Parameter Identification Near Periodic Orbits of Hybrid Dynamical Systems ⋆

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Abstract: We present a novel identification framework that enables the use of first-order methods when estimating model parameters near a periodic orbit of a hybrid dynamical system. The proposed method reduces the space of initial conditions to a smooth manifold that contains the hybrid dynamics near the periodic orbit while maintaining the parametric dependence of the original hybrid model. First-order methods apply on this subsystem to minimize average prediction error, thus identifying parameters for the original hybrid system. We implement the technique and provide simulation results for a hybrid model relevant to terrestrial locomotion.

Keywords: parameter identification; hybrid systems; model reduction; periodic motion

1. INTRODUCTION

Many physical processes are fundamentally *hybrid*, such as phase transition in a chemical reaction, gene activation in a biochemical reaction network, and impact of a limb with a substrate during terrestrial locomotion. Faithful models for each of these phenomena are constructed from smooth submodels by specifying rules that switch between the submodels when certain logical propositions are satisfied (e.g. "gene is activated" or "limb impacts substrate").

The simplest non-equilibrium attractor of a dynamical system is a periodic orbit. Such orbits are important for regulating concentrations of nutrients and proteins (Atkinson et al. [2003]) and for describing the steady-state running gaits of legged robots and animals (Holmes et al. [2006]). Studying these dynamical behaviors requires identifying unknown parameters in a given hybrid dynamical model. Unfortunately, there are few identification tools applicable to the dynamics encountered in legged locomotion and biochemistry other than global search. Recently, Burden et al. [2011b] developed a method to reduce a hybrid system to a smooth dynamical system near a periodic orbit. In this paper, we apply this model reduction technique to enable the use of first-order methods to solve the parameter identification problem. In addition to reducing the computation required to identify the hybrid model, this approach overcomes technical limitations of prior work.

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2. RELATED WORK

Estimation and identification for hybrid dynamical systems has been an active area of research for several decades; refer to the surveys Tugnait [1981], Paoletti et al. [2007] and references therein. Most contributions address the identification of piecewise affine ARX models. Unfortunately, these techniques are not applicable to the nonlinear dynamics and deterministic hybrid transitions generally encountered in biochemistry and biomechanics.

This paper considers the parameter estimation problem for hybrid dynamical systems in state space form. In general, the identification of hybrid systems is a combinatorial problem: one considers all possible discrete state sequences by hypothesizing, at each observation time, that the system has evolved to any of the known discrete modes. For each discrete state sequence, continuous model parameters are estimated using classical methods. Finally, the model is selected that gives the best agreement with the observations; see for instance Cinquemani et al. [2008], Balakrishnan et al. [2004], Op Den Buijs et al. [2005], van Riel et al. [2005]. Unless additional structure is imposed—non-zero *dwell time* of the switching dynamics or *a priori* knowledge of the discrete mode of some observations—this combinatorial approach is computationally intractable, necessitating approximate methods.

One approximation to the combinatorial problem is obtained by applying a change detection algorithm to determine the discrete state, then using classical methods to estimate the continuous dynamics; see for instance Ferrari-Trecate et al. [2001], Altendorfer et al. [2001], Srinivasan and Holmes [2008], Vries et al. [2009]. An alternative approximation to the combinatorial problem that imposes no *a priori* assumptions on the discrete state transitions is *prediction error minimization*; see Ljung [1999] for details.

Starting from an initial parameter estimate, one minimizes the squared residuals between predicted and observed output by iteratively stepping in the direction opposite the prediction error gradient. This method is not directly applicable in the present context since the hybrid flow (and hence the prediction error) is generally not differentiable over the entire parameter space due to discrete transitions. The gradient of the flow can be propagated forward through discrete transitions using the methods of Hiskens and Pai [2000], Phipps et al. [2005], Hoffman et al. [2009]. However, since the maps that reset the state during a discrete transition are not invertible in general, such computations cannot be applied in backward time through a discrete transition. Therefore existing approximate methods require some information about the discrete mode to be known *a priori*. A primary aim of the present work is to overcome this fundamental technical limitation. Recently, Burden et al. [2011b] provided an analytical technique to extract a reduced-order smooth dynamical system near a periodic orbit of a hybrid dynamical system. In this paper, we use this smooth subsystem to identify parameters for the original model.

3. HYBRID DYNAMICAL SYSTEMS

3.1 Background from Differential Geometry

We assume familiarity with the tools and terminology of differential geometry. If any of the concepts we discuss are unfamiliar, we refer the reader to Marsden and Ratiu [1999], Lee [2002] for more details.

A *smooth n-dimensional manifold M with boundary ∂M* is an n-dimensional topological manifold covered by a collection of *smooth coordinate charts*, and we write $\dim M$ for the dimension of M; elementary examples include n-dimensional Euclidean space $\mathbb{R}^n$ and the n-dimensional unit sphere S^n. The topological boundary $\partial M \subset M$ is a smooth $(n-1)$-dimensional submanifold. Each $x \in M$ has an associated *tangent space* T_xM, and the disjoint union of the tangent spaces at each point comprises the *tangent bundle* $TM := \coprod_{x\in M} T_xM$; note that any element in TM may be regarded as a pair (x, v) where $x \in M$ and $v \in T_xM$. We let $\mathfrak{T}(M)$ denote the set of *smooth vector fields* on M, i.e. smooth maps $G : M \to TM$ for which $G(x) = (x, v)$ for some $v \in T_xM$ and all $x \in M$. If $f : M \to N$ is a smooth map between smooth manifolds, we write $\operatorname{dom} f = M$ and $\operatorname{im} f = N$. At each $x \in M$ we let $Df(x) : T_xM \to T_{f(x)}N$ denote the Jacobian linearization of f. The *rank* of $f : M \to N$ at the point $x \in M$ is $\operatorname{rank}_x f = \operatorname{rank} Df(x)$. If $\operatorname{rank}_x f = r$ for all $x \in M$, we write $\operatorname{rank} f = r$. If $\operatorname{rank} f = \dim M$ and f is a homeomorphism onto its image, then f is a *smooth embedding*, and $f(M)$ is a *smooth embedded submanifold.* In this case, any smooth vector field $G \in \mathfrak{T}(M)$ determines a unique $Df(G) \in \mathfrak{T}(f(M))$. A vector field $G \in \mathfrak{T}(M)$ is *tangent* to a k-dimensional submanifold S at $x \in S$ if $G(x) \in T_xS \subset T_xM$; otherwise G is *transverse.*

3.2 Smooth & Hybrid Dynamical Systems

Smooth dynamical systems are comprised of a state space together with a vector field that encodes the dynamics.

Definition 1. A *smooth dynamical system* is specified by a pair (M, G):

M is a smooth manifold with boundary ∂M;
G is a smooth vector field on M, i.e. $G \in \mathfrak{T}(M)$.

Hybrid dynamical systems are obtained from smooth systems by specifying (i) regions of the state space that trigger transitions and (ii) rules to reinitialize the state once a transition occurs. We consider the following class of hybrid systems, described in detail in Burden et al. [2011b].

Definition 2. A *hybrid dynamical system* is specified by a tuple (D, F, G, R):

$D = \coprod_{j\in J} D_j$ is a smooth hybrid manifold;
$F \in \mathfrak{T}(D)$ is a smooth hybrid vector field on D;
$G \subset \partial D$ is a smooth open hybrid submanifold;
$R : G \to D$ is a smooth hybrid map.

We call R the *reset map* and G the *guard* and assume F is transverse to G to ensure trajectories are well-defined.

3.3 Example Hybrid Dynamical System: Vertical Hopper

We illustrate the components of a hybrid system in the *vertical hopper* shown in Fig. 1. This system evolves through an *aerial* mode and a *ground* mode. The aerial mode consists of the set of configurations D_a where the lower mass is above the ground (see Fig. 1 for notation),

$$(\sigma, y, \dot{y}, x, \dot{x}) \in D_a = S^1 \times T\mathbb{R} \times T\mathbb{R}_{\geq 0}.$$

In this mode, the dynamics are governed by Newton's laws together with the linear clock dynamics $\dot{\sigma} = \omega$,

$$F|_{D_a} = \begin{cases} \dot{\sigma} &= \omega, \\ \mu\ddot{y} &= k(\ell_0 - (y - x)) + a\sin\sigma - \mu g, \\ m\ddot{x} &= -k(\ell_0 - (y - x)) - a\sin\sigma - b\dot{x} - mg. \end{cases}$$

The boundary ∂D_a contains the states where the lower mass has just impacted the ground,

$$\partial D_a = \{(\sigma, y, \dot{y}, x, \dot{x}) \in D_a : x = 0\}.$$

A hybrid transition occurs on the subset of the boundary $G_a \subset \partial D_a$ where the lower mass has negative velocity,

$$G_a = \{(\sigma, y, \dot{y}, 0, \dot{x}) \in \partial D_a : \dot{x} < 0\}.$$

In this case, the state is reinitialized in the ground mode by annihilating the velocity of the lower mass,

$$R|_{G_a} : G_a \to D_g,\ R|_{G_a}(\sigma, y, \dot{y}, 0, \dot{x}) = (\sigma, y, \dot{y}).$$

In the ground mode, the lower mass is pressed into the ground and the boundary consists of the set of configurations where the forces acting on this mass balance,

$$D_g = \left\{(\sigma, y, \dot{y}) \in S^1 \times T\mathbb{R} : -k(\ell_0 - y) - a\sin\sigma \leq mg\right\},$$
$$\partial D_g = \{(\sigma, y, \dot{y}) \in D_g : -k(\ell_0 - y) - a\sin\sigma = mg\},$$
$$F|_{D_g} = \begin{cases} \dot{\sigma} &= \omega, \\ \mu\ddot{y} &= k(\ell_0 - y) + a\sin\sigma - \mu g. \end{cases}$$

A hybrid transition occurs when the forces balance and will instantaneously increase to pull the mass off the ground,

$$G_g = \left\{(\sigma, y, \dot{y}) \in \partial D_g : \frac{\partial}{\partial t}\left(ky(t) - a\sin\sigma(t)\right) > 0\right\},$$

and the state is reset in the aerial mode by initializing the position and velocity of the lower mass to zero,

$$R|_{G_g} : G_g \to D_a,\ R|_{G_g}(\sigma, y, \dot{y}) = (\sigma, y, \dot{y}, 0, 0).$$

This defines a hybrid dynamical system (D, F, G, R) where

$$D = D_a \coprod D_g,\ F \in \mathfrak{T}(D),\ G = G_a \coprod G_g,\ R : G \to D.$$

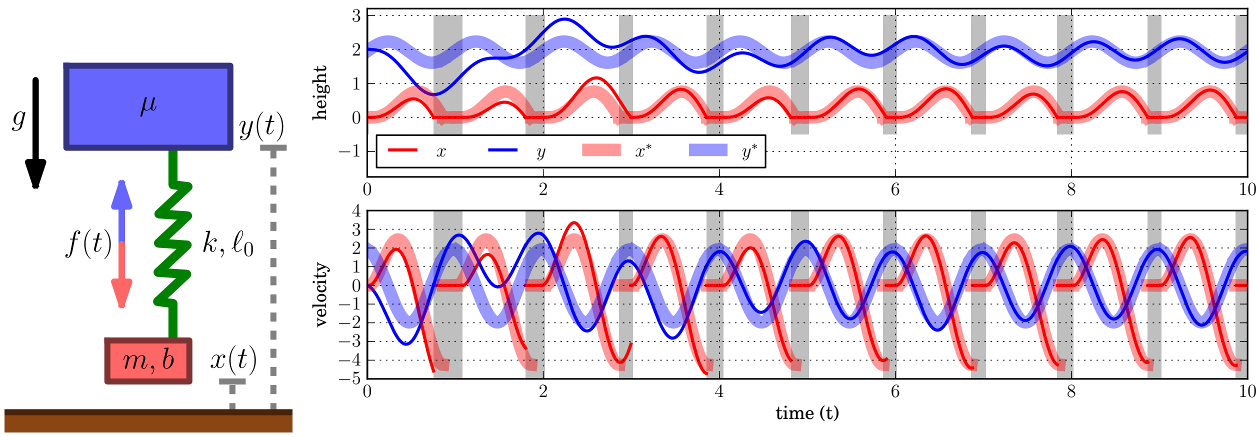

Fig. 1. Schematic of vertical hopper and trajectory converging to periodic orbit. Two masses m and μ, constrained to move vertically above a ground plane in a gravitational field with magnitude g, are connected by a linear spring with stiffness k and nominal length ℓ_0. The spring is equipped with an actuator that exerts a periodically-varying force $f(t) := a \sin \sigma(t)$ where $\sigma \in S^1$ and $\dot{\sigma} = \omega$. The lower mass experiences viscous drag proportional to velocity with constant b when it is in the air, and impacts plastically with the ground (i.e. it is not permitted to penetrate the ground and its velocity is instantaneously set to zero whenever a collision occurs). With parameters $(m, \mu, k, b, \ell_0, a, \omega, g) = (1, 2, 10, 5, 2, 2, 2\pi, 2)$, the vertical hopper possesses a stable periodic orbit $\gamma = (\sigma^*, y^*, \dot{y}^*, x^*, \dot{x}^*)$ to which nearby trajectories $(\sigma, y, \dot{y}, x, \dot{x})$ converge asymptotically. In the trajectory plot, filled vertical gray regions indicate when the transient trace is in the ground mode, and the line styles in the velocity plot match the styles for the corresponding state variables in the position plot.

4. PARAMETER IDENTIFICATION

If each component of the hybrid dynamical system (D, F, G, R) depends smoothly on a parameter $\theta \in \Theta$ where Θ is a smooth manifold, then the parameters may be appended to the continuous state to obtain the hybrid dynamical system $(D \times \Theta, F_\Theta, G \times \Theta, R_\Theta)$ where

$$F_\Theta = (F, 0_\Theta) \in \mathcal{T}(D \times \Theta),$$
$$R_\Theta = (R, \mathrm{id}_\Theta) : G \times \Theta \to D \times \Theta;$$

here, $0_\Theta \in \mathcal{T}(\Theta)$ denotes the zero vector field and $\mathrm{id}_\Theta : \Theta \to \Theta$ the identity map on Θ. In the sequel we will suppress parametric dependence and refer to the continuous state of a hybrid system alternately as an *initial condition* or *parameter*. Note that if the original hybrid system has a periodic orbit, the augmented system does as well, hence the model reduction result in Burden et al. [2011b] generalizes to parameterized hybrid systems (see also Section 4.2).

4.1 Parameter Identification

Given a hybrid system (D, F, G, R), let $\phi(t, x) \in D$ denote the point on the trajectory for the system at time t from initial condition $x \in D$. Then given an observation function $Y : D \to \mathbb{R}^m$ and data $\{\eta_i\}_{i=1}^N \subset \mathbb{R}^m$ with sampling period $\tau \in \mathbb{R}$, form the *average prediction error*

$$\varepsilon\left(x, \{\eta_i\}_{i=1}^N\right) = \frac{1}{N} \sum_{i=1}^N \|Y(\phi(i\tau, x)) - \eta_i\|^2 \qquad (1)$$

and pose the parameter identification problem as

$$x^* = \arg\min_{x \in D} \varepsilon\left(x, \{\eta_i\}_{i=1}^N\right). \qquad (2)$$

Note that the prediction error ε can be discontinuous in x due to discontinuities in the hybrid flow ϕ during discrete transitions, whence in general we must resort to global optimization techniques to solve the problem in (2). This is clear in the vertical hopper example since observations of the velocity of the lower mass $\dot{x}$ are discontinuous in time as shown in Fig. 1.

Even if ε is piecewise smooth, first-order methods may fail to converge to parameters near the transition surface $G \subset \partial D$. To see this, consider the problem of identifying an initial condition for the vertical hopper near G_a, $(\sigma^*, y^*, \dot{y}^*, x, \dot{x}^*) \in D_a$, $0 < x \ll 1$, $\dot{x} < 0$. A first-order method initialized in D_g near $R(G_a)$ will converge to a point near $R(\sigma^*, y^*, \dot{y}, 0, \dot{x}^*) \in D_g$. The reset map $R|_{G_a}$ is not invertible since $\dim \mathrm{dom}\, R|_{G_a} > \dim \mathrm{im}\, R|_{G_a}$, therefore it is not obvious how the method could be modified to step back to D_a and recover the correct parameters.

4.2 Model Reduction Near Periodic Orbits

Under a non-degeneracy condition on rank loss in the Poincaré map associated with a periodic orbit of a hybrid system, there exists a smooth system that (i) embeds in the hybrid system near the periodic orbit and (ii) attracts all nearby trajectories in finite time. We formalize these facts as follows.

Definition 3. A *hybrid dynamical embedding* of a smooth system (M, G) into a hybrid system (D, F, G, R) is a hybrid embedding $f : M \to D$ for which $Df(G) = F|_{f(M)}$ and $R|_{f(M) \cap G}$ is a hybrid diffeomorphism.

Theorem 4. (Theorem 1 in Burden et al. [2011b]). Let γ be a periodic orbit of the hybrid dynamical system (D, F, G, R) and suppose the composition of a Poincaré map for γ with itself at least $\min_{j \in J} \dim D_j$ times has constant rank equal to r on a neighborhood of its fixed point. Then there is an $(r+1)$-dimensional dynamical system (M, G), a hybrid dynamical embedding $f : M \to D$, and an open hybrid set $W \subset D$ so that $\gamma \subset f(M) \cap W$ and trajectories starting in W flow into $f(M)$ in finite time.

4.3 Example of Model Reduction: Vertical Hopper

Choosing a Poincaré section Σ in the ground domain D_g at phase $\sigma = \pi$, $\Sigma := \{(\sigma, y, \dot{y}) : \sigma = \pi\} \subset D_g$, we find numerically that the hopper possesses a stable periodic orbit γ that intersects the Poincaré section at $\gamma \cap \Sigma = \{\xi\}$ where $\xi = (y, \dot{y}) \approx (1.95, 1.91)$. The linearization DP of the associated Poincaré map $P : \Sigma \to \Sigma$ has eigenvalues $\operatorname{spec} DP(\xi) \approx -0.27 \pm 0.73j$ at the fixed point $P(\xi) = \xi$. Since neither eigenvalue is near the origin, we conclude the Poincaré map P is a local diffeomorphism, hence the rank hypothesis of Theorem 4 is trivially satisfied. Thus the dynamics of the hopper collapse to a one degree-of-freedom mechanical system after a single hop.

4.4 Parameter Identification Near Periodic Orbits

Given a hybrid system (D, F, G, R) with periodic orbit γ and associated reduced-order subsystem (M, G) with hybrid dynamical embedding $f : M \to D$, the smooth subsystem inherits the parametric dependence of the original hybrid system, thus the identification problem may be posed on this subsystem. To that end, let $\varphi(t, z) \in M$ denote the point on the trajectory for the subsystem at time t from initial condition $z \in M$. Form the average prediction error confined to the subsystem

$$\epsilon\left(z, \{\eta_i\}_{i=1}^N\right) = \frac{1}{N} \sum_{i=1}^{N} \|Y \circ f(\varphi(i\tau, z)) - \eta_i\|^2, \qquad (3)$$

and pose the parameter identification problem as

$$z^* = \arg\min_{z \in M} \epsilon\left(z, \{\eta_i\}_{i=1}^N\right). \qquad (4)$$

Now ϵ is smooth if and only if $Y \circ f : M \to \mathbb{R}^m$ is smooth. In examples relevant to the study of legged locomotion, the observed states (e.g. body center-of-mass trajectory) are affected by hybrid transitions only indirectly through the change in the vector field, hence $Y \circ f$ is smooth. In this case, (4) can in principle be solved using any first-order method applicable to the smooth manifold M, e.g. the trust-region method of Absil et al. [2007].

In general, obtaining coordinates for the smooth subsystem M requires solving the nonlinear ODE in each hybrid domain explicitly so that the invariant submanifold of the Poincaré section can be extracted. However, it is common in models of terrestrial locomotion for the center-of-mass state variables to give coordinates for this invariant submanifold near the periodic orbit.

4.5 Example of Parameter Identification: Vertical Hopper

In the vertical hopper, the invariant subsystem coordinates are comprised of the actuator state σ together with the position and velocity of the upper mass $(y, \dot{y})$. In this case, it is possible to obtain explicit coordinates for the Poincaré section $\Sigma \subset D \cap M$, and then every $z \in M$ can be obtained by following the dynamics on M from a point $\zeta \in \Sigma$ for a certain amount of time $t \in \mathbb{R}_{\geq 0}$ so that $\varphi(t, \zeta) = z$. Therefore in practice, we reformulate the problem in (4) to search over initial conditions in the Poincaré section and initial simulation time:

$$(t^*, \zeta^*) = \operatorname*{argmin}_{(t,\zeta) \in \mathbb{R}_{\geq 0} \times \Sigma} \epsilon\left(\varphi(t, \zeta), \{\eta_i\}_{i=1}^N\right). \qquad (5)$$

We implemented a steepest descent algorithm with step sizes chosen using the Armijo rule (refer to Bertsekas [1999] for details) and the error gradient approximated using finite differences to solve the problem in (5) for simulated data from the vertical hopper. In particular, we identified initial conditions and parameters from noisy observations of the height of the upper mass. The algorithm reliably converges to a local minimum of the prediction error that is near the correct parameter values; results from one particular experiment are shown in Figs. 2 & 3. Our method recovers the correct initial condition from a randomized starting point chosen near the hopper's periodic orbit using 40 noisy observations sampled over two cycles, as shown in Fig. 2; accurately estimating model parameters requires data from significantly more cycles, as shown in Fig. 3.

5. DISCUSSION

We presented a first-order method for the identification of parameters for a class of hybrid dynamical models. By formally reducing the hybrid model to a smooth subsystem that arises naturally near a hybrid periodic orbit, the proposed method removes the requirement of prior work that information about the the discrete state be known *a priori*. Though applicable to the study of oscillatory behavior in chemistry, biology, and robotics, the proposed technique has some drawbacks; we conclude by discussing these issues and future research efforts to address them.

First, the proposed method is only applicable in a neighborhood of a periodic orbit. In practice, many models of biochemical reaction networks and terrestrial locomotion are deliberately designed to possess periodic orbits, and the neighborhood over which the model reduction results apply can be large; we plan to develop analytical tools to discover periodic orbits and establish the size of the reduction neighborhood. If multiple periodic orbits exist, the technique presented here is only applicable near each orbit individually; the practitioner must select amongst the available orbits before identifying parameters.

Second, to enable the use of first-order methods, it was necessary to restrict the initial conditions to the smooth invariant subsystem M; the technique presented here cannot identify initial conditions off this submanifold. However, the model reduction result of Theorem 4 guarantees that all initial conditions near the periodic orbit will collapse to M after a finite number of cycles, n. Therefore removing the first n cycles of data guarantees that the observations are generated from an initial condition on the subsystem, justifying the restriction to M. We note that it is possible for parameters in the original system to become unidentifiable under the restriction to M; we are seeking conditions that ensure identifiability is preserved.

Third, as noted in Section 4.4, we cannot expect in general to obtain analytical coordinates for the invariant subsystem whose existence is guaranteed by Theorem 4. Therefore applying our identification procedure to a general problem will require a numerical approximation of the coordinates for the subsystem. Finally, for our technique to accommodate process noise in addition to measurement error, the model reduction result of Burden et al. [2011b] must be generalized to stochastic hybrid systems.

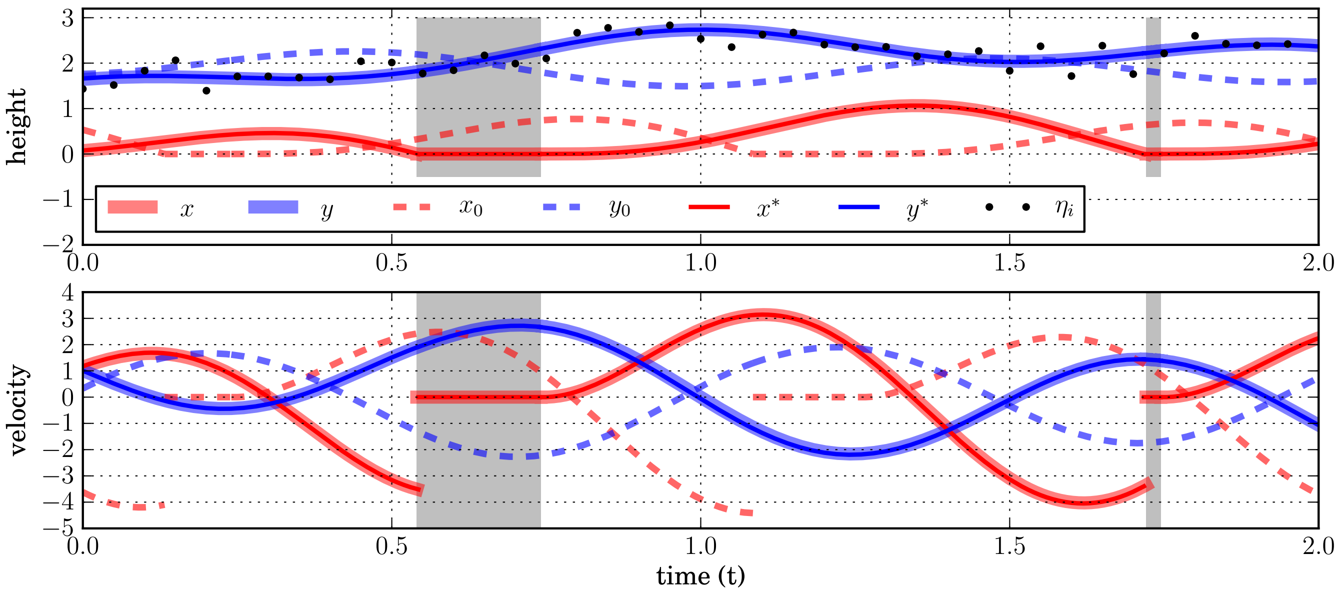

Fig. 2. Identification of initial conditions for the vertical hopper. The model is simulated from the initial condition $z = (\sigma, y, \dot{y}) \approx (4.7, 1.6, 1.0)$ and the position of the upper mass is measured at 20Hz with independent and identically distributed zero-mean Gaussian noise with variance 0.2. The initial conditions are identified by solving the problem in (4) from the initial guess $z_0 = (\sigma_0, y_0, \dot{y}_0) \approx (8.0, 1.5, 1.1)$ using the first-order method described in Section 4.5 to obtain the estimate $z^* = (\sigma^*, y^*, \dot{y}^*) \approx (4.6, 1.6, 1.1)$. Our method decreases the prediction error from $\epsilon(z_0) \approx 3.0 \times 10^{-1}$ to $\epsilon(z^*) \approx 3.9 \times 10^{-2}$, lower than the error obtained from the correct parameters $\epsilon(z) = 4.0 \times 10^{-2}$.

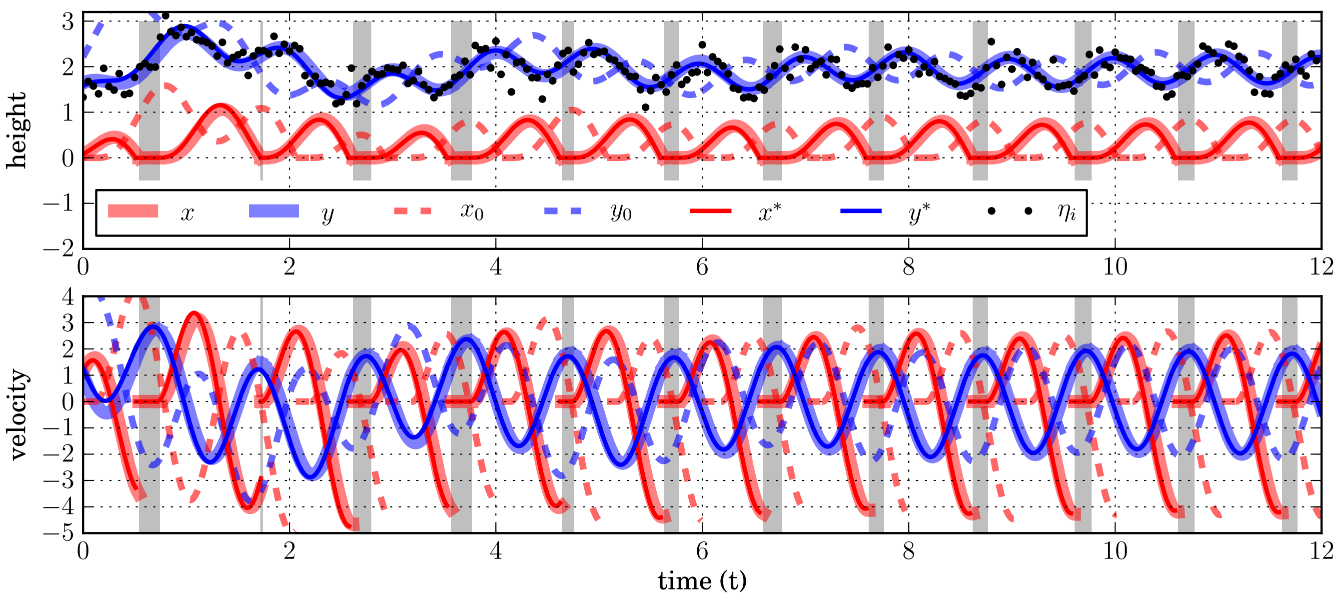

Fig. 3. Identification of initial conditions and parameters for the vertical hopper. In practice, some model parameters may be known *a priori* or estimated from the data. When identifying the vertical hopper using data from a physical system, the upper mass μ, nominal leg length ℓ_0, actuator frequency ω, and gravitational constant g can be accurately estimated from the data or from a separate experiment, hence we assume they are known and identify the lower mass m, spring stiffness k, damping coefficient b, and actuator amplitude a. The initial condition $z = (\sigma, y, \dot{y}, m, k, b, a) \approx (4.7, 1.6, 1.0, 1, 10, 5, 2)$ is sampled and identified as described in the caption for Fig. 2 from the initial guess $z_0 = (\sigma_0, y_0, \dot{y}_0, m_0, k_0, b_0, a_0) \approx (8.0, 1.5, 1.1, 0.97, 11.2, 5.3, 2.3)$ to obtain the estimate $z^* = (\sigma^*, y^*, \dot{y}^*, m^*, k^*, b^*, a^*) \approx (4.8, 1.6, 1.2, 0.95, 10.1, 5.2, 2.0)$. Our method decreases the prediction error from $\epsilon(z_0) \approx 3.9 \times 10^{-1}$ to $\epsilon(z^*) \approx 4.19 \times 10^{-2}$, lower than the error from the correct parameters $\epsilon(z) = 4.22 \times 10^{-2}$.

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Appendix A. NUMERICAL METHODS

To simulate the vertical hopper, we use the algorithm developed by Burden et al. [2011a] with step size $h = 1 \times 10^{-2}$ and relaxation parameter $\varepsilon = 1 \times 10^{-10}$. The sourcecode for simulations in this paper is available online at `http://purl.org/sburden/sysid2012`.

Appendix B. DISCLAIMER

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