ is a contraction for all $\omega \in \Omega$ (i.e. for all $\omega \in \Omega$ the induced norm $\|DP_\omega(\rho, \dot{\rho})\| < 1$), then the periodic orbit $\phi^{(\rho,\dot{\rho})}$ is exponentially stable [7, Prop. 15]. (Note that it does not suffice to find a different norm $\|\cdot\|_\omega$ for each $\omega \in \Omega$ with respect to which $DP_\omega(\rho, \dot{\rho})$ is a contraction [3, Ex. 2.1].) If instead for some $\omega \in \Omega$ there exists an eigenvector ν for $DP_\omega(\rho, \dot{\rho})$ with eigenvalue λ such that $|\lambda| > 1$ and $\nu \in R_\omega$, then $(\rho, \dot{\rho})$ is exponentially unstable; this instability test is illustrated in Fig. 4.

6.3 Assessing controllability

In this section we consider the problem of assessing (*small–time*, *local* [35]) controllability along a trajectory in a mechanical system subject to unilateral constraints. The local control problem has been solved quite satisfactorily along trajectories in such systems that undergo constraint activation and deactivation at distinct instants in time for cases where the control input influences the discrete–time [24] or continuous–time [29] portions of (1). We concern ourselves here with the controlled dynamics in (28), and focus our attention on trajectories that activate and/or deactivate multiple constraints simultaneously since (to the best of our knowledge) this case has not previously been addressed in the literature.

Toward that end, let $\widetilde{\phi} : \widetilde{\mathcal{F}} \to TA$ be the flow of (28) (a mechanical system subject to unilateral constraints with input parameter $u \in U = \mathbb{R}^m$), and let $\widetilde{\phi}^{((\rho,\dot{\rho}),\mu)}$ be a trajectory initialized at $(\rho, \dot{\rho}) \in TA$ with input parameter $\mu \in U$. If $\widetilde{\phi}$ were C^1 at $(t, (\rho, \dot{\rho}), \mu) \in \widetilde{\mathcal{F}}$, then (small–time) local controllability about $\widetilde{\phi}^{((\rho,\dot{\rho}),\mu)}$ could be determined using an invertibility condition on the (Jacobian) matrix $D\widetilde{\phi}(t, (\rho, \dot{\rho}), \mu)$. Indeed, a straightforward application of the Implicit Function Theorem [20, Thm. C.40] shows that if the subblock $D_U\widetilde{\phi}(t, (\rho, \dot{\rho}), \mu)$, which transforms first–order variations in the input parameter u into the resulting first–order variations in the state $(q, \dot{q})$ at time t, is invertible, then (28) is (small–time) locally controllable along $\widetilde{\phi}^{((\rho,\dot{\rho}),\mu)}$ [21, Thm. 8].[21]

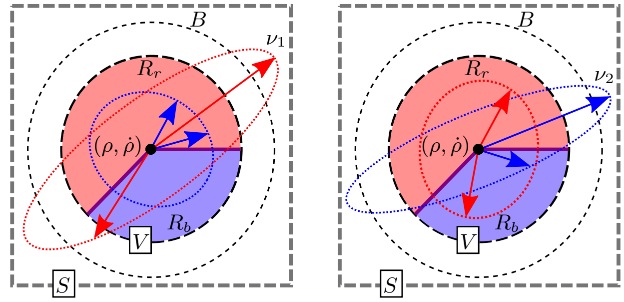

Figure 4: A Poincaré section S (outer grey box) and the neighborhood V (inner circle) containing $(\rho, \dot{\rho})$ over which the piecewise–differentiable Poincaré (or first–return) map $P : V \to S$ is defined. In this example: the set $\Omega = \{b, r\}$ indexes the colored (blue,red) region $R_\omega \subset V$ where the selection function P_ω is active for $\omega \in \Omega$; $B \subset S$ is the unit ball centered at $(\rho, \dot{\rho})$; dotted ellipses indicate unit balls transformed by the respective selection functions; and arrows indicate the principle axes of these ellipses. (*left*) P is exponentially unstable since the eigenvector ν_1 of DP_r has an eigenvalue $\lambda > 1$ and $\nu_1 \in R_r$. (*right*) The given instability check from Sec. 6.2 is inconclusive since there is no $\omega \in \Omega$ and eigenvector ν of DP_ω, for which $\nu \in R_\omega$ and $\|DP_\omega(\rho, \dot{\rho})\nu\| > 1$. Although the eigenvector v_2 of DP_b has eigenvalue $\lambda > 1$, v_2 is not active, i.e. $v_2 \notin R_b$.

In contrast to the preceding discussion, suppose now that $\widetilde{\phi}^{((\rho,\dot{\rho}),\mu)}$ undergoes simultaneous constraint activations in the time interval $(0, t) \subset [0, \infty)$. In this case $\widetilde{\phi}$ will not be C^1 at $(t, (\rho, \dot{\rho}), \mu)$, so the classical test for controllability is not applicable. If all constraint activations and deactivations are admissible for $\widetilde{\phi}^{((\rho,\dot{\rho}),\mu)}$ and nearby trajectories, then Thm. 1 (piecewise differentiability across contact mode sequences) implies that $\widetilde{\phi}$ is PC^r at $(t, (\rho, \dot{\rho}), \mu)$ and hence possesses a B–derivative $D\widetilde{\phi}(t, (\rho, \dot{\rho}), \mu)$, that is, a continuous and piecewise–linear first–order approximation. By analogy with the classical test [21, Thm. 8], a variant of the Implicit Function Theorem applicable to PC^r functions [33, Thm. 4.2.3] can be used to derive a sufficient condition for small–time local controllability along $\widetilde{\phi}^{((\rho,\dot{\rho}),\mu)}$: if the piecewise–linear function that transforms first–order variations in (an appropriately–chosen subspace of) input parameters u into the resulting first–order variations in the state $(q, \dot{q})$ at time t is a (piecewise–linear) homeomorphism, then (28) is (small–time) locally controllable along $\widetilde{\phi}^{((\rho,\dot{\rho}),\mu)}$.

7. REFERENCES

[1] M. A. Aizerman and F. R. Gantmacher. Determination of stability by linear approximation of a periodic solution of a system of differential equations with discontinuous right–hand sides. *The Quarterly Journal of Mechanics and Applied Mathematics*, 11(4):385–398, 1958.

[21] It will be useful in what follows to note that this invertibility condition is equivalent to the existence of a linear homeomorphism relating variations in (an appropriately–chosen subspace of) input parameters to variations in system states.

[2] P. Ballard. The dynamics of discrete mechanical systems with perfect unilateral constraints. *Archive for Rational Mechanics and Analysis*, 154(3):199–274, 2000.

[3] M. S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):475–482, 1998.

[4] S. A. Burden. *A Hybrid Dynamical Systems Theory for Legged Locomotion*. PhD thesis, EECS Department, University of California, Berkeley, 2014.

[5] S. A. Burden, H. Gonzalez, R. Vasudevan, R. Bajcsy, and S. S. Sastry. Metrization and Simulation of Controlled Hybrid Systems. *IEEE Transactions on Automatic Control*, 60(9):2307–2320, 2015.

[6] S. A. Burden, S. Revzen, and S. S. Sastry. Model reduction near periodic orbits of hybrid dynamical systems. *IEEE Transactions on Automatic Control*, 60(10):2626–2639, 2015.

[7] S. A. Burden, S. S. Sastry, D. E. Koditschek, and S. Revzen. Event–selected vector field discontinuities yield piecewise–differentiable flows. *SIAM Journal on Applied Dynamical Systems*, 15(2):1227–1267, 2016.

[8] M. Di Bernardo, C. J. Budd, P. Kowalczyk, and A. R. Champneys. *Piecewise–smooth dynamical systems: theory and applications*. Springer, 2008.

[9] A. F. Filippov. *Differential equations with discontinuous righthand sides*. Springer, 1988.

[10] R. Goebel, J. Hespanha, A. R. Teel, C. Cai, and R. Sanfelice. Hybrid systems: Generalized solutions and robust stability. In *IFAC Symposium On Nonliear Control Systems*, pages 1–12, 2004.

[11] J. W. Grizzle, G. Abba, and F. Plestan. Asymptotically stable walking for biped robots: Analysis via systems with impulse effects. *IEEE Transactions on Automatic Control*, 46(1):51–64, 2002.

[12] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. Springer, 1983.

[13] N. Hinrichs, M. Oestreich, and K. Popp. Dynamics of oscillators with impact and friction. *Chaos, Solitons & Fractals*, 8(4):535–558, 1997.

[14] I. A. Hiskens and M. A. Pai. Trajectory sensitivity analysis of hybrid systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 47(2):204–220, 2000.

[15] Y. Hürmüzlü and D. B. Marghitu. Rigid body collisions of planar kinematic chains with multiple contact points. *The International Journal of Robotics Research*, 13(1):82–92, 1994.

[16] A. M. Johnson, S. A. Burden, and D. E. Koditschek. A hybrid systems model for simple manipulation and self–manipulation systems. *The International Journal of Robotics Research*, 35(11):1289–1327, 2016.

[17] K. C. Kiwiel. *Methods of descent for nondifferentiable optimization*. Springer, 1985.

[18] S. Kuindersma, R. Deits, M. Fallon, A. Valenzuela, H. Dai, F. Permenter, T. Koolen, P. Marion, and R. Tedrake. Optimization–based locomotion planning, estimation, and control design for the atlas humanoid robot. *Autonomous Robots*, pages 1–27, 2015.

[19] V. Kumar, E. Todorov, and S. Levine. Optimal control with learned local models: Application to dexterous manipulation. In *IEEE International Conference on Robotics and Automation (ICRA)*, pages 378–383, May 2016.

[20] J. M. Lee. *Introduction to smooth manifolds*. Springer–Verlag, 2012.

[21] A. U. Levin and K. S. Narendra. Control of nonlinear dynamical systems using neural networks: controllability and stabilization. *IEEE Transactions on Neural Networks*, 4(2):192–206, 1993.

[22] S. Levine, C. Finn, T. Darrell, and P. Abbeel. End–to–end training of deep visuomotor policies. *Journal of Machine Learning Research*, 17(1):1334–1373, 2016.

[23] H. Lin and P. J. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 54(2):308–322, 2009.

[24] A. Long, T. Murphey, and K. Lynch. Optimal motion planning for a class of hybrid dynamical systems with impacts. In *IEEE International Conference on Robotics and Automation*, pages 4220–4226, 2011.

[25] J. Lygeros, K. H. Johansson, S. N. Simic, J. Zhang, and S. S. Sastry. Dynamical properties of hybrid automata. *IEEE Transactions on Automatic Control*, 48(1):2–17, Jan 2003.

[26] K. D. Mombaur, R. W. Longman, H. G. Bock, and J. P. Schlöder. Open–loop stable running. *Robotica*, 23(1):21–33, 2005.

[27] Y. Or and A. D. Ames. Stability and completion of Zeno equilibria in Lagrangian hybrid systems. *IEEE Transactions on Automatic Control*, 56(6):1322–1336, 2011.

[28] C. D. Remy, K. Buffinton, and R. Siegwart. Stability analysis of passive dynamic walking of quadrupeds. *The International Journal of Robotics Research*, 29(9):1173–1185, 2010.

[29] M. Rijnen, A. Saccon, and H. Nijmeijer. On optimal trajectory tracking for mechanical systems with unilateral constraints. In *IEEE Conference on Decision and Control*, pages 2561–2566, 2015.

[30] S. M. Robinson. Local structure of feasible sets in nonlinear programming, part iii: Stability and sensitivity. In *Nonlinear Analysis and Optimization*, volume 30 of *Mathematical Programming Studies*, pages 45–66. Springer Berlin Heidelberg, 1987.

[31] R. Rockafellar. A property of piecewise smooth functions. *Computational Optimization and Applications*, 25(1/3):247–250, 2003.

[32] M. Schatzman. Uniqueness and continuous dependence on data for one–dimensional impact problems. *Mathematical and Computer Modelling*, 28(4—8):1–18, 1998.

[33] S. Scholtes. *Introduction to piecewise differentiable equations*. Springer–Verlag, 2012.

[34] D. E. Stewart. Rigid–body dynamics with friction and impact. *SIAM Review*, 42(1):3–39, 2000.

[35] H. J. Sussmann. A General Theorem on Local Controllability. *SIAM Journal on Control and Optimization*, 25:158–194, 1987.

[36] E. D. Wendel and A. D. Ames. Rank deficiency and superstability of hybrid systems. *Nonlinear Analysis: Hybrid Systems*, 6(2):787–805, 2012.