Reduction and Identification for Hybrid Dynamical Models of Terrestrial Locomotion

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ABSTRACT

The study of terrestrial locomotion has compelling applications ranging from design of legged robots to development of novel prosthetic devices. From a first-principles perspective, the dynamics of legged locomotion seem overwhelmingly complex as nonlinear rigid body dynamics couple to a granular substrate through viscoelastic limbs. However, a surfeit of empirical data demonstrates that animals use a small fraction of their available degrees-of-freedom during locomotion on regular terrain, suggesting that a reduced–order model can accurately describe the dynamical variation observed during steady–state locomotion. Exploiting this emergent phenomena has the potential to dramatically simplify design and control of micro–scale legged robots. We propose a paradigm for studying dynamic terrestrial locomotion using empirically–validated reduced–order models.

Keywords: model reduction, parameter identification, hybrid systems, terrestrial locomotion

1. INTRODUCTION

Careful study of biological terrestrial locomotion yields an apparent paradox: nonlinear, intermittent interaction between the neuromusculoskeletal system[1] and terrain[2] can reduce to a strikingly low–dimensional behavior.[3] The mechanisms underlying this empirical observation are partially understood: neural pattern generators driving locomotion synchronize;[4] physiology and gait exhibit symmetries;[5] muscles are recruited synergistically;[6] granular substrata exhibit solidification effects.[7] Taken together, this evidence suggests the dynamics of terrestrial locomotion can be accurately described using a mathematical model of significantly reduced order.[8]

Exploiting low–dimensional dynamics inherent in locomotion has the potential to dramatically simplify design and control of terrestrial locomotion. Specifically, given a low–dimensional mathematical model that accurately predicts the behavior of a legged robot or animal, problems of controller synthesis or performance enhancement can be tractably solved in the reduced–order model and the resulting feedback laws or design changes applied to the original physical system. However, there are two major obstacles to applying this approach to any particular locomotor: first, a reduced–order model must be developed that describes the observed low–dimensional dynamics; second, this model must be rigorously validated using empirical data from the physical system. Both challenges must be overcome in the context of the *non–linear*, *hybrid* dynamics arising from intermittent contact of the locomotor's limbs with its environment. In this paper, we propose a paradigm for extracting reduced–order models for terrestrial locomotion and subsequently identifying free parameters in the models using empirical data.

1.1 Related Work: Model Reduction

Most existing techniques for model reduction of hybrid dynamics are generalizations of techniques applicable to classical dynamical systems. Examples include feedback linearization in underactuated bipeds,[9] symmetric reduction in piecewise–Lagrangian systems,[10] and averaging of coupled oscillators in neuromechanical models.[11] Though powerful, these tools impose assumptions on the dynamics that are difficult to verify for an artibrary legged locomotor; we focus on a reduction technique that is unique to hybrid dynamical systems.[12]

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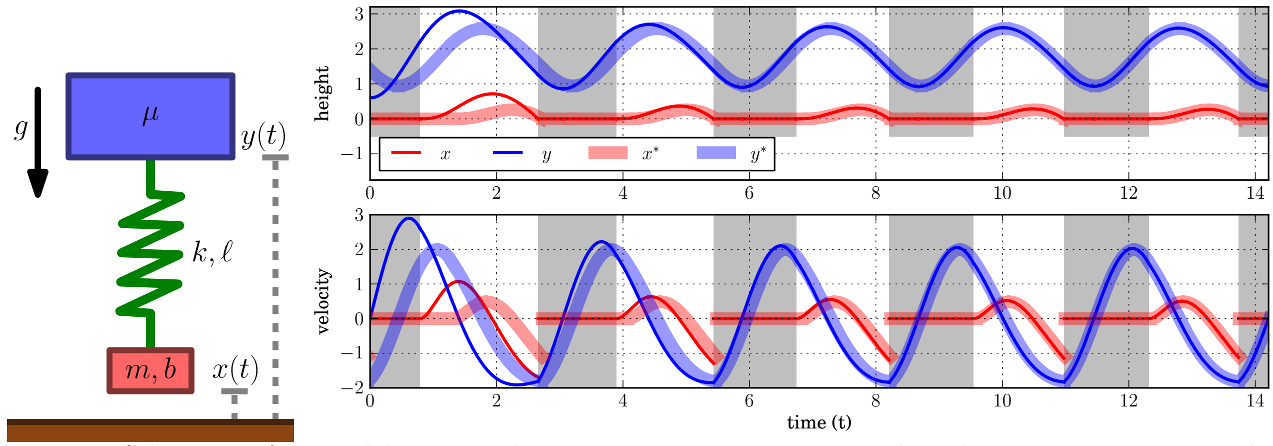


Figure 1: Schematic of vertical hopper and trajectory converging to periodic orbit. Two masses m and μ, constrained to move vertically above a ground plane in a gravitational field with magnitude g, are connected by a linear spring with stiffness k and nominal length ℓ. The lower mass experiences viscous drag proportional to velocity with constant b when it is in the air, and impacts plastically with the ground (i.e. it is not permitted to penetrate the ground and its velocity is instantaneously set to zero whenever a collision occurs). When the lower mass is in contact with the ground, the spring stiffens by a factor a. With parameters $(m, \mu, k, b, \ell, a, g) = (1, 3, 10, 5, 2, 2, 2)$, the vertical hopper possesses a stable periodic orbit $\gamma = (y^*, \dot{y}^*, x^*, \dot{x}^*)$ to which nearby trajectories $(y, \dot{y}, x, \dot{x})$ converge asymptotically. Filled gray regions indicate when the transient trace is in the ground mode, and the line styles in the velocity plot match the styles in the position plot.

1.2 Related Work: System Identification

Although there has been significant recent interest in system identification for hybrid systems, the majority of the work focuses on classes of systems that do not naturally model terrestrial locomotion. Techniques have been developed for systems possessing stochastic discrete transitions,[13] piecewise-linear dynamics;[14] or controlled discrete transitions.[15] We pursue a technique applicable to the reduced–order dynamics that emerge in models of legged locomotion.[16]

1.3 Example: Vertical Hopper

The proposed framework will be illustrated throughout the paper through application to the *vertical hopper* shown in Figure 1. This hybrid mechanical system may be regarded as a stylized model of the interaction between a body and a single appendage intermittently impacting the ground.

2. MODEL REDUCTION

2.1 Preliminaries

We begin by defining the class of hybrid systems considered in this paper. Our results are most naturally cast in the framework of differential geometry and topology,[17,18] but we make efforts to provide concrete interpretations in terms of classical ordinary differential equations defined on Euclidean space. The justification for the general setting of calculus on manifolds will become apparent when we demonstrate the appearance of proper hybrid subsystems that have a natural *intrinsic* description as a submanifold but generally no *extrinsic* interpretation.

Definition 2.1. *A* continuous–time dynamical system *is a pair* (M, F) *where:*

M *is a smooth manifold with boundary* ∂M*;*

F *is a smooth vector field on* M*, i.e.* $F \in \mathfrak{T}(M)$*.*

In local coordinates on the manifold, the vector field F defines a classical ordinary differential equation (ODE).

DEFINITION 2.2. *A* discrete–time dynamical system *is a pair* (Σ, P) *where:*

Σ *is a smooth manifold;*

P *is a smooth endomorphism of* Σ*, i.e.* $P : \Sigma \to \Sigma$.

Iterating the map P determines the dynamics of the discrete–time system (Σ, P).

For our purposes, it is expedient to define hybrid dynamical systems over a finite disjoint union of connected smooth manifolds, $M = \coprod_{j \in J} M_j$, a set we endow with the natural (piecewise-defined) topology and smooth structure. We will refer to such spaces as *hybrid manifolds.* Note that the dimensions of the constituent manifolds are not required to be equal. Several differential geometric constructions have natural generalizations to such spaces; we will prepend the modifier 'hybrid' to make it clear when this generalization is being invoked. For instance, the *hybrid tangent bundle* TM is the disjoint union of the tangent bundles TM_j, the *hybrid boundary* ∂M is the disjoint union of the boundaries ∂M_j.

Let $M = \coprod_{j \in J} M_j$ and $N = \coprod_{\ell \in L} N_\ell$ be two hybrid manifolds. Note that if a map $f : M \to N$ is continuous, then for each $j \in J$ there exists $\ell \in L$ such that $f(M_j) \subset N_\ell$ and hence $f|_{M_j} : M_j \to N_\ell$. Using this observation, there is a natural way to define differentiability of continuous maps between hybrid manifolds. Namely, a map $f : M \to N$ will be called *smooth* if f is continuous and $f|_{M_j} : M_j \to N$ is smooth for each $j \in J$. A smooth map $F : M \to TM$ will be called a *hybrid vector field* if for all $x \in M$ there exists $v \in T_x M$ such that $F(x) = (x, v)$.

DEFINITION 2.3. *A* hybrid dynamical system *is a tuple* $H = (D, F, G, R)$ *where:*

$D = \coprod_{j \in J} D_j$ *is a smooth hybrid manifold;*

$F : D \to TD$ *is a smooth hybrid vector field;*

$G \subset \partial D$ *is open;*

$R : G \to D$ *is a smooth hybrid map.*

Obeying convention,[19] *we call* R *the* reset map *and* G *the* guard.

Roughly speaking, an *execution* or *trajectory* of a hybrid dynamical system is determined from an initial condition in D by following the continuous-time dynamics determined by the vector field F until the trajectory reaches the guard G, at which point the reset map R is applied to obtain a new initial condition. We will let $\phi : \mathbb{R}_{\geq 0} \times D \to D$ denote the *hybrid flow* obtained by applying this procedure from every initial condition, i.e. $\phi(t, x)$ is the point obtained by following the hybrid dynamics for $t \in \mathbb{R}_{\geq 0}$ units of time from $x \in D$.

Note that if F is tangent to G at $x \in G$, there is a possible ambiguity in determining a trajectory from x since one may either follow the flow of F on D or apply the reset map to obtain a new initial condition $y = R(x)$.

ASSUMPTION 1. *To ensure that trajectories are uniquely defined, we assume that* F *is outward–pointing on* G.

We are particularly interested in a *periodic orbit* γ of a hybrid dynamical system, which is a nonequilibrium trajectory that intersects itself. Since the hybrid dynamics are deterministic, this implies there exists $T < \infty$ such that for any $t \in \mathbb{R}_{\geq 0}$ and $x \in \gamma$ we have $\phi(t + T, x) = \phi(t, x)$. To study the hybrid dynamics near such an orbit, we require that γ is not a *Zeno* execution,[19] i.e. that it does not undergo infinitely many discrete transitions in finite time.

ASSUMPTION 2. *We assume that every periodic orbit* γ *undergoes only finitely many discrete transitions.*

This assumption is justified in models of dynamic multi–legged locomotion, where limbs are generally compliant and hence impact plastically with the terrain. However, it is worth noting that models of biped walking have been constructed that possess Zeno periodic orbits;[20] our analysis does not easily extend to such phenomena.

2.2 Poincaré Map

The Poincaré map is a classical tool for studying dynamics near a periodic orbit in a dynamical system. In essence, the map integrates the continuous–time flow for one cycle to yield a discrete–time map whose dynamics govern the behavior of the original system. A formal construction is given elsewhere.[12]

Definition 2.4. *Let γ be a periodic orbit of a hybrid dynamical system $H = (D, F, G, R)$ and let $\Sigma \subset D_j$ be a $(\dim D_j - 1)$–dimensional submanifold that intersects γ at a single point $\{\xi\} = \Sigma \cap \gamma$, some j. The* Poincaré map *$P : \Sigma \to \Sigma$ is defined by integrating the hybrid flow from initial conditions in Σ until the trajectory intersects Σ:*

$$\forall x \in \Sigma : \textit{if } \tau(x) = \inf\{t > 0 : \phi(t, x) \in \Sigma\} < \infty, \ \textit{then } P(x) = \phi(\tau(x), x).$$

Under Assumptions 1 & 2, this map is well–defined and smooth in a neighborhood of its fixed point $P(\xi) = \xi$.

Theorem 2.5. *(Grizzle* et al.*[21]) There exists an open set $U \subset \Sigma$ containing ξ such that the restriction $P|_U : U \to \Sigma$ is a well–defined and smooth map.*

Although P is technically only guaranteed to be defined in a neighborhood U of its fixed point by this Theorem, for notational simplicity we may regard P as a (partial) function $P : \Sigma \to \Sigma$.

A straightforward application of Sylvester's inequality[22] shows that the rank of the Poincaré map is bounded above by the minimum dimension of all hybrid domains; more precise bounds are pursued elsewhere.[23]

Corollary 1. *If a periodic orbit γ passes through domains indexed by I and $P : U \to \Sigma$ is a Poincaré map associated with γ, then $\forall x \in U : \operatorname{rank} DP(x) < \min_{i \in I} \dim D_i$.*

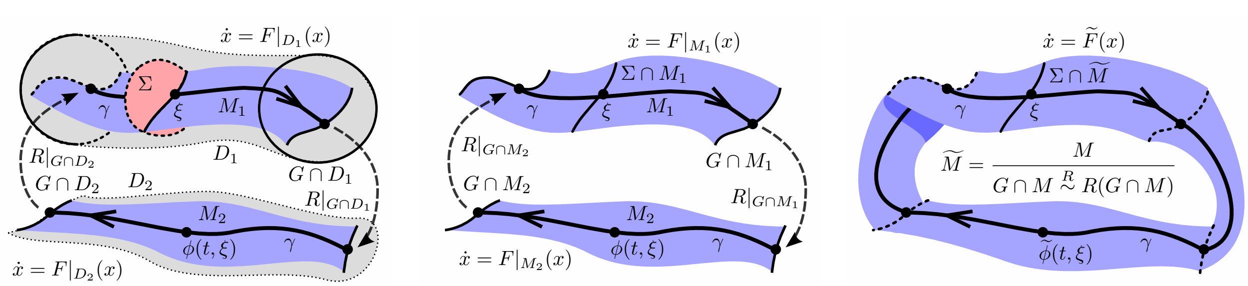
2.3 Exact Reduction to Hybrid Subsystem

When iterates of a Poincaré map associated with a periodic orbit of a hybrid dynamical system have constant rank, trajectories starting near the orbit converge in finite time to a constant-dimensional subsystem.

Theorem 2.6. *(Burden* et al.*[12]) Let γ be a periodic orbit for a hybrid dynamical system $H = (D, F, G, R)$ with Poincaré map $P : U \to \Sigma$. Suppose that $\operatorname{rank} DP^n(x) = r$ for all $x \in U$, where $n = \min_{i \in I} \dim D_i$ and I indexes all domains γ passes through. Then there is an $(r + 1)$–dimensional hybrid–invariant submanifold $M \subset D$ and a hybrid open set $W \subset D$ for which $\gamma \subset M \cap W$ and trajectories starting in W contract to M in finite time.*

Since M is invariant under both the continuous and discrete hybrid dynamics, restricting the vector field and reset map to the subsystem yields a hybrid dynamical system of the form in Definition 2.3 that satisfies Assumptions 1 & 2 and governs the behavior of the original system near the periodic orbit γ.

Corollary 2. *$H|_M = (M, F|_M, G \cap M, R|_{G \cap M})$ is a hybrid dynamical system that contains the periodic orbit γ. The periodic orbit γ is Lyapunov (resp. asymptotically, exponentially) stable in H if and only if it is Lyapunov (resp. asymptotically, exponentially) stable in $H|_M$.*



(a) $H = (D, F, G, R)$ (b) $H|_M = (M, F|_M, G \cap M, R|_{G \cap M})$ (c) $(\widetilde{M}, \widetilde{F})$

Figure 2: Illustration of Theorems 2.6 & 2.7. (a) A hybrid dynamical system $H = (D, F, G, R)$ containing a periodic orbit γ with associated Poincaré map $P : \Sigma \to \Sigma$. (b) An invariant subsystem M emerges; nearby trajectories contract to M in finite time. (c) The subsystem may be smoothed to yield a continuous-time dynamical system $(\widetilde{M}, \widetilde{F})$.

The hypothesis that DP^n is constant rank is difficult to verify in general. There is a special case in which it is straightforward to check: namely, when the rank at the fixed point ξ achieves the upper bound stipulated by Corollary 1. This is important since it is possible to approximate rank $DP^n(\xi)$ via numerical simulation.[24]

Corollary 3. *If* $\operatorname{rank} DP^n(\xi) = \min_{i\in I} \dim D_i - 1$, *then there exists an open set* $V \subset U$ *containing* ξ *such that* $\operatorname{rank} DP^n(x) = \min_{i\in I} \dim D_i - 1$ *for all* $x \in V$.

2.4 Smoothing the Reduced Hybrid Subsystem

The subsystem M yielded by Theorem 2.6 has four important properties: the constituent manifolds have the same dimension; the reset map $R|_{M\cap G}$ is a diffeomorphism between disjoint portions of the boundary ∂M; and the vector field $F|_M$ points inward along the range of the reset map $R(M \cap G) \subset \partial M$. Given any hybrid system satisfying these properties, we can globally *smooth* the hybrid transitions using tools from differential topology[18] to obtain a single continuous-time dynamical system; details are provided elsewhere.[12] Executions of the hybrid subsystem are preserved as integral curves of the continuous–time system.

Theorem 2.7. *(Burden* et al.[12]*) Let* $H = (M, F, G, R)$ *be a hybrid dynamical system with* $M = \coprod_{j\in J} M_j$. *Suppose* $\dim M_j = n$ *for all* $j \in J$, $R(G) \subset \partial M$, $\partial M = G \coprod R(G)$, R *is a diffeomorphism, and* F *is inward-pointing along* $R(G)$. *Then the topological quotient* $\widetilde{M} = \frac{M}{G \overset{R}{\sim} R(G)}$ *has a smooth manifold structure such that:*

1. *the quotient projection* $\pi : M \to \widetilde{M}$ *restricts to a smooth embedding* $\pi|_{M_j} : M_j \hookrightarrow \widetilde{M}$ *for each* $j \in J$;
2. *there is a smooth vector field* $\widetilde{F} \in \mathfrak{T}(\widetilde{M})$ *such that any execution of* H *is mapped by* $\pi : M \to \widetilde{M}$ *to yield an integral curve of* $\widetilde{F}$:
$$\forall t \in T : \frac{\partial}{\partial t}\pi \circ \phi(t, x) = \widetilde{F}\left(\pi \circ \phi(t, x)\right).$$

2.5 Example: Reduction in the Vertical Hopper

In this section, we apply Theorem 2.6 to the *vertical hopper* example shown in Fig. 1. This system evolves through an *aerial* mode and a *ground* mode. The aerial mode D_a consists of the set of configurations where the lower mass is above the ground (see Fig. 1 and its caption for notation),

$$(y, \dot{y}, x, \dot{x}) \in D_a = T\mathbb{R} \times T\mathbb{R}_{\geq 0}.$$

The dynamics are governed by Newton's laws,

$$F|_{D_a} = \begin{cases} \mu\ddot{y} & = k(\ell - (y - x)) - \mu g, \\ m\ddot{x} & = -k(\ell - (y - x)) - b\dot{x} - mg. \end{cases}$$

The boundary ∂D_a contains the states where the lower mass has just impacted the ground,

$$\partial D_a = \{(y, \dot{y}, x, \dot{x}) \in D_a : x = 0\}.$$

A hybrid transition occurs on the subset of the boundary $G_a \subset \partial D_a$ where the lower mass has negative velocity,

$$G_a = \{(y, \dot{y}, 0, \dot{x}) \in \partial D_a : \dot{x} < 0\}.$$

In this case, the state is reinitialized in the ground mode by annihilating the velocity of the lower mass,

$$R|_{G_a} : G_a \to D_g,\ R|_{G_a}(y, \dot{y}, 0, \dot{x}) = (y, \dot{y}).$$

In the ground mode, the lower mass is pressed into the ground but has no dynamics, and the boundary consists of the set of configurations where the forces acting on this mass balance,

$$\begin{aligned} D_g &= \{(y, \dot{y}) \in T\mathbb{R} : -k(\ell - y) \leq mg\}, \\ \partial D_g &= \{(y, \dot{y}) \in D_g : -k(\ell - y) = mg\}, \\ F|_{D_g} &= \{\ \mu\ddot{y} = ak(\ell - y) - \mu g. \end{aligned}$$

A hybrid transition occurs when the forces balance and will instantaneously increase to pull the mass off the ground,

$$G_g = \left\{ (y, \dot{y}) \in \partial D_g : \frac{\partial}{\partial t} y(t) > 0 \right\},$$

and the state is reset in the aerial mode by initializing the position and velocity of the lower mass to zero,

$$R|_{G_g} : G_g \to D_a, \; R|_{G_g}(y, \dot{y}) = (y, \dot{y}, 0, 0).$$

This defines a hybrid dynamical system (D, F, G, R) where

$$D = D_a \coprod D_g, \; F \in \mathcal{T}(D), \; G = G_a \coprod G_g, \; R : G \to D.$$

Choosing a Poincaré section Σ in the ground domain D_g at mid-stance, $\Sigma := \{(y, \dot{y}) : \dot{y} = 0\} \subset D_g$, we find numerically* that the hopper possesses a stable periodic orbit γ that intersects the Poincaré section at $\gamma \cap \Sigma = \{\xi\}$ where $\xi = (y, \dot{y}) \approx (0.94, 0)$. The linearization DP of the associated scalar–valued Poincaré map $P : \Sigma \to \Sigma$ has eigenvalue $\operatorname{spec} DP(\xi) \approx 0.57$ at the fixed point $P(\xi) = \xi$. The rank of the Poincaré map P attains the upper bound of Corollary 1, hence Corollary 3 implies the rank hypothesis of Theorem 2.6 is satisfied. Thus the dynamics of the hopper collapse to a one degree–of–freedom mechanical system after a single hop. One interpretation of this finding is that the unilateral (Lagrangian) constraint that arises from the impact of the lower mass with the ground persists when the system returns to the aerial mode.

3. SYSTEM IDENTIFICATION

If each component of the hybrid dynamical system $H = (D, F, G, R)$ depends smoothly on a parameter $\theta \in \Theta$ where Θ is a smooth manifold without boundary, then the parameters may be appended to the continuous state to obtain the hybrid dynamical system $H_\Theta = (D \times \Theta, F_\Theta, G \times \Theta, R_\Theta)$ where

$$F_\Theta = (F, 0_\Theta) \in \mathcal{T}(D \times \Theta), \; R_\Theta = (R, \mathrm{id}_\Theta) : G \times \Theta \to D \times \Theta;$$

here, $0_\Theta \in \mathcal{T}(\Theta)$ denotes the zero vector field and $\mathrm{id}_\Theta : \Theta \to \Theta$ the identity map on Θ. In the sequel we will suppress parametric dependence and refer to the continuous state of a hybrid system alternately as an *initial condition* or *parameter*. Note that if $\gamma \subset D$ is a periodic orbit for H with parameter $\theta \in \Theta$, then $\gamma \times \{\theta\} \subset D_\Theta$ is a periodic orbit for H_Θ, hence the model reduction and smoothing results developed in the previous section generalize to parameterized hybrid systems.

3.1 Problem Formulation

Given a hybrid system $H = (D, F, G, R)$, recall that $\phi(t, x) \in D$ denotes the point on the trajectory for the system at time t from initial condition $x \in D$. Then given an observation function $Y : D \to \mathbb{R}^m$ and data $\{\eta_k\}_{k=1}^N \subset \mathbb{R}^m$ with sampling period $\tau \in \mathbb{R}$, define the *mean square prediction error* $\varepsilon : D \to \mathbb{R}$ by

$$\varepsilon(x) = \frac{1}{N} \sum_{k=1}^{N} \| Y(\phi(k\tau, x)) - \eta_k \|^2 \tag{1}$$

and pose the parameter identification problem in the framework of prediction error minimization[25] as

Problem 1. *(Parameter Identification on D)*

$$\widehat{x} = \arg\min_{x \in D} \varepsilon(x). \tag{2}$$

Note that the prediction error ε can be discontinuous in x due to discontinuities in the hybrid flow ϕ during discrete transitions, whence in general one must resort to global optimization techniques to solve Problem 1. However, even smooth observations do not generally lead to a scalable computational approach to Problem 1, leading us to consider a tractable reformulation of the problem.[16]

*For numerical simulations, we use a recently–developed algorithm[24] with step size $h = 1 \times 10^{-2}$ and relaxation parameter $\varepsilon = 1 \times 10^{-10}$. The sourcecode for simulations in this paper is available online at `http://purl.org/sburden/spie2013`.

3.2 Restriction to Smoothed Hybrid Subsystem

Given a hybrid system $H = (D, F, G, R)$ with periodic orbit γ whose Poincaré maps satisfy the hypotheses of Theorem 2.6, we obtain a reduced-order subsystem $H|_M = (M, F|_M, G \cap M, R|_{G\cap M})$. The corresponding smoothed subsystem $(\widetilde{M}, \widetilde{F})$ yielded by Theorem 2.7 inherits the parametric dependence of the original hybrid system, thus the identification problem may be posed on this subsystem. To make identification tractable, we require observations to be smooth functions of time. This is reasonable in examples relevant to the study of terrestrial locomotion, since the observed states (e.g. center–of–mass motion of the body) are affected by hybrid transitions only indirectly through the change in the vector field.

Assumption 3. *For each $x \in D$, the observation $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ defined by $y(t) = Y(\phi(t, x))$ is smooth.*

Let $\widetilde{\phi}(t, z) \in \widetilde{M}$ denote the point on the trajectory for the smoothed subsystem at time t from initial condition $z \in \widetilde{M}$. Note that under Assumption 3, there exists a unique smooth observation function $\widetilde{Y} : \widetilde{M} \to \mathbb{R}^m$ satisfying $\forall x \in M : Y(x) = \widetilde{Y}(\pi(x))$ where $\pi : M \to \widetilde{M}$ is the canonical quotient projection—the discontinuities present in the hybrid execution do not appear in the observations. We use this observation function to form the mean square prediction error confined to the smoothed subsystem $\widetilde{\varepsilon} : \widetilde{M} \to \mathbb{R}$ as

$$\widetilde{\varepsilon}(z) = \frac{1}{N} \sum_{k=1}^{N} \left\| \widetilde{Y}(\widetilde{\phi}(k\tau, z)) - \eta_k \right\|^2 \tag{3}$$

and pose the parameter identification problem as

Problem 2. *(Parameter Identification on $\widetilde{M}$)*

$$\widehat{z} = \arg \min_{z \in \widetilde{M}} \widetilde{\varepsilon}(z). \tag{4}$$

Now local optima for Problem 2 can in principle be approximated using any first-order method applicable on the smooth manifold $\widetilde{M}$.[26] However, practical implementation of such an algorithm would require an explicit coordinate representation for the hybrid subsystem M, which in turn necessitates analytical solution of nonlinear ordinary differential equations. Since it is not generally feasible to solve the ODEs to obtain these coordinates, we pursue a technique to circumvent this requirement.

3.3 Smooth Covering via Poincaré Map

Although it is not generally possible to obtain an explicit representation for the reduced-order subsystem generated by Theorem 2.6, the Poincaré map construction can be used to obtain a computational surrogate. Specifically, the Theorem shows that every point $z \in \widetilde{M}$ can be reached by flowing forward in time from some point $u \in \Sigma$ on the Poincaré section for some amount of time $t \in \mathbb{R}_{\geq 0}$, i.e. $z = \pi \circ \phi(t, u)$. Moreover, so long as t is large enough to ensure that the trajectory undergoes n cycles as it passes from $u \in \Sigma$ to $z \in \widetilde{M}$, the map $\pi \circ \phi$ is smooth and constant–rank, i.e. it is a smooth covering map.[17]

Now to differentiate the prediction error (4) with respect to $z \in \widetilde{M}$, we find $(t, u) \in \mathbb{R}_{\geq 0} \times \Sigma$ such that $z = \pi \circ \phi(t, u)$ and then compute derivatives in coordinates on $\mathbb{R}_{\geq 0} \times \Sigma$. It is generally possible to choose Σ locally as a subspace in a coordinate chart on a hybrid domain, so $\mathbb{R}_{\geq 0} \times \Sigma$ may be regarded as a vector space and hence these derivatives may be computed as in classical calculus. This is a significant advancement beyond previous work[16] where we required the Poincaré map to be a diffeomorphism.

3.4 Scalable Algorithm for Parameter Identification

We use a (first–order) steepest descent algorithm with step sizes chosen using the Armijo rule[27] and the error gradient approximated using finite differences to solve Problem 2; details are provided in Algorithm 1.

3.5 Example: Identification of Initial Conditions for Vertical Hopper

Figure 3 demonstrates the results of applying Algorithm 1 to identify the initial condition for the vertical hopper model using noisy observations of the upper mass.

Algorithm 1: Parameter Identification on $\widetilde{M}$

Given:

Hybrid dynamical system $H = (D, F, G, R)$ with periodic orbit γ having period $T \in \mathbb{R}$;
Poincaré map $P : \Sigma \to \Sigma$ for γ with fixed point $P(\xi) = \xi$ satisfying hypotheses of Theorem 2.6;
Observation function $Y : D \to \mathbb{R}^m$ and observation data $\{\eta_i\}_{i=1}^{N} \subset \mathbb{R}^m$ with sampling period $\tau \in \mathbb{R}$;
Armijo parameters $\alpha, \beta \in (0, 1)$, and termination tolerance σ;

Initialize:

$n = \dim \Sigma$; $U = \operatorname{span} DP^n(\xi)$;
Define $\epsilon : \mathbb{R}_{\geq 0} \times U \to \mathbb{R}$ as $\epsilon(t, u) = \widetilde{\varepsilon}(\phi(t + nT, u))$ where $\widetilde{\varepsilon}$ is given in (4);
$k = 0$; $(t_0, u_0) = (0, \xi)$; $d_0 = -\nabla\epsilon(t_0, u_0)$;

Compute:

while $\|d_k\| > \sigma$ **do**
  $\ell = \min\left\{ i \in \mathbb{N} : \epsilon(t_0, u_0) - \epsilon\left((t_0, u_0) + \beta^i d_k\right) \geq \alpha\beta^i \|d_k\|^2 \right\}$;
  $(t_{k+1}, u_{k+1}) = (t_k, u_k) + \beta^\ell d_k$;
  $d_{k+1} = -\nabla\epsilon(t_{k+1}, u_{k+1})$;
  $k = k + 1$;
end

Return:

$\widehat{z} = \phi(t_k + nT, u_k)$;

4. CONCLUSION

We presented a unified framework for model reduction and system identification in a class of hybrid dynamical models of terrestrial locomotion. Although the technique is agnostic to the phenomenological origin of the locomotor's dynamics, the present work is applicable only to the dynamics near a periodic orbit that undergoes isolated discrete transitions. In future work we seek to generalize these tools to aperiodic maneuvers that may undergo multiple simultaneous hybrid transitions.

ACKNOWLEDGMENTS

This research was supported by the Army Research Laboratory under Cooperative Agreement W911NF-08-2-0004 and by an NSF Graduate Research Fellowship to S.A.B.. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Laboratory or the U.S. Government. The U.S. Government is authorized to reproduce and distribute for Government purposes notwithstanding any copyright notation herein.

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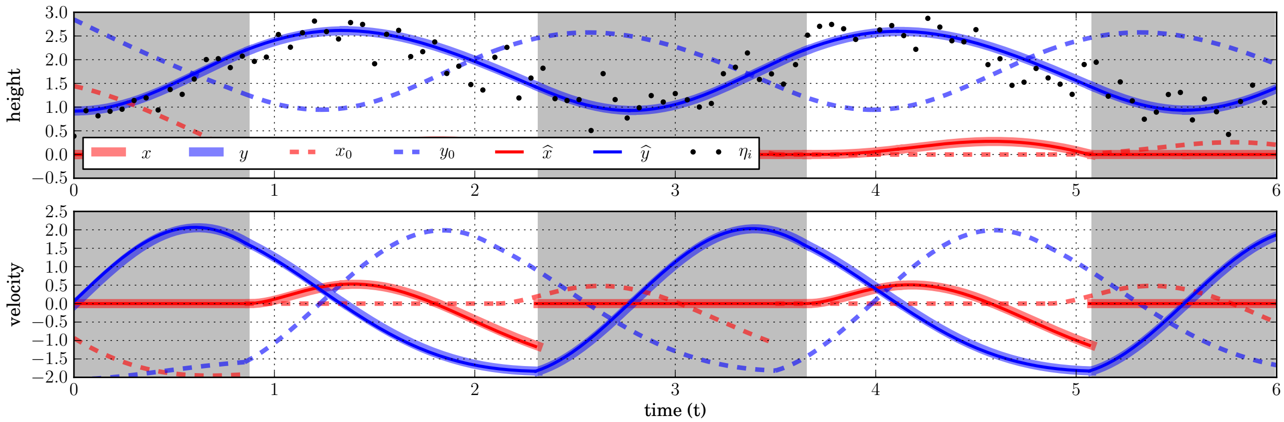

Figure 3: Identification of initial conditions for the vertical hopper. The model is simulated from the initial condition $z = \phi(t, u)$ with $(t, u) \approx (3.41, 0.14)$ and the position of the upper mass is measured at 20Hz with independent and identically distributed zero–mean Gaussian noise with variance 0.3. The initial conditions are identified by solving Problem 2 from the initial guess $z_0 = \phi(t_0, y_0)$ where $(t_0, y_0) \approx (2.29, 0.03)$ using Algorithm 1 to obtain the estimate $\widehat{z} = \phi(\widehat{t}, \widehat{y})$ where $(\widehat{t}, \widehat{y}) \approx (3.42, 0.16)$. Our method decreases the prediction error from $\widetilde{\varepsilon}(z_0) \approx 1.43$ to $\widetilde{\varepsilon}(\widehat{z}) \approx 8.91 \times 10^{-2}$ (lower than the error obtained from the correct parameters $\widetilde{\epsilon}(z) = 8.94 \times 10^{-2}$) and reproduces the unobserved velocity trace with high fidelity.

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