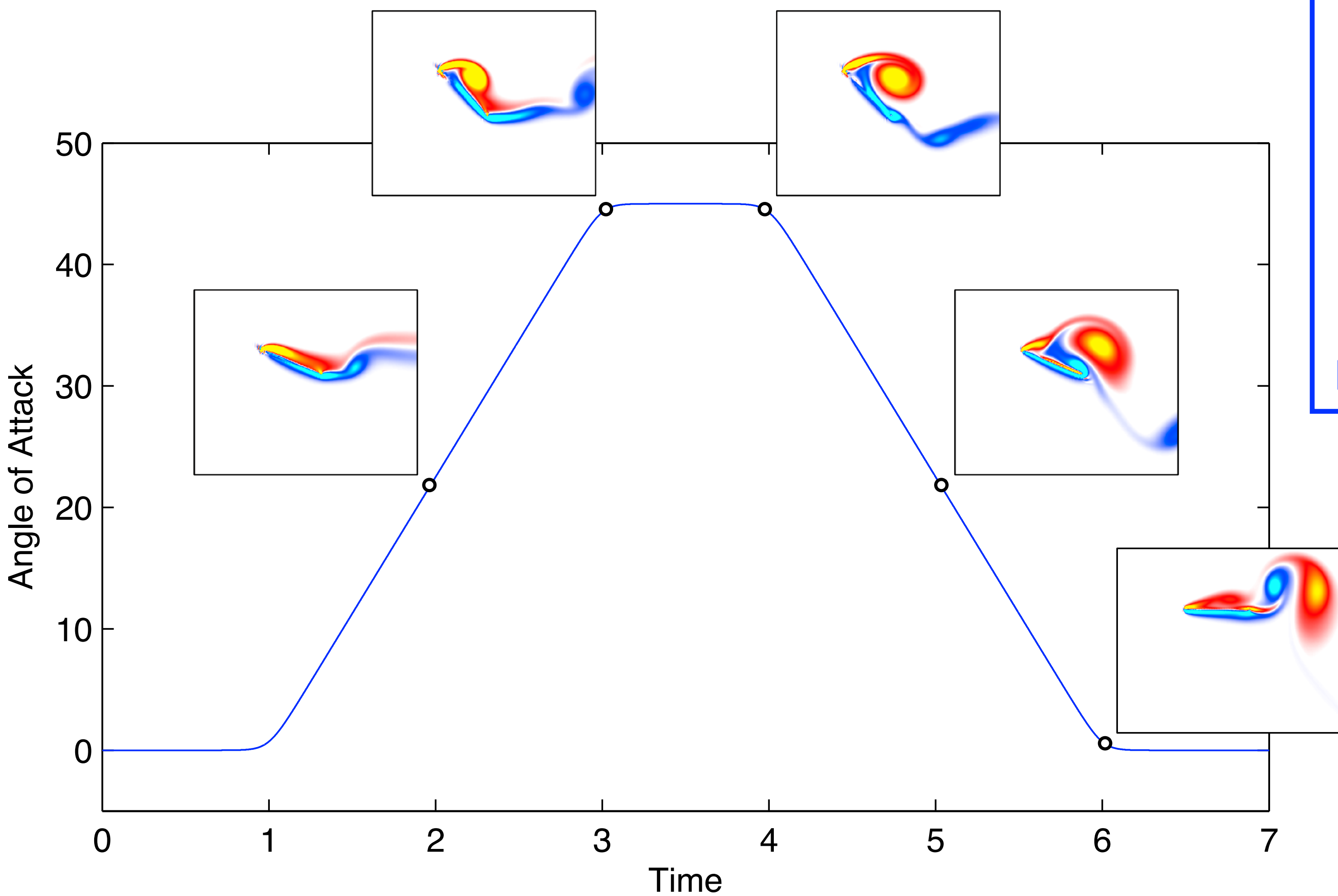
Low-dimensional state-space representations for classical unsteady aerodynamic models



$$\frac{d}{dt}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} = \begin{bmatrix}A_r & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}B_r\\ 0\\ 1\end{bmatrix}\ddot{\alpha}$$

input

$$C_L = \begin{bmatrix}C_r & C_{L_\alpha} & C_{L_{\dot{\alpha}}}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} + C_{L_{\ddot{\alpha}}}\ddot{\alpha}$$

ERA Model

quasi-steady and added-mass

Steve Brunton & Clancy Rowley
Princeton University

49th AIAA ASM January 6, 2011

Applications of Unsteady Models

- Conventional UAVs (performance/robustness)
- Micro air vehicles (MAVs)
- Flow control, flight dynamic control
- Autopilots / Flight simulators
- Gust disturbance mitigation
- Understand bird/insect flight

Need for State-Space Models

- Need models suitable for control
- Combining with flight models

FLYIT Simulators, Inc.

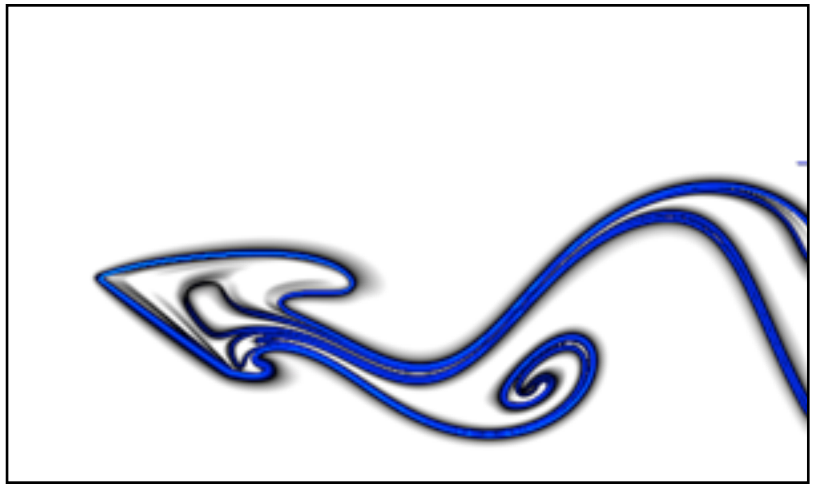
Predator (General Atomics)

Flexible Wing
(University of Florida)

Bio-locomotion

3 Types of Unsteadiness

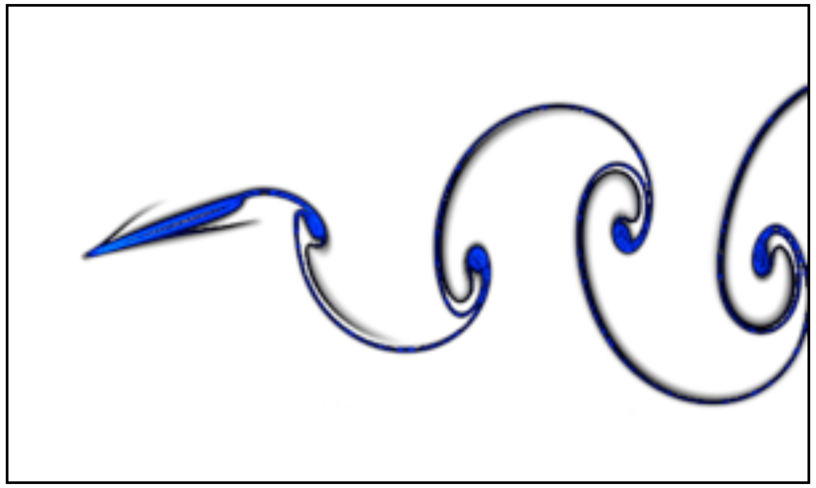
1. High angle-of-attack



$$\alpha > \alpha_{\text{stall}}$$

Large amplitude, slow

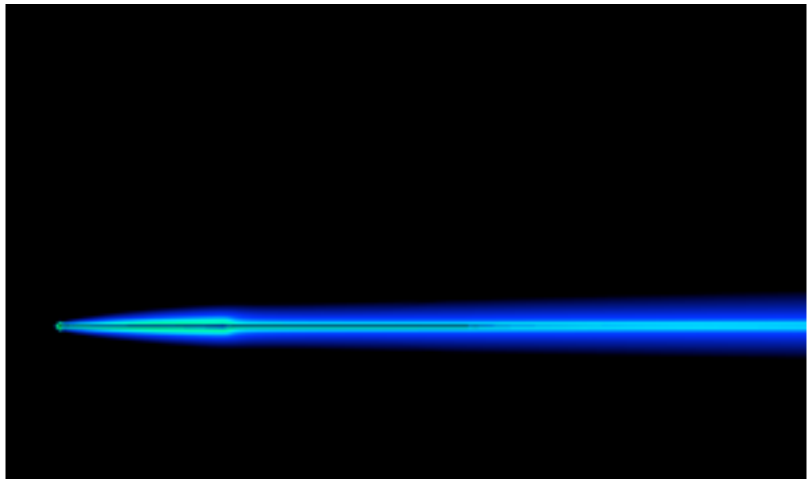
2. Strouhal number



$$St = \frac{Af}{U_\infty}$$

Moderate amplitude, fast

3. Reduced frequency



$$k = \frac{\pi f c}{U_\infty}$$

Small amplitude, very fast

Closely related

$$\alpha_{\text{eff}} = \tan^{-1}(\pi St)$$

Brunton and Rowley, AIAA ASM 2009

3 Types of Unsteadiness

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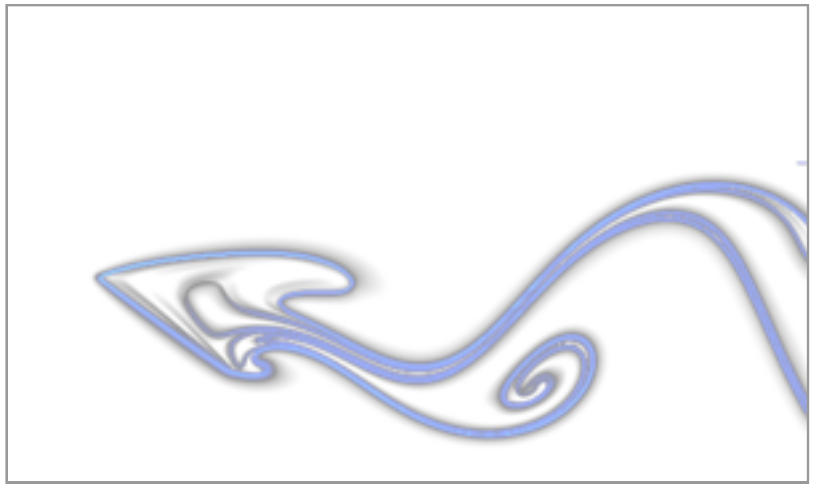

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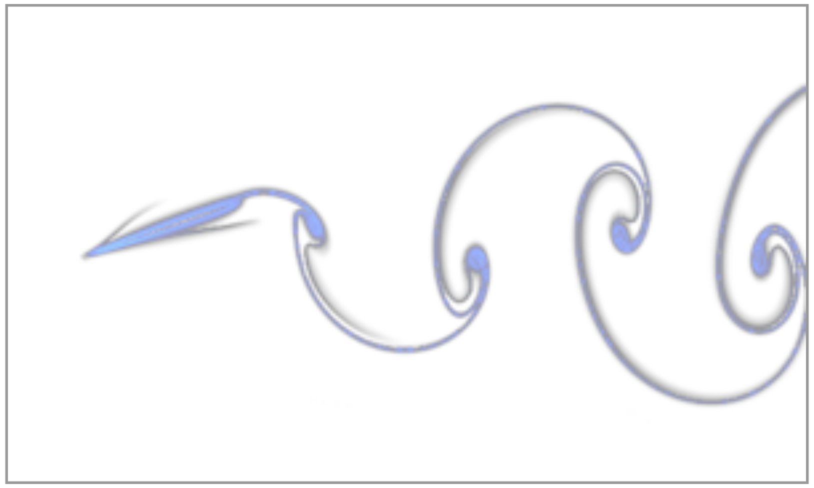

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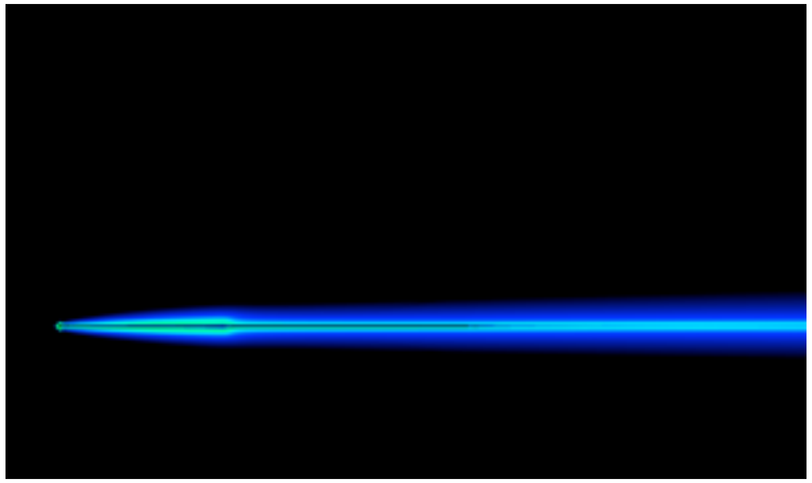


$$k = \frac{\pi f c}{U_\infty}$$

Small amplitude, very fast

Closely related

$$\alpha_{\text{eff}} = \tan^{-1}(\pi St)$$

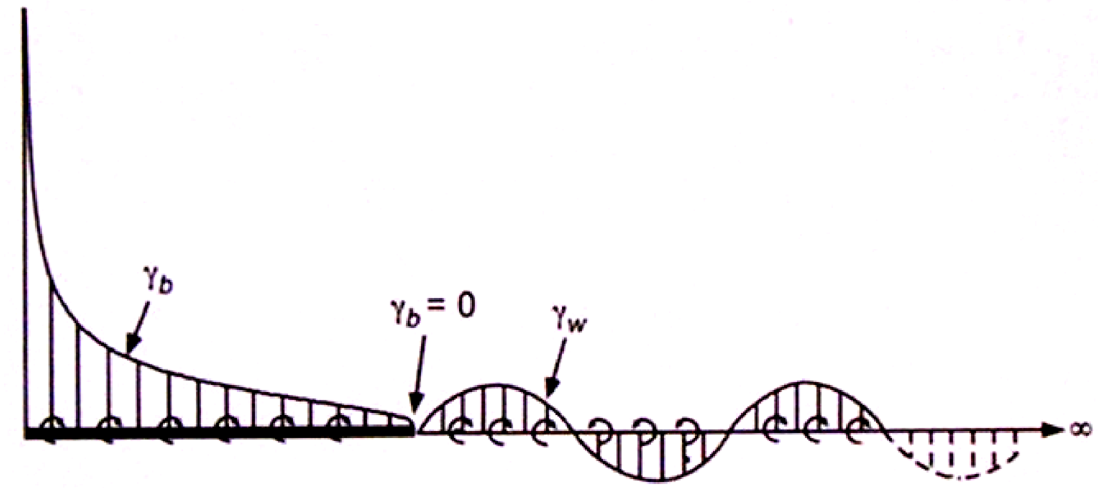
Brunton and Rowley, AIAA ASM 2009

Candidate Lift Models

$$C_L = 2\pi\alpha$$

$$C_L = C_{L_\alpha}\alpha$$

$$C_L = C_L(\alpha)$$



$$C_L(t) = C_L^\delta(t)\alpha(0) + \int_0^t C_L^\delta(t-\tau)\dot{\alpha}(\tau)d\tau$$

Wagner's Indicial Response

$$C_L = \underbrace{\frac{\pi}{2}\left[\ddot{h} + \dot{\alpha} - \frac{a}{2}\ddot{\alpha}\right]}_{\text{Added-Mass}} + \underbrace{2\pi\left[\alpha + \dot{h} + \frac{1}{2}\dot{\alpha}\left(\frac{1}{2} - a\right)\right]}_{\text{Circulatory}} C(k)$$

Theodorsen's Model

Motivation for State-Space Models

Captures input output dynamics accurately

Computationally tractable

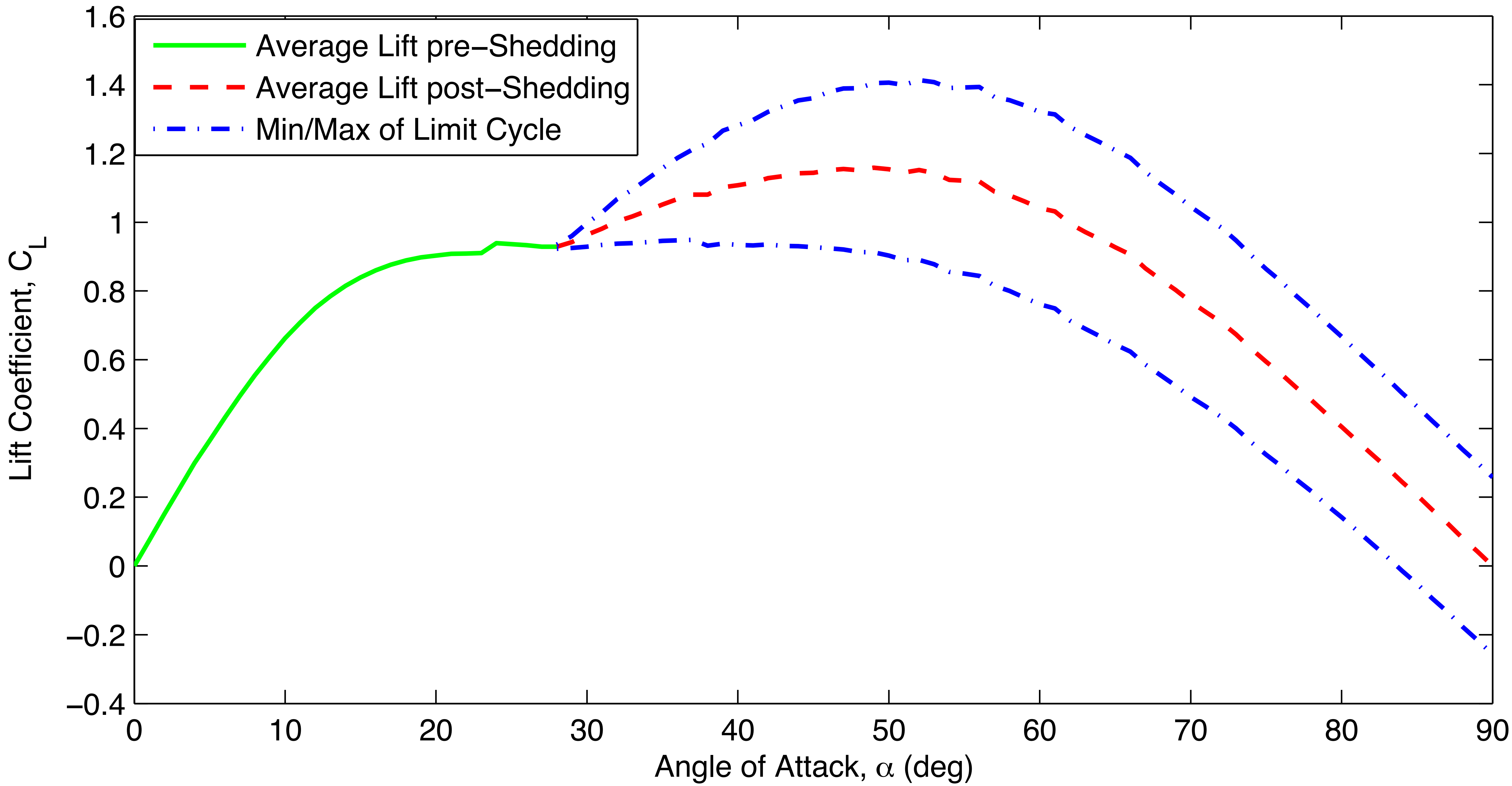
fits into control framework

Wagner, 1925.

Theodorsen, 1935.

Leishman, 2006.

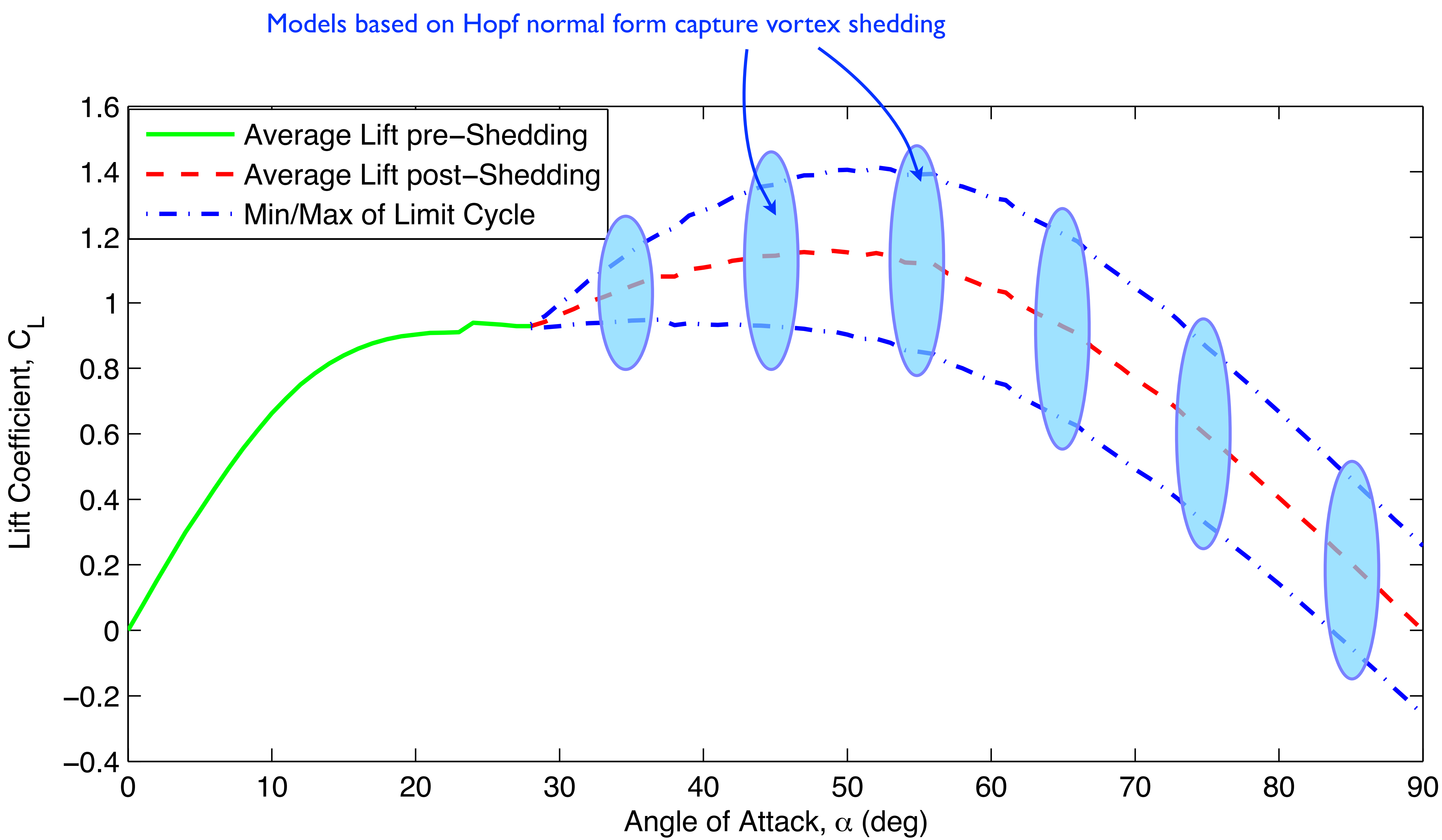
Lift vs Angle of Attack



Low Reynolds number, (Re=100)

Hopf bifurcation at $\alpha_{\text{crit}} \approx 28^\circ$ **(pair of imaginary eigenvalues pass into right half plane)**

Lift vs Angle of Attack



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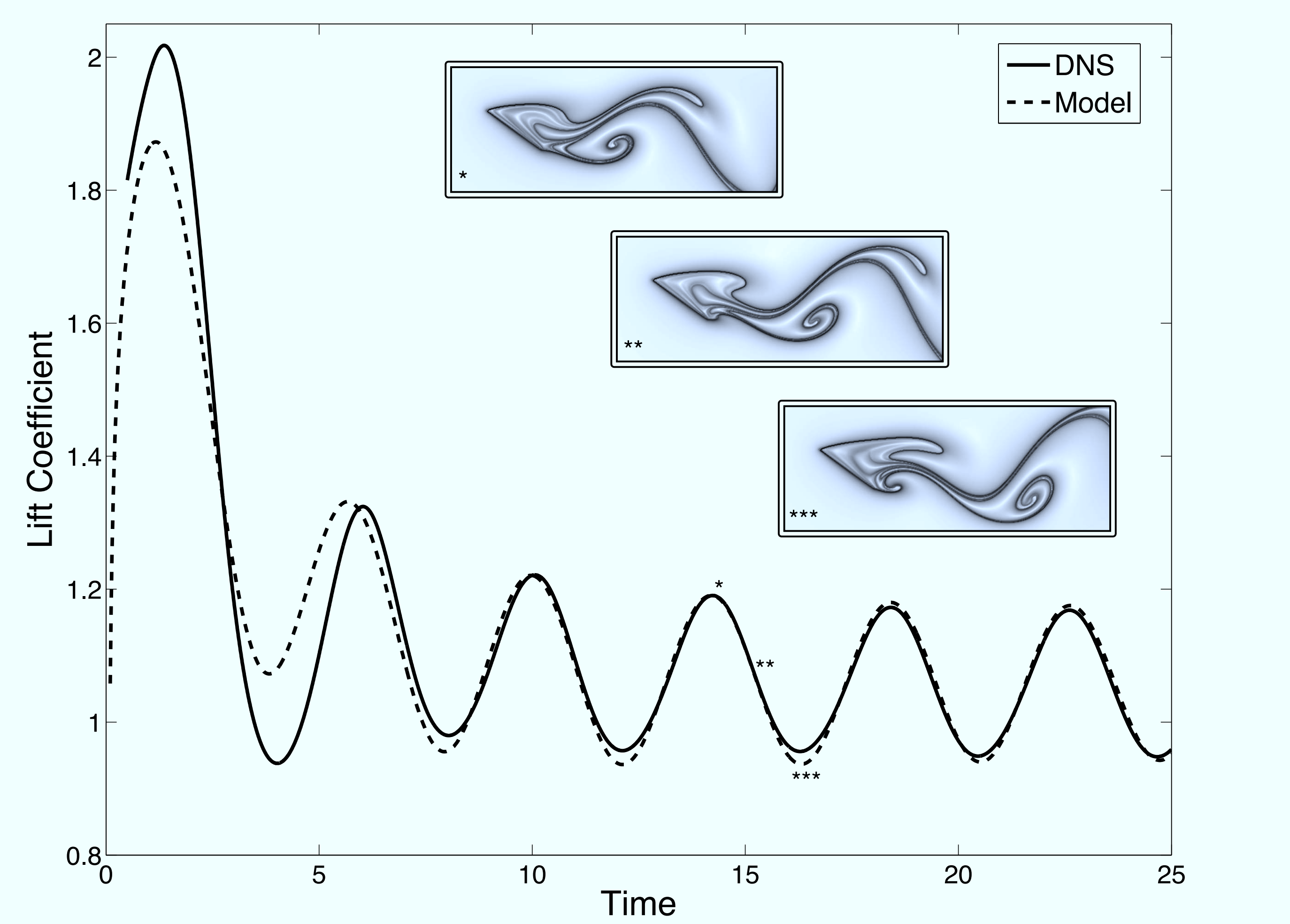
High angle of attack models

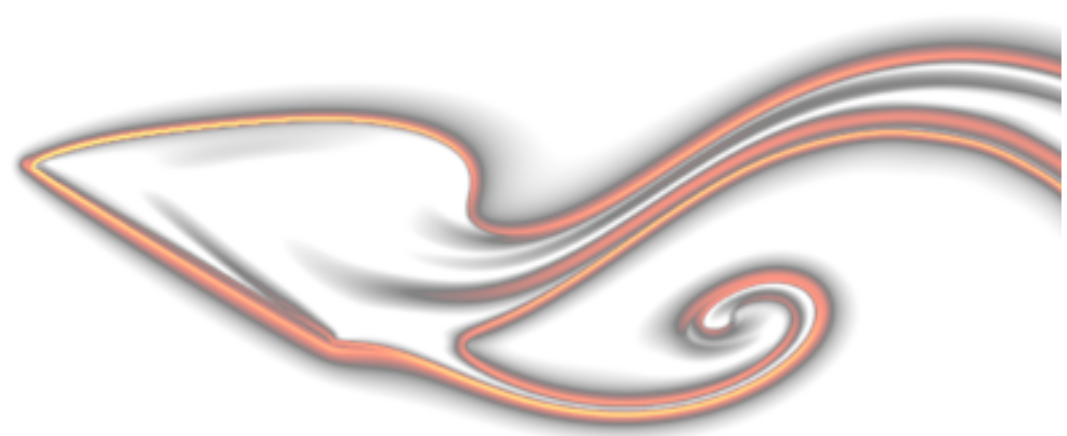
Heuristic Model



$$
\left.\begin{aligned}
\dot{x} &= (\alpha - \alpha_c)\mu x - \omega y - a x (x^2 + y^2) \\
\dot{y} &= (\alpha - \alpha_c)\mu y + \omega x - a y (x^2 + y^2) \\
\dot{z} &= -\lambda z
\end{aligned}\right\} \implies
\begin{aligned}
\dot{r} &= r\left[(\alpha - \alpha_c)\mu - a r^2\right] \\
\dot{\theta} &= \omega \\
\dot{z} &= -\lambda z
\end{aligned}
$$

Galerkin Projection onto POD

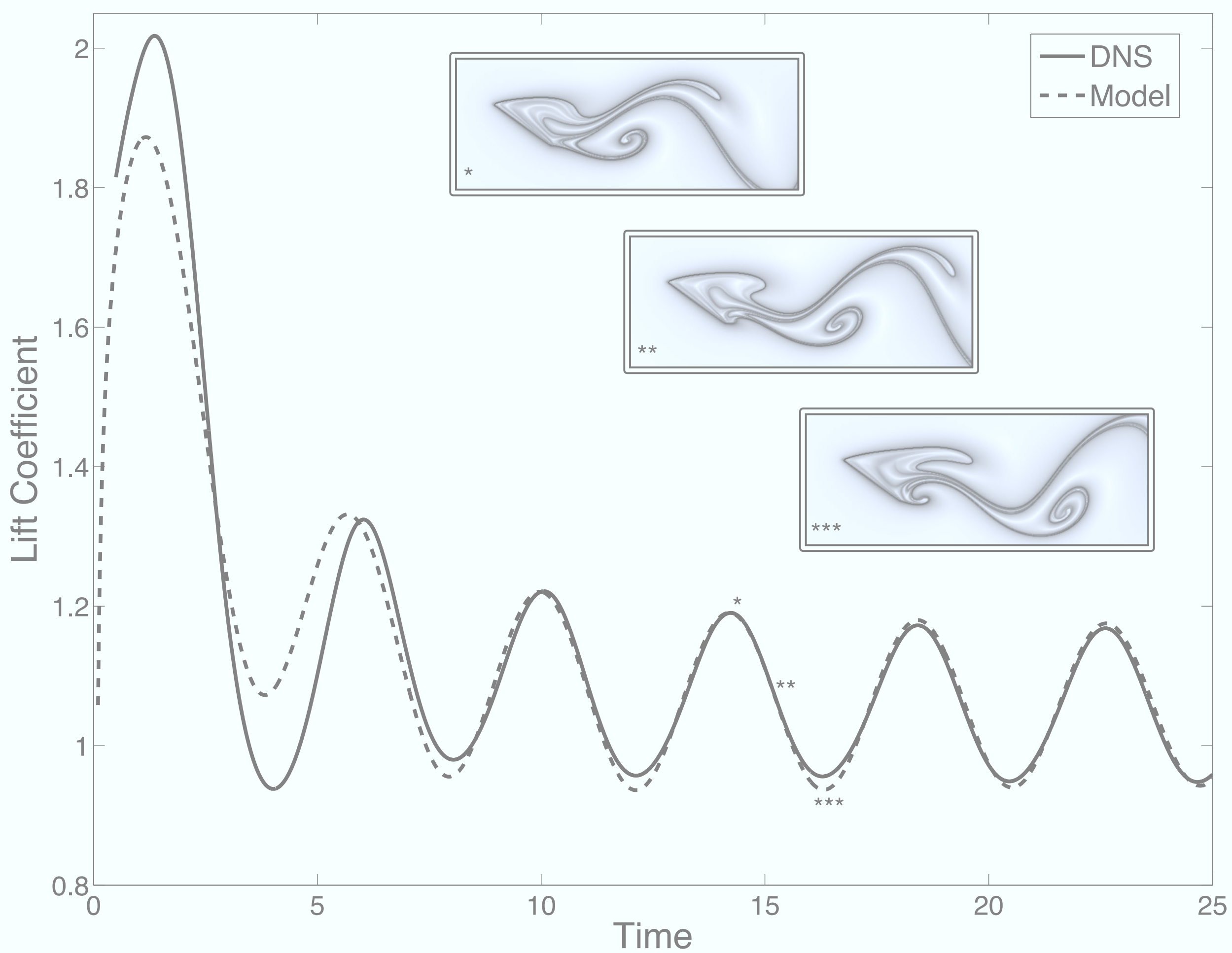
Full DNS



Reconstruction

High angle of attack models

Heuristic Model



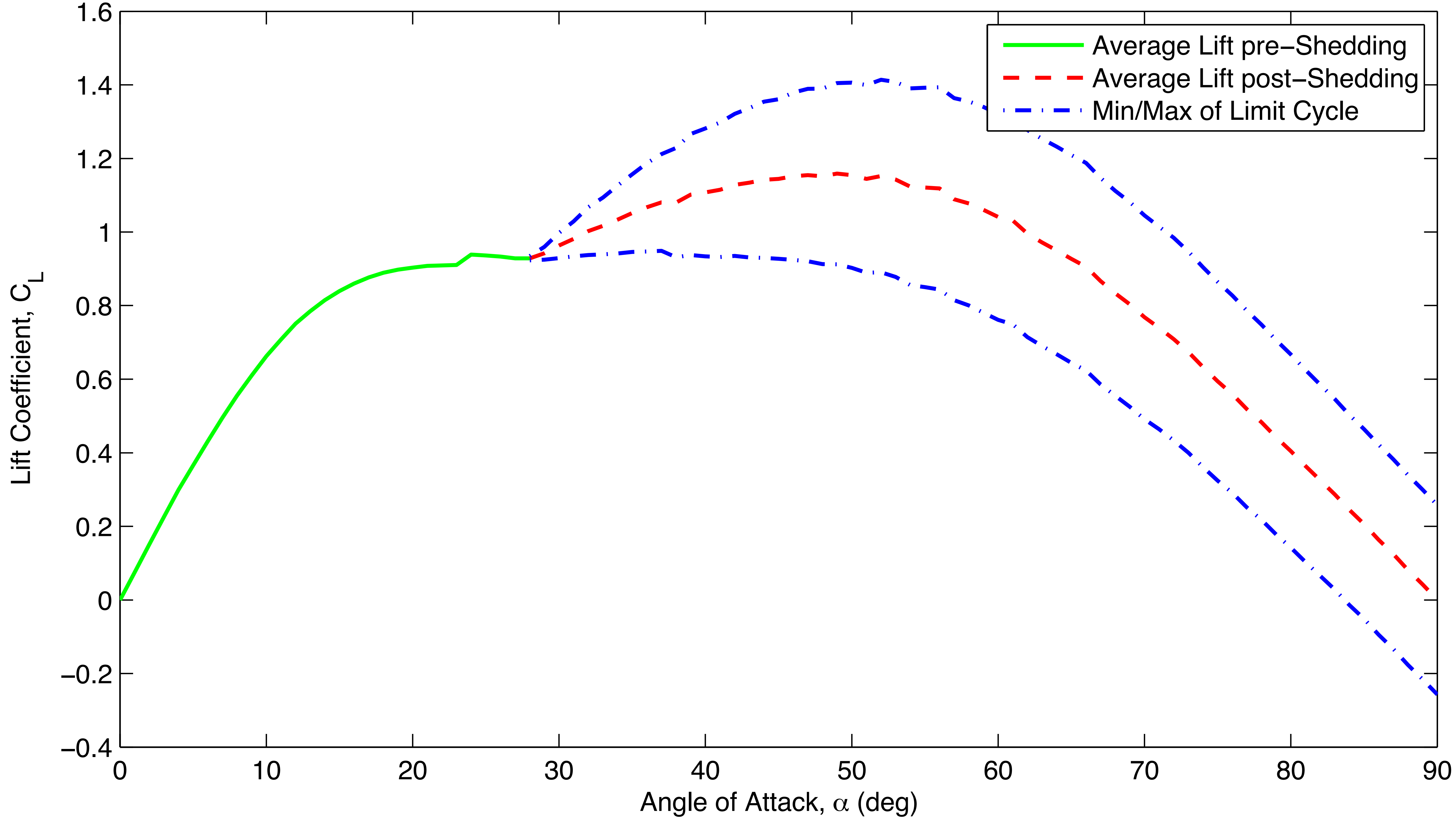
$$\begin{aligned}\dot{x} &= (\alpha-\alpha_c)\mu x - \omega y - a x(x^2+y^2)\\ \dot{y} &= (\alpha-\alpha_c)\mu y + \omega x - a y(x^2+y^2)\\ \dot{z} &= -\lambda z\end{aligned}\Bigg\} \implies \begin{aligned}\dot{r} &= r\left[(\alpha-\alpha_c)\mu - a r^2\right]\\ \dot{\theta} &= \omega\\ \dot{z} &= -\lambda z\end{aligned}$$

Galerkin Projection onto POD

Full DNS

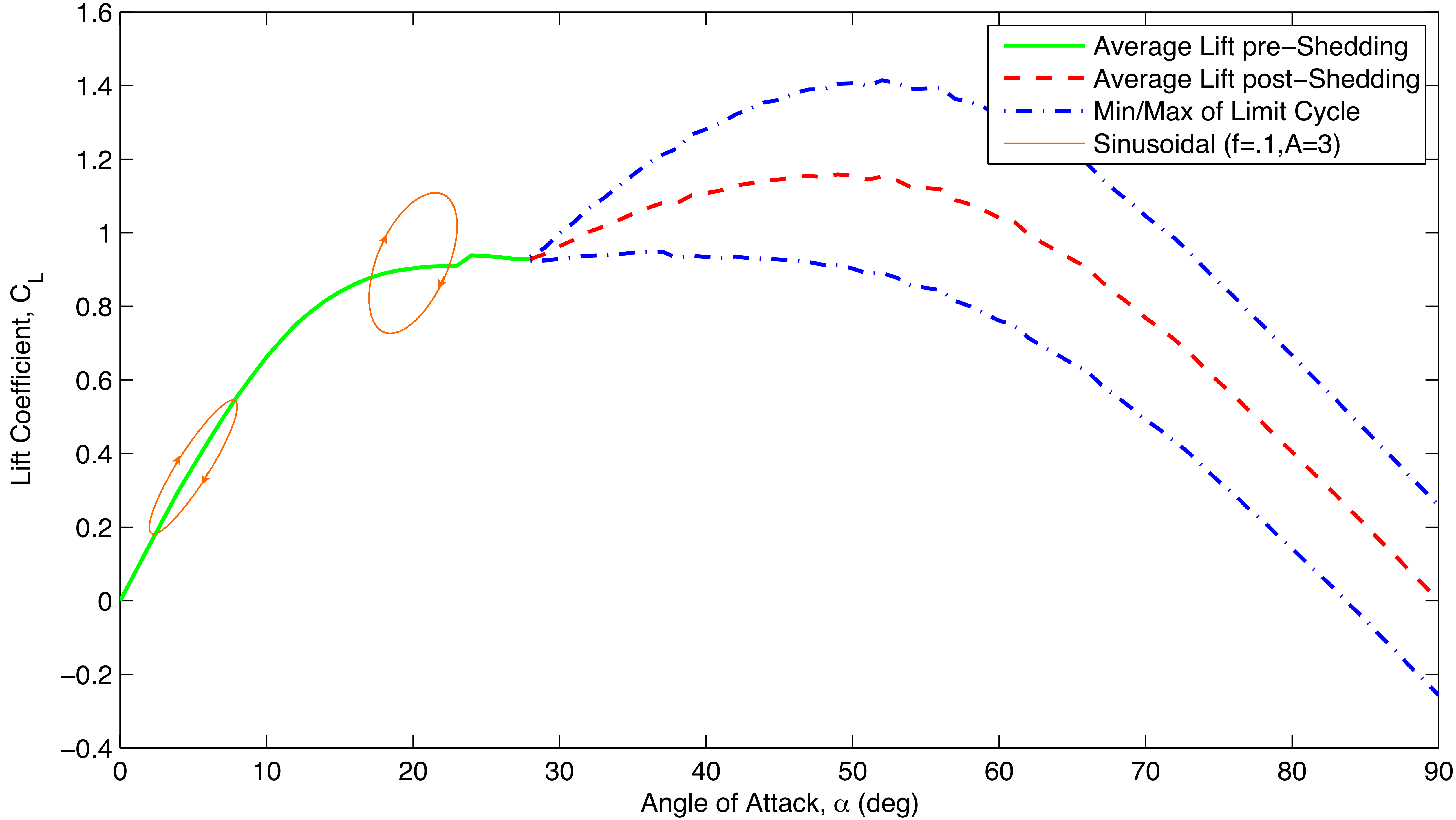
Reconstruction

Lift vs. Angle of Attack



Need model that captures lift due to moving airfoil!

Lift vs. Angle of Attack



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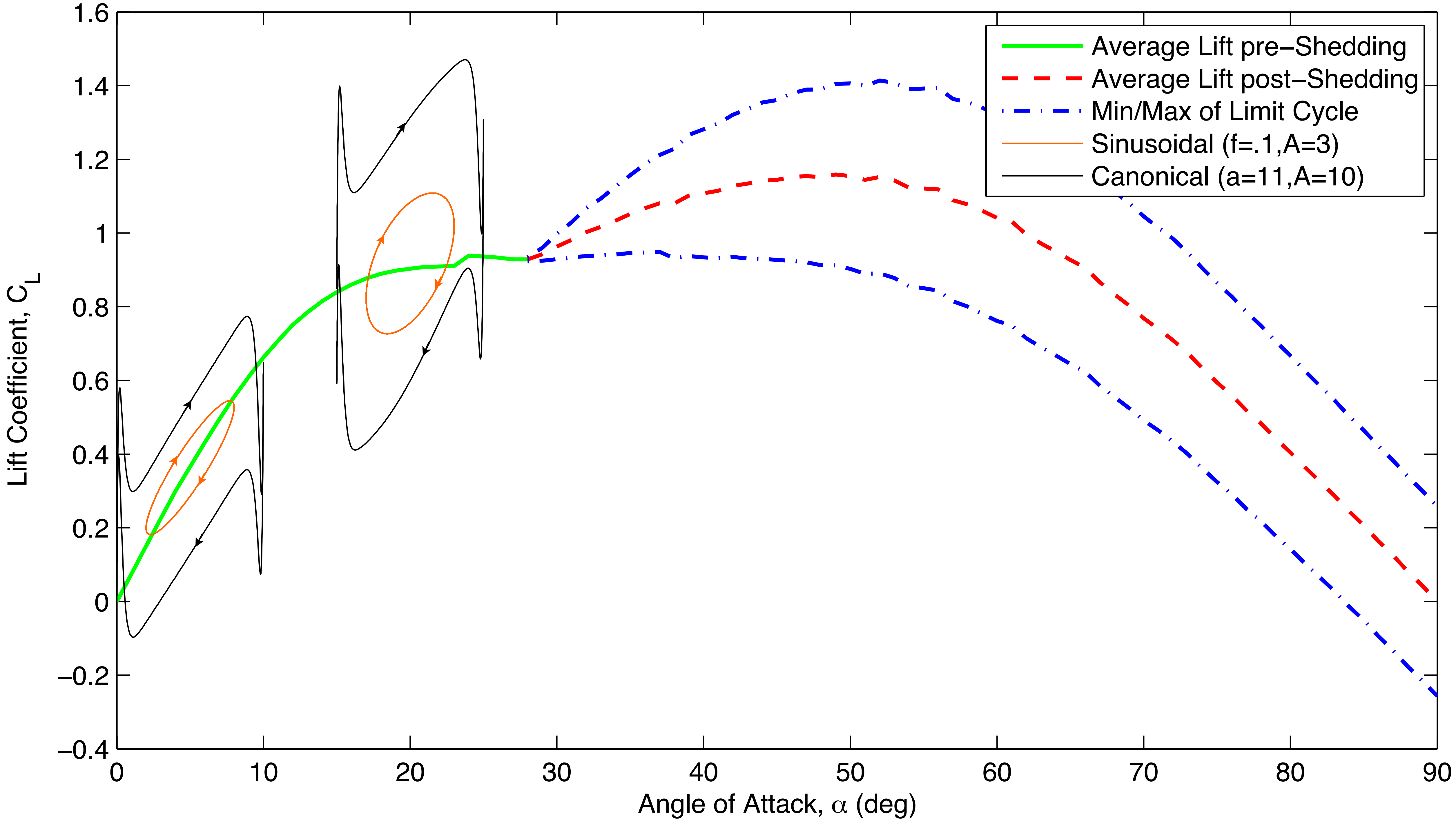

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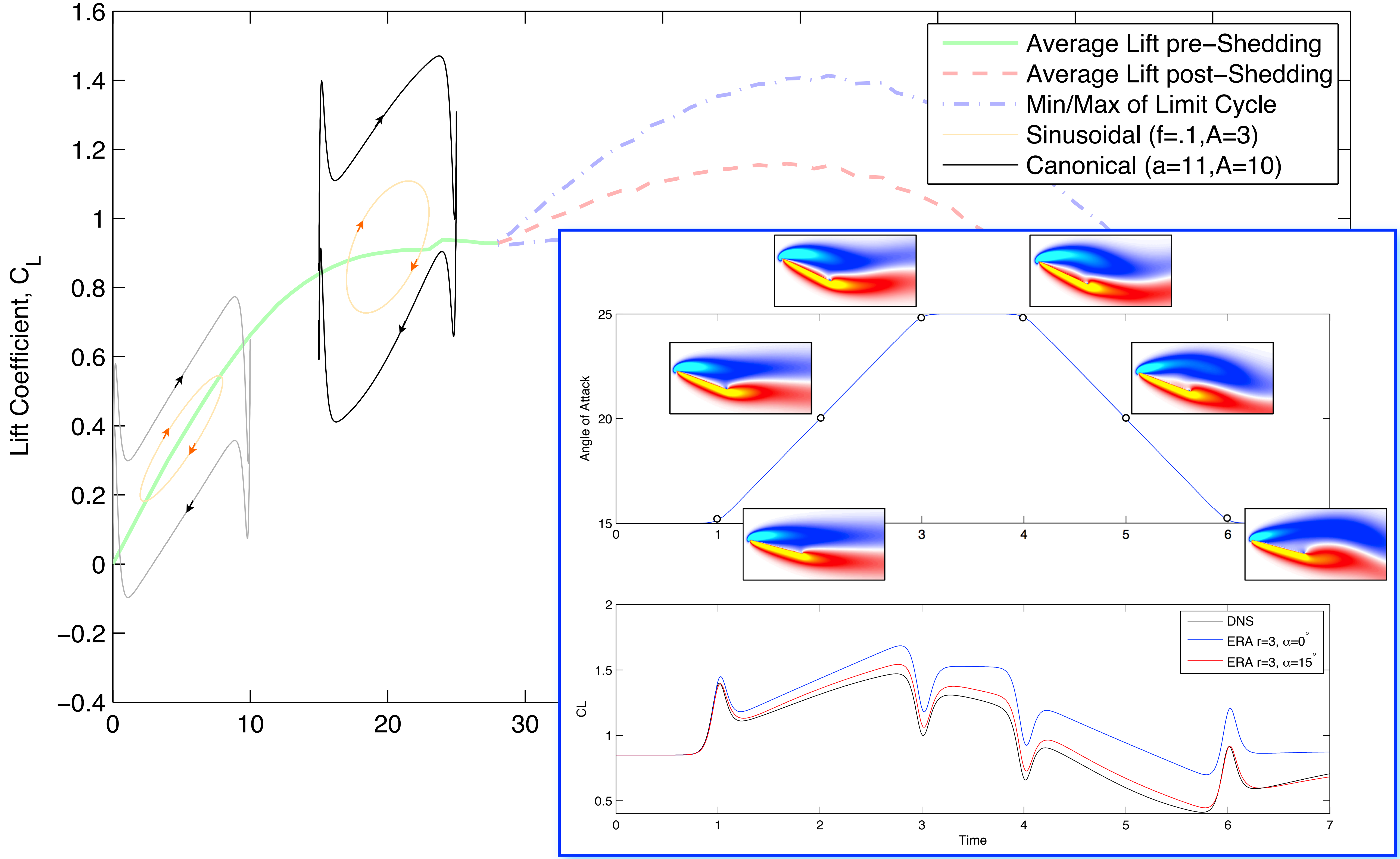

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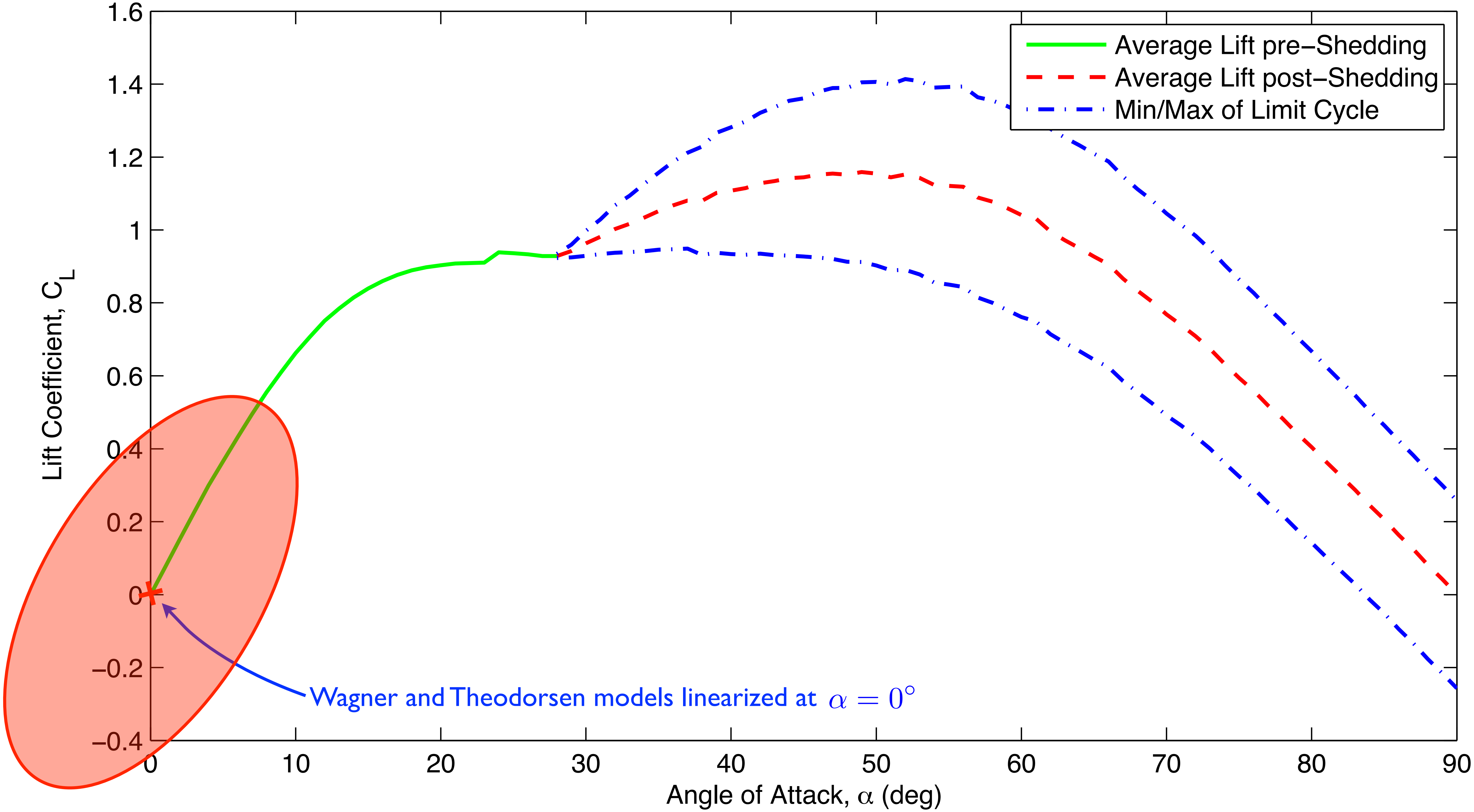
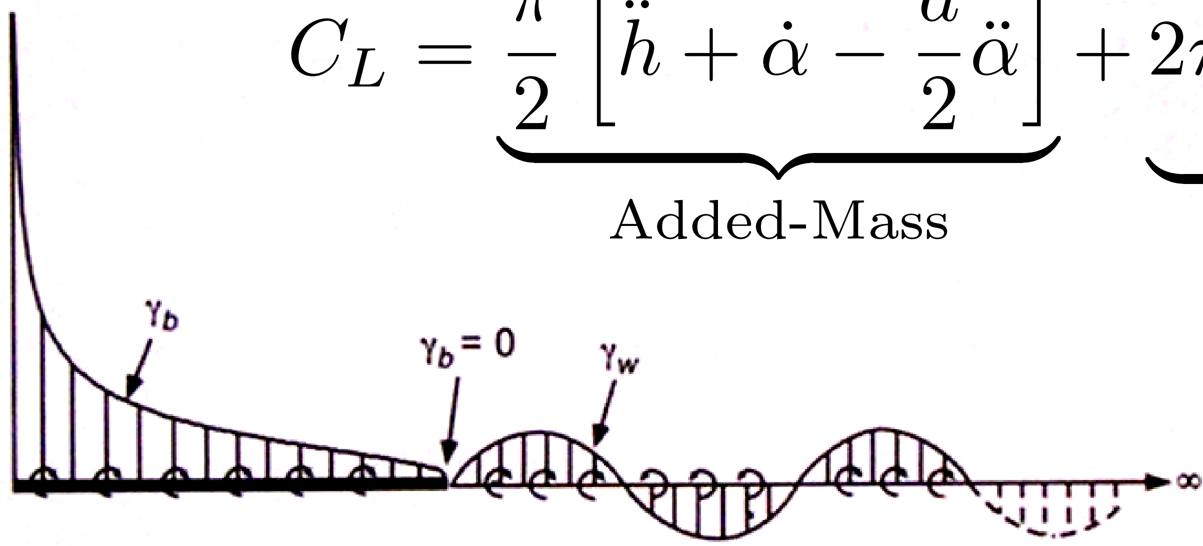


Theodorsen's Model

$$C_L = \underbrace{\frac{\pi}{2}\left[\ddot{h} + \dot{\alpha} - \frac{a}{2}\ddot{\alpha}\right]}_{\text{Added-Mass}} + \underbrace{2\pi\left[\alpha + \dot{h} + \frac{1}{2}\dot{\alpha}\left(\frac{1}{2} - a\right)\right]}_{\text{Circulatory}} C(k)$$



$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)}$$

2D Incompressible, inviscid model

Unsteady potential flow (w/ Kutta condition)

Linearized about zero angle of attack

$$\boxed{k = \frac{\pi f c}{U_\infty}}$$

Apparent Mass

Increasingly important for lighter aircraft

Not trivial to compute, but essentially solved

force needed to move air as plate accelerates

Circulatory Lift

Captures separation effects

Need improved models here

source of all lift in steady flight

Theodorsen, 1935.

Leishman, 2006.

Empirical Theodorsen

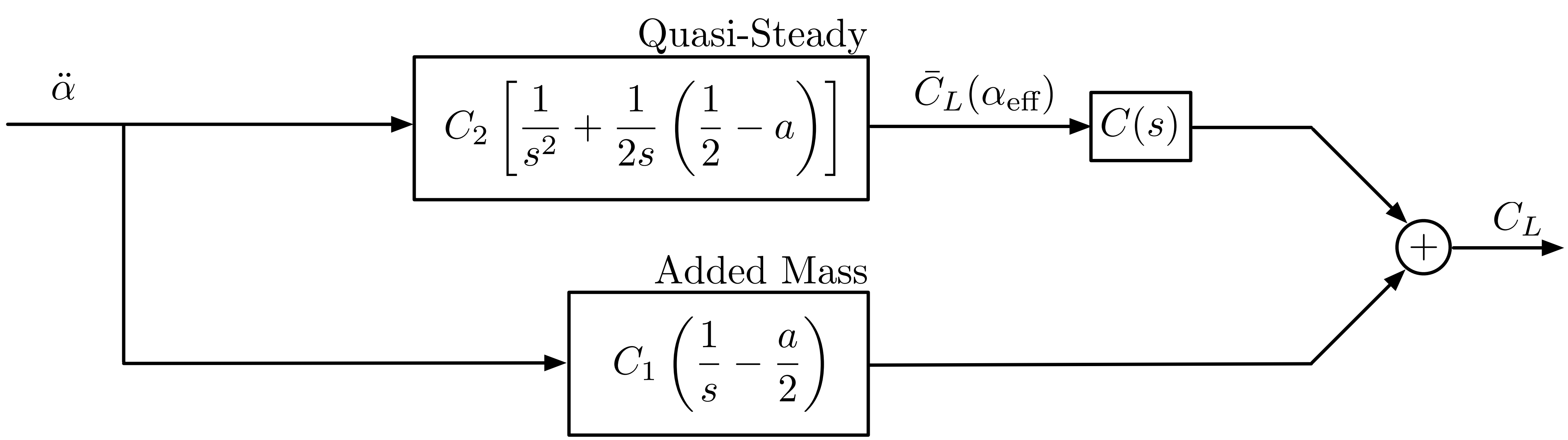
$$C_L = \underbrace{\frac{\pi}{2}\left[\ddot{h} + \dot{\alpha} - \frac{a}{2}\ddot{\alpha}\right]}_{\text{Added-Mass}} + \underbrace{2\pi\left[\alpha + \dot{h} + \frac{1}{2}\dot{\alpha}\left(\frac{1}{2} - a\right)\right]}_{\text{Circulatory}} C(k)$$

Generalized Coefficients

$$C_L = C_1\left[\dot{\alpha} - \frac{a}{2}\ddot{\alpha}\right] + C_2\left[\alpha + \frac{1}{2}\dot{\alpha}\left(\frac{1}{2} - a\right)\right] C(k)$$

Transfer Function

$$\frac{\mathcal{L}\left[C_L\right]}{\mathcal{L}\left[\ddot{\alpha}\right]} = C_1\left(\frac{1}{s} - \frac{a}{2}\right) + C_2\left[\frac{1}{s^2} + \frac{1}{2s}\left(\frac{1}{2} - a\right)\right] C(s)$$



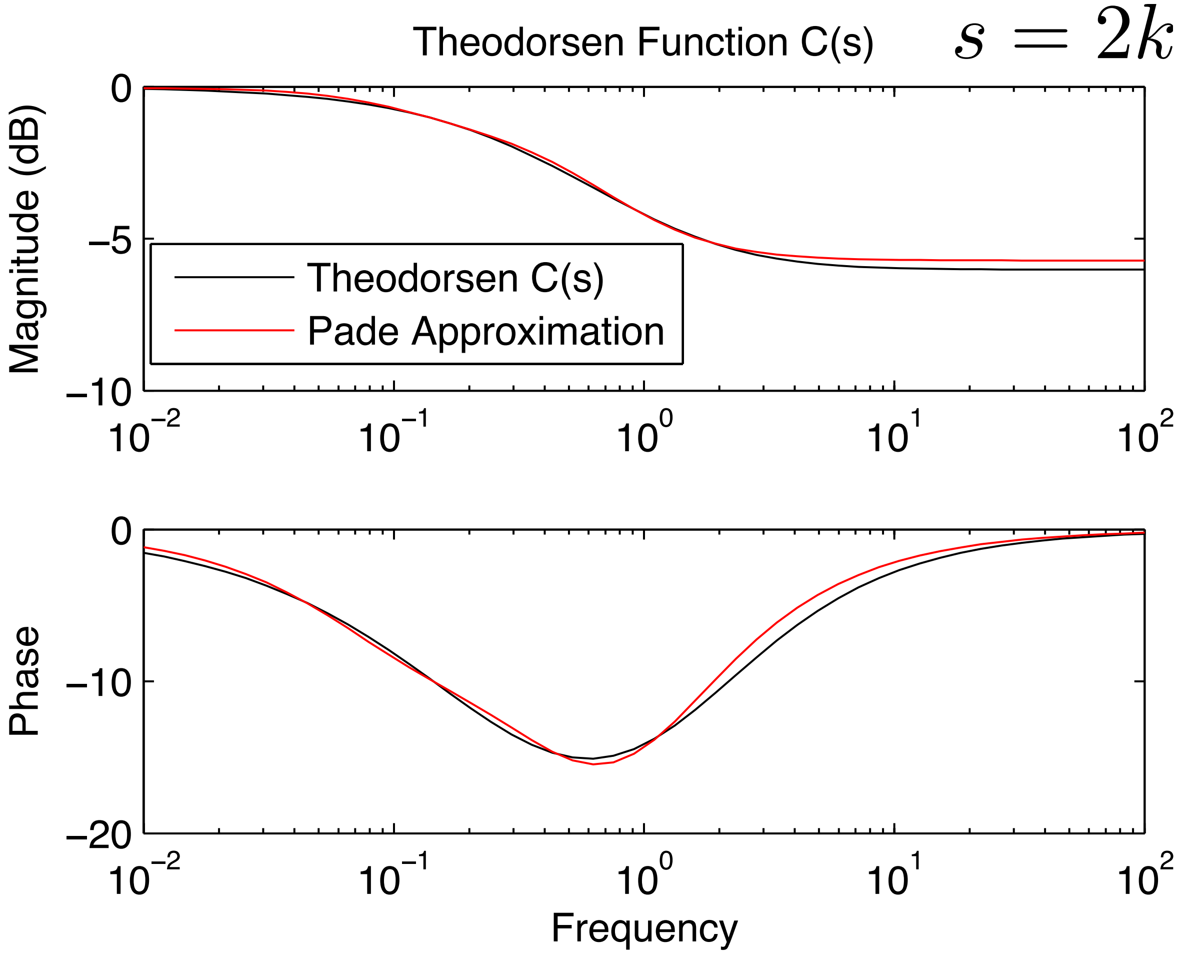
Pade Approximate C(k)

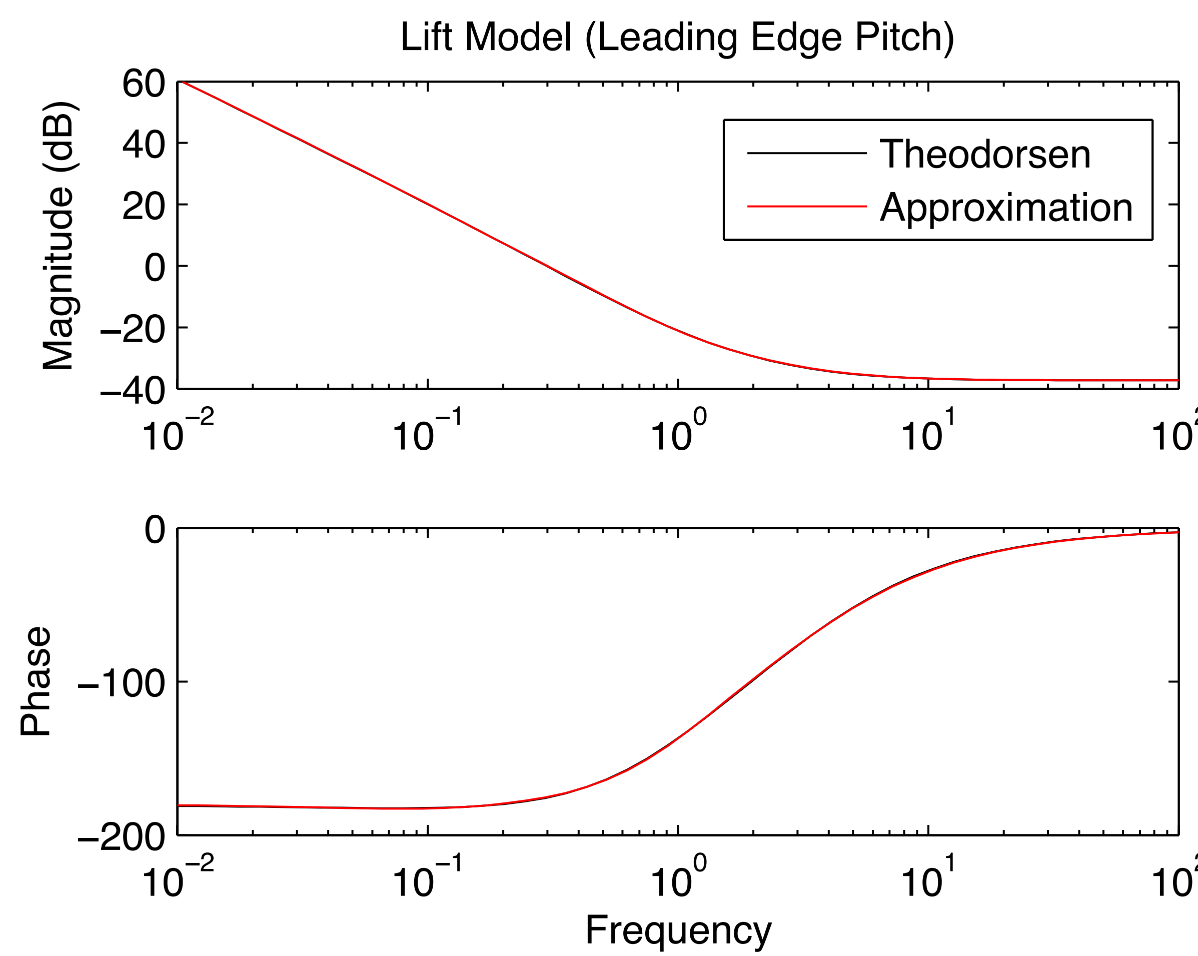
$$C(k) \approx .99612 - .1666\frac{k}{k+.0553} - .3119\frac{k}{k+.28606}$$

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)}$$

$$C(s) \approx \frac{.1294s^2 + .1376s + .01576}{.25s^2 + .1707s + .01582}$$

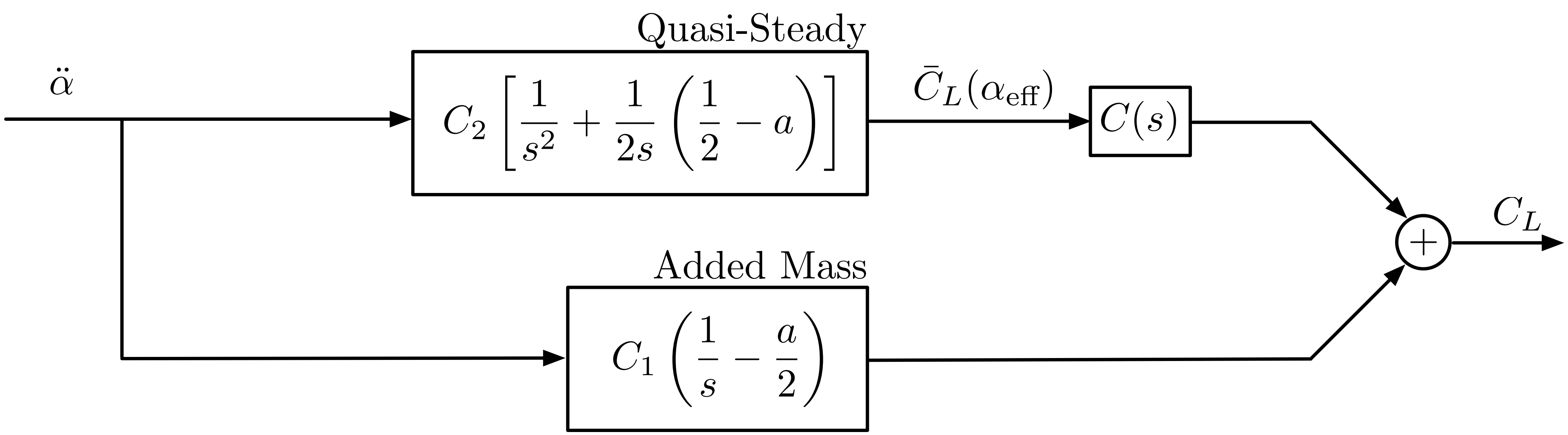
$$s = 2k$$





Breuker, Abdalla, Milanese, and Marzocca,
AIAA *Structures, Structural Dynamics, and Materials Conference* 2008.

Empirical C(s)

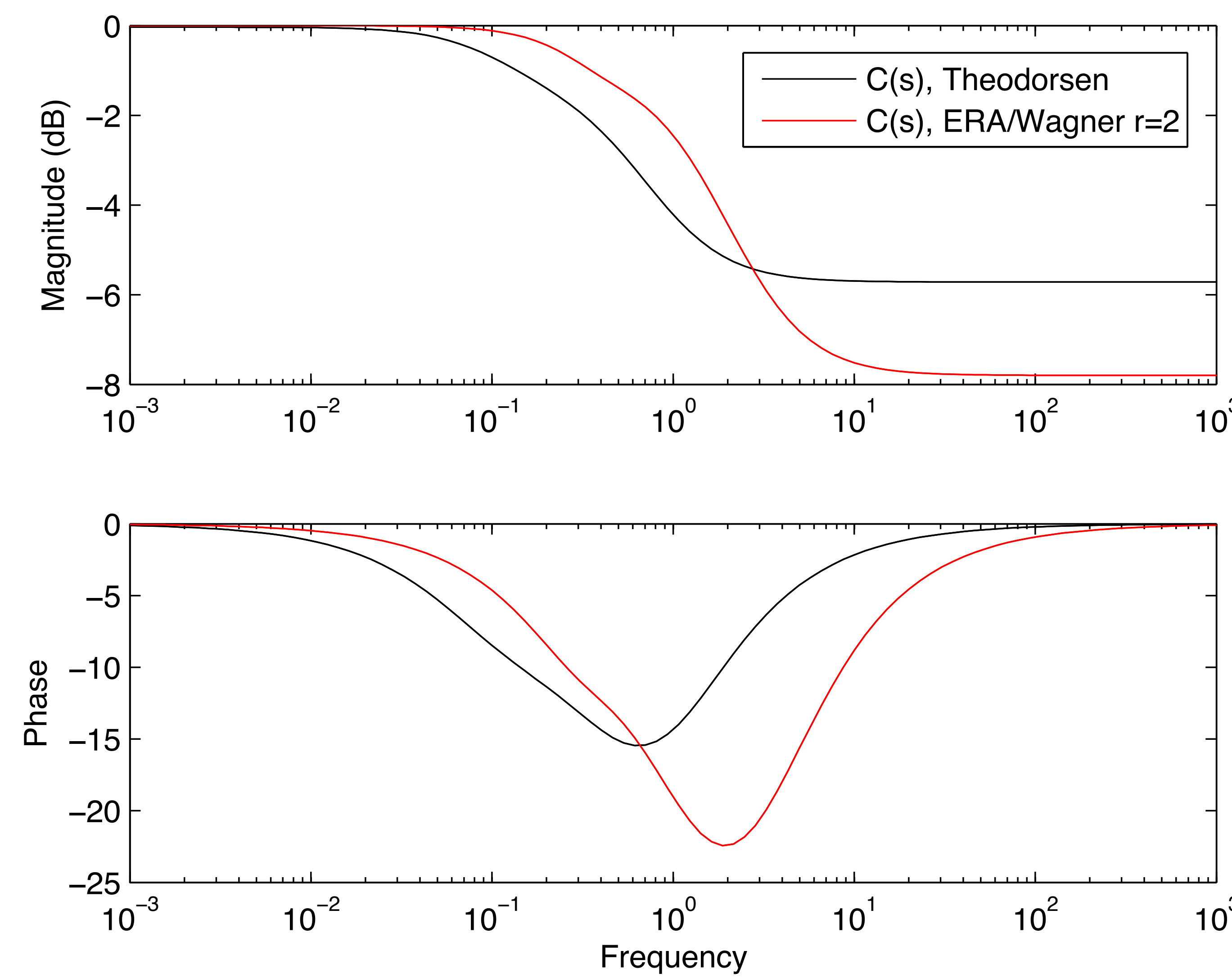


Isolating C(k)

Start with empirical ERA model

Subtract off quasi-steady and divide through by added-mass

Remainder is C(k)



Alternative Representation

Generalized Theodorsen

$$C_L = C_1\left[\dot\alpha - \frac{a}{2}\ddot\alpha\right] + C_2\left[\alpha + \frac{1}{2}\dot\alpha\left(\frac{1}{2}-a\right)\right]C(k)$$

$$C_L = \underbrace{-\frac{a}{2}C_1}_{C_{L_{\ddot\alpha}}}\ddot\alpha + \underbrace{\left[C_1 + \frac{C_2}{2}\left(\frac{1}{2}-a\right)\right]}_{C_{L_{\dot\alpha}}}\dot\alpha + \underbrace{C_2}_{C_{L_\alpha}}\alpha - \underbrace{C_2 C'(k)\left[\alpha + \frac{1}{2}\dot\alpha\left(\frac{1}{2}-a\right)\right]}_{\text{fast dynamics}}$$

$$C_L(\alpha,\dot\alpha,\ddot\alpha,\mathbf{x}) = C_{L_\alpha}\alpha + C_{L_{\dot\alpha}}\dot\alpha + C_{L_{\ddot\alpha}}\ddot\alpha + C\mathbf{x}$$

Stability derivatives plus fast dynamics

State-Space Representation

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \alpha \\ \dot\alpha \end{bmatrix} = \begin{bmatrix} -.6828 & -.0633 & C_2 & C_2(1-2a)/4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \alpha \\ \dot\alpha \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\ddot\alpha$$

$$C_L = \begin{bmatrix} .197 & .0303 & .5176C_2 & C_1 + .5176C_2(1-2a)/4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \alpha \\ \dot\alpha \end{bmatrix} - \frac{aC_1}{2}\ddot\alpha$$

Bode Plot of Theodorsen

$$C_L = \underbrace{\frac{\pi}{2}\left[\ddot{h} + \dot{\alpha} - \frac{a}{2}\ddot{\alpha}\right]}_{\text{Added-Mass}} + \underbrace{2\pi\left[\alpha + \dot{h} + \frac{1}{2}\dot{\alpha}\left(\frac{1}{2} - a\right)\right]}_{\text{Circulatory}} C(k)$$

$$\boxed{k = \frac{\pi f c}{U_\infty}}$$

Frequency response

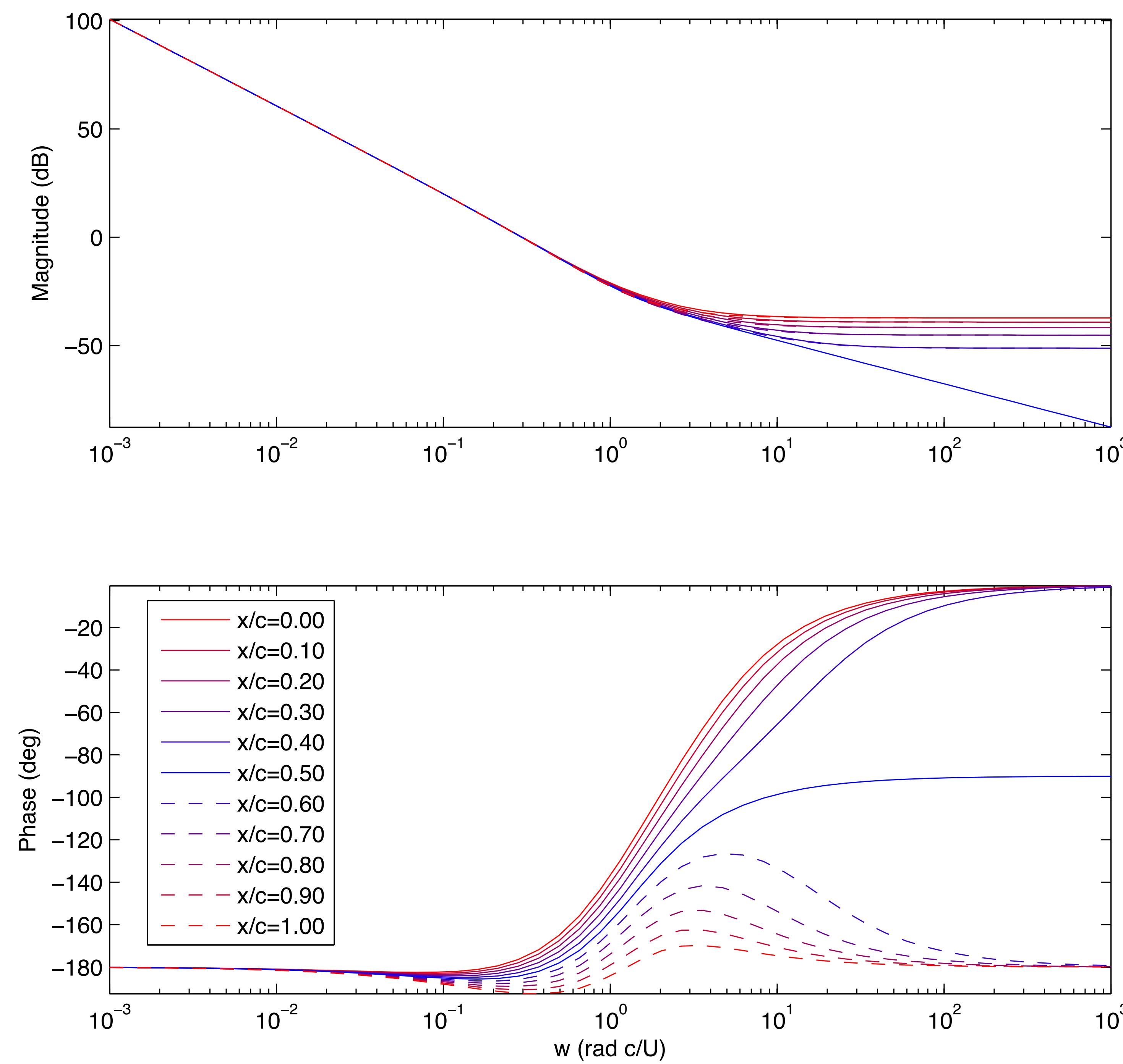
input is $\ddot{\alpha}$ **(** α **is angle of attack)**

output is lift coefficient $\mathrm{C_L}$

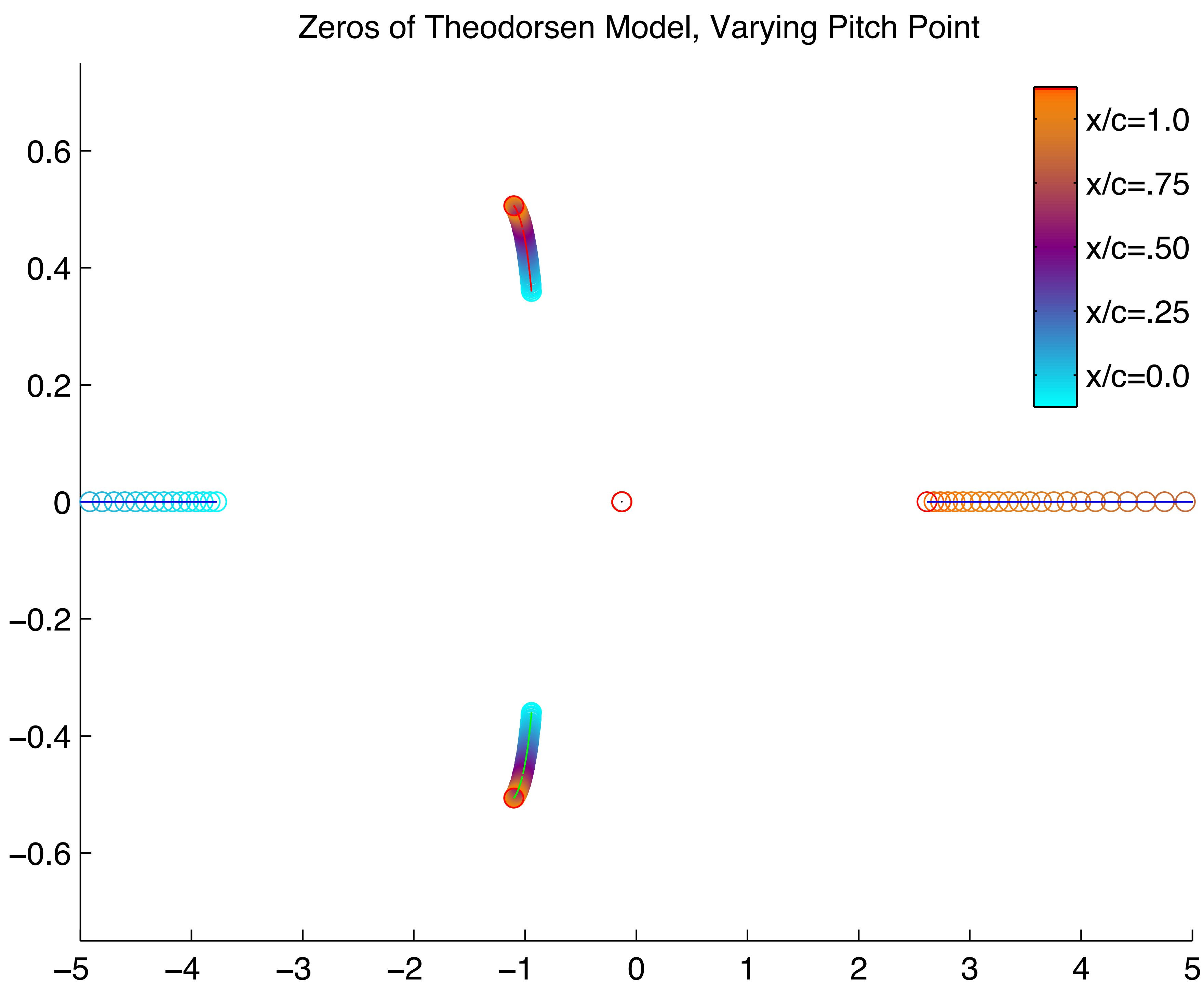
Low frequencies dominated by quasi-steady forces

High frequencies dominated by added-mass forces

Crossover region determined by Theodorsen's function $\mathrm{C(k)}$



Zeros of Theodorsen's Model



As pitch point moves aft of center, zero enters RHP at +infinity.

Wagner's Indicial Response

Given an impulse in angle of attack, $\alpha = \delta(t)$, the time history of Lift is $C_L^\delta(t)$

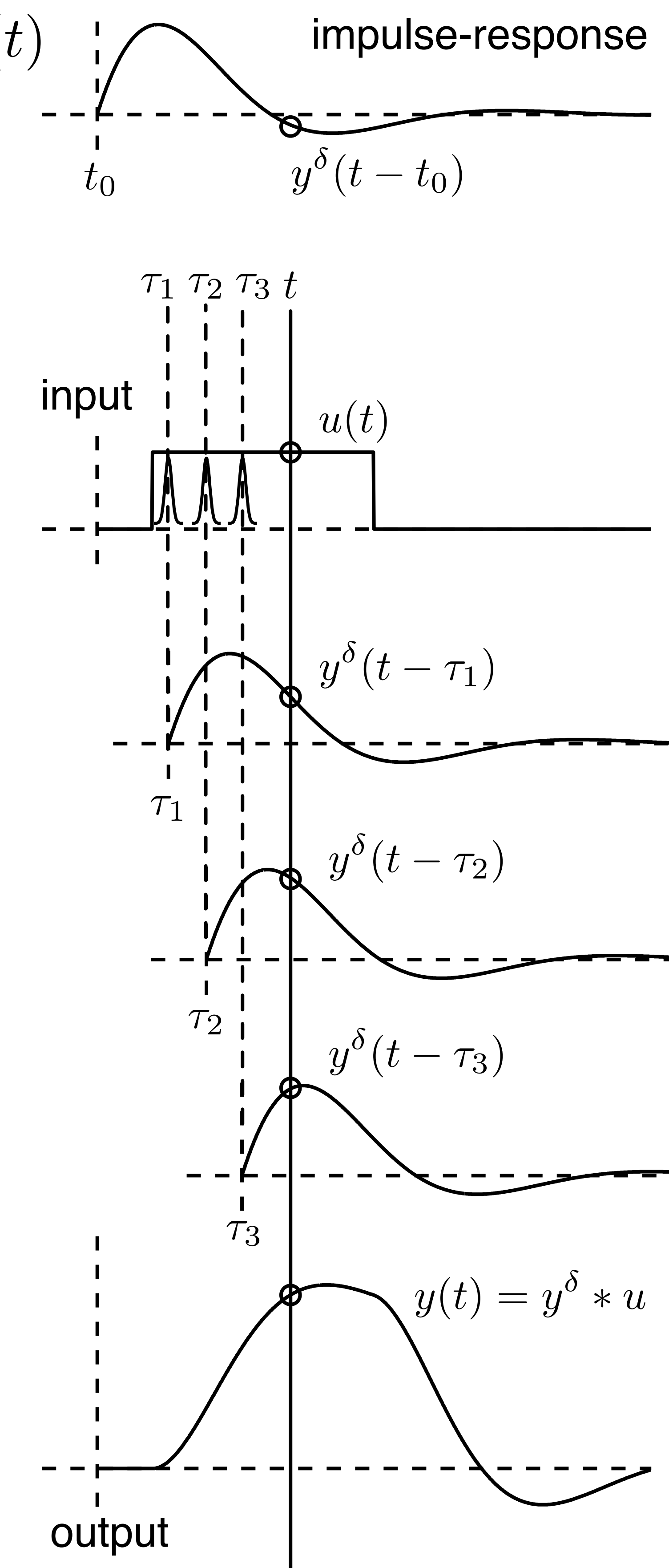
The response to an arbitrary input $\alpha(t)$ is given by linear superposition:

$$C_L(t) = \int_0^t C_L^\delta(t-\tau)\alpha(\tau)d\tau = \left(C_L^\delta * \alpha\right)(t)$$

Given a step in angle of attack, $\dot{\alpha} = \delta(t)$, the time history of Lift is $C_L^S(t)$

The response to an arbitrary input $\alpha(t)$ is given by:

$$C_L(t) = C_L^S(t)\alpha(0) + \int_0^t C_L^S(t-\tau)\dot{\alpha}(\tau)d\tau$$



Model Summary

Reconstructs Lift for arbitrary input

Linearized about $\alpha = 0$

Based on experiment, simulation or theory

convolution integral inconvenient for feedback control design

Wagner, 1925.

Leishman, 2006.

Reduced Order Wagner

Stability derivatives plus fast dynamics

$$C_L(\alpha, \dot{\alpha}, \ddot{\alpha}, \mathbf{x}) = C_{L_\alpha}\alpha + C_{L_{\dot{\alpha}}}\dot{\alpha} + C_{L_{\ddot{\alpha}}}\ddot{\alpha} + C\mathbf{x}$$

Quasi-steady and added-mass

Fast dynamics

Transfer Function

$$Y(s) = \left[\frac{C_{L_\alpha}}{s^2} + \frac{C_{L_{\dot{\alpha}}}}{s} + C_{L_{\ddot{\alpha}}} + G(s)\right] s^2 U(s)$$

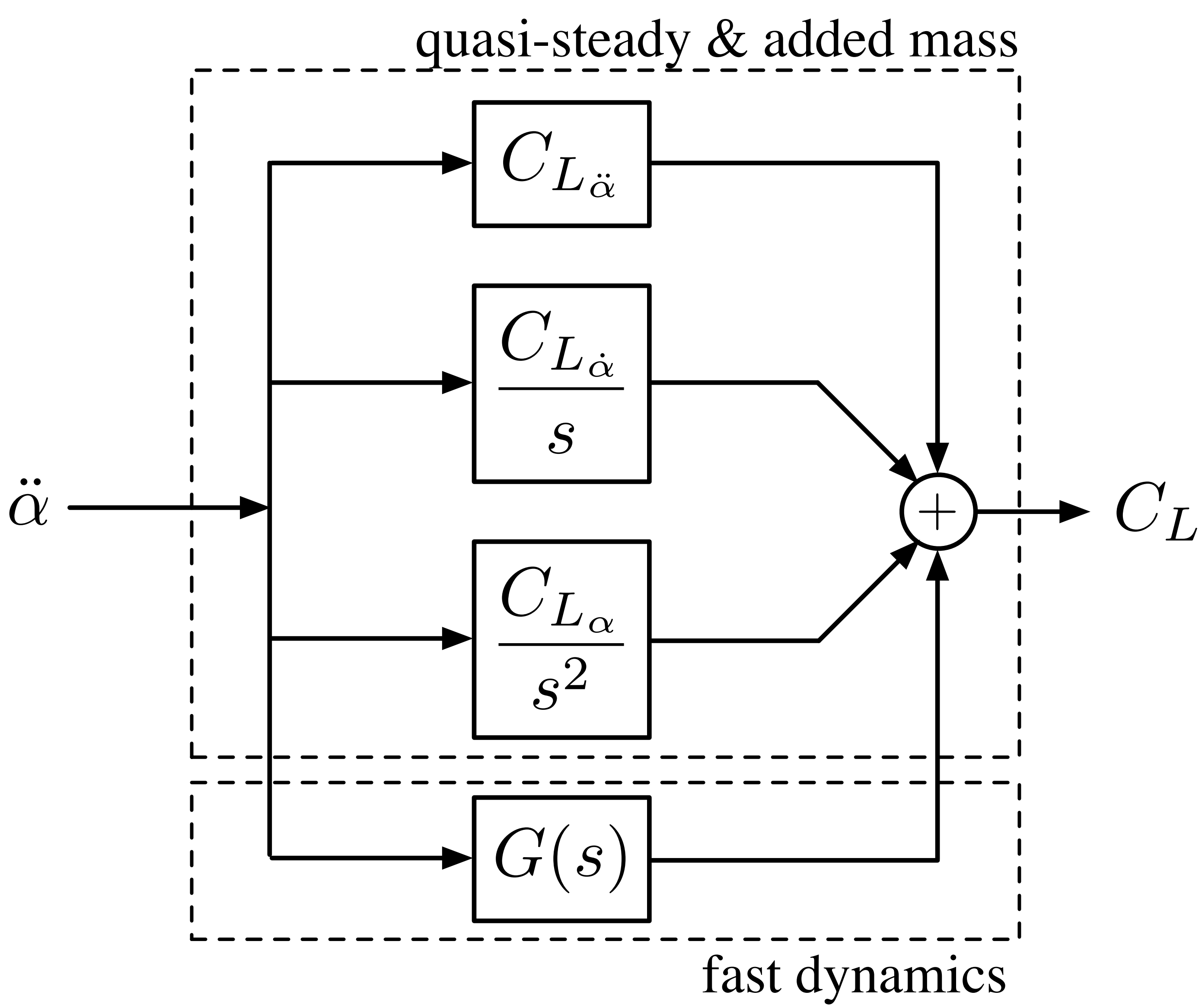
State-Space Form

$$\frac{d}{dt}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} = \begin{bmatrix}A_r & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}B_r\\ 0\\ 1\end{bmatrix}\ddot{\alpha}$$

$$C_L = \begin{bmatrix}C_r & C_{L_\alpha} & C_{L_{\dot{\alpha}}}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} + C_{L_{\ddot{\alpha}}}\ddot{\alpha}$$

Reduced Order Wagner

$$C_L(t) = C_L^S(t)\alpha(0) + \int_0^t C_L^S(t-\tau)\dot{\alpha}(\tau)d\tau$$



fast dynamics

$$\frac{d}{dt}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} = \begin{bmatrix}A_r & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}B_r\\ 0\\ 1\end{bmatrix}\ddot{\alpha}$$

input

$$C_L = \begin{bmatrix}C_r & C_{L_\alpha} & C_{L_{\dot{\alpha}}}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} + C_{L_{\ddot{\alpha}}}\ddot{\alpha}$$

ERA Model

quasi-steady and added-mass

Model Summary

Linearized about $\alpha = 0$

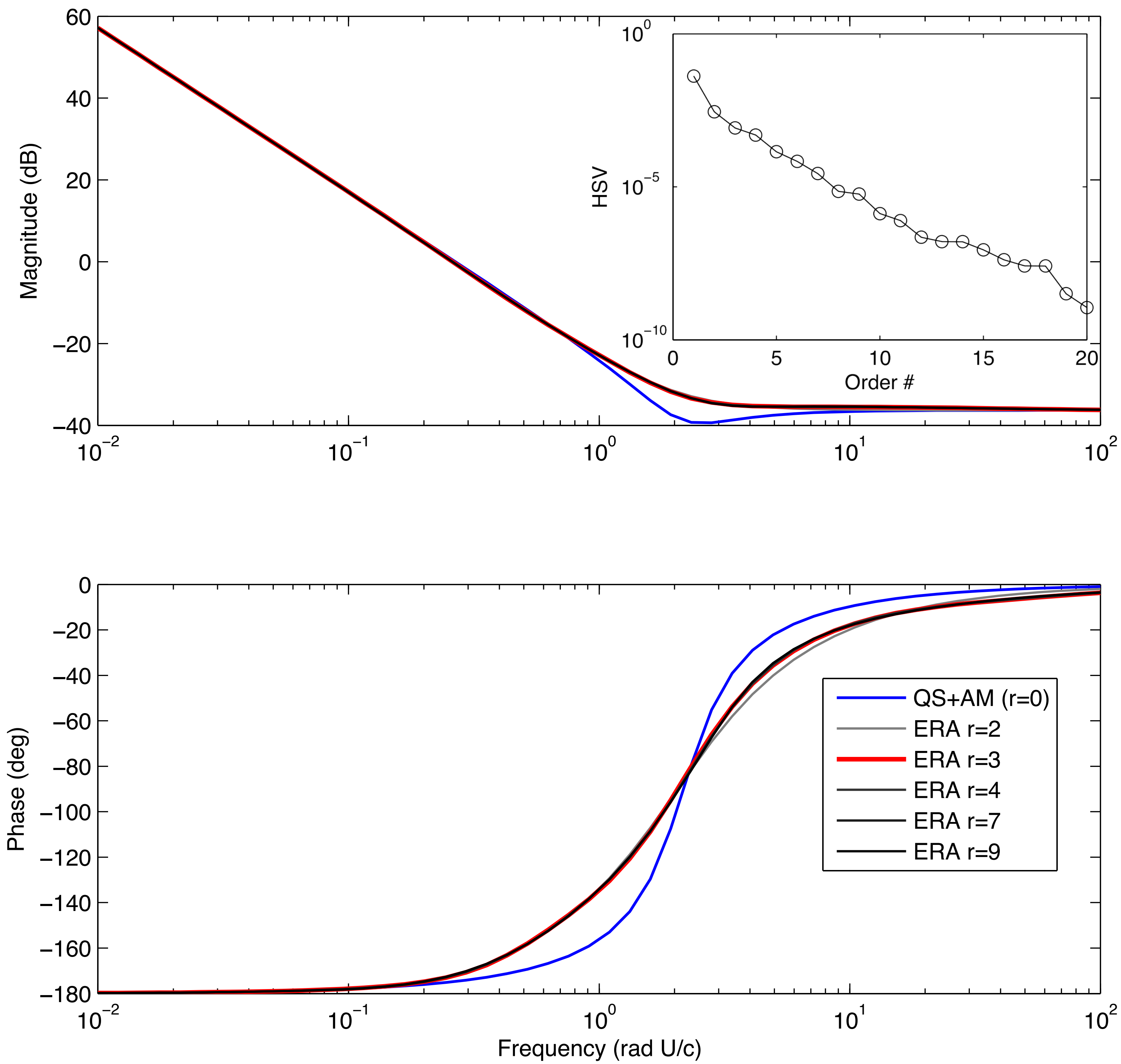
Based on experiment, simulation or theory

Recovers stability derivatives $C_{L_\alpha}, C_{L_{\dot{\alpha}}}, C_{L_{\ddot{\alpha}}}$ associated with quasi-steady and added-mass

ODE model ideal for control design

Brunton and Rowley, *in preparation.*

Bode Plot - Pitch (LE)



Frequency response

input is $\ddot{\alpha}$ **(** α **is angle of attack)**

output is lift coefficient C_L

Pitching at leading edge

Model without additional fast dynamics [QS+AM (r=0)] is inaccurate in crossover region

Models with fast dynamics of ERA model order >3 are converged

Punchline: additional fast dynamics (ERA model) are essential

Brunton and Rowley, *in preparation.*

Bode Plot - Pitch (QC)

Frequency response

input is $\ddot{\alpha}$ (α is angle of attack)

output is lift coefficient $\mathrm{C_L}$

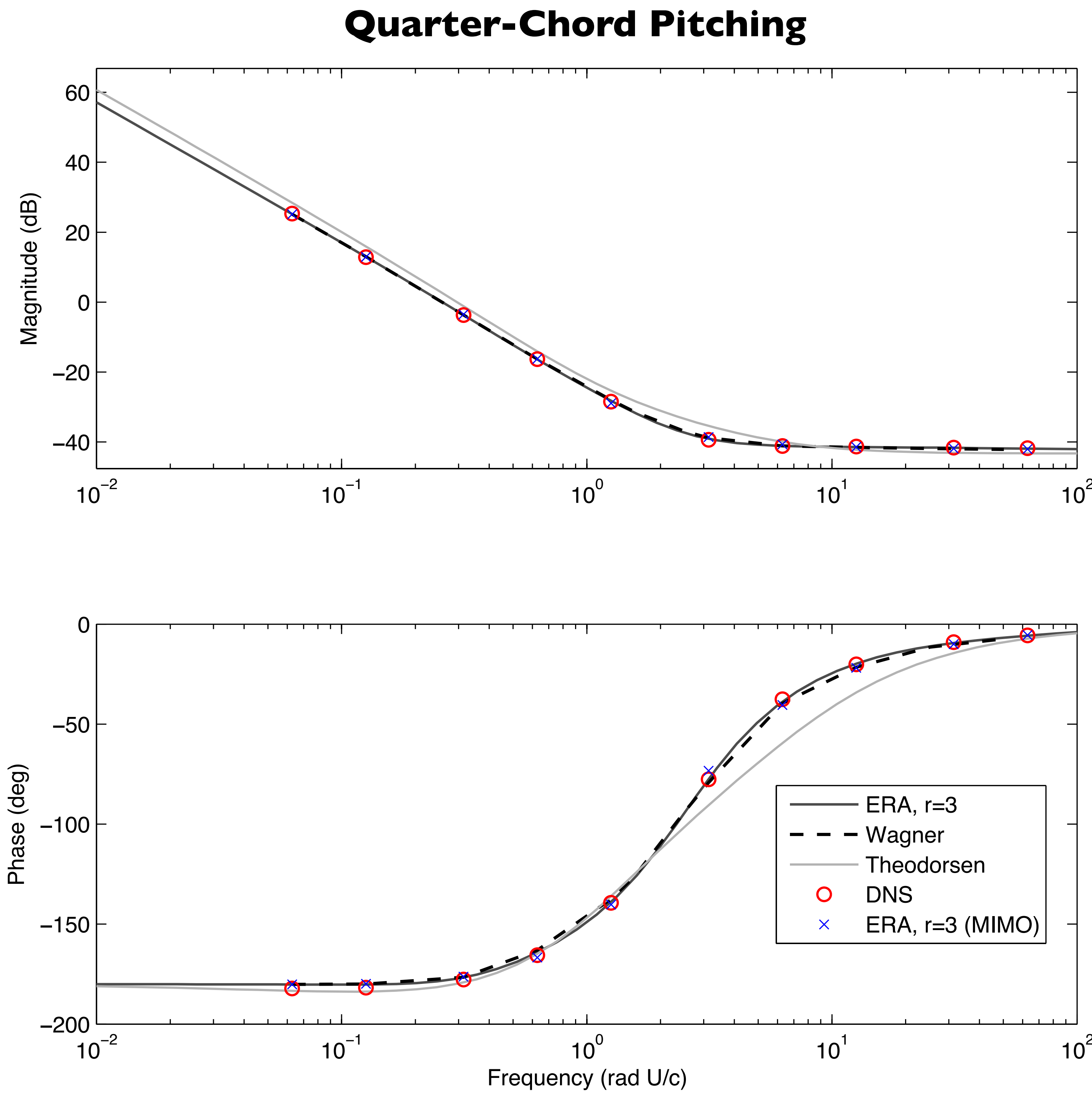
Pitching at quarter chord

Reduced order model with ERA r=3 accurately reproduces Wagner

Wagner and ROM agree better with DNS than Theodorsen's model.

Asymptotes are correct for Wagner because it is based on experiment

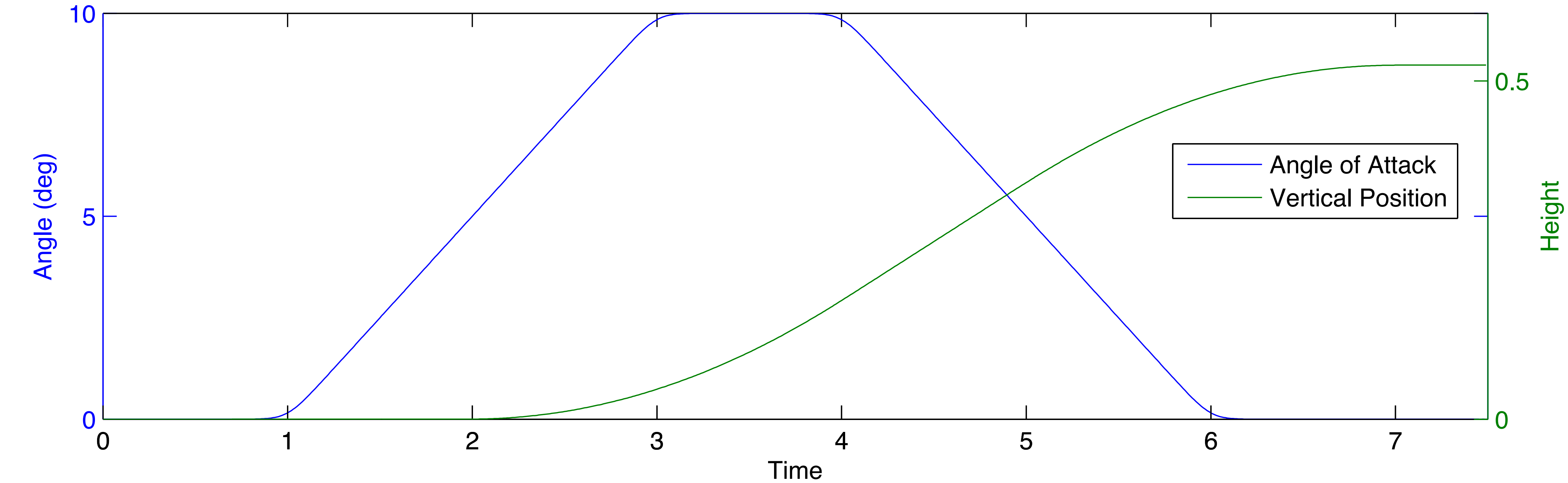
Model for pitch/plunge dynamics [ERA, r=3 (MIMO)] works as well, for the same order model

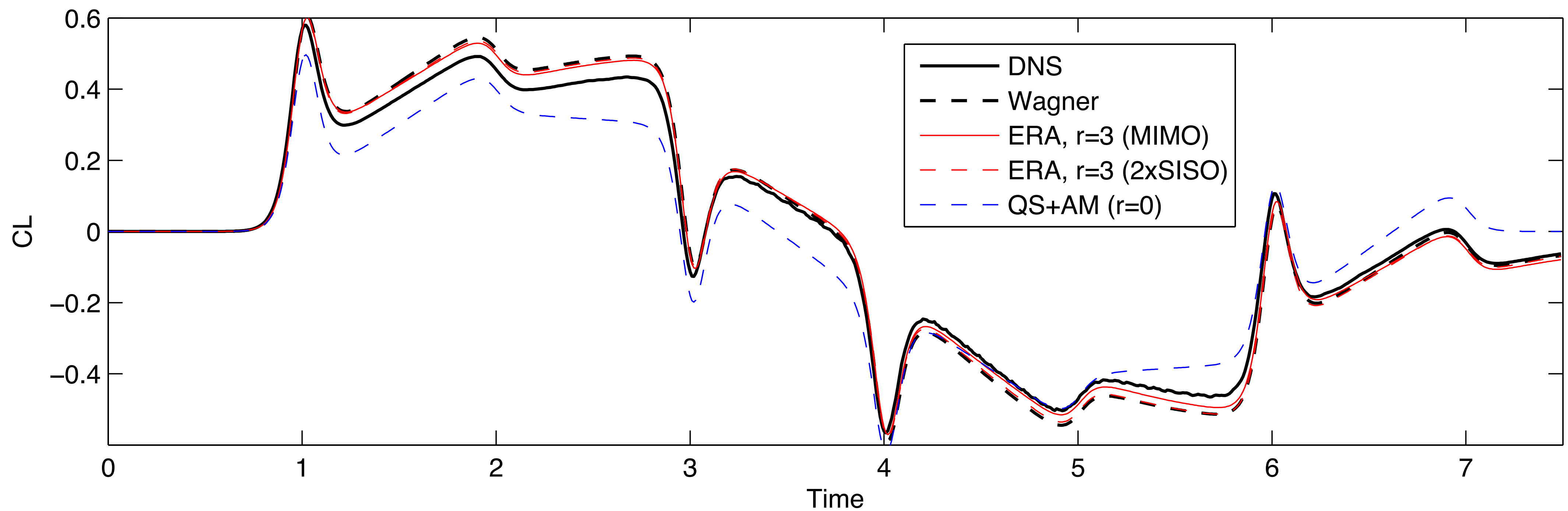


Brunton and Rowley, *in preparation.*

Pitch/Plunge Maneuver

Canonical pitch-up, hold, pitch-down maneuver, followed by step-up in vertical position



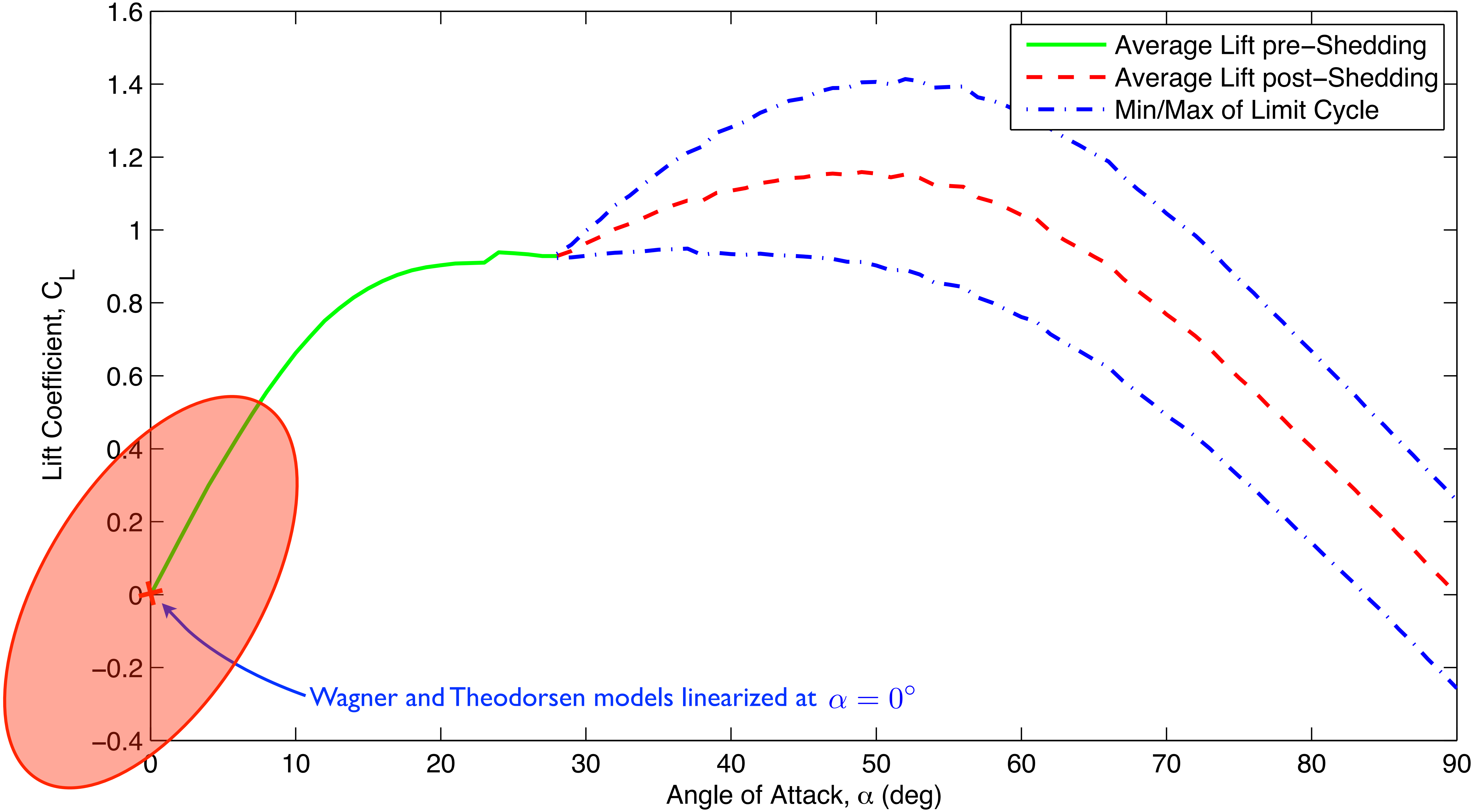


OL, Altman, Eldredge, Garmann, and Lian, 2010

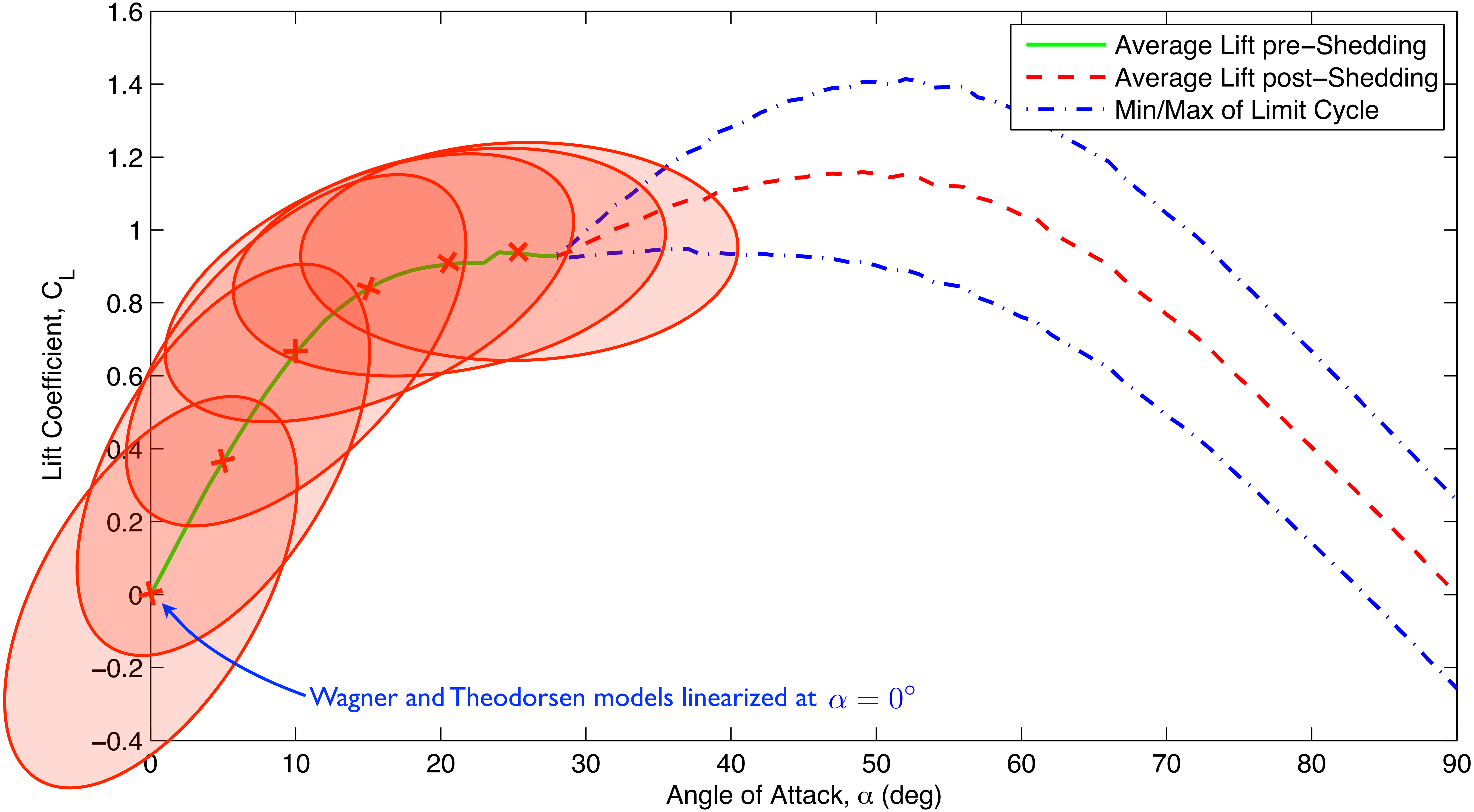
Brunton and Rowley, *in preparation*.

Reduced order model for Wagner's indicial response accurately captures lift coefficient history from DNS

Lift vs. Angle of Attack



Lift vs. Angle of Attack



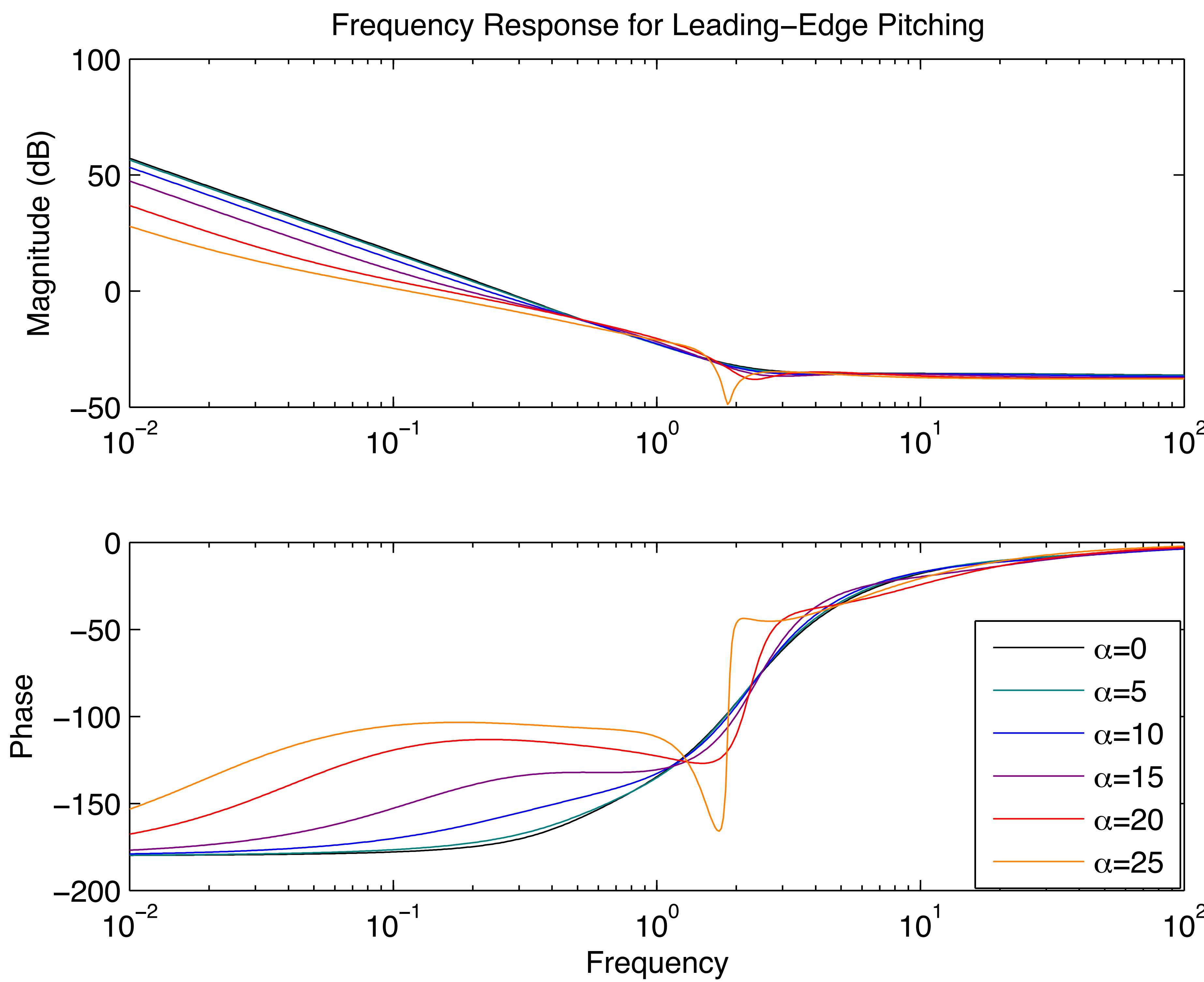
Bode Plot of ERA Models

Results

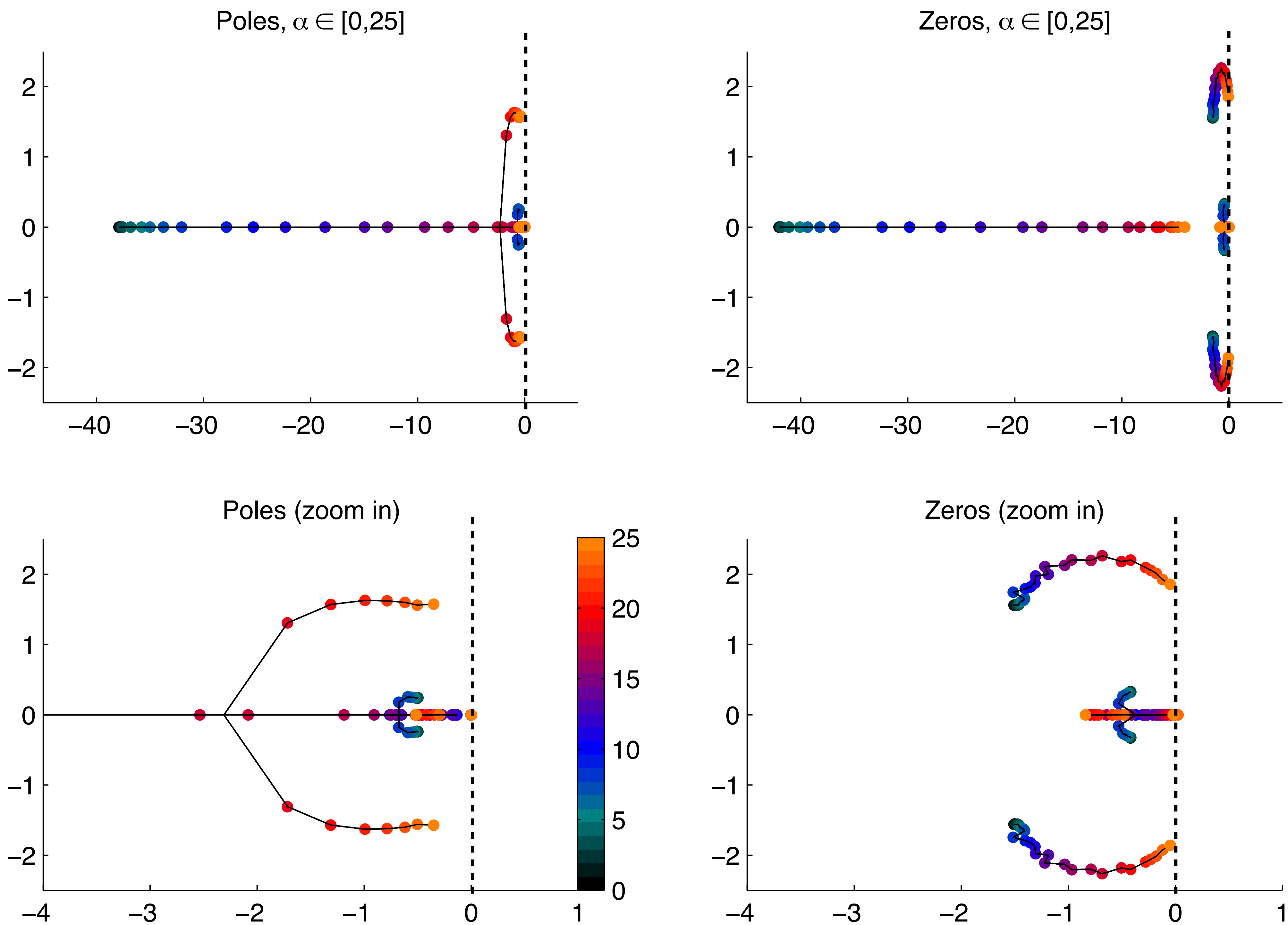
Lift slope decreases for increasing angle of attack, so magnitude of low frequency motions decreases for increasing angle of attack.

At larger angle of attack, phase converges to -180 at much lower frequencies. I.e., solutions take longer to reach equilibrium in time domain.

Consistent with fact that for large angle of attack, system is closer to Hopf instability, and a pair of eigenvalues are moving closer to imaginary axis.



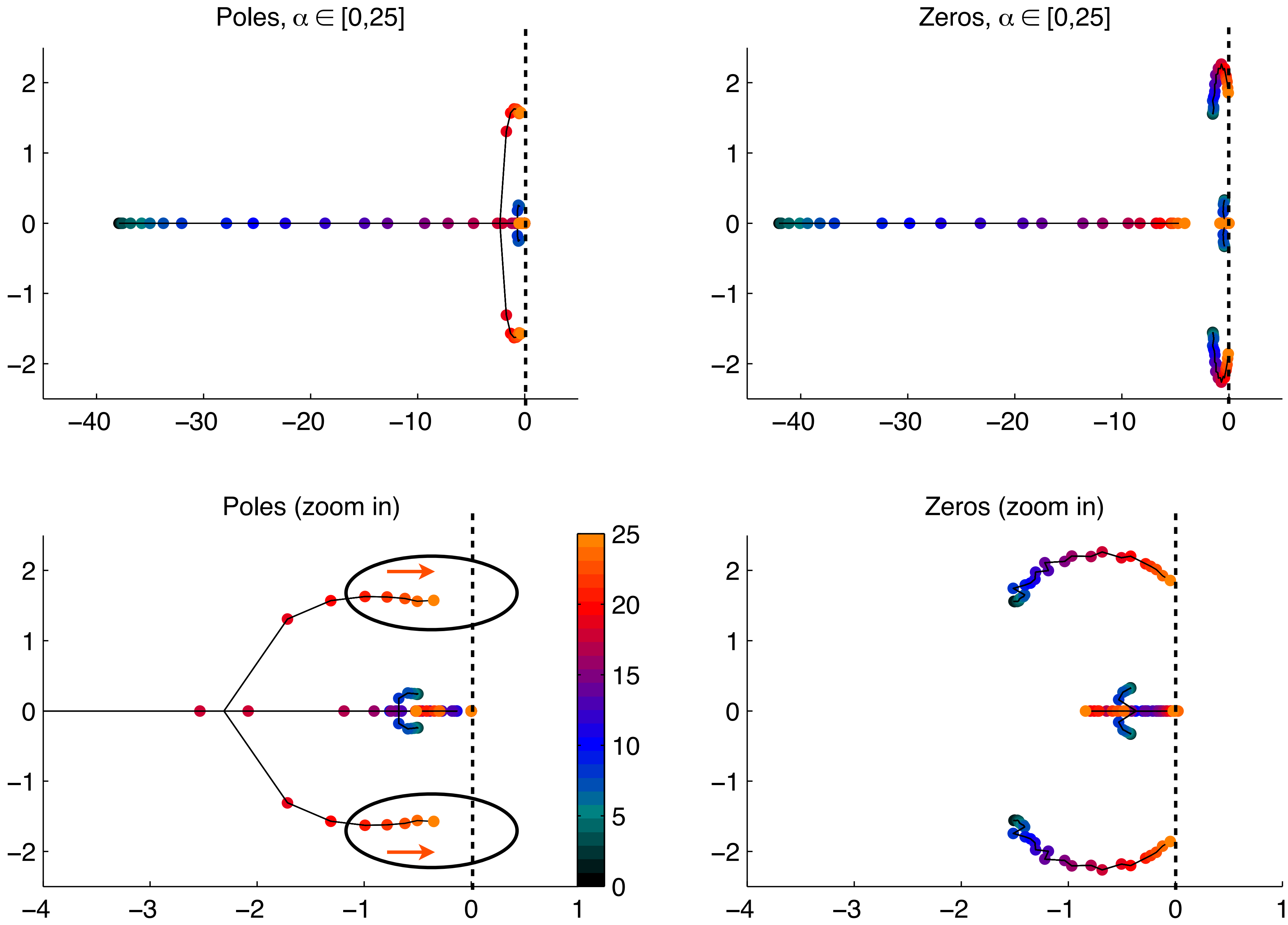
Poles and Zeros of ERA Models



As angle of attack increases, pair of poles (and pair of zeros) march towards imaginary axis.

This is a good thing, because a Hopf bifurcation occurs at $\alpha_{\text{crit}} \approx 28^\circ$

Poles and Zeros of ERA Models



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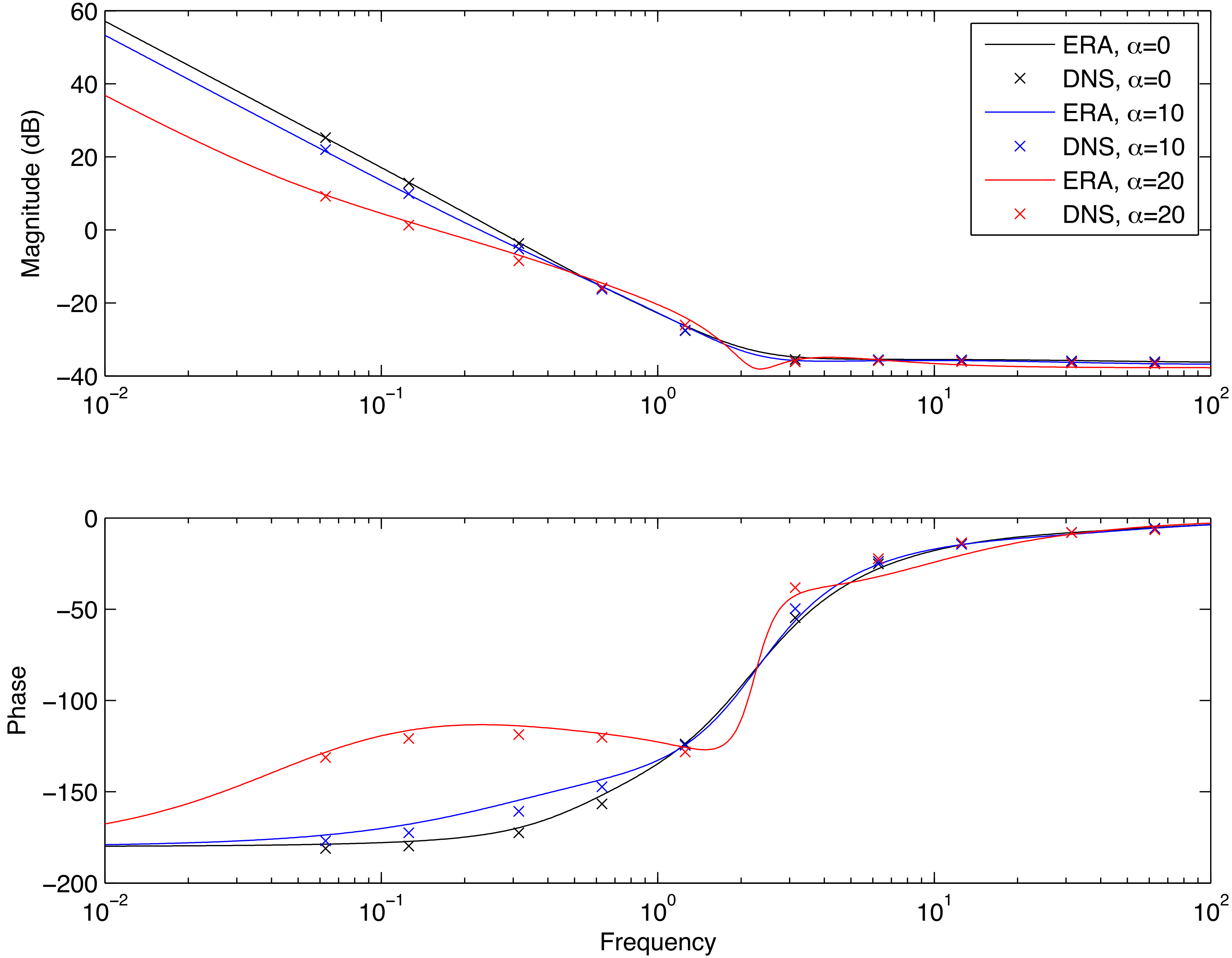
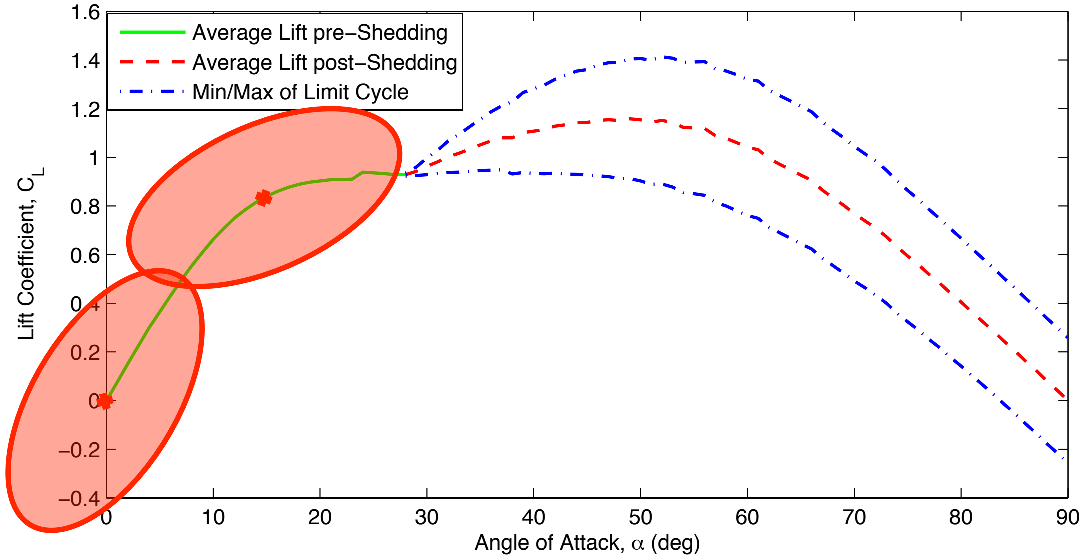
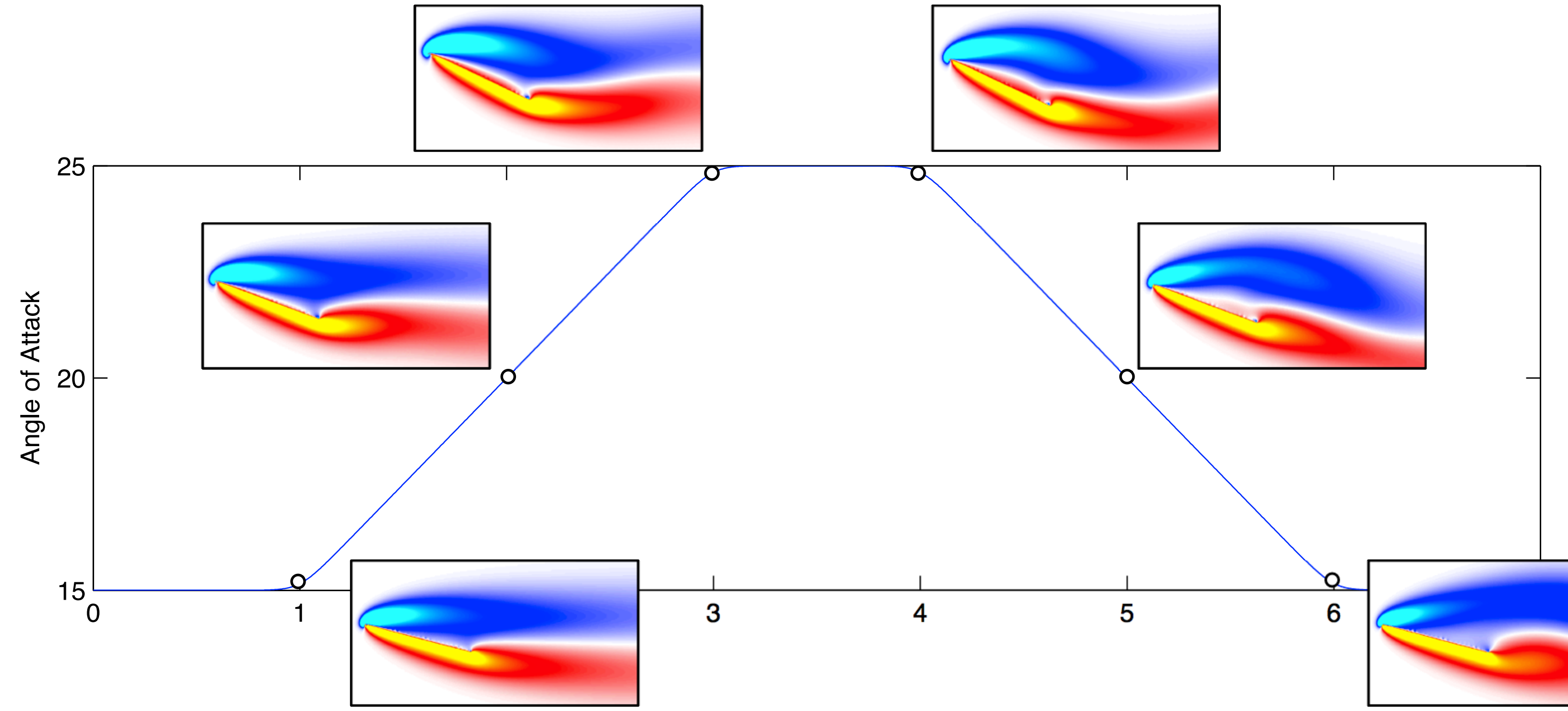
Bode Plot of Model (-) vs Data (x)

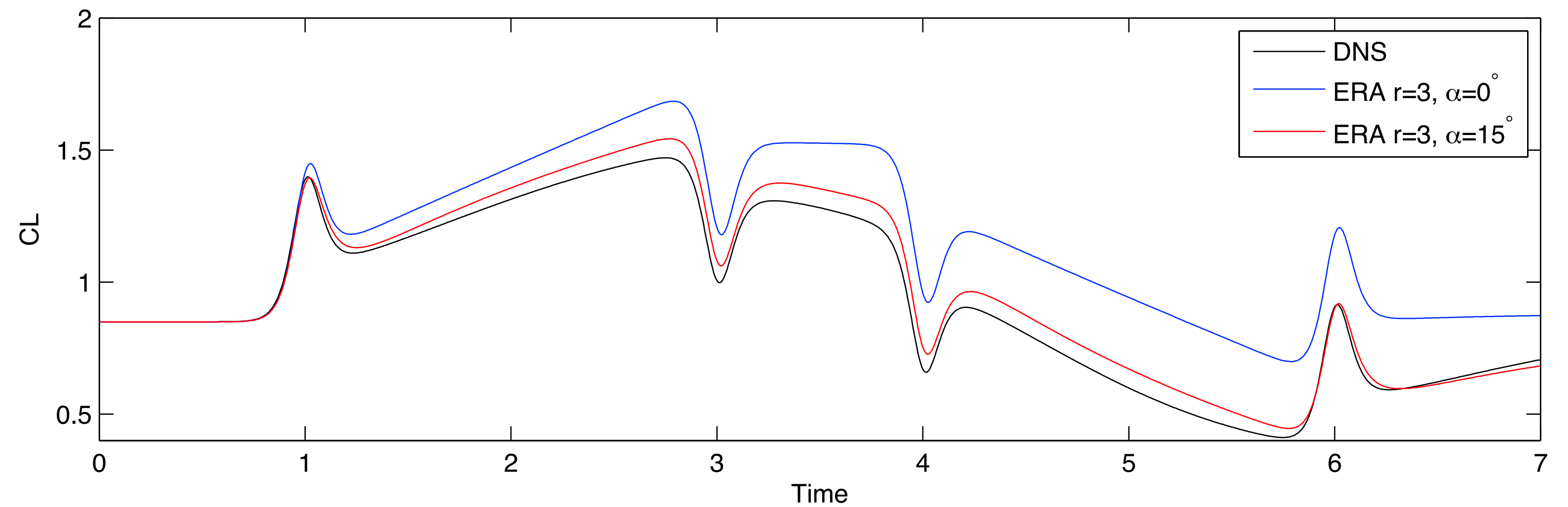


Direct numerical simulation confirms that local linearized models are accurate for small amplitude sinusoidal maneuvers

Large Amplitude Maneuver







Compare models linearized at

$\alpha = 0^\circ$ **and** $\alpha = 15^\circ$

For pitching maneuver with

$$\alpha \in [15^\circ, 25^\circ]$$

Model linearized at $\alpha = 15^\circ$

captures lift response more accurately

$$G(t) = \log\left[\frac{\cosh(a(t-t_1))\cosh(a(t-t_4))}{\cosh(a(t-t_2))\cosh(a(t-t_3))}\right] \qquad \alpha(t) = \alpha_0 + \alpha_{\max}\frac{G(t)}{\max(G(t))}$$

OL, Altman, Eldredge, Garmann, and Lian, 2010

Conclusions

Reduced order model based on indicial response at non-zero angle of attack

- Based on eigensystem realization algorithm (ERA)
- Models appear to capture dynamics near Hopf bifurcation
- Locally linearized models outperform models linearized at $\alpha = 0^\circ$

Empirically determined Theodorsen model

- Theodorsen's C(k) may be approximated, or determined via experiments
- Models are cast into state-space representation
- Pitching about various points along chord is analyzed

Future Work:

- Combine models linearized at different angles of attack
- Add large amplitude effects such as LEV and vortex shedding

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Wagner, 1925.

Theodorsen, 1935.

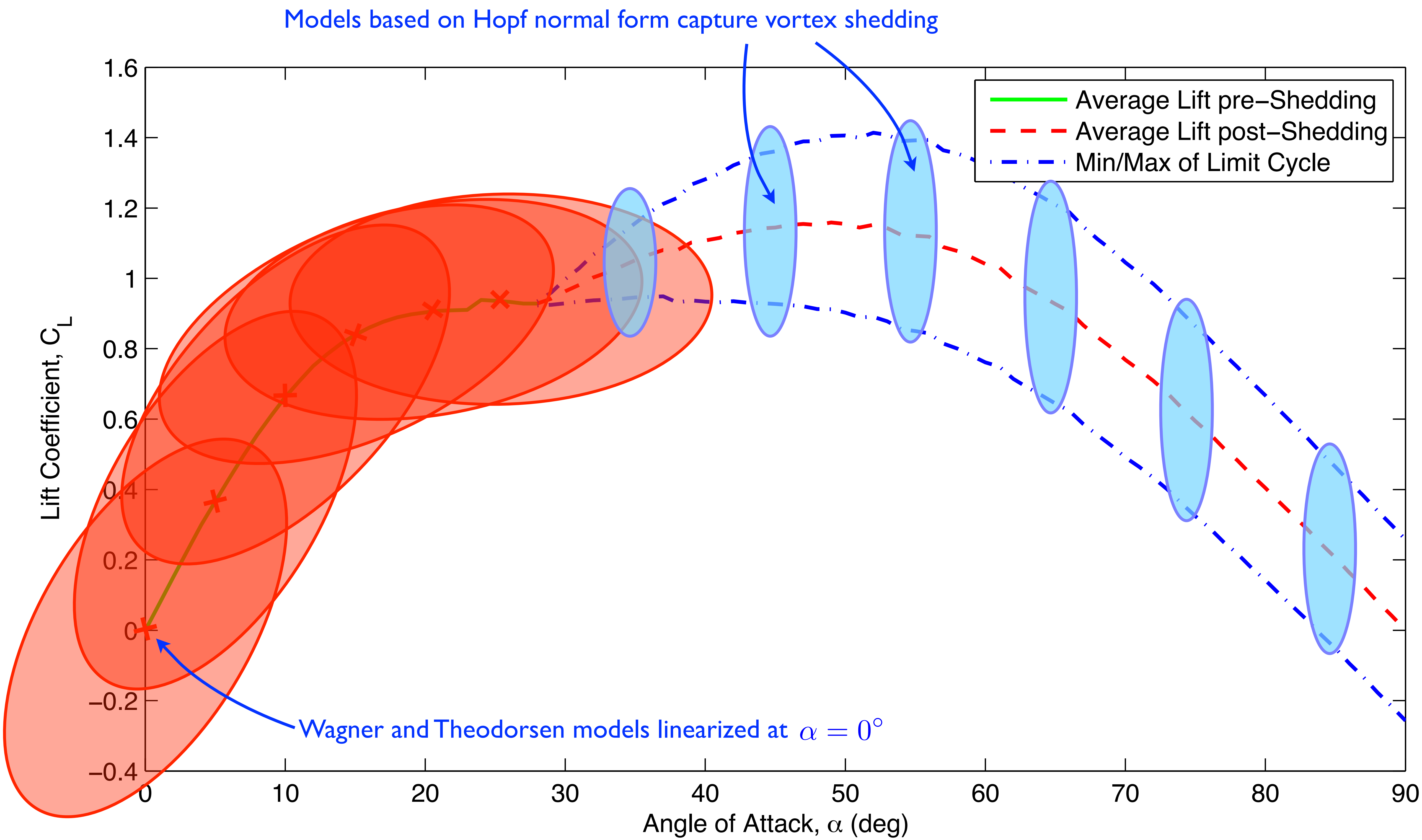
Leishman, 2006.

Brunton and Rowley, AIAA ASM 2009-2011

OL, Altman, Eldredge, Garmann, and Lian, 2010

Breuker, Abdalla, Milanese, and Marzocca, AIAA 2008.

Lift vs. Angle of Attack



Bode Plot - Plunge

Frequency response

input is $\ddot{y}$ (vertical acceleration)

output is lift coefficient C_L

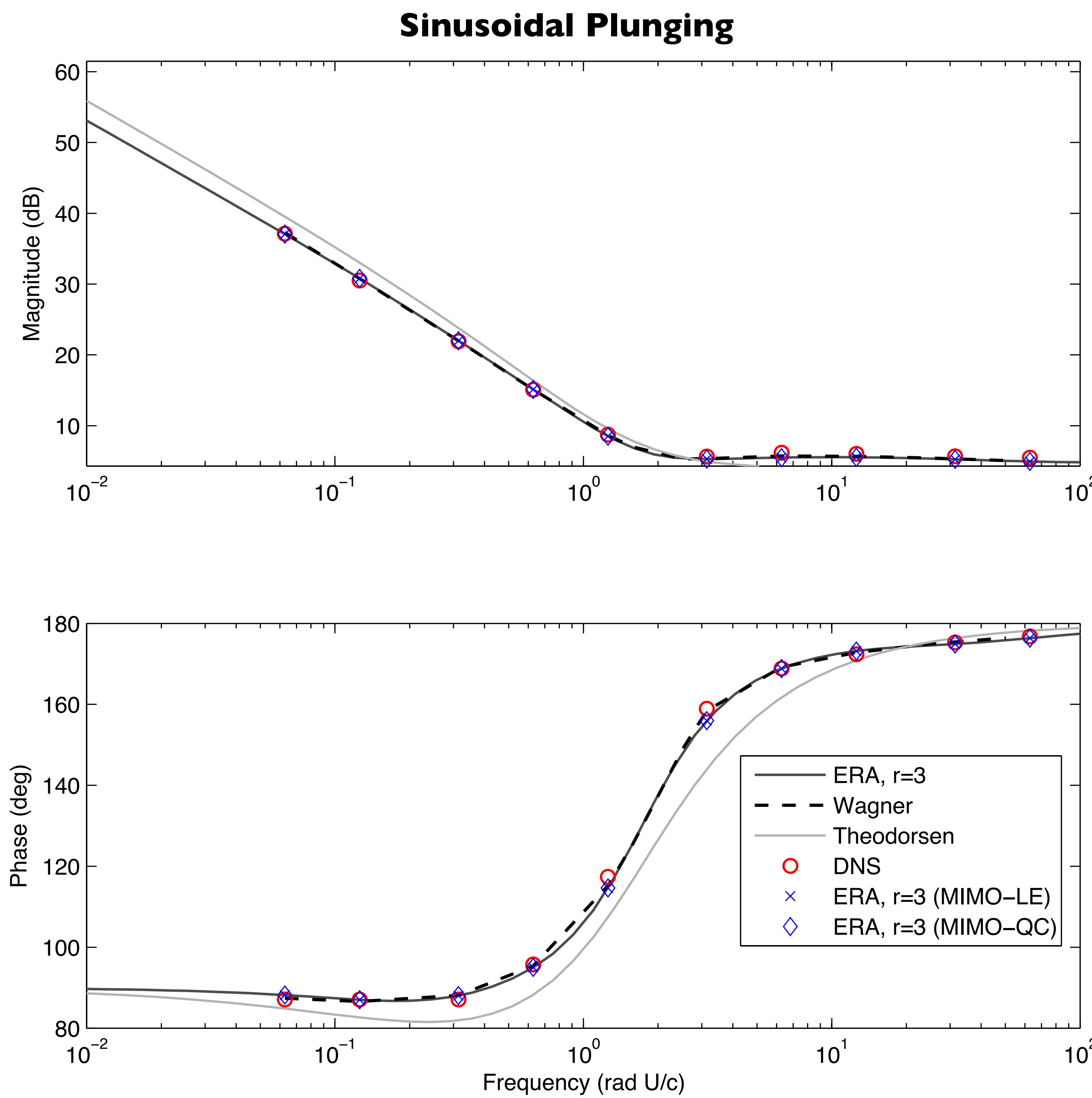
Plunging changes flight path angle and free stream velocity

Reduced order model with ERA r=3 accurately reproduces Wagner

Wagner and ROM agree better with DNS than Theodorsen's model.

Asymptotes are correct for Wagner because it is based on experiment

Model for pitch/plunge dynamics [ERA, r=3 (MIMO)] works as well, for the same order model



Brunton and Rowley, *in preparation*.