# Assessing stability and controllability of multi–legged gaits

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## 1 Motivation and State–of–the–Art

Parsimonious dynamical models for legged locomotion are piecewise–defined. The state flows in continuous–time according to an ordinary differential equation (ODE) until a *touchdown* or *liftoff* event occurs, triggering an instantaneous reset at a discrete time instant [5]. We've shown that models for periodic gaits with footfalls isolated in time (e.g. bipedal *walk* or *run*, quadrupedal *walk* or *gallop*) reduce to classical dynamical systems—smooth ODEs on smooth manifolds [3]. However, footfalls are not isolated for general behaviors.

During other typical gaits (*pace*, *trot*, *hop*, *pronk*, *alternating tripod*), footfalls occur simultaneously. In recent work [2], we developed analytical and computational tools that can accommodate arbitrary event times in nonsmooth dynamical systems whose states evolve continuously in time. However, as discussed in detail in other recent work [5], legged locomotors encounter both nonsmooth and discontinuous transitions during footfalls. In the present work, we are working to extend the analytical and computational tools in [2] to apply to the dynamics of multi–legged locomotion in [5].

## 2 Our Approach and Results

We represent a given locomotor's gait as a periodic orbit in a hybrid dynamical system [3, 5]. To assess stability and controllability of the gait, it is common to employ the Poincaré map associated with the orbit. Specifically, standard computational techniques for assessing stability and controllability of the gait utilize the (Fréchet or Jacobian) derivative of the Poincaré map: eigenvalue properties determine (exponential) stability of the orbit, and rank properties determine (local) controllability. If footfalls occur at isolated instants in time along the gait's periodic orbit, the associated Poincaré map is smooth [3, §III-B] and these techniques apply; if multiple footfalls occur simultaneously, the Poincaré map is generally nonsmooth [2, §4.2] and hence these techniques are inapplicable. We provide new computational techniques for assessing stability and controllability applicable in the presence of simultaneous footfalls.

Although the Poincaré map $P : S \to \Sigma$ associated with a multi–legged gait like a trot is generally not classically differentiable, it is nevertheless piecewise–differentiable in a sense discussed in detail in [2, §4.2]. This implies that, although there does not exist a single linear map (i.e. a matrix) that provides a first–order approximation for the behavior of the map near its fixed point, there does exist a *piecewise–linear* map (i.e. a collection of matrices) that provides a first–order approximation. This collection of matrices can be computed by augmenting the classical variational equation with discontinuous updates[1] that occur at touchdown and liftoff events; this idea was originally introduced for the case of isolated events in [1], conjectured to extend to two simultaneous events in [4], and rigorously proven to apply in the presence of an arbitrary number of simultaneous events in [2]. We report here on how to apply of this computational procedure to assess stability and controllability of multi–legged gaits.

For concreteness, Fig. 2 illustrates a trot gait for the model represented in Fig. 1. Given a gait with known footfalls and liftoffs like this trot, we construct updates to the variational equation associated with each simultaneous touchdown or liftoff event. Each update corresponds to a possible sequence $\omega \in \Omega$ of limb touchdown or liftoff transitions, and contributes one "piece" $DP_\omega(\alpha)$ to the piecewise–linear first–order approximation to $P$ at its fixed point $\alpha$. To assess *stability* of the gait, we can employ [2, Prop 14] by finding a norm with respect to which the induced norm of each $DP_\omega(\alpha)$ is a contraction.[2] To assess *instability* of the gait, we can employ [4, Prop. 4] by finding a footfall sequence $\omega \in \Omega$ such that $DP_\omega(\alpha)$ has an unstable eigenvector in its "piece" of the tangent space $T_\alpha\Sigma$. To assess *controllability* of the gait, we can employ an implicit function Theorem [6, Corollary 20] using the derivatives of each $P_\omega$ with respect to a finitely–parameterized family of control inputs.

We will present concrete results applying the stability, instability, and controllability tests described above to the saggital–plane quadruped model illustrated in Fig. 1, and discuss how to generalize these tests and apply them to other locomotors and gaits.

### Notes and Support

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[1] provided by so–called *saltation matrices* [1]

[2] In the case where footfalls occur at isolated instants in time, $\Omega = \{\omega\}$, i.e. $P$ is differentiable at $\alpha$, hence this stability criterion is equivalent to the commonly–employed eigenvalue test.

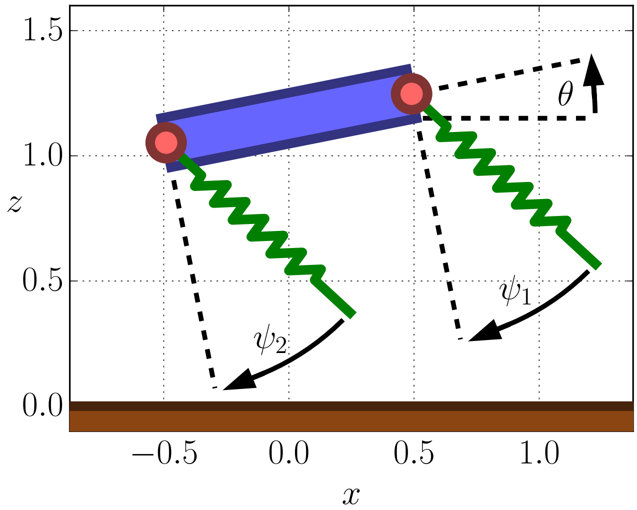


Figure 1: Schematic of saggital–plane quadruped model with three mechanical degrees–of–freedom $(x, y, \theta)$; for clarity, only two limbs shown.

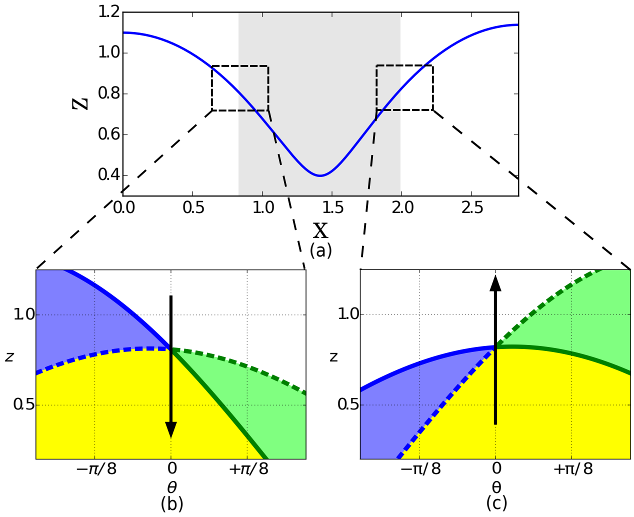


Figure 2: Illustration of periodic trot gait for the model in Fig. 1 undergoing simultaneous limb touchdown and liftoff transitions. (a) center–of–mass trajectory in horizontal ($x$) and vertical ($z$) position coordinates; gray region indicates *stance* interval when both feet are on the ground. (b,c) show portion of the trajectory (indicated with a vertical arrow) near touchdown (b) and liftof (c) transitions in height ($z$) and pitch ($\theta$) coordinates; colored regions indicate where different subsets of feet are on the ground.


**References**

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