On infinitesimal contraction analysis for hybrid systems

Samuel A. Burden[1] and Samuel D. Coogan[2]

Abstract— **Infinitesimal contraction analysis, wherein global convergence results are obtained from properties of local dynamics, is a powerful analysis tool. In this letter, we generalize infinitesimal contraction analysis to hybrid systems in which state-dependent guards trigger transitions defined by reset maps between modes that may have different norms and need not be of the same dimension. In contrast to existing literature, we do not restrict mode sequence or dwell time.**

We work in settings where the hybrid system flow is differentiable almost everywhere and its derivative is the solution to a jump-linear-time-varying differential equation whose jumps are defined by a saltation matrix determined from the guard, reset map, and vector field. Our main result shows that if the vector field is infinitesimally contracting, and if the saltation matrix is non-expansive, then the intrinsic distance between any two trajectories decreases exponentially in time. When bounds on dwell time are available, our approach yields a bound on the intrinsic distance between trajectories regardless of whether the dynamics are expansive or contractive. We illustrate our results using two examples: a constrained mechanical system and an electrical circuit with an ideal diode.

I. Introduction

A dynamical system is *contractive* if all trajectories converge to one another [1]. Contractive systems enjoy strong asymptotic properties, e.g. any equilibrium or periodic orbit is globally asymptotically stable. Provocatively, these global results can sometimes be obtained by analyzing local (or *infinitesimal*) properties of the system's dynamics. In smooth differential (or difference) equations, for instance, a bound on a matrix measure (or induced norm) of the derivative of the equation can be used to prove global contractivity [1], [2], [3]; this approach has been successfully applied to biological [4], [5], [6], mechanical [7], [8], and transportation [9], [10] systems.

Recent work has extended contraction analysis to certain classes of nonsmooth systems. Contraction for systems with a continuous vector field that is piecewise-differentiable was first suggested in [11] and rigorously characterized in [12]. Contraction of switched systems, potentially with sliding modes, is studied in [13] by explicitly considering contraction of the sliding vector field in [14] via a regularization approach that does not require explicit computation of the sliding vector field. The paper [15] considers contraction of Carathéodory switched systems for which the time-varying switching signal is piecewise-continuous and allows for different norms for each mode of the switched system.

The present paper complements and, in some cases, extends these prior works by considering a more general class of *hybrid* systems in which state-dependent *guards* trigger instantaneous transitions defined by *reset* maps between distinct *modes*. Different norms in each mode are allowed, and modes need not even be of the same dimension. In contrast to previous work generalizing contraction analysis to hybrid systems, our generalization of infinitesimal contraction analysis does not restrict mode sequence as in [16] or dwell time as in [16], [17].

This paper generalizes infinitesimal contraction analysis to hybrid systems by leveraging local dynamical properties of continuous-time flow *and* discrete-time reset to bound the time rate of change of the *intrinsic* distance between trajectories without imposing restrictions on mode sequence or dwell time. The intrinsic distance we employ is derived in [18] from the natural condition that the distance between a point in a guard and the point it resets to is zero. This intrinsic distance is distinct from the Skorohod [19] or Tavernini [20] *trajectory metrics* [18, Sec. V-A] and from the *distance function* introduced in [21]; it is an instantiation of the class of *distance functions* defined in [22] we found particularly useful in the present context. Importantly, the use of this intrinsic distance avoids restrictive conclusions regarding the closely-related notion of incremental stability [23, Prop. 1].

The conditions we obtain for infinitesimal contraction (Theorem 1 in Section IV) have intuitive appeal: the derivative of the vector field, which captures the infinitesimal dynamics of continuous-time flow, must be infinitesimally contractive with respect to the matrix measure determined by the vector norm used in each mode (4); the *saltation matrix*, which captures the infinitesimal dynamics of discrete-time reset, must be contractive with respect to the induced norm determined by the vector norms used on either side of the reset (5). If upper and lower bounds on *dwell time* are available, we can bound the intrinsic distance between trajectories, regardless of whether this distance is expanding or contracting in continuous- or discrete-time (Corollary 1 in Section IV). We illustrate our results using two examples: a constrained mechanical system and an electrical circuit with an ideal diode. Additional results and applications are provided in a technical report [24], including a proof that the continuous- and discrete-time from infinitesimal contractivity conditions from our Theorem 1 are necessary for contraction with respect to the intrinsic distance defined in Section III-C.

[1]S. A. Burden is with the Department of Electrical & Computer Engineering, University of Washington, Seattle, WA, USA, sburden@uw.edu.

[2]S. D. Coogan is with the School of Electrical and Computer Engineering and the School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA, USA, sam.coogan@gatech.edu.

Support is provided by the U. S. Army Research Laboratory and the U. S. Army Research Office under contract/grant #W911NF-16-1-0158, the Air Force Office of Scientific Research under award #FA9550-19-1-0015, and Awards #1836819,1836932 from the National Science Foundation.

II. Notation

The *disjoint union* of a collection of sets $\{S_\alpha\}_{\alpha\in A}$ is defined $\coprod_{\alpha\in A} S_\alpha = \bigcup_{\alpha\in A}(\{\alpha\}\times S_\alpha)$. Given $(a,x) \in \coprod_{\alpha\in A} S_\alpha$, we will simply write $x \in \coprod_{\alpha\in A} S_\alpha$ when a is clear from context. For a function γ with scalar argument, we denote limits from the left and right (when these exist) by $\gamma(t^-) = \lim_{\sigma\uparrow t}\gamma(\sigma)$ and $\gamma(t^+) = \lim_{\sigma\downarrow t}\gamma(\sigma)$. Given a smooth function $f : X \times Y \to Z$, we let $D_x f : TX \times Y \to TZ$ denote the *derivative of f with respect to $x \in X$* and $Df = (D_x f, D_y f) : TX \times TY \to TZ$ denote the derivative of f with respect to both $x \in X$ and $y \in Y$. Here, TX denotes the *tangent bundle* of X; when $X \subset \mathbb{R}^d$ we have $TX = X \times \mathbb{R}^d$. The *induced norm* of a linear function $M : \mathbb{R}^{n_j} \to \mathbb{R}^{n_{j'}}$ is $\|M\|_{j,j'} = \sup_{x\in\mathbb{R}^{n_j}} |Mx|_{j'}/|x|_j$ where $|\cdot|_j$ and $|\cdot|_{j'}$ denote the vector norms on $\mathbb{R}^{n_j}$ and $\mathbb{R}^{n_{j'}}$, respectively; when the norms are clear from context, we omit the subscripts. The *matrix measure* of $A \in \mathbb{R}^{n\times n}$, denoted $\mu(A)$, is $\mu(A) = \lim_{h\downarrow 0}(\|I + hA\| - 1)/h$.

III. Preliminaries

A *hybrid system* is a tuple $\mathcal{H} = (\mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R})$ where:

- $\mathcal{D} = \coprod_{j\in\mathcal{J}} \mathcal{D}_j$ is a set of states where $\mathcal{J}$ is a finite set of discrete states or *modes* and $\mathcal{D}_j = \mathbb{R}^{n_j}$ is a set of continuous states for each $j \in \mathcal{J}$ equipped with a norm $|\cdot|_j$ for some $n_j \in \mathbb{N}$;
- $\mathcal{F} : [0,\infty)\times\mathcal{D} \to T\mathcal{D}$ is a time-varying vector field, and we write $\mathcal{F}_j = \mathcal{F}|_{[0,\infty)\times\mathcal{D}_j} : [0,\infty)\times\mathcal{D}_j \to \mathbb{R}^{n_j}$;
- $\mathcal{G} = \coprod_{j\in\mathcal{J}} \mathcal{G}_j$ is a guard set with $\mathcal{G}_j \subset \mathcal{D}_j$ for all $j \in \mathcal{J}$;
- $\mathcal{R} : \mathcal{G} \to \mathcal{D}$ is a reset map.

$\mathcal{G}_{j,j'} = \mathcal{G}_j \cap \mathcal{R}^{-1}(\mathcal{D}_{j'})$ denotes states in $\mathcal{D}_j$ that reset to $\mathcal{D}_{j'}$.

Before we assess infinitesimal contractivity, we first impose restrictions on the *components* of the hybrid system as well as its *flow*, that is, the collection of trajectories it specifies. To help motivate and contextualize the assumptions, we provide expository remarks following each assumption that explain how each condition is employed in what follows and what specific dynamical phenomena it precludes.

A. Hybrid system components and constructions

We begin by stating and discussing assumptions on the hybrid system components.

Assumption 1 (hybrid system components)**.** *Given a hybrid system $\mathcal{H} = (\mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R})$:*

- *1.1 (vector field is Lipschitz and differentiable) $\mathcal{F}_j = \mathcal{F}|_{\mathcal{D}_j} : [0,\infty)\times\mathcal{D}_j \to \mathcal{D}_j$ is globally Lipschitz continuous and continuously differentiable (C^1) for all $j \in \mathcal{J}$;*
- *1.2 (discrete transitions are isolated) $\mathcal{R}(\mathcal{G}) \cap \mathcal{G} = \emptyset$;*
- *1.3 (guards and resets are differentiable) there exists C^1 and nondegenerate $g_{j,j'} : \mathcal{D}_j \to \mathbb{R}$ such that $\mathcal{G}_{j,j'} \subseteq \{x \in \mathcal{D}_j : g_{j,j'}(x) \le 0\} \subseteq \mathcal{G}_j$ and C^1 $\mathcal{R}_{j,j'} : \{x \in \mathcal{D}_j : g_{j,j'}(x) \le 0\} \to \mathcal{D}_{j'}$ such that $\mathcal{R}_{j,j'}|_{\mathcal{G}_{j,j'}} = \mathcal{R}|_{\mathcal{G}_{j,j'}}$ for each $j,\ j' \in \mathcal{J}$;*
- *1.4 (vector field is transverse to guard) $Dg_{j,j'}(x)\cdot\mathcal{F}_j(t,x) < 0$ for all $j, j' \in \mathcal{J}$, $t \ge 0$, and $x \in \mathcal{G}_{j,j'}$.*

Before we proceed, we make a number of remarks about the preceding Assumption.

Remark 1 (vector fields generate differentiable global flows). Assumption 1.1 ensures there exists a continuously differentiable *flow* $\phi_j : [0,\infty)\times[0,\infty)\times\mathcal{D}_j \to \mathcal{D}_j$ for $\mathcal{F}_j$. In other words, if $\chi : [\tau,\infty) \to \mathcal{D}_j$ denotes the trajectory for $\mathcal{F}_j$ initialized at $\chi(\tau) \in \mathcal{D}_j$, then $\chi(t) = \phi(t,\tau,\chi(\tau))$ for all $t \in [\tau,\infty)$. This condition enables application of classical infinitesimal contractivity analysis for continuous-time flows.

Remark 2 (discrete transitions are isolated). Since Assumption 2.1 below will (in particular) prevent an infinite number of discrete transitions from occurring at the same time instant, Assumption 1.2 is imposed without loss of generality. Indeed, given a hybrid system that permitted at most m discrete transitions at the same instant of time (an example with $m = 2$ can be found in [25, Thm. 8]), the reset map could be replaced with its m-fold composition to yield a hybrid system with isolated discrete transitions that has the same set of trajectories (as defined below).

Remark 3 (closed guards). Noting that continuity of $g_{j,j'}$ ensures $\overline{\mathcal{G}}_{j,j'} = \{x \in \mathcal{D}_j : g_{j,j'}(x) \le 0\}$ is closed, we observe that $\mathcal{G}_j = \bigcup_{j'\in\mathcal{J}} \mathcal{G}_{j,j'} \subset \bigcup_{j'\in\mathcal{J}} \overline{\mathcal{G}}_{j,j'} \subset \mathcal{G}_j$, whence Assumption 1.3 ensures $\mathcal{G}_j = \bigcup_{j'\in\mathcal{J}} \overline{\mathcal{G}}_{j,j'} \subset \mathcal{D}_j$ is a closed set. Note that the disjoint components of the guard, $\mathcal{G}_{j,j'}$, are not required to be closed.

B. Hybrid system trajectories and flow

Informally, a trajectory of a hybrid system $\mathcal{H}$ is a right-continuous function of time that satisfies the continuous-time dynamics specified by $\mathcal{F}$ on $\mathcal{D}$ and the discrete-time dynamics specified by $\mathcal{R}$ on $\mathcal{G}$. Formally, a function $\chi : [\tau, T) \to \mathcal{D}$ with $\tau \ge 0$ is a *trajectory* of $\mathcal{H}$ if:

1) $D\chi(t) = \mathcal{F}(t,\chi(t))$ for almost all $t \in [\tau, T)$;
2) $\chi(t^+) = \chi(t)$ for all $t \in [\tau, T)$;
3) $\chi(t^-) = \chi(t)$ if and only if $\chi(t) \notin \mathcal{G}$;
4) $\chi(t^-) \ne \chi(t) \implies \chi(t^-) \in \mathcal{G}$ and $\chi(t) = \mathcal{R}(\chi(t^-))$.

Note that it is allowed, but not required, that $T = \infty$ (although we will shortly impose additional assumptions that ensure trajectories are defined for all positive time). If the domain of χ cannot be extended in forward time to define a trajectory on a larger time domain, then χ is termed *maximal*. The following Proposition ensures that maximal trajectories exist and are unique under the conditions in Assumption 1; its proof is standard [26, Thm. III-1].

Proposition 1 (existence and uniqueness of trajectories)**.** *Under the conditions in Assumption 1, there exists a unique maximal trajectory $\chi : [\tau, T) \to \mathcal{D}$ satisfying $\chi(\tau) = x$ if $x \in \mathcal{D}\backslash\mathcal{G}$ or $\chi(\tau) = \mathcal{R}(x)$ if $x \in \mathcal{G}$ for any initial state $x \in \mathcal{D}$ and initial time $\tau \ge 0$.*

We will restrict the class of trajectories exhibited by the hybrid system in Assumption 2 below. Before imposing these restrictions, we first develop tools that enable analysis of how trajectories vary with respect to initial conditions. The reset map induces an equivalence relation $\overset{\mathcal{R}}{\sim}$ on $\mathcal{D}$ defined as the smallest equivalence relation containing $\{(x,y) \in \mathcal{G}\times\mathcal{D} : \mathcal{R}(x) = y\} \subset \mathcal{D}\times\mathcal{D}$, for which we write $x \overset{\mathcal{R}}{\sim} y$ to indicate x and y are related. The equivalence

class for $x \in \mathcal{D}$ is defined as $[x]_{\mathcal{R}} = \left\{ y \in \mathcal{D} | x \overset{\mathcal{R}}{\sim} y \right\}$. The *quotient space* induced by the equivalence relation is denoted $\mathcal{M} = \{[x]_{\mathcal{R}} | x \in \mathcal{D}\}$ endowed with the quotient topology [27, Appendix A]; we note that such quotient spaces have been studied repeatedly in the hybrid systems literature [28], [29], [30], [18].

To define a distance on the quotient $\mathcal{M}$, we will adopt the approach in [18] and use the length of paths that are continuous in the quotient. A path $\gamma : [0,1] \to \mathcal{D}$ is *smoothly $\mathcal{R}$-connected* if there exists an open set $\mathcal{O} \subset [0,1]$ such that: the relative complement $\mathcal{O}^C \subset [0,1]$ is countable (so that, in particular, the closure $\overline{\mathcal{O}} = [0,1]$); γ is continuously differentiable on $\mathcal{O}$; $\lim_{r' \uparrow r} \gamma(r') \overset{\mathcal{R}}{\sim} \lim_{r' \downarrow r} \gamma(r')$ for all $r \in (0,1)$; and $\gamma(0) \overset{\mathcal{R}}{\sim} \lim_{r' \downarrow 0} \gamma(r')$, $\gamma(1) \overset{\mathcal{R}}{\sim} \lim_{r' \uparrow 1} \gamma(r')$.

A set $\mathcal{O}$ satisfying the above conditions is termed a *support set* for γ at time t. Intuitively, a smoothly $\mathcal{R}$-connected path γ is a path through the modes $\{\mathcal{D}_j\}_{j \in \mathcal{J}}$ of the hybrid system that is allowed to jump through the reset map $\mathcal{R}$ (forward or backward) and is smooth almost everywhere. With a slight abuse of notation,[1] we consider γ a path in $\mathcal{M}$; with this identification, all $\mathcal{R}$-connected paths are continuous paths in the quotient space $\mathcal{M}$. Any support set $\mathcal{O}$ for a smoothly $\mathcal{R}$-connected path γ is a countable union of (disjoint) open intervals (cf. [31, Prop. 0.21]); let $\mathcal{O} = \bigcup_{i=1}^{k} (u_i, v_i)$ with possibly $k = \infty$. Because each segment $\gamma|_{(u_i,v_i)}$ is continuously differentiable, the segment is (in particular) continuous, so its image must necessarily belong to a single $\mathcal{D}_j$ for some $j \in \mathcal{J}$.

Assumption 2 (hybrid system flow)**.** *Given a hybrid system $\mathcal{H} = (\mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R})$:*

- *2.1 (Zeno) no trajectory has infinite resets in finite time;*
- *2.2 (continuity of hybrid system flow) with $\phi : \mathcal{F} \to \mathcal{D}$ denoting the hybrid system flow, i.e. $\phi(t, \tau, x) = \chi(t)$ where[2] $\chi : [\tau, \infty) \to \mathcal{D}$ is the unique trajectory initialized at $\chi(\tau) = x$, the projection $\pi \circ \phi(t, \tau, x)$, regarded as a function $\mathcal{D} \to \mathcal{M}$, is continuous;*
- *2.3 (support sets of smoothly $\mathcal{R}$-connected paths) for all $t \geq \tau \geq 0$ and all smoothly $\mathcal{R}$-connected paths γ, there exists a support set $\mathcal{O}$ of $\phi(t, \tau, \gamma(\cdot))$ such that, for all $r' \in \mathcal{O}$, there exists $\epsilon > 0$ such that all trajectories $\phi(\cdot, \tau, \gamma(r))$ with $r \in (r' - \epsilon, r' + \epsilon)$ undergo the same sequence of discrete state transitions as $\phi(\cdot, \tau, \gamma(r'))$ on the time interval $[\tau, t]$.*

Before we proceed, we make a number of remarks about the preceding Assumption.

Remark 4 (Zeno and forward completeness)*.* Since our results below will strongly leverage the fact that the hybrid system flow is everywhere locally a composition of a finite number of differentiable flows and resets, we cannot easily extend our approach to Zeno trajectories.

Remark 5 (continuity of hybrid system flow)*.* Continuous flow is necessary for infinitesimal contraction with respect to the intrinsic distance defined below, as the intrinsic distance between trajectories on either side of a flow discontinuity will grow linearly with time.

Remark 6 (support sets of $\mathcal{R}$-connected paths)*.* Note that Assumption 1 already suffices to ensure the conditions in Assumptions 2.2-2.3 hold in regions where guards do not "overlap", i.e. where the intersection of their closures is empty, $\overline{\mathcal{G}}_{j,j'} \cap \overline{\mathcal{G}}_{j,j''} = \emptyset$. Where guards do overlap ($\overline{\mathcal{G}}_{j,j'} \cap \overline{\mathcal{G}}_{j,j''} \neq \emptyset$), Assumption 2.3 does not require that *all* trajectories along a path undergo the same sequence of discrete state transitions, only that the path's domain contains an open dense subset wherein each connected component undergoes the same sequence of discrete state transitions on finite time horizons. We emphasize that this condition *does not* require that all trajectories visit the same sequence of modes.

It is well-known [32] that, under favorable conditions, the hybrid system flow ϕ is differentiable almost everywhere and, moreover, its derivative can be computed by solving a jump-linear-time-varying differential equation. The preceding assumptions are favorable enough to ensure the flow has these properties so that, in particular, the derivative along a path can be computed using the jump-linear-time-varying differential equation. These facts are summarized in the following Proposition, whose proof is standard [32].

Proposition 2. *Under Assumptions 1 and 2, given an initial time $\tau \geq 0$ and a smoothly $\mathcal{R}$-connected path γ, let $\psi(t, r) = \phi(t, \tau, \gamma(r))$ for all $t \geq \tau$ and define $w(t, r) = D_r \psi(t, r)$ whenever the derivative exists. Then $w(\tau^-, r) = D_r \gamma(r)$ and $w(\cdot, r)$ satisfies a linear-time-varying differential equation*

$$D_t w(t, r) = D_x \mathcal{F}(t, \psi(t, r)) w(t, r), \ \psi(t^-, r) \in \mathcal{D} \backslash \mathcal{G}, \quad (1)$$

with jumps $w(t, r) = \Xi(t, \psi(t^-, r)) w(t^-, r), \ \psi(t^-, r) \in \mathcal{G}$, where $\Xi(t, x)$ is the saltation matrix*: $\forall t \geq 0, x \in \mathcal{G}_{j,j'}$*

$$\Xi(t, x) = D\mathcal{R}(x) + \frac{(\mathcal{F}^+ - D\mathcal{R}(x)\mathcal{F}^-) \, Dg_{j,j'}(x)}{Dg_{j,j'}(x)\mathcal{F}^-} \quad (2)$$

where $\mathcal{F}^+ = \mathcal{F}_{j'}(t, \mathcal{R}(x))$, $\mathcal{F}^- = \mathcal{F}_j(t, x)$.

C. Hybrid system intrinsic distance

As the final preliminary construction, we define the length of a smoothly $\mathcal{R}$-connected path $\gamma : [0,1] \to \mathcal{D}$ as the sum of the lengths of its segments, and use this *length structure* [33, Ch. 2] to define an *intrinsic distance* on $\mathcal{M}$. To that end, define the length of a continuously differentiable path segment $\gamma|_{(u_i,v_i)} : (u_i, v_i) \to \mathcal{D}_j$ in the usual way using the norm $|\cdot|_j$ in $\mathcal{D}_j$, namely, $L_j(\gamma|_{(u_i,v_i)}) = \int_{u_i}^{v_i} |D\gamma_j(r)|_j dr$ (we drop the subscript for L when the mode is clear from context), and define length of γ at time t as the sum of the lengths of its segments,

$$L(\gamma) = \sum_{i=1}^{k} L(\gamma|_{(u_i,v_i)}) = \int_0^1 |D\gamma(r)|_{j(r)} dr,$$

where $j(r) \in \mathcal{J}$ denotes the mode satisfying $\gamma(r) \in \mathcal{D}_{j(r)}$ for each $r \in [0,1]$. With Γ denoting the set of smoothly

[1] $\pi \circ \gamma$ is a path in $\mathcal{M}$, where $\pi : \mathcal{D} \to \mathcal{M}$ is the *quotient projection*.

[2] Note that Assumption 2.1 ensures that the maximal time interval in Proposition 1 is $[\tau, \infty)$, i.e. $T = \infty$.

$\mathcal{R}$-connected paths in $\mathcal{M}$, and letting

$$\Gamma(x,y) = \{\gamma \in \Gamma : \gamma(0) = x \text{ and } \gamma(1) = y,\ x, y \in \mathcal{D}\}$$

denote the subset of paths that start at $x \in \mathcal{D}$ and end at $y \in \mathcal{D}$, we define the distance $d(x,y)$ between x and y by

$$d(x,y) = \inf_{\gamma \in \Gamma(x,y)} L(\gamma). \tag{3}$$

We note that $d : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ belongs to the class of *distance function* in [22, Def. 1], but it has a stronger *intrinsic* relationship to the hybrid system [18, Thm. 13].

IV. Results

The main contribution of this paper is the provision of local (or *infinitesimal*) conditions under which the distance between any pair of trajectories in a hybrid system (as measured by the intrinsic distance defined in (3)) is bounded by an exponential envelope. These conditions are made precise in Theorem 1 and Corollary 1. In what follows, we will only check infinitesimal contractivity conditions on a *contraction region* $\mathcal{C} \subset \mathcal{D}$, that is, a forward-invariant subset ($\xi \in \mathcal{C}$ implies $\phi(t,0,\xi) \in \mathcal{C}$ for all $t \geq 0$) that descends to a simply-connected subset in the quotient.

Theorem 1. *Under Assumptions 1 and 2, if there exists $c \in \mathbb{R}$ and contraction region $\mathcal{C} \subset \mathcal{D}$ such that, for all $j, j' \in \mathcal{J}$, $t \geq 0$,*

$$\mu_j\left(D_x \mathcal{F}_j(t,x)\right) \leq c \quad \forall x \in \mathcal{C} \cap \mathcal{D}_j \backslash \mathcal{G}_j \text{ and} \tag{4}$$

$$\|\Xi(t,x)\|_{j,j'} \leq 1 \quad \forall x \in \mathcal{C} \cap \mathcal{G}_{j,j'}, \tag{5}$$

then $d(\phi(t,0,\xi),\phi(t,0,\zeta)) \leq e^{ct} d(\xi,\zeta)$ for all $t \geq 0$ and $\xi, \zeta \in \mathcal{C} \cap \mathcal{D}$.

Proof: Given $x(0) = \xi$, $z(0) = \zeta$, and $s \in [0,t]$, for fixed $\epsilon > 0$, let $\gamma \in \Gamma(\xi,\zeta)$ be such that $L(\gamma) < d(\xi,\zeta) + \epsilon$, and let $\psi(t,r) = \phi(t,0,\gamma(r))$. Since $\phi(t,0,\cdot)$ is piecewise-differentiable, it follows from Assumption 2.2 and 2.3 that $\psi(t,\cdot)$ is a smoothly $\mathcal{R}$-connected path for all $t \geq 0$. Let $w(t,r) = D_r \psi(t,r)$ whenever the derivative exists. By Proposition 2, $w(t,r)$ satisfies the equations

$$\dot{w}(t,r) = D_x \mathcal{F}(t,\psi(t,r)) w(t,r),\ \psi(t,r) \in \mathcal{D} \backslash \mathcal{G}, \tag{6}$$

$$w(t^+,r) = \Xi(t,\psi(t^-,r)) w(t^-,r),\ \psi(t^-,r) \in \mathcal{G}. \tag{7}$$

For fixed r, let $\{t_i\}_{i=1}^k \subset [0,\infty)$ with $t_0 \leq t_1 \leq \cdots$ and possibly $k = \infty$ be the set of times at which the trajectory $\phi(t,s,\gamma(r))$ intersects a guard so that $\psi(\cdot,r)|_{[t_i,t_{i+1})}$ is continuous for all $i \in \{0,1,\ldots,k-1\}$ where $t_0 = s$ by convention, and, additionally, $\psi(\cdot,r)|_{[t_k,\infty)}$ is continuous if $k < \infty$. Now consider some fixed time $T > 0$. If $k < \infty$ and $t_k \leq T$, let $i = k$; otherwise, let i be such that $t_i \leq T < t_{i+1}$. Let j be the active mode of the system during the interval $[t_i, t_{i+1})$, i.e. $\psi(t,r) \in \mathcal{D}_j$ for all $t \in [t_i, t_{i+1})$. With $J(t) = D_x \mathcal{F}_j(\psi(t,r))$ for $t \in [t_i, t_{i+1})$ we have

$$\begin{aligned} |w(T,r)| &\leq e^{\int_{t_i}^T \mu(J(\tau))d\tau} |w(t_i^+,r)| \leq e^{c(T-t_i)} |w(t_i^+,r)| \\ &\leq e^{c(T-t_i)} \|\Xi(t,\psi(t_i^-,r))\| |w(t_i^-,r)| \leq e^{c(T-t_i)} |w(t_i^-,r)| \end{aligned}$$

where the inequalities follow from, respectively, Coppel's inequality [34, p. 34], (4), (7), and (5). Since this holds for any $T < t_{i+1}$, $|w(t_{i+1}^-,r)| \leq e^{c(t_{i+1}-t_i)} |w(t_i^-,r)|$ whenever $i \leq k$. By recursion, $|w(T,r)| \leq e^{cT} |w(s^-,r)|$. Since T was arbitrary, we have proved $|w(t,r)| \leq e^{ct} |w(0^-,r)|$.

Because $\psi(T,\cdot)$ for fixed $T > 0$ is a smoothly $\mathcal{R}$-connected path, there exists a support set $\mathcal{O} = \bigcup_{i=1}^k (u_i, v_i)$ of $\psi(T,\cdot)$ such that $\psi(T,\cdot)|_{(u_i,v_i)}$ is continuously-differentiable for all $i \in \{0,1,\ldots,k\}$. It follows that $L\left(\psi(T,\cdot)|_{(u_i,v_i)}\right) = \int_{u_i}^{v_i} |w(T,\sigma)| d\sigma$, whence

$$\begin{aligned} L(\psi(T,\cdot)) &= \sum_{i=1}^k L\left(\psi(T,\cdot)|_{(u_i,v_i)}\right) \leq \int_0^1 |w(T,\sigma)| d\sigma \\ &\leq e^{cT} \int_0^1 |w(0^-,\sigma)| d\sigma = e^{cT} L(\gamma) \leq e^{cT} (d(\xi,\zeta) + \epsilon). \end{aligned}$$

In addition, observe $d(\phi(T,0,\xi),\phi(T,0,\zeta)) \leq L(\psi(T,\cdot))$. Noting that T was arbitrary and ϵ can be arbitrarily small concludes the proof. □

Now suppose that uniform upper or lower bounds on the *dwell time* between successive resets are known. Then the number of discrete state transitions is upper or lower bounded on any compact time horizon, and the proof of Theorem 1 can be adapted to derive an exponential bound on the intrinsic distance between any pair of trajectories.

Corollary 1. *Under Assumptions 1 and 2, suppose the dwell time between resets is at most $\overline{\tau} \in (0,\infty]$ and at least $\underline{\tau} \in [0,\infty)$ and there exists $c \in \mathbb{R}$, $K \in \mathbb{R}_{\geq 0}$, and forward-invariant $\mathcal{C} \subset \mathcal{D}$ such that, for all $j, j' \in \mathcal{J}$, $t \geq 0$,*

$$\mu_j(D_x \mathcal{F}_j(t,x)) \leq c \quad \forall x \in \mathcal{C} \cap \mathcal{D}_j \backslash \mathcal{G}_j \text{ and} \tag{8}$$

$$\|\Xi(t,x)\|_{j,j'} \leq K \quad \forall x \in \mathcal{C} \cap \mathcal{G}_{j,j'}, \tag{9}$$

then $d(\phi(t,s,\xi),\phi(t,s,\zeta)) \leq \mathcal{K} e^{c(t-s)} d(\xi,\zeta)$ for all $t \geq s \geq 0$ and $\xi, \zeta \in \mathcal{C} \cap \mathcal{D}$ where $\mathcal{K} = \max\{K^{\lceil (t-s)/\underline{\tau} \rceil}, K^{\lfloor (t-s)/\overline{\tau} \rfloor}\}$. In particular, if $\max\{K e^{c\underline{\tau}}, K e^{c\overline{\tau}}\} < 1$ then $\lim_{t\to\infty} d(\phi(t,s,\xi),\phi(t,s,\zeta)) = 0$.

Remark 7 (summary of main result). Our main contribution is Theorem 1, where we generalize infinitesimal contractivity analysis to the class of hybrid systems satisfying Assumptions 1 and 2. This generalization has intuitive appeal, since it combines infinitesimal conditions on continuous-time flow (via the matrix measure of the vector field derivative, (4)) and discrete-time reset (via the induced norm of the saltation matrix, (5)) that parallel the conditions imposed separately in prior work on smooth continuous-time and discrete-time systems, and establishes contraction with respect to the hybrid system's intrinsic distance d. With bounds on *dwell time*, i.e. the time between discrete transitions, our approach yields a bound in Corollary 1 on the intrinsic distance between trajectories regardless of whether the dynamics are contractive or expansive. We say a hybrid system is *contractive* if $\lim_{t\to\infty} d(\phi(t,s,\xi),\phi(t,s,\zeta)) = 0$ holds for all trajectories.

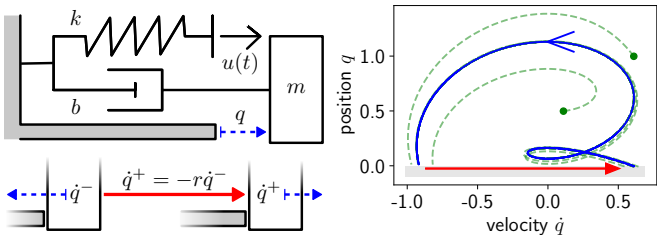

Fig. 1: *Mechanical system subject to unilateral constraint (Sec. V-A). Left:* schematic of continuous-time dynamics (top) and discrete-time dynamics (bottom). *Right:* exponentially stable periodic orbit (solid blue line) with nearby trajectories (dashed green lines) produced by $u(t) = \sin(t) + 1/4$, $m = b = k = 1$, $r = 2/3$.

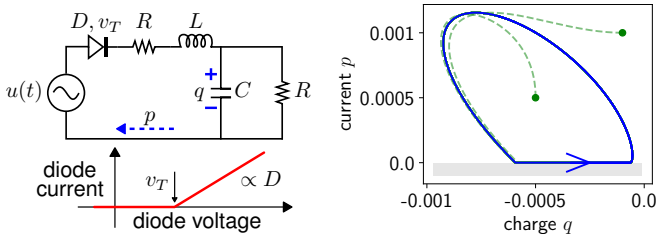

Fig. 2: *Electrical circuit with ideal diode (Sec. V-B). Left:* schematic of circuit (top) and current-voltage characteristic of diode (bottom). *Right:* exponentially stable periodic orbit (solid blue line) with nearby trajectories (dashed green lines) produced by $u(t) = 2\sin(t) + 1$, $R = 10^3, L = 10^3, C = 10^{-3}, D = 10^{-3}, v_T = 1$.

V. Examples

We now present examples that highlight key advances of our approach, namely, non-identity resets (Section V-A) and non-constant mode dimension (Section V-B). Although academic, the examples illustrate important contraction phenomena of interest in mechanical and electrical systems; extensions and other applications are provided in a technical report [24]. We note that stability arises in complementary ways in the two examples: whereas the mechanical system is non-expansive in continuous-time and contractive in discrete-time (so we apply Corollary 1), the electrical circuit is contractive in continuous-time and non-expansive in discrete-time (so we apply Theorem 1).

A. Mechanical system subject to unilateral constraint

1) Hybrid system: Consider $\mathcal{H} = (\mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R})$ from Fig. 1 (assume $m, b, k, r > 0$):

$$\mathcal{D} = \left\{(q, \dot{q}) \in \mathbb{R}^2\right\}; \qquad \mathcal{F} : m\ddot{q} + b\dot{q} + kq = u(t);$$
$$\mathcal{G} = \left\{(q, \dot{q}) : q \leq 0, \dot{q} < 0\right\}; \quad \mathcal{R} : (q, \dot{q}) \mapsto (q, -r\dot{q}).$$

This system can be regarded as a particular case of the class of hybrid systems that arise when modeling mechanical systems subject to unilateral constraints [25]; specifically, $\mathcal{H}$ is a linear *impact oscillator*. Although such systems cannot globally satisfy Assumptions 1 and 2 due to the possibility of grazing (i.e. zero-velocity impact) and since the guard is not closed, they do satisfy these Assumptions along trajectories that impact at non-zero velocity.

2) Infinitesimal contraction analysis: We use total energy $e = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2$ to define a norm $\forall x = (q, \dot{q}) \in \mathcal{D}$:

$$|x| = \frac{1}{2}x^\top E x, \;\; E = D_x^2 e = \operatorname{diag}(m, k). \tag{10}$$

Computing the matrix measure of $D_x\mathcal{F}$ using the norm in (10) yields $\mu(D_x\mathcal{F}) = \max \operatorname{spec} \frac{1}{2}\left(D_x\mathcal{F}^\top \cdot E + E \cdot D_x\mathcal{F}\right) = 0$ since

$$\frac{1}{2}D_x\mathcal{F}^\top \cdot E + E \cdot D_x\mathcal{F} = \operatorname{diag}(0, -b), \tag{11}$$

so (4) is satisfied with $c = 0$. The saltation matrix

$$\Xi(t) = \begin{bmatrix} -r & 0 \\ \frac{(1+r)u(t)}{m\dot{q}} & -r \end{bmatrix} \tag{12}$$

has induced norm $|\Xi| = r$ when $u(t) = 0$, so (5) is satisfied when $r \in (0, 1)$ and $\dot{q} < 0$ for all sufficiently small $u(t)$.

3) Interpretation: We interpret the preceding analysis in the context of the tracking problem for mechanical systems subject to unilateral constraints [35]. Given a periodic input $u : [0, \infty) \to \mathbb{R}$ that produces a non-Zeno non-grazing periodic trajectory $q : [0, \infty) \to \mathbb{R}$ for $\mathcal{H}$, we can apply Corollary 1 to conclude that all trajectories initialized sufficiently close to q will converge to q at an exponential rate.

B. Electrical circuit with ideal diode

1) Hybrid system: Consider $\mathcal{H} = (\mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R})$ from Fig. 2 (assume $R, L, C, D, v_T > 0$):

$$\mathcal{D} = \mathcal{D}_0 \coprod \mathcal{D}_1 = \{q \in \mathbb{R}\} \coprod \left\{(p, q) \in \mathbb{R}^2\right\};$$
$$\mathcal{F} : \dot{q} = -p - \tfrac{1}{RC}q, L\dot{p} + (R + 1/D)p + q/C + v_T = u(t);$$
$$\mathcal{G} = \{q : u(t) - q/C \geq v_T\} \coprod \{(p, q) : p \leq 0, \dot{p} < 0\};$$
$$\mathcal{R} : q \in \mathcal{G} \cap \mathcal{D}_0 \mapsto (0, q), \;\; (p, q) \in \mathcal{G} \cap \mathcal{D}_1 \mapsto q.$$

This system can be regarded as a particular case of the class of hybrid systems that arise when modeling electrical circuits with diodes; specifically, $\mathcal{H}$ models an AC-DC converter with an ideal diode; the two modes correspond to whether the diode is off (0) or on (1). Although such systems cannot globally satisfy Assumptions 1 and 2 due to the possibility of grazing, they do satisfy these Assumptions along trajectories of interest like the one illustrated in Fig. 2.

2) Infinitesimal contraction analysis: We use total energy $e = \frac{L}{2}p^2 + \frac{1}{2C}q^2$ to define a norm $\forall x = (p, q)$:

$$|x| = \frac{1}{2}x^\top E x, \;\; E = D_x^2 e = \operatorname{diag}(L, 1/C). \tag{13}$$

Computing the matrix measure of $D_x\mathcal{F}$ using the norm in (13) yields $\mu(D_x\mathcal{F}) = \max \operatorname{spec} \frac{1}{2}\left(D_x\mathcal{F}^\top \cdot E + E \cdot D_x\mathcal{F}\right) < 0$ since

$$\frac{1}{2}D_x\mathcal{F}^\top \cdot E + E \cdot D_x\mathcal{F} = \begin{bmatrix} -R - 1/D & -1/C \\ -1/C & -1/RC^2 \end{bmatrix},$$

so (4) is satisfied with $c < 0$. Since the vector field is continuous, the saltation matrices (2) simply equal the derivative of the resets[3], which both have unity induced norms, so (5) is satisfied.

[3]Since the guard is time-varying, this conclusion technically falls outside the scope of the present paper – see the technical report [24] for details.

3) Interpretation: We interpret the preceding analysis in the context of the application domain, where the hybrid system $\mathcal{H}$ models an AC-DC converter. Theorem 1 ensures the system is infinitesimally contractive for sinusoidally-fluctuating (AC) voltage inputs $u(t)$, hence there exists an exponentially stable limit cycle, whence the capacitor voltage $q(t)/C$ fluctuates periodically around a well-defined average (DC) output voltage.

VI. Conclusion

We generalized infinitesimal contraction analysis to hybrid systems by leveraging local properties of continuous-time flow and discrete-time reset to bound the time rate of change of the intrinsic distance between trajectories. Furthermore, we showed that contraction with respect to this intrinsic distance implies infinitesimal contraction in continuous- and discrete-time. Our results expand the toolkit for stability analysis in hybrid systems. Importantly, although we had to introduce new techniques to generalize infinitesimal contraction to the hybrid setting, our approach leverages the key idea in classical analyses: a system is contractive if path lengths decrease in time [1], [3], [36]. This close parallel may facilitate generalization of a variety of classical techniques to hybrid systems including state-varying distance metrics defined by Riemannian [1] or Finsler [36] structures.

Acknowledgments

We thank Matthew D. Kvalheim for correcting an error in an earlier version of this paper and Joseph G. Sullivan for providing the circuit example.

References

[1] W. Lohmiller and J.-J. Slotine, "On contraction analysis for non–linear systems," *Automatica*, vol. 34, no. 6, pp. 683–696, 1998.

[2] A. Pavlov, A. Pogromsky, N. van de Wouw, and H. Nijmeijer, "Convergent dynamics, a tribute to Boris Pavlovich Demidovich," *Systems & Control Letters*, vol. 52, no. 3, pp. 257–261, 2004.

[3] E. D. Sontag, "Contractive systems with inputs," in *Perspectives in Mathematical System Theory, Control, and Signal Processing*, vol. 398, pp. 217–228, Springer Berlin Heidelberg, 2010.

[4] A. Raveh, M. Margaliot, E. D. Sontag, and T. Tuller, "A model for competition for ribosomes in the cell," *Journal of the Royal Society, Interface / the Royal Society*, vol. 13, no. 116, 2016.

[5] G. Russo and M. di Bernardo, "How to synchronize biological clocks," *Journal of computational biology: a journal of computational molecular cell biology*, vol. 16, no. 2, pp. 379–393, 2009.

[6] W. Wang and J.-J. E. Slotine, "On partial contraction analysis for coupled nonlinear oscillators," *Biological Cybernetics*, vol. 92, no. 1, pp. 38–53, 2005.

[7] W. Lohmiller and J.-J. E. Slotine, "Control system design for mechanical systems using contraction theory," *IEEE Transactions on Automatic Control*, vol. 45, no. 5, pp. 984–989, 2000.

[8] I. R. Manchester and J.-J. E. Slotine, "Transverse contraction criteria for existence, stability, and robustness of a limit cycle," *Systems & Control Letters*, vol. 63, no. Supplement C, pp. 32–38, 2014.

[9] S. Coogan and M. Arcak, "A compartmental model for traffic networks and its dynamical behavior," *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2698–2703, 2015.

[10] G. Como, E. Lovisari, and K. Savla, "Throughput optimality and overload behavior of dynamical flow networks under monotone distributed routing," *IEEE Transactions on Control of Network Systems*, vol. 2, no. 1, pp. 57–67, 2015.

[11] W. Lohmiller and J.-J. E. Slotine, "Nonlinear process control using contraction theory," *AIChE Journal*, vol. 46, no. 3, pp. 588–596, 2000.

[12] M. di Bernardo, D. Liuzza, and G. Russo, "Contraction analysis for a class of nondifferentiable systems with applications to stability and network synchronization," *SIAM Journal on Control and Optimization*, vol. 52, no. 5, pp. 3203–3227, 2014.

[13] M. di Bernardo and D. Fiore, "Incremental stability of bimodal Filippov systems in $\mathbb{R}^n$," in *IEEE Conference on Decision and Control*, pp. 4679–4684, 2014.

[14] D. Fiore, S. Hogan, and M. di Bernardo, "Contraction analysis of switched systems via regularization," *Automatica*, vol. 73, pp. 279–288, 2016.

[15] W. Lu and M. di Bernardo, "Contraction and incremental stability of switched Carathéodory systems using multiple norms," *Automatica*, vol. 70, pp. 1–8, 2016.

[16] J. Z. Tang and I. R. Manchester, "Transverse contraction criteria for stability of nonlinear hybrid limit cycles," in *IEEE Conference on Decision and Control*, pp. 31–36, 2014.

[17] K. E. Rifai and J. J. Slotine, "Compositional contraction analysis of resetting hybrid systems," *IEEE transactions on automatic control*, vol. 51, no. 9, pp. 1536–1541, 2006.

[18] S. A. Burden, H. Gonzalez, R. Vasudevan, R. Bajcsy, and S. S. Sastry, "Metrization and Simulation of Controlled Hybrid Systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2307–2320, 2015.

[19] D. Gokhman, "Topologies for hybrid solutions," *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 2, pp. 468–473, 2008.

[20] L. Tavernini, "Differential automata and their discrete simulators," *Nonlinear Analysis, Theory, Methods & Applications*, vol. 11, no. 6, pp. 665–683, 1987.

[21] J. J. B. Biemond, W. P. M. H. Heemels, R. G. Sanfelice, and N. van de Wouw, "Distance function design and Lyapunov techniques for the stability of hybrid trajectories," *Automatica*, vol. 73, pp. 38–46, 2016.

[22] J. J. B. Biemond, N. v. de Wouw, W. P. M. H. Heemels, and H. Nijmeijer, "Tracking control for hybrid systems with State-Triggered jumps," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 876–890, 2013.

[23] J. J. B. Biemond, R. Postoyan, W. P. M. H. Heemels, and N. van de Wouw, "Incremental stability of hybrid dynamical systems," *IEEE transactions on automatic control*, vol. 63, no. 12, pp. 4094–4109, 2018.

[24] S. A. Burden, T. Libby, and S. D. Coogan, "On infinitesimal contraction analysis for hybrid systems," arXiv:1811.03956v2, 2020.

[25] A. M. Johnson, S. A. Burden, and D. E. Koditschek, "A hybrid systems model for simple manipulation and self-manipulation systems," *The International Journal of Robotics Research*, vol. 35, no. 11, pp. 1354–1392, 2016.

[26] J. Lygeros, K. H. Johansson, S. N. Simic, J. Zhang, and S. S. Sastry, "Dynamical properties of hybrid automata," *IEEE Transactions on Automatic Control*, vol. 48, no. 1, pp. 2–17, 2003.

[27] J. Lee, *Introduction to Smooth Manifolds*. Springer, 2012.

[28] M. Schatzman, "Uniqueness and continuous dependence on data for one–dimensional impact problems," *Mathematical and Computer Modelling*, vol. 28, no. 4—8, pp. 1–18, 1998.

[29] S. N. Simic, K. H. Johansson, J. Lygeros, and S. Sastry, "Towards a geometric theory of hybrid systems," *Dynamics of Continuous, Discrete & Impulsive Systems. Series B. Applications & Algorithms*, vol. 12, no. 5-6, pp. 649–687, 2005.

[30] A. Ames, *A Categorical Theory of Hybrid Systems*. PhD thesis, University of California at Berkeley, 2006.

[31] G. B. Folland, *Real analysis*. Wiley, 1999.

[32] M. A. Aizerman and F. R. Gantmacher, "Determination of stability by linear approximation of a periodic solution of a system of differential equations with discontinuous Right–Hand sides," *Quarterly Journal of Mechanics and Applied Mathematics*, vol. 11, no. 4, pp. 385–398, 1958.

[33] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*. American Mathematical Society Providence, RI, 2001.

[34] C. Desoer and M. Vidyasagar, *Feedback systems: Input-output properties*. Society for Industrial and Applied Mathematics, 2008.

[35] L. Menini and A. Tornambe, "Asymptotic tracking of periodic trajectories for a simple mechanical system subject to nonsmooth impacts," *IEEE Transactions on Automatic Control*, vol. 46, pp. 1122–1126, July 2001.

[36] F. Forni and R. Sepulchre, "A differential Lyapunov framework for contraction analysis," *IEEE Transactions on Automatic Control*, vol. 59, no. 3, pp. 614–628, 2014.