Reduced order models for unsteady aerodynamic forces at low Reynolds number

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Motivation

Applications of Unsteady Models

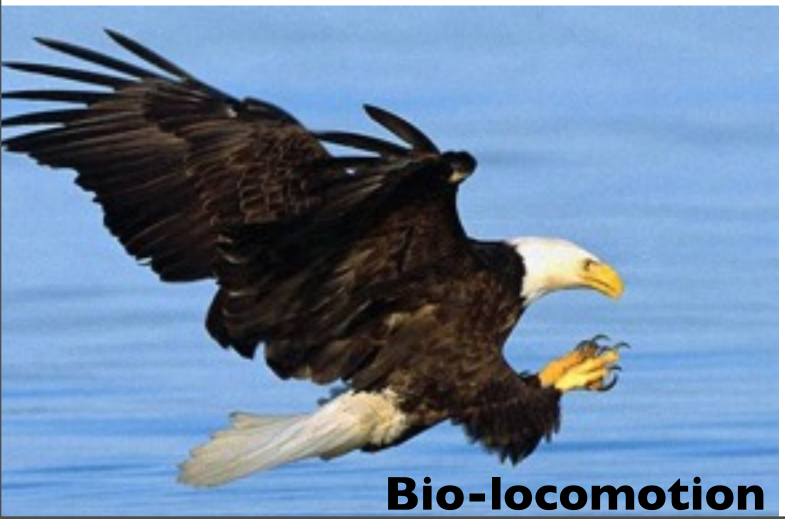
- Conventional UAVs (performance/robustness)
- Micro air vehicles (MAVs)
- Flow control, flight dynamic control
- Autopilots / Flight simulators
- Gust disturbance mitigation
- Understand bird/insect flight

Need for State-Space Models

- Need models suitable for control
- Combining with flight models

Predator (General Atomics)

FLYIT Simulators, Inc.



Bio-locomotion

Flexible Wing
(University of Florida)

Flow Control (expert)

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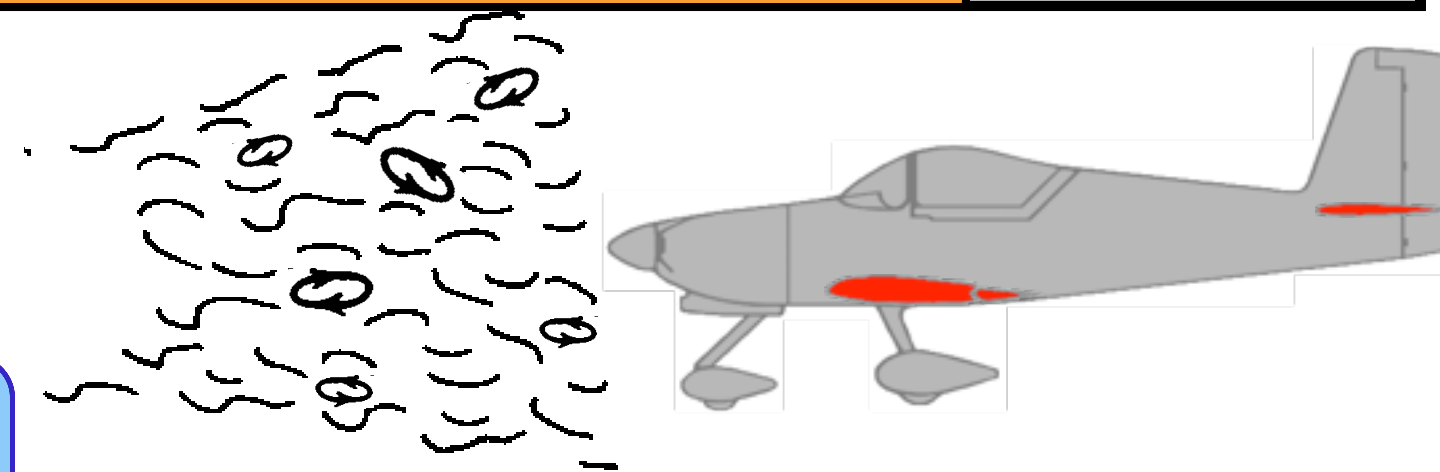
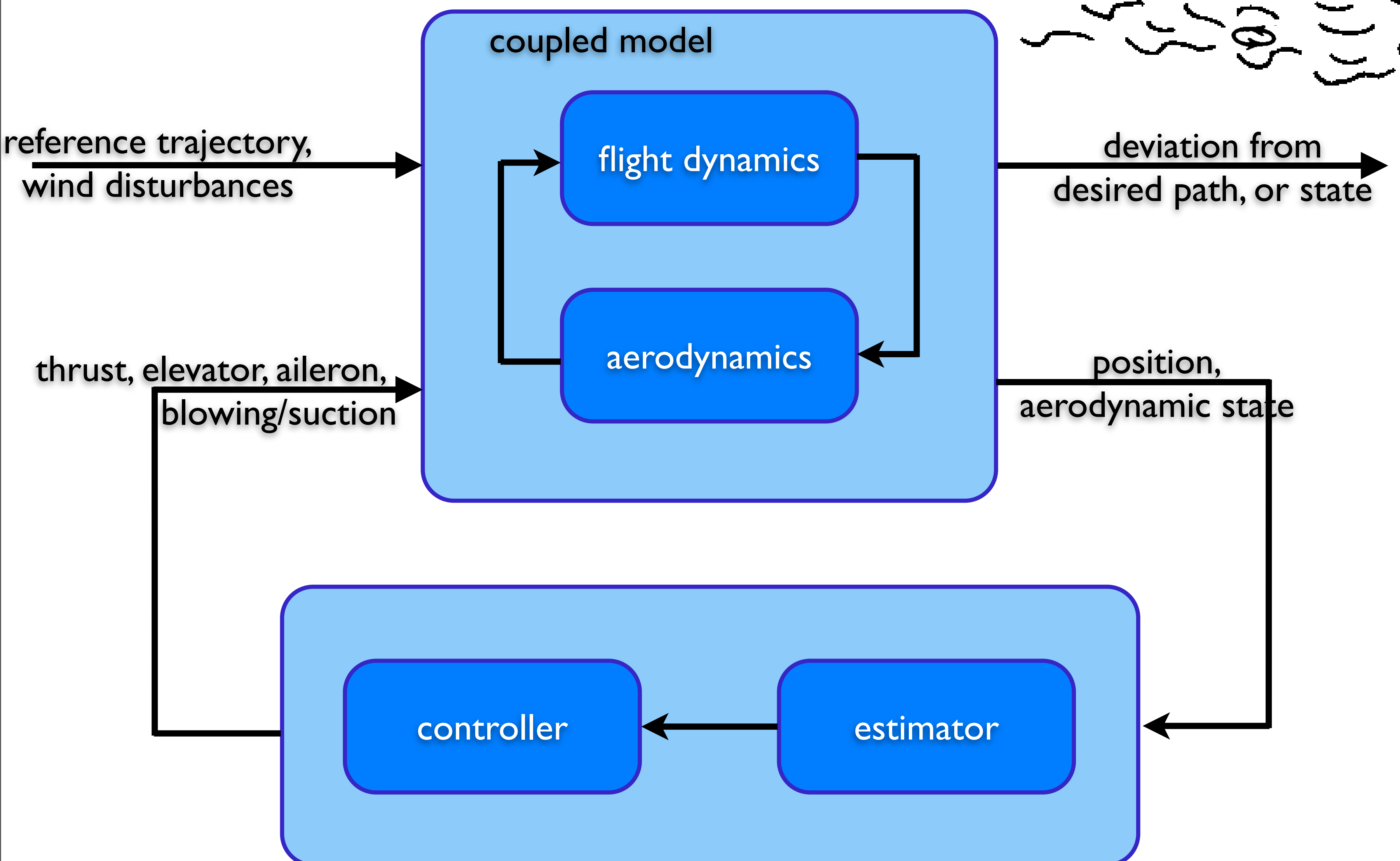
Flight Dynamic Control





2D Incompressible Flow, (Re=100)

Stationary, AoA = 25

Pitch

Stationary, AoA = 35

Plunge

Immersed boundary method

- Multi-domain approach
- Boundary forces computed as Lagrange-multipliers to enforce no slip

Colonius & Taira, 2008.

2D Incompressible Navier-Stokes:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2 \mathbf{u} + \int_s \mathbf{f}\left(\xi(s,t)\right)\delta(\xi - \mathbf{x})ds$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u}\left(\xi(s,t)\right) = \int_{\mathbf{x}} \mathbf{u}(\mathbf{x})\delta(\mathbf{x} - \xi)d\mathbf{x} = \mathbf{u}_B\left(\xi(s,t)\right)$$

Finite Time Lyapunov Exponents

Finite Time Lyapunov Exponents (FTLE)

Measure of stretching between neighboring particles

σ **is time-dependent for unsteady flows**

$$\sigma(\Phi_0^T;\mathbf{x_0}) = \frac{1}{|T|}\log\sqrt{\lambda_{\max}(\Delta(\mathbf{x_0}))}$$

where $$\Delta = \left(\mathbf{D}\Phi_0^T\right)^* \mathbf{D}\Phi_0^T$$

Lagrangian Coherent Structures (LCS)

LCS are hyperbolic ridges in the FTLE field

Generalize invariant manifolds for time varying flows

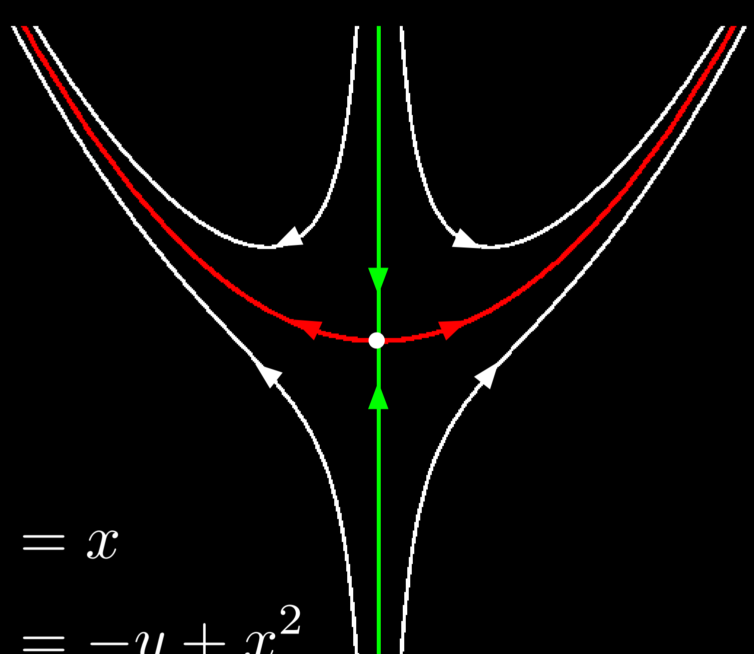
$\mathbf{\Phi_0^T}$ **- particle flow map**

pLCS - positive-time LCS (repelling)

nLCS - negative-time LCS (attracting)

Haller, 2002;
Shadden *et al.*, 2005

Attracting nLCS



$\dot{x} = x$

$\dot{y} = -y + x^2$

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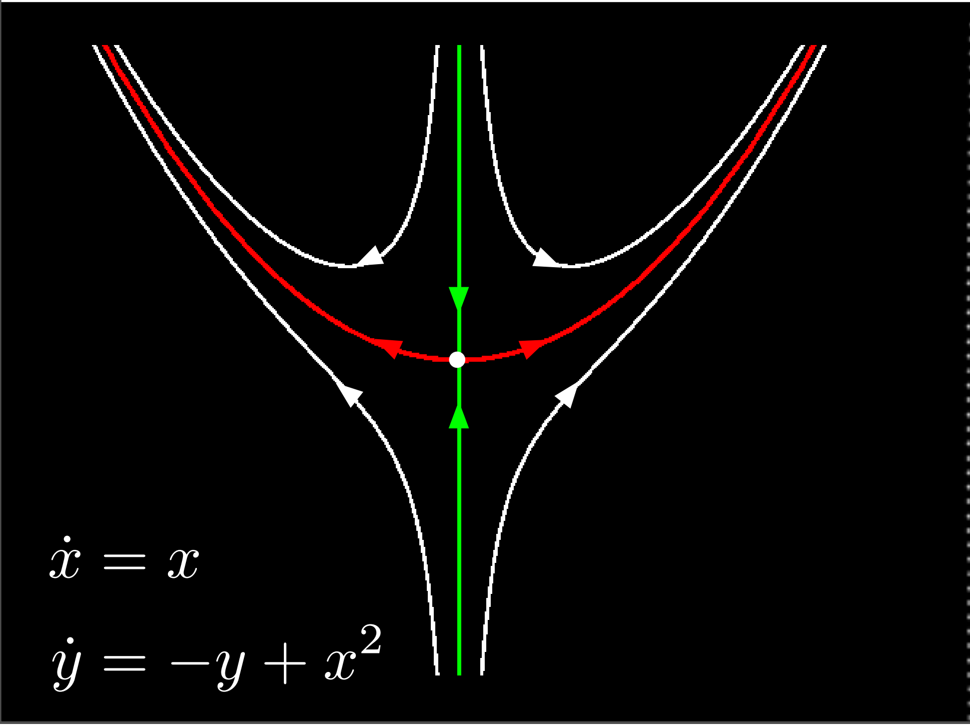

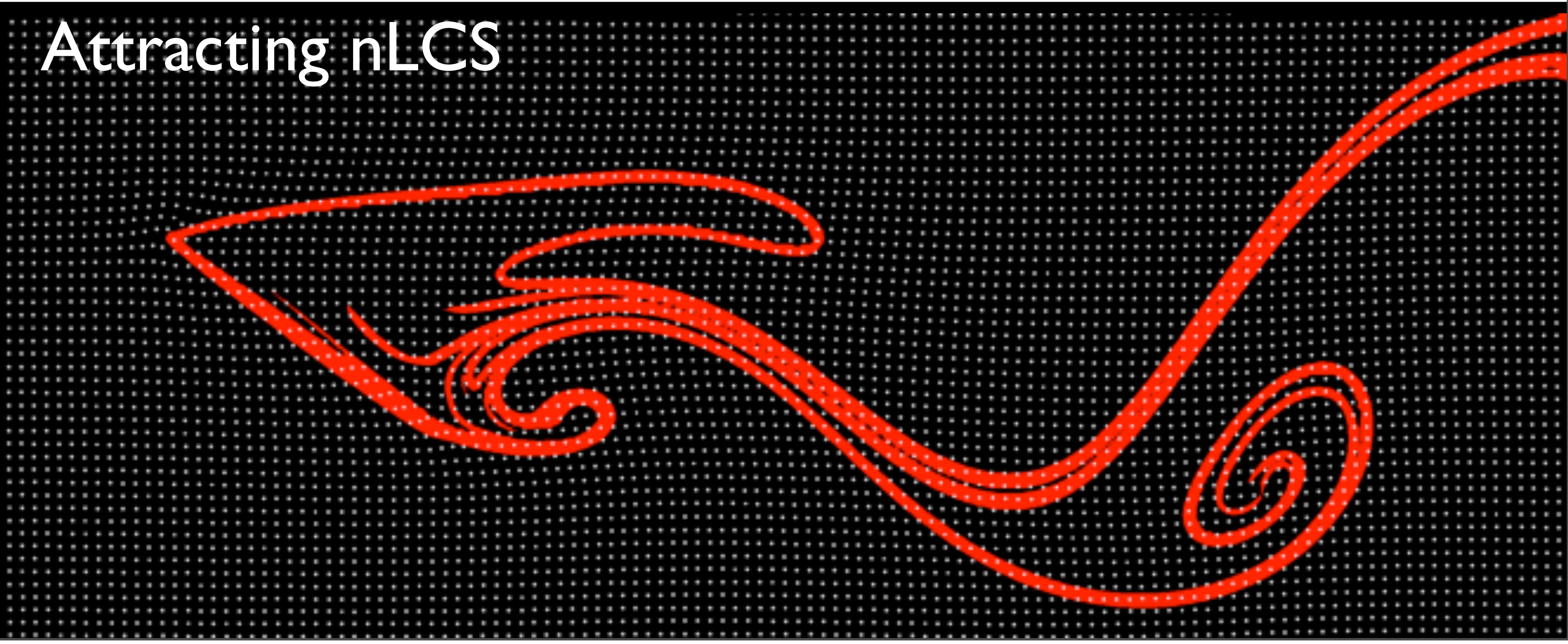

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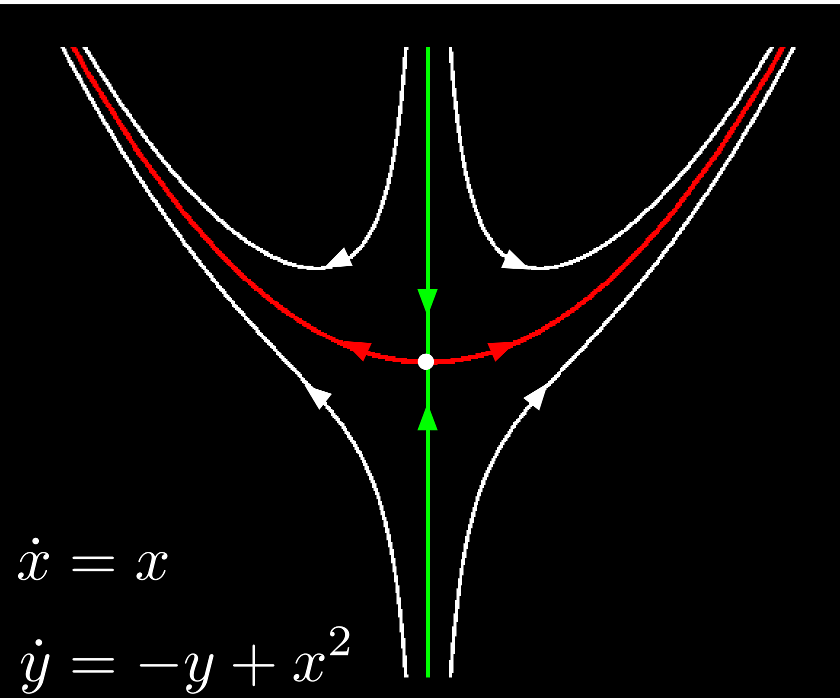

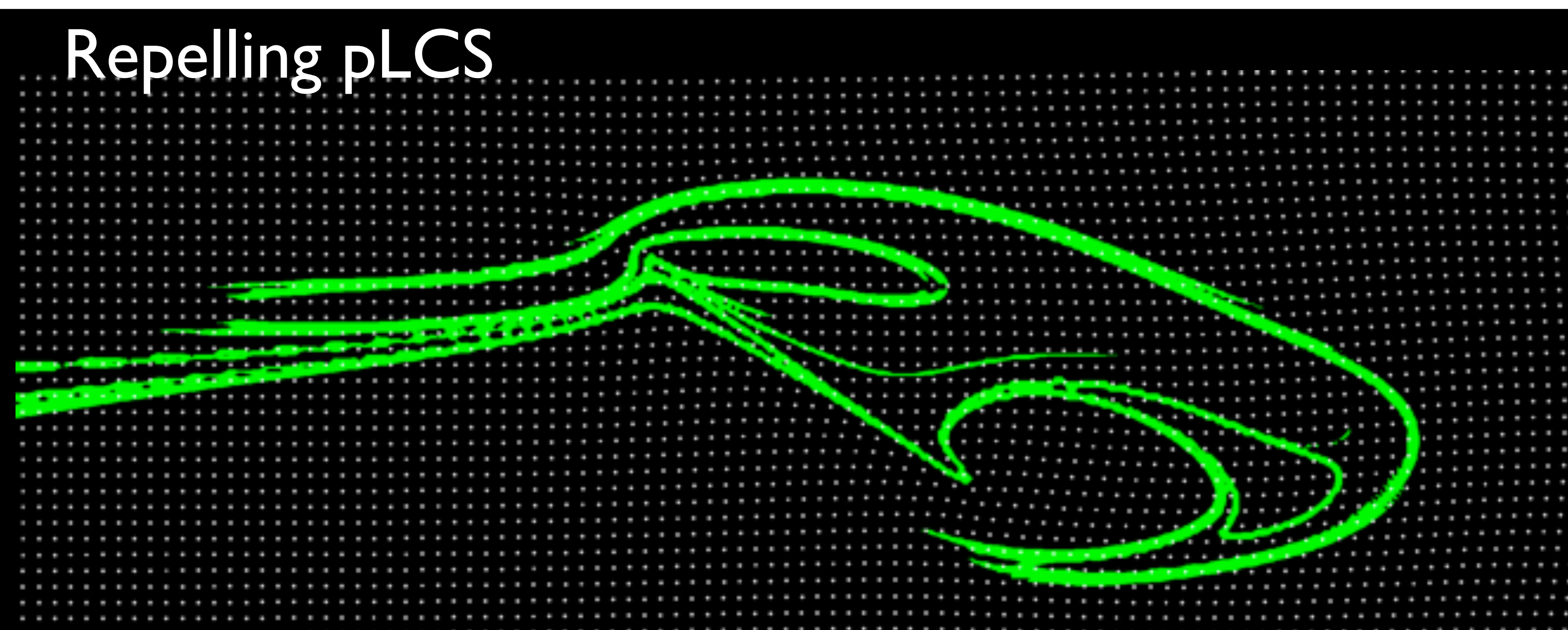

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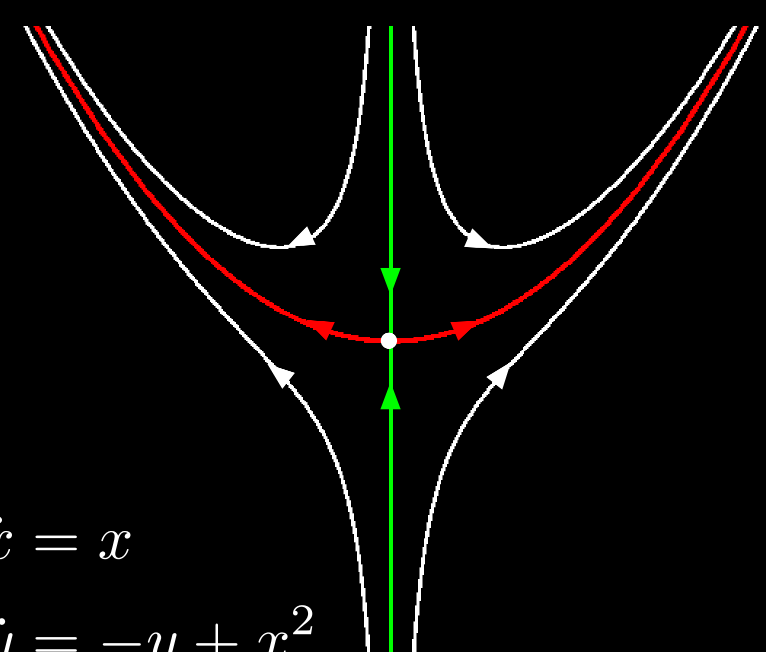

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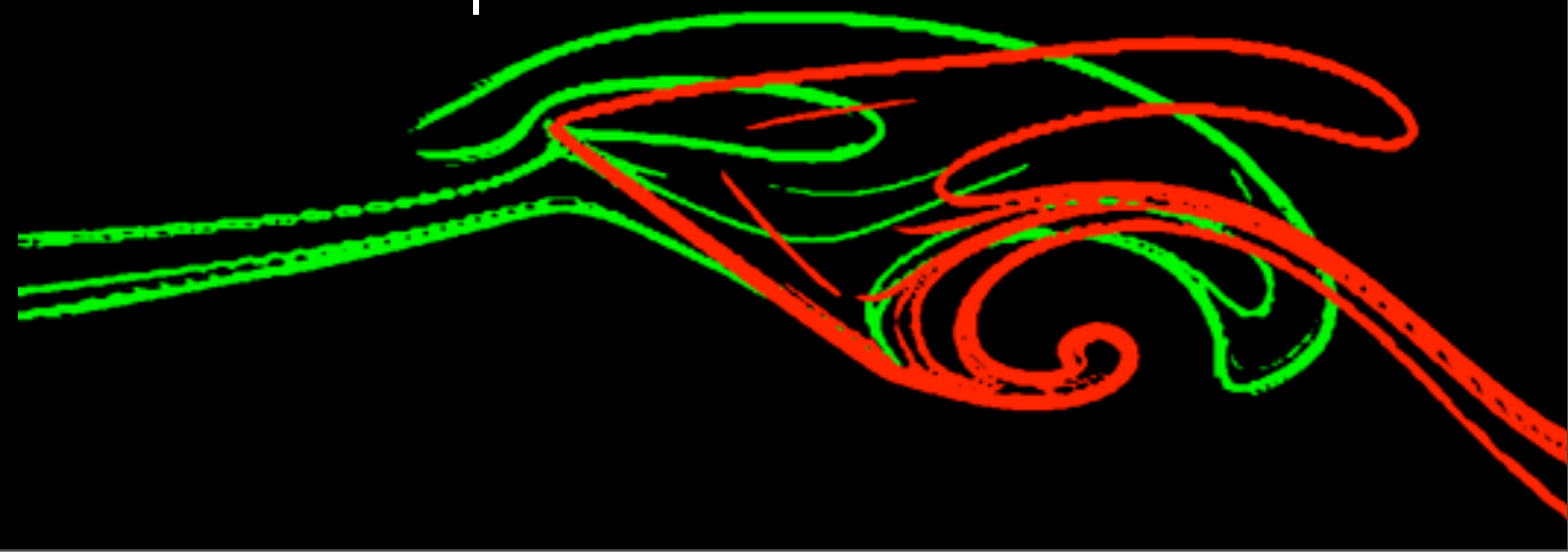
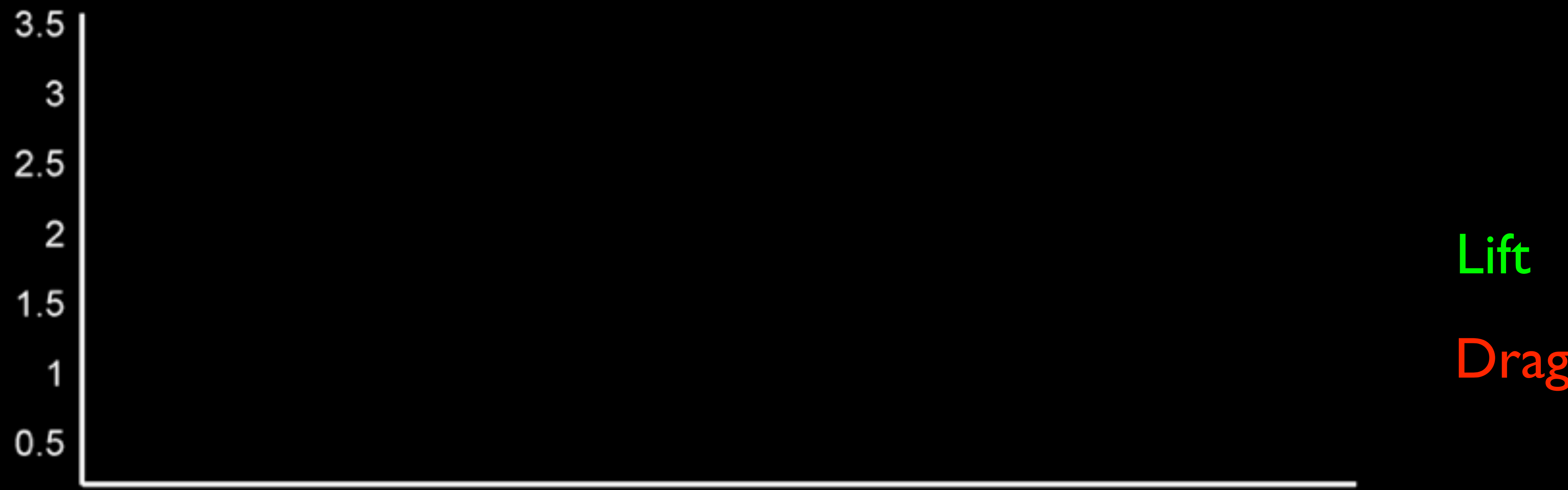
Both nLCS & pLCS



2D Model Problem



$\mathrm{Re} = 300$

$\alpha = 32^\circ$

2D Model Problem

Added-Mass

Transient

Periodic Vortex Shedding

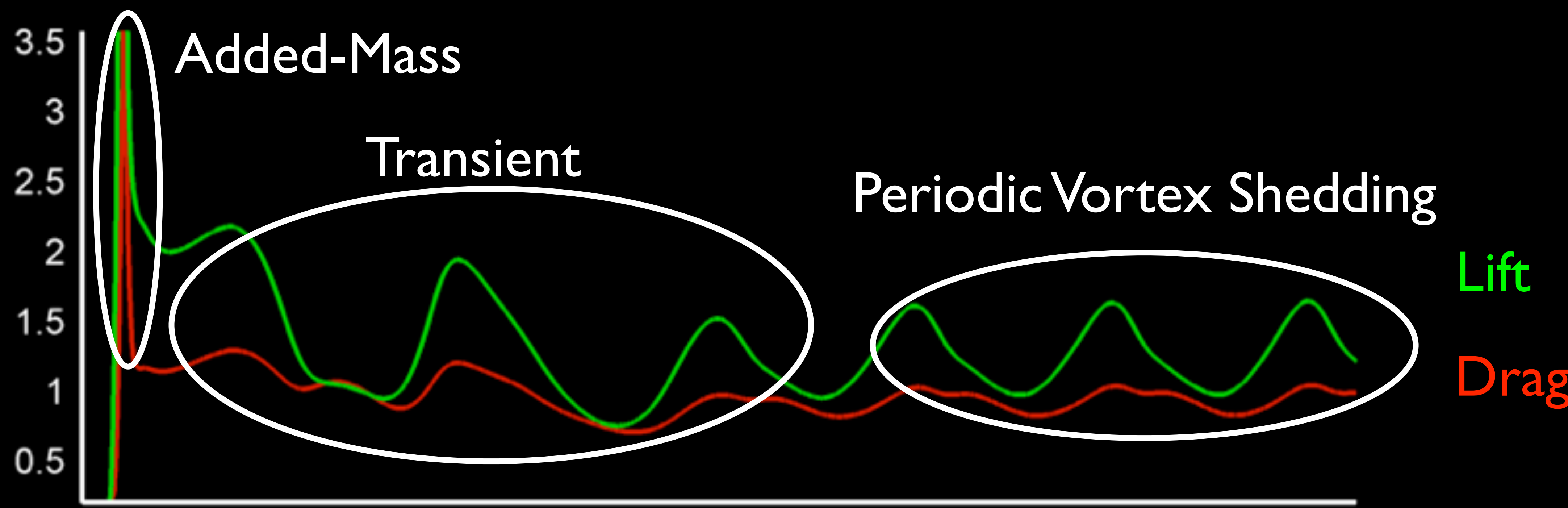
Lift

Drag

$\mathrm{Re} = 300$

$\alpha = 32^\circ$

2D Model Problem



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Unsteady Aerodynamic Forces

Added Mass

Increasingly important for small/light aircraft

Unsteady potential flow forces (F=ma)

force needed to move air as plate accelerates

Circulatory/Viscous

Captures separation effects

Need improved models here

source of all lift in steady flight... and more

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The mass of the body and surrounding fluid are being accelerated, to different extents.

Kinetic energy T will be in some manner proportional to U (for potential and Stokes flows)

$$T = \rho \frac{I}{2} U^2 \qquad \text{where} \qquad I = \int_V \frac{u_i}{U} \cdot \frac{u_i}{U} dV$$

If body accelerates, T probably increases, and energy must be supplied:

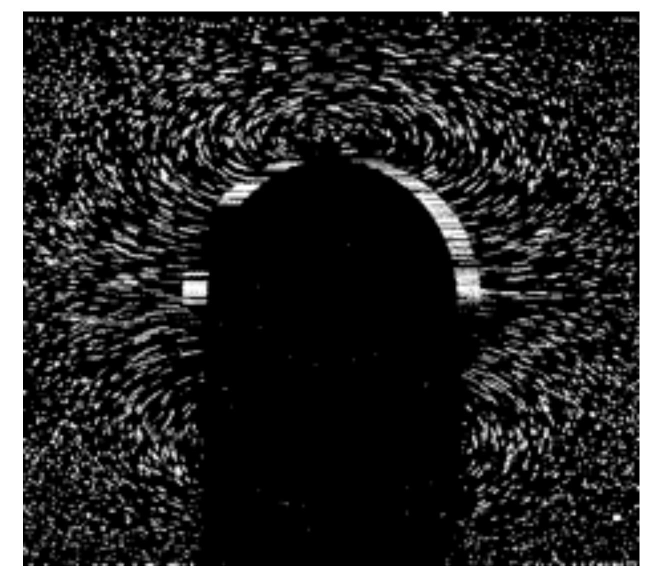
$$\frac{dT}{dt} = -FU \quad \Longrightarrow \quad F_i = -\underbrace{\rho I_{ij}}_{\text{AM}} \dot{U}_j$$

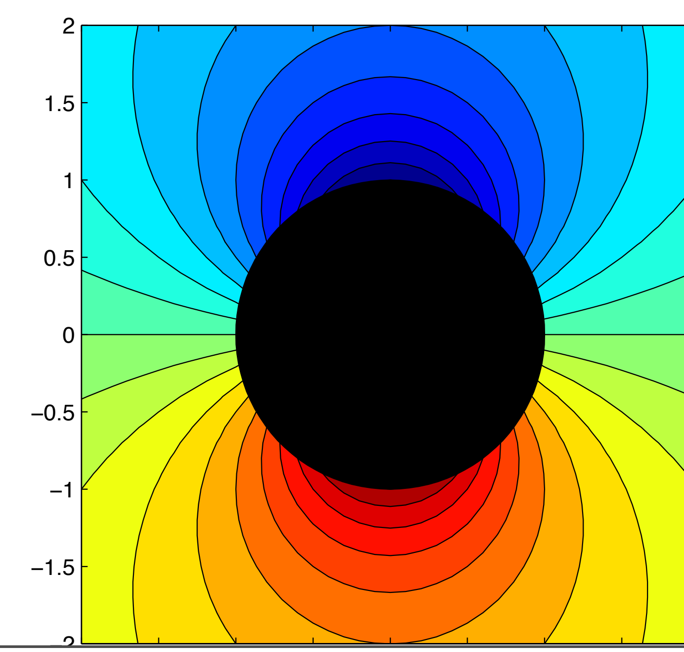
Lamb, 1945.

Milne-Thompson, 1962

Newman, 1977.

cylinder moving in Lab frame





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Boundary layer

Laminar separation bubble

Leading edge vortex

Periodic Vortex Shedding

Milne-Thompson, 1973.

Stengel, 2004.

Theodorsen’s Model - 1935

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Increasingly important for small/light aircraft

Unsteady potential flow forces (F=ma)

force needed to move air as plate accelerates

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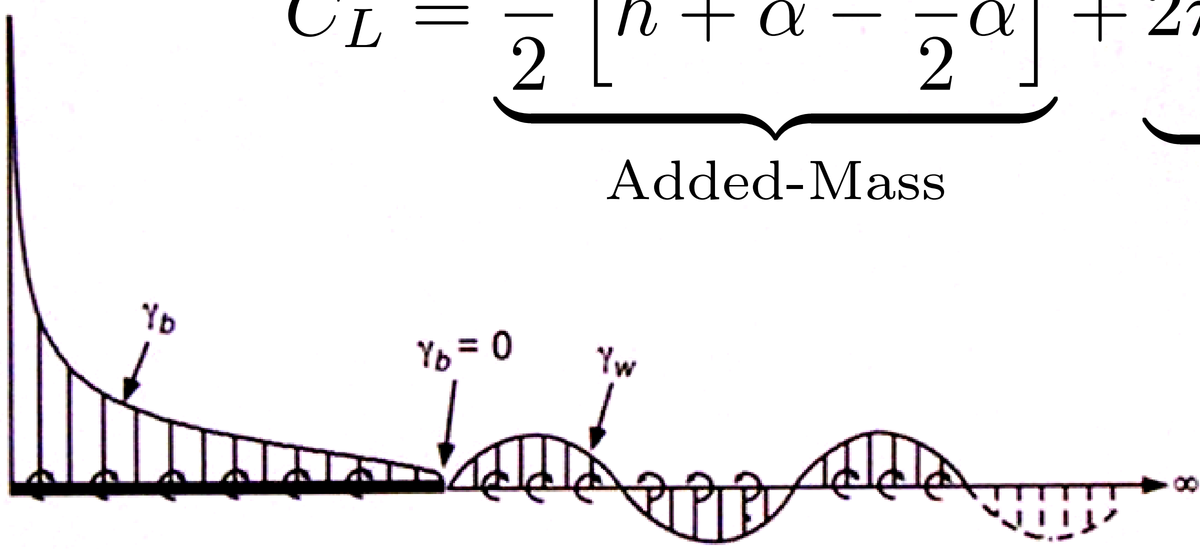
$$C_L = \underbrace{\frac{\pi}{2}\left[\ddot{h} + \dot{\alpha} - \frac{a}{2}\ddot{\alpha}\right]}_{\text{Added-Mass}} + \underbrace{2\pi\left[\alpha + \dot{h} + \frac{1}{2}\dot{\alpha}\left(\frac{1}{2} - a\right)\right]C(k)}_{\text{Circulatory}}$$



$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)}$$

2D Incompressible, inviscid model

Unsteady potential flow (w/ Kutta condition)

Linearized about zero angle of attack

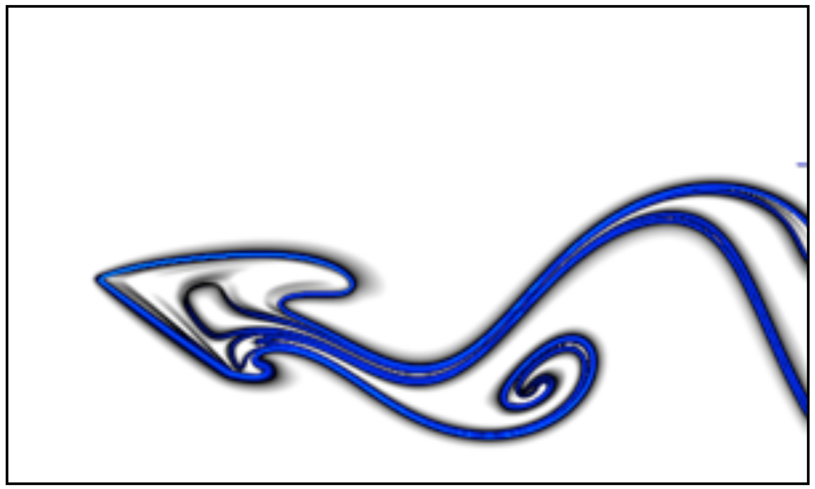
$$\boxed{k = \frac{\pi f c}{U_\infty}}$$

Theodorsen, 1935.

Leishman, 2006.

Three Types of Unsteadiness

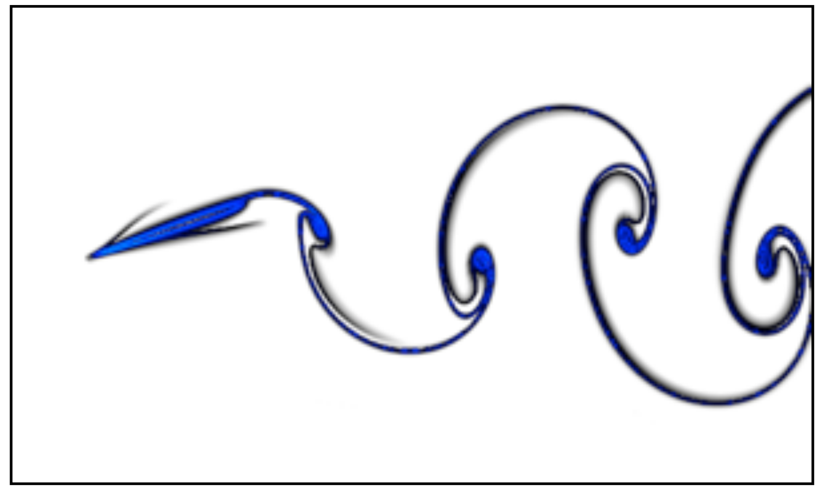
1. High angle-of-attack



$$\alpha > \alpha_{\text{stall}}$$

Large amplitude, slow

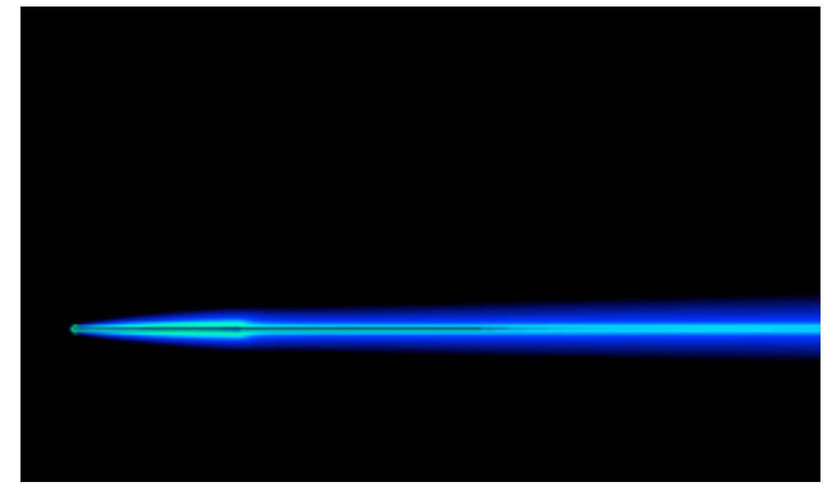
2. Strouhal number



$$St = \frac{Af}{U_\infty}$$

Moderate amplitude, fast

3. Reduced frequency



$$k = \frac{\pi f c}{U_\infty}$$

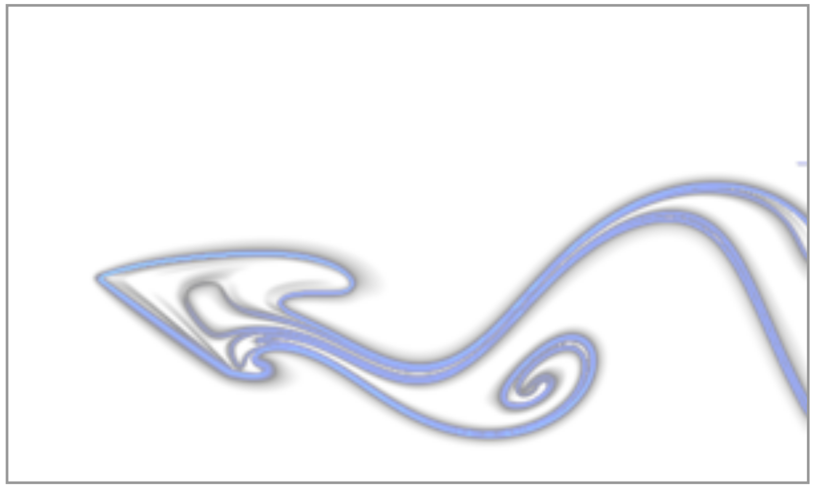
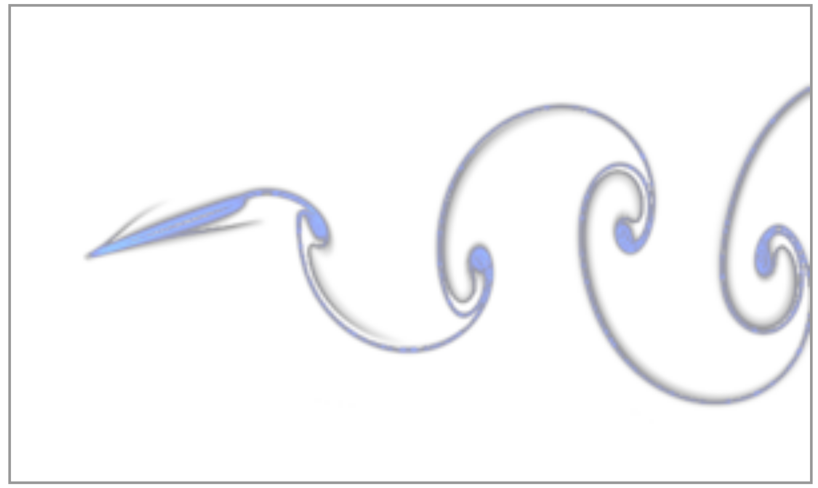
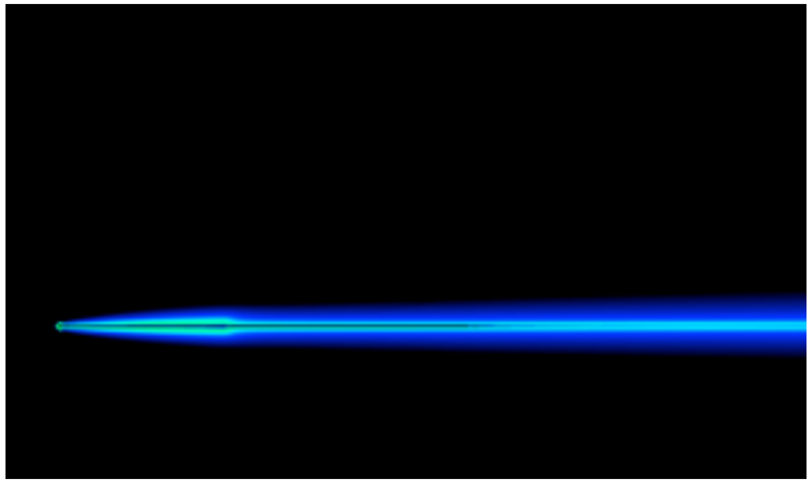
Small amplitude, very fast

Closely related

$$\alpha_{\text{eff}} = \tan^{-1}(\pi St)$$

Brunton and Rowley, AIAA ASM 2009

Three Types of Unsteadiness

1. High angle-of-attack	2. Strouhal number	3. Reduced frequency
		
$\alpha > \alpha_{\text{stall}}$	$St = \dfrac{Af}{U_\infty}$	$k = \dfrac{\pi f c}{U_\infty}$
Large amplitude, slow	**Moderate amplitude, fast**	**Small amplitude, very fast**

Closely related

$$\alpha_{\text{eff}} = \tan^{-1}(\pi St)$$

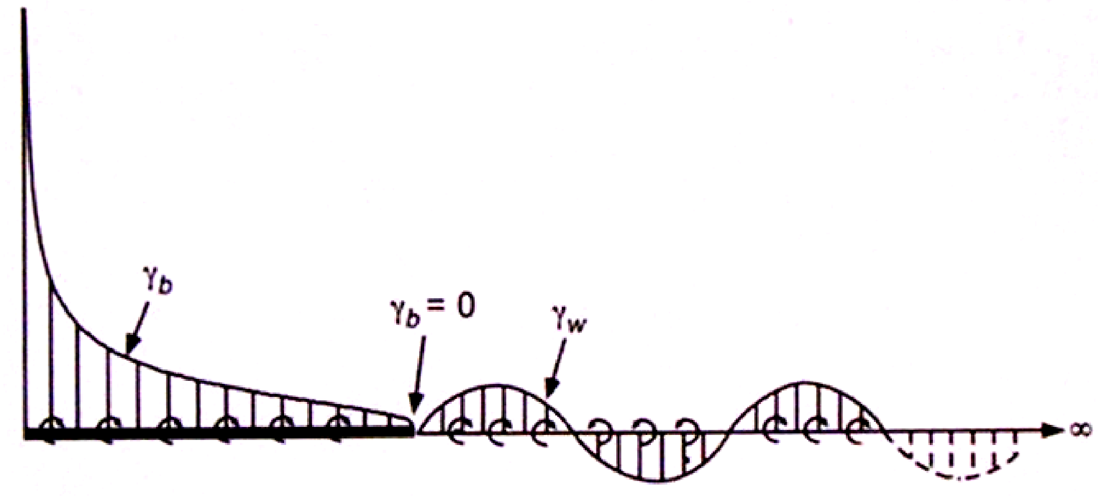
Brunton and Rowley, AIAA ASM 2009

Candidate Lift Models

$$C_L = 2\pi\alpha$$

$$C_L = C_{L_\alpha}\alpha$$

$$C_L = C_L(\alpha)$$



$$C_L(t) = C_L^\delta(t)\alpha(0) + \int_0^t C_L^\delta(t-\tau)\dot{\alpha}(\tau)d\tau$$

Indicial Response (Wagner)

$$C_L = \underbrace{\frac{\pi}{2}\left[\ddot{h} + \dot{\alpha} - \frac{a}{2}\ddot{\alpha}\right]}_{\text{Added-Mass}} + \underbrace{2\pi\left[\alpha + \dot{h} + \frac{1}{2}\dot{\alpha}\left(\frac{1}{2} - a\right)\right]}_{\text{Circulatory}} C(k)$$

Theodorsen's Model

Motivation for State-Space Models

Captures input output dynamics accurately

Computationally tractable

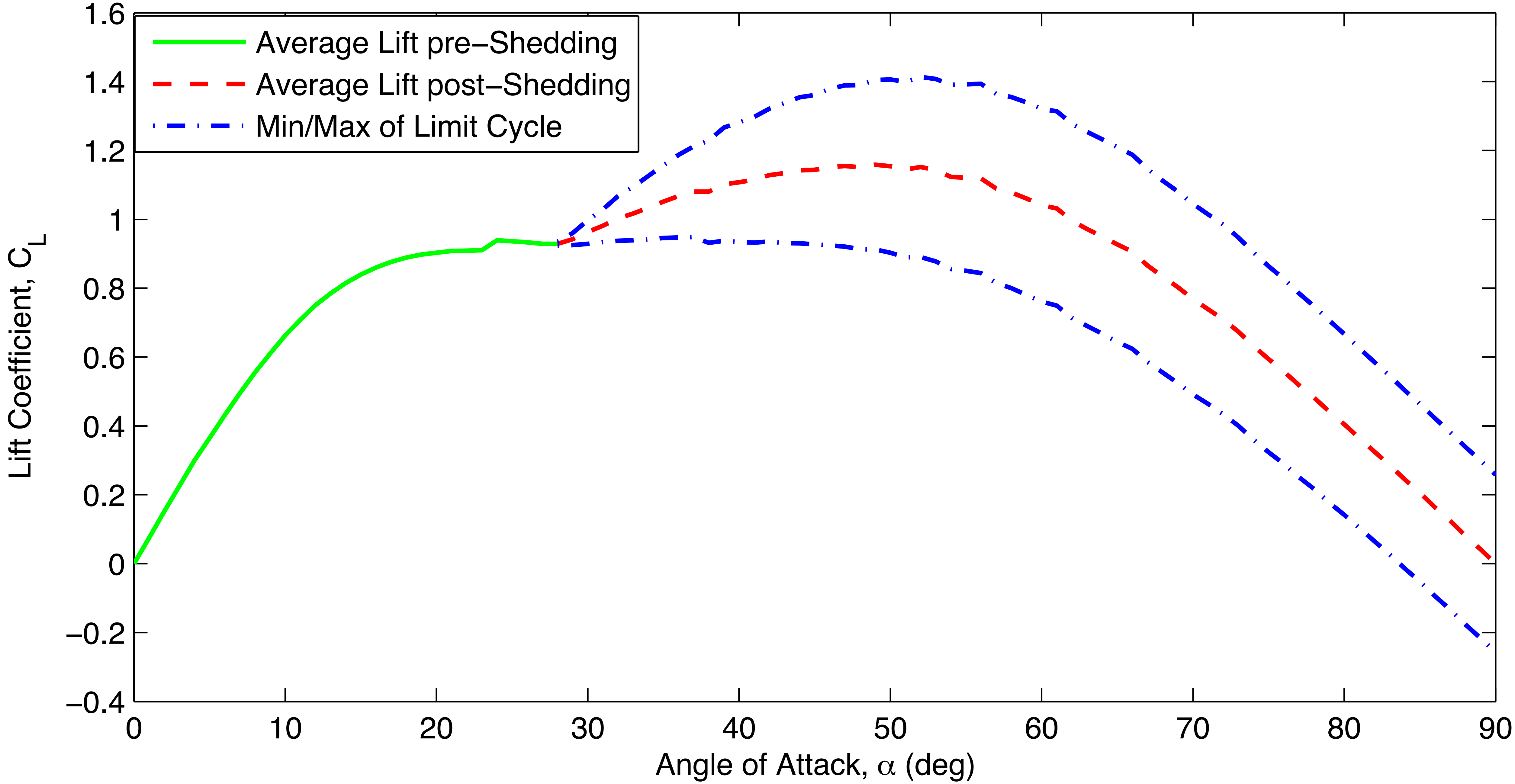
fits into control framework

Wagner, 1925.

Theodorsen, 1935.

Leishman, 2006.

Lift vs Angle of Attack

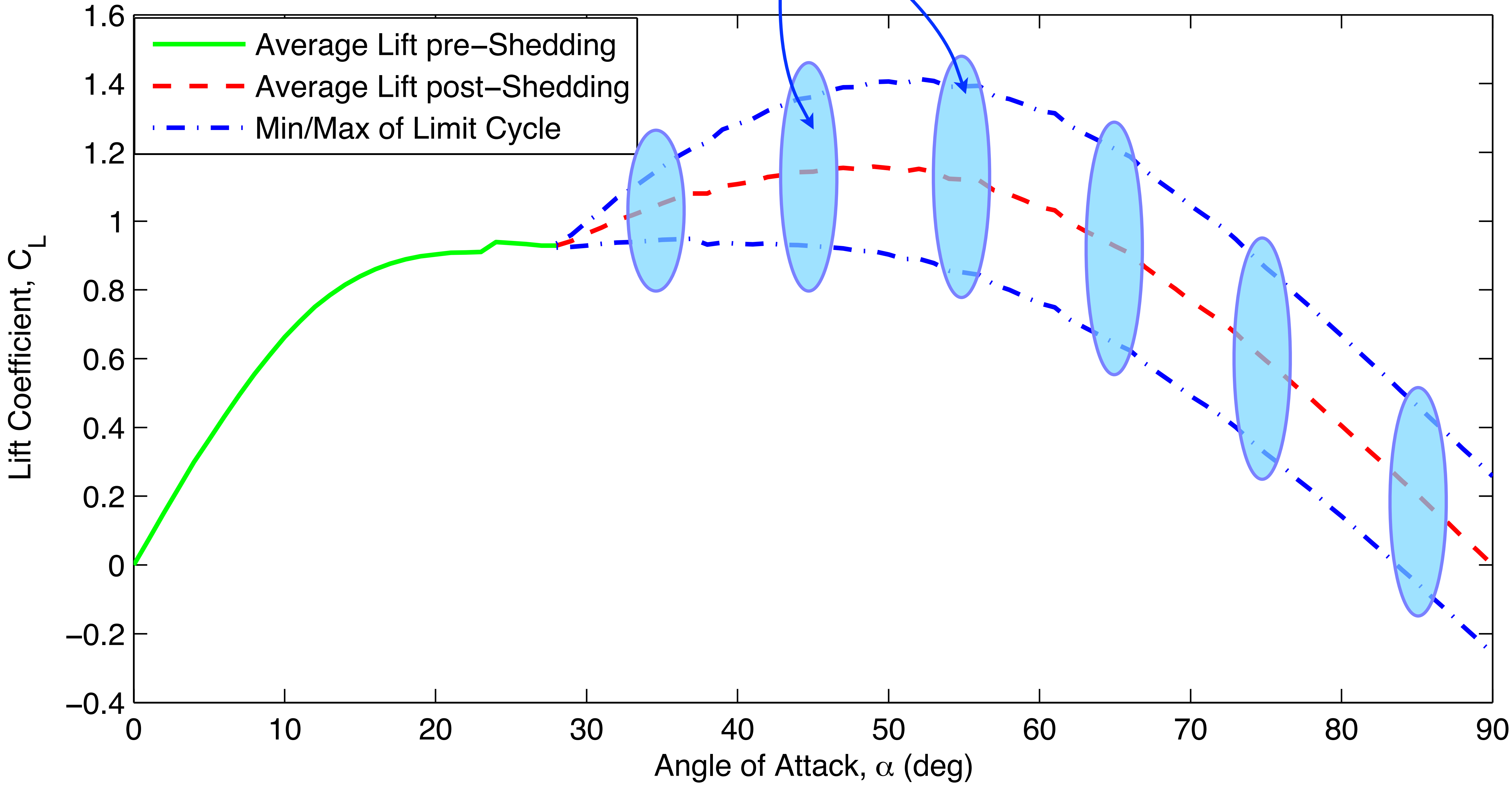


Low Reynolds number, (Re=100)

Hopf bifurcation at $\alpha_{\rm crit} \approx 28^\circ$ **(pair of imaginary eigenvalues pass into right half plane)**

Lift vs Angle of Attack

Models based on Hopf normal form capture vortex shedding



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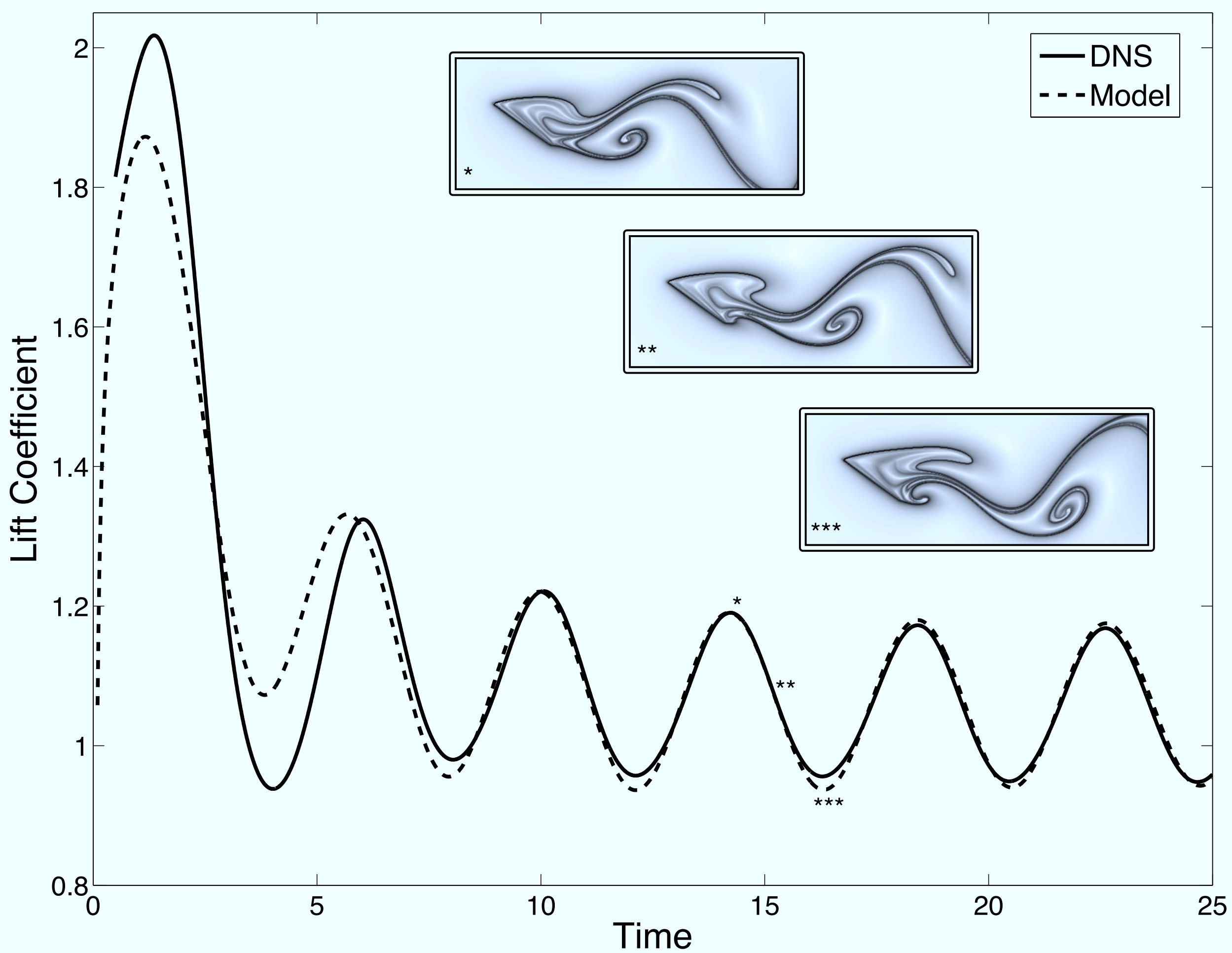
High angle of attack models

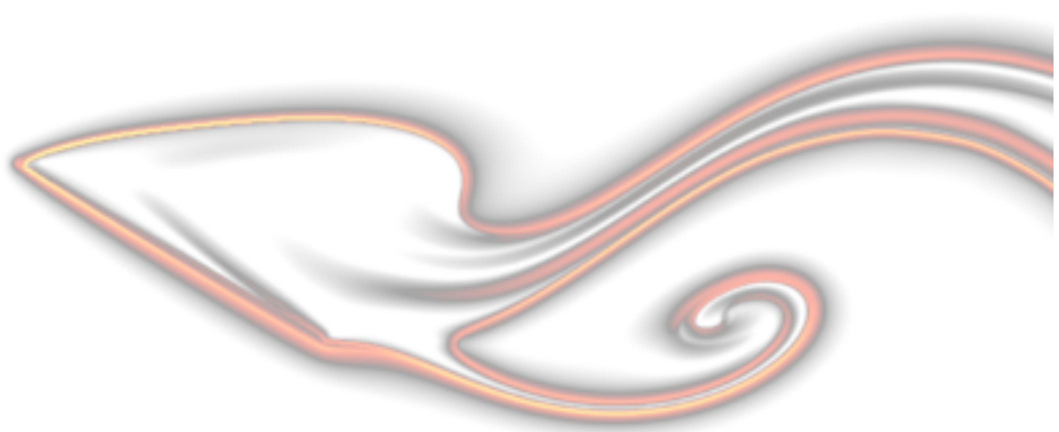
Heuristic Model



$$
\left.\begin{aligned}
\dot{x} &= (\alpha - \alpha_c)\mu x - \omega y - a x(x^2 + y^2) \\
\dot{y} &= (\alpha - \alpha_c)\mu y + \omega x - a y(x^2 + y^2) \\
\dot{z} &= -\lambda z
\end{aligned}\right\} \implies
\begin{aligned}
\dot{r} &= r\left[(\alpha - \alpha_c)\mu - a r^2\right] \\
\dot{\theta} &= \omega \\
\dot{z} &= -\lambda z
\end{aligned}
$$

Galerkin Projection onto POD

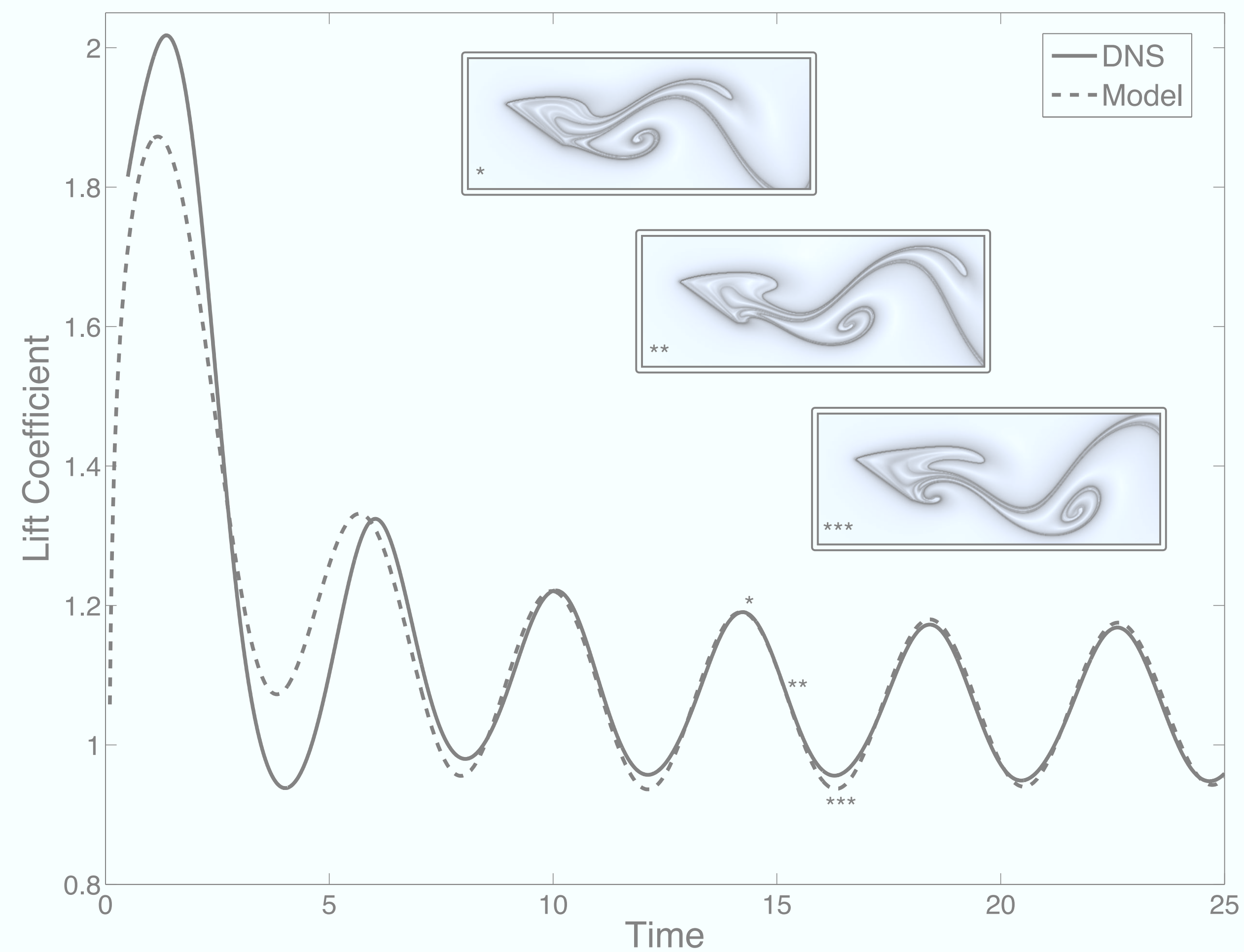
Full DNS



Reconstruction

High angle of attack models

Heuristic Model



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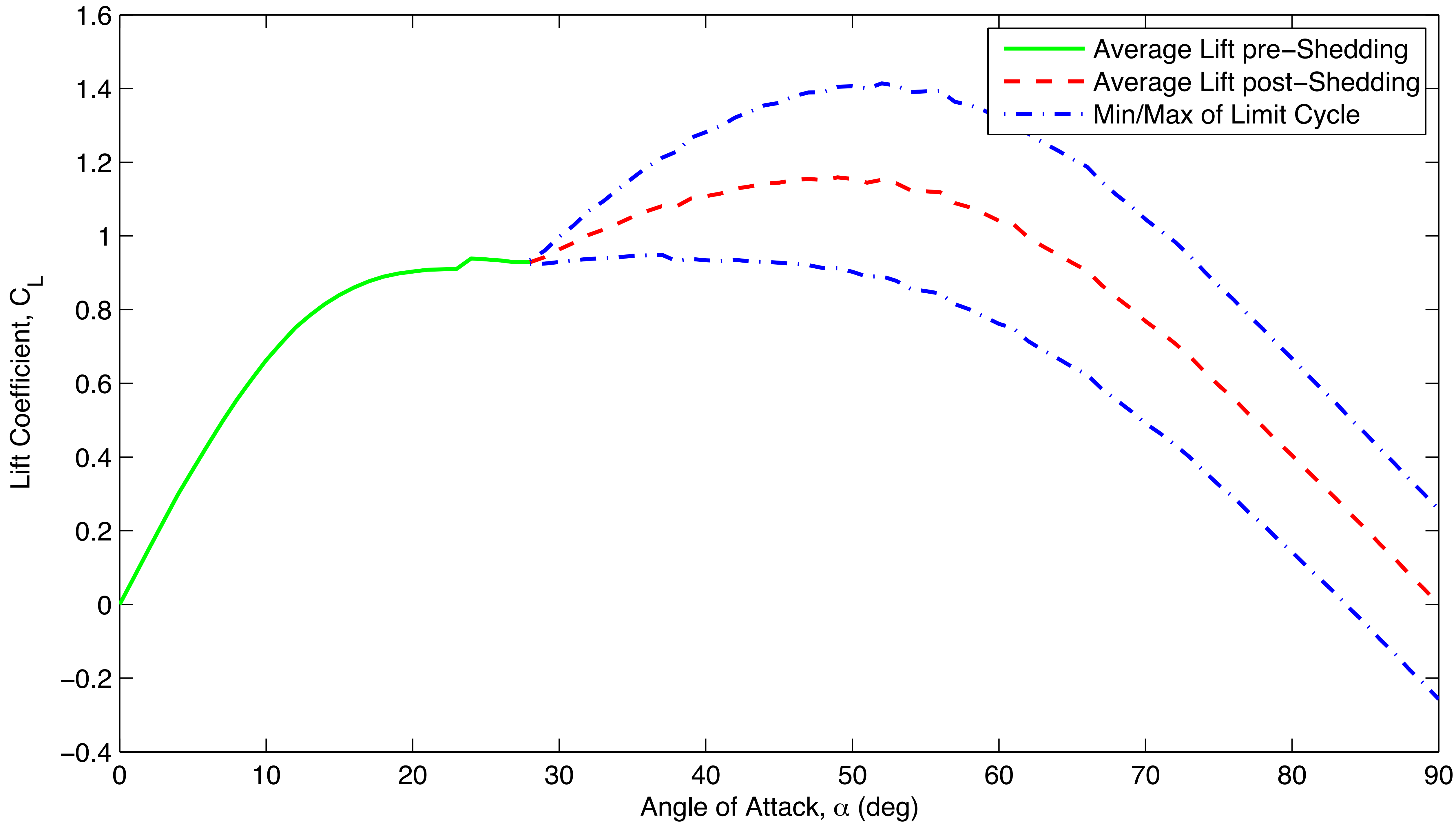
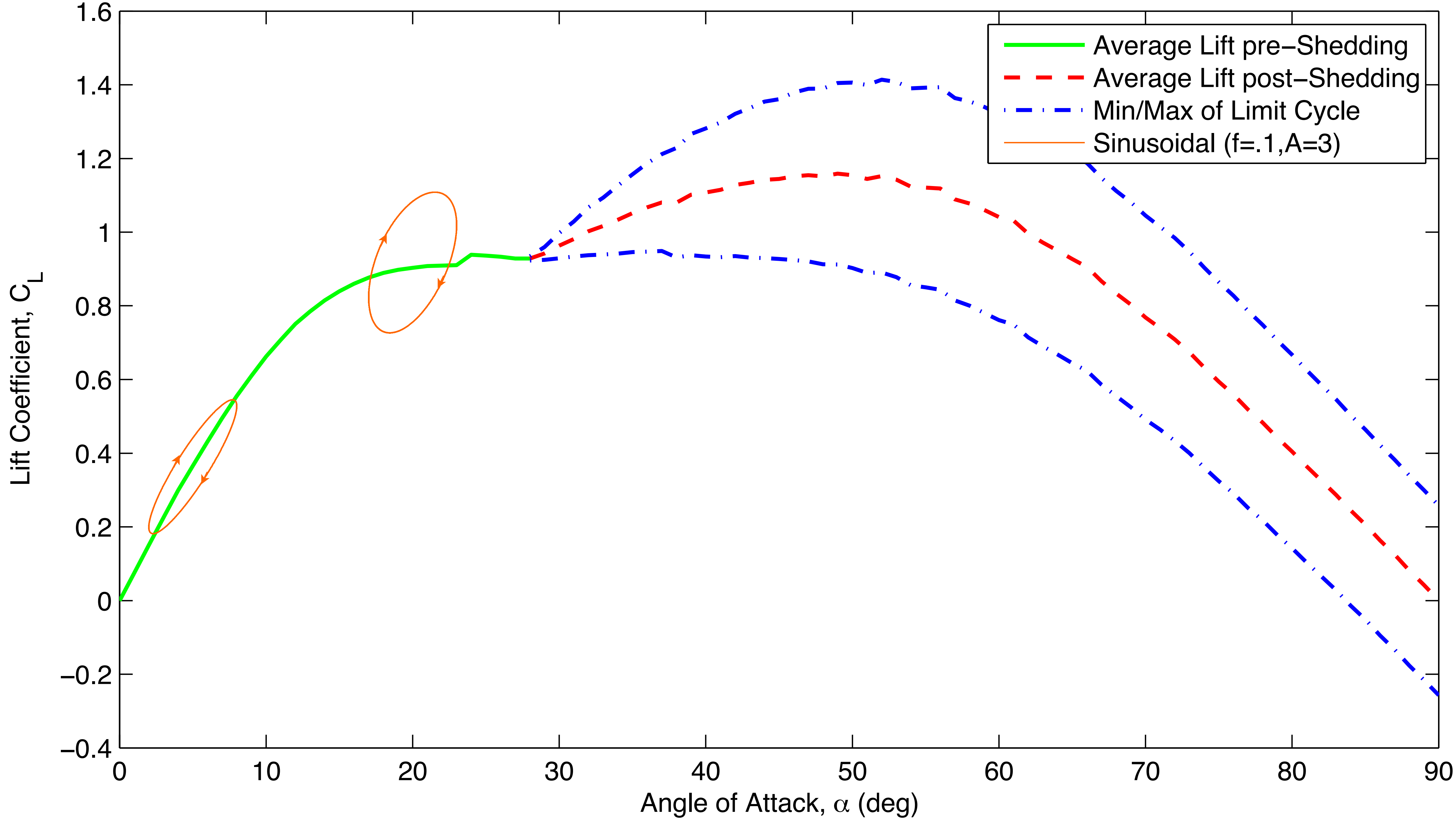


Need model that captures lift due to moving airfoil!

Lift vs. Angle of Attack



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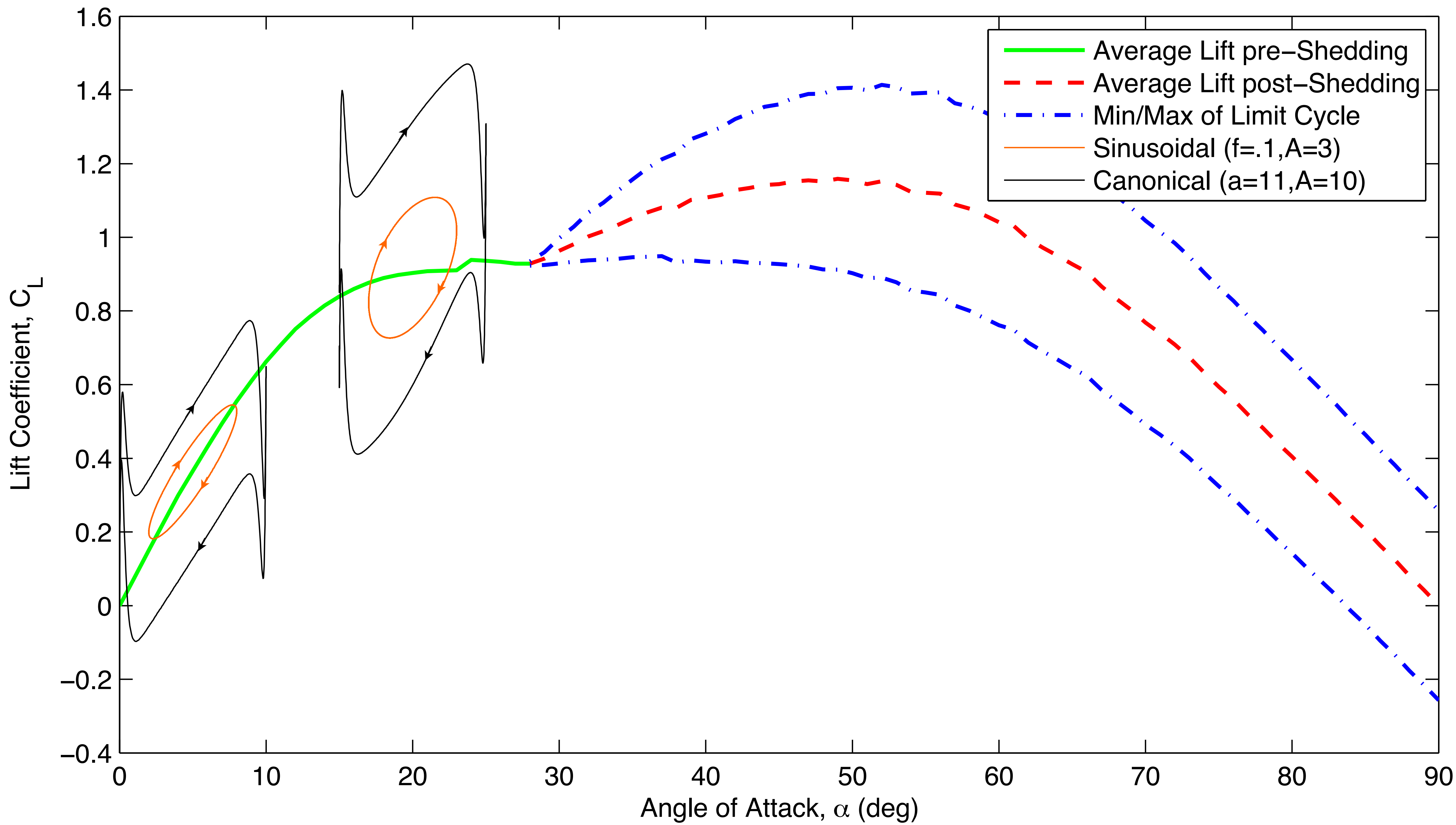

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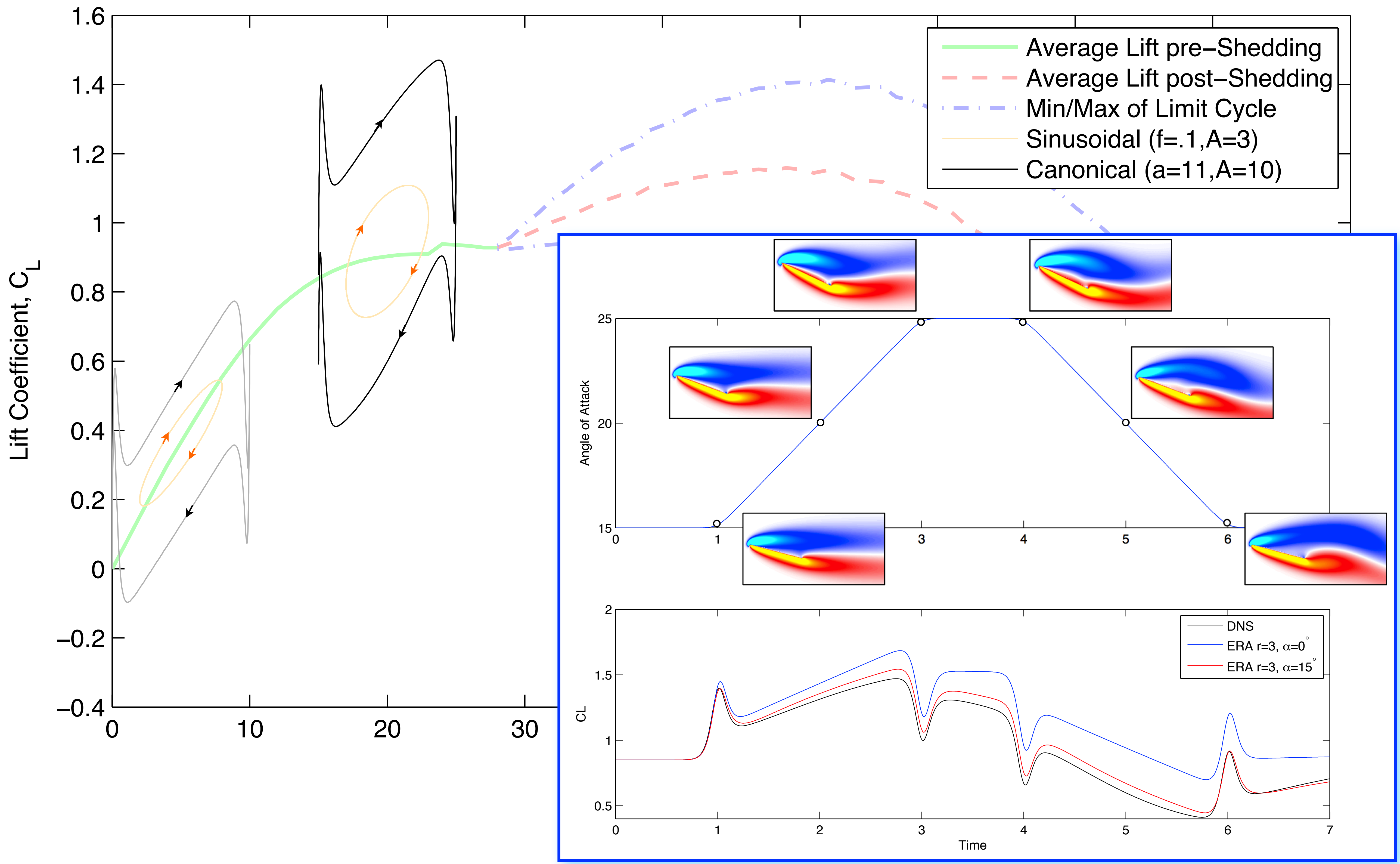

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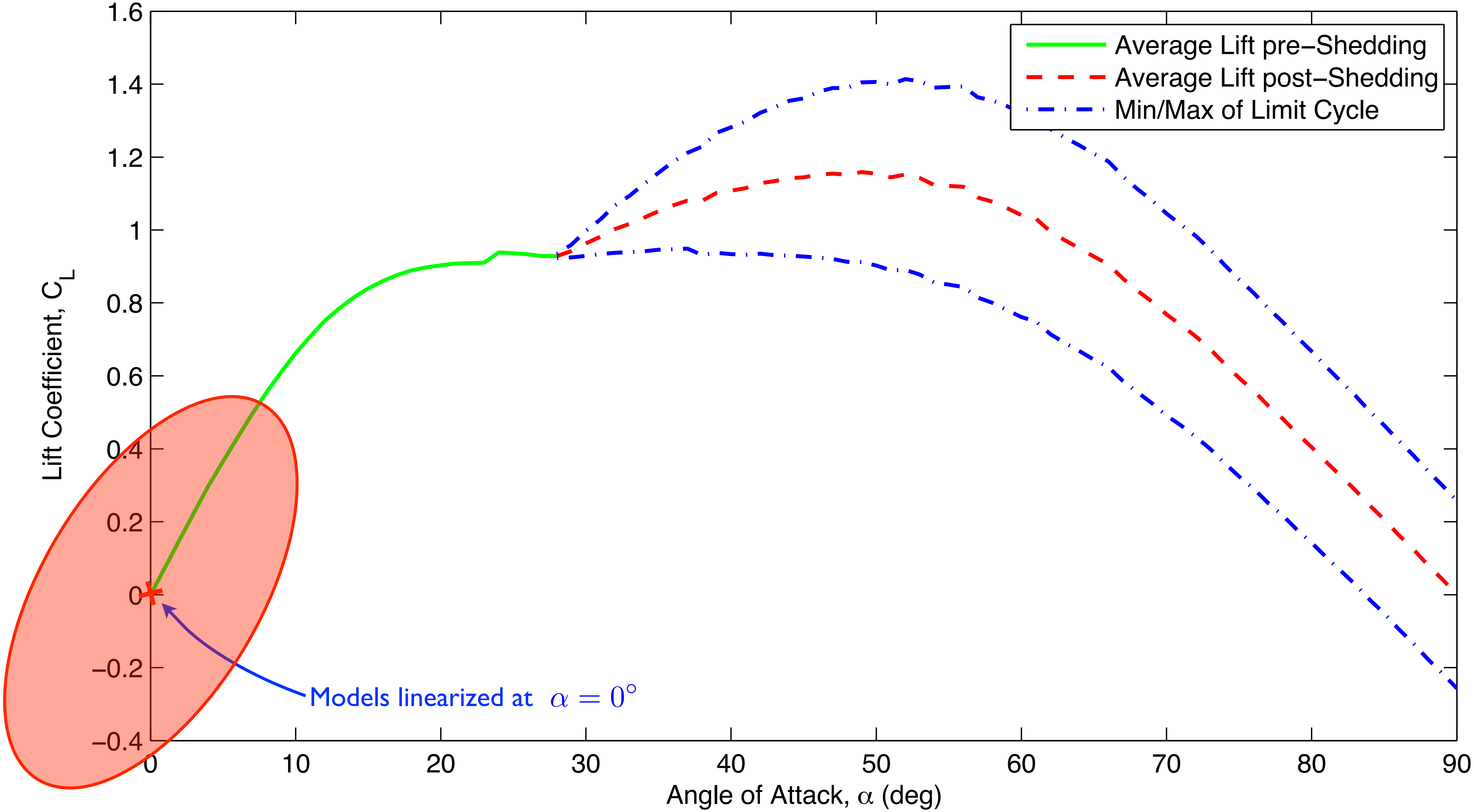

Theodorsen's Model

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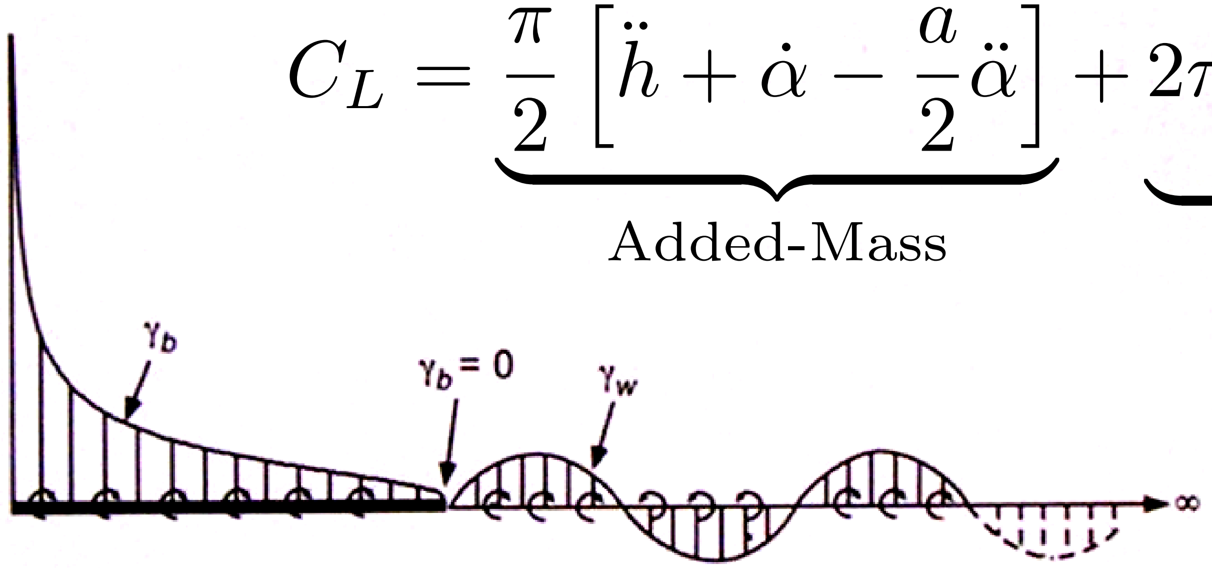

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2D Incompressible, inviscid model

Unsteady potential flow (w/ Kutta condition)

Linearized about zero angle of attack

$$\boxed{k = \frac{\pi f c}{U_\infty}}$$

Apparent Mass

Increasingly important for lighter aircraft

Not trivial to compute, but essentially solved

force needed to move air as plate accelerates

Circulatory Lift

Captures separation effects

Need improved models here

source of all lift in steady flight

Theodorsen, 1935.

Leishman, 2006.

Bode Plot of Theodorsen

$$C_L = \underbrace{\frac{\pi}{2}\left[\ddot{h} + \dot{\alpha} - \frac{a}{2}\ddot{\alpha}\right]}_{\text{Added-Mass}} + \underbrace{2\pi\left[\alpha + \dot{h} + \frac{1}{2}\dot{\alpha}\left(\frac{1}{2} - a\right)\right]}_{\text{Circulatory}} C(k)$$

$$\boxed{k = \frac{\pi f c}{U_\infty}}$$

Frequency response

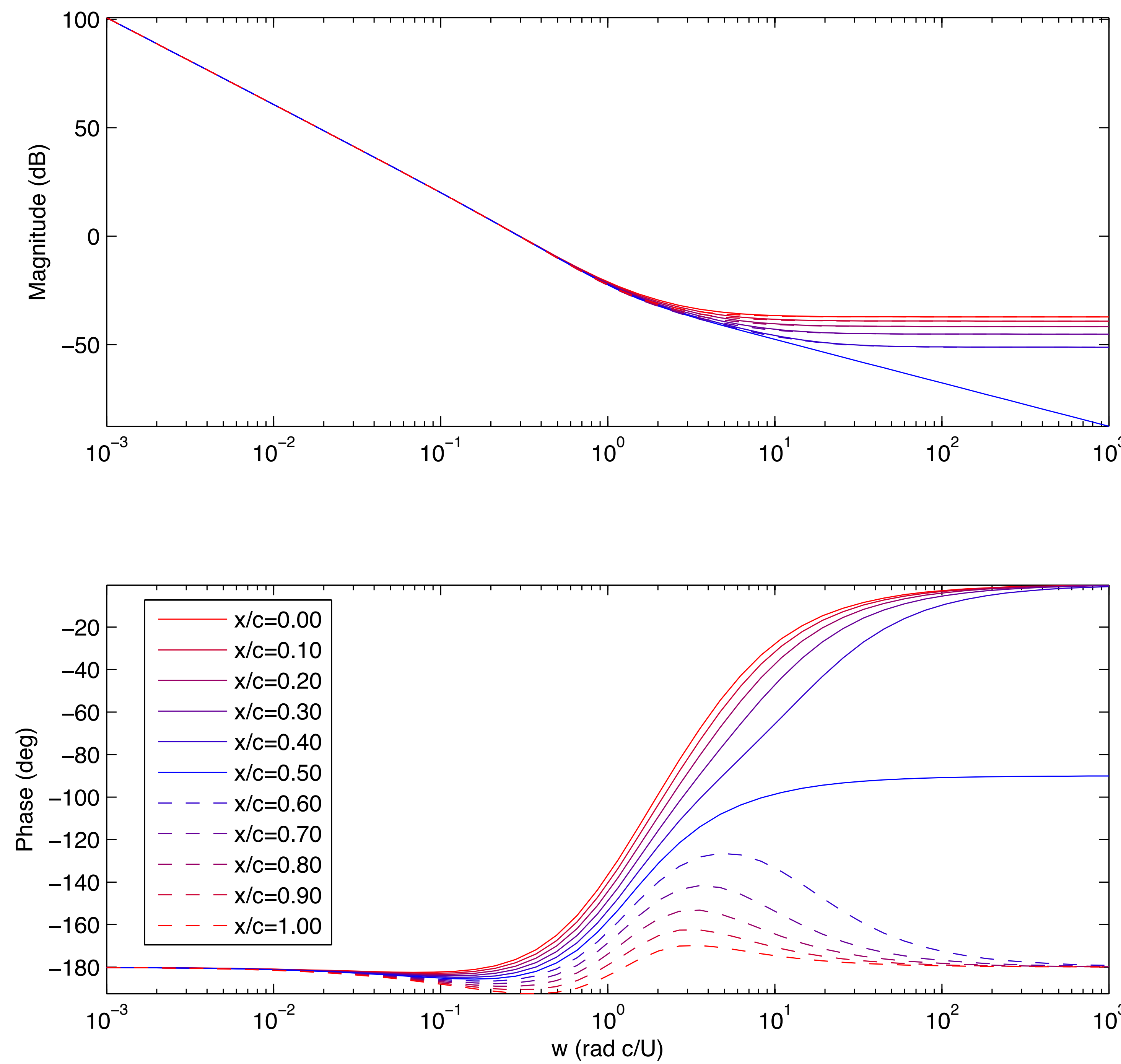
input is $\ddot{\alpha}$ (α is angle of attack)

output is lift coefficient $\mathrm{C_L}$

Low frequencies dominated by quasi-steady forces

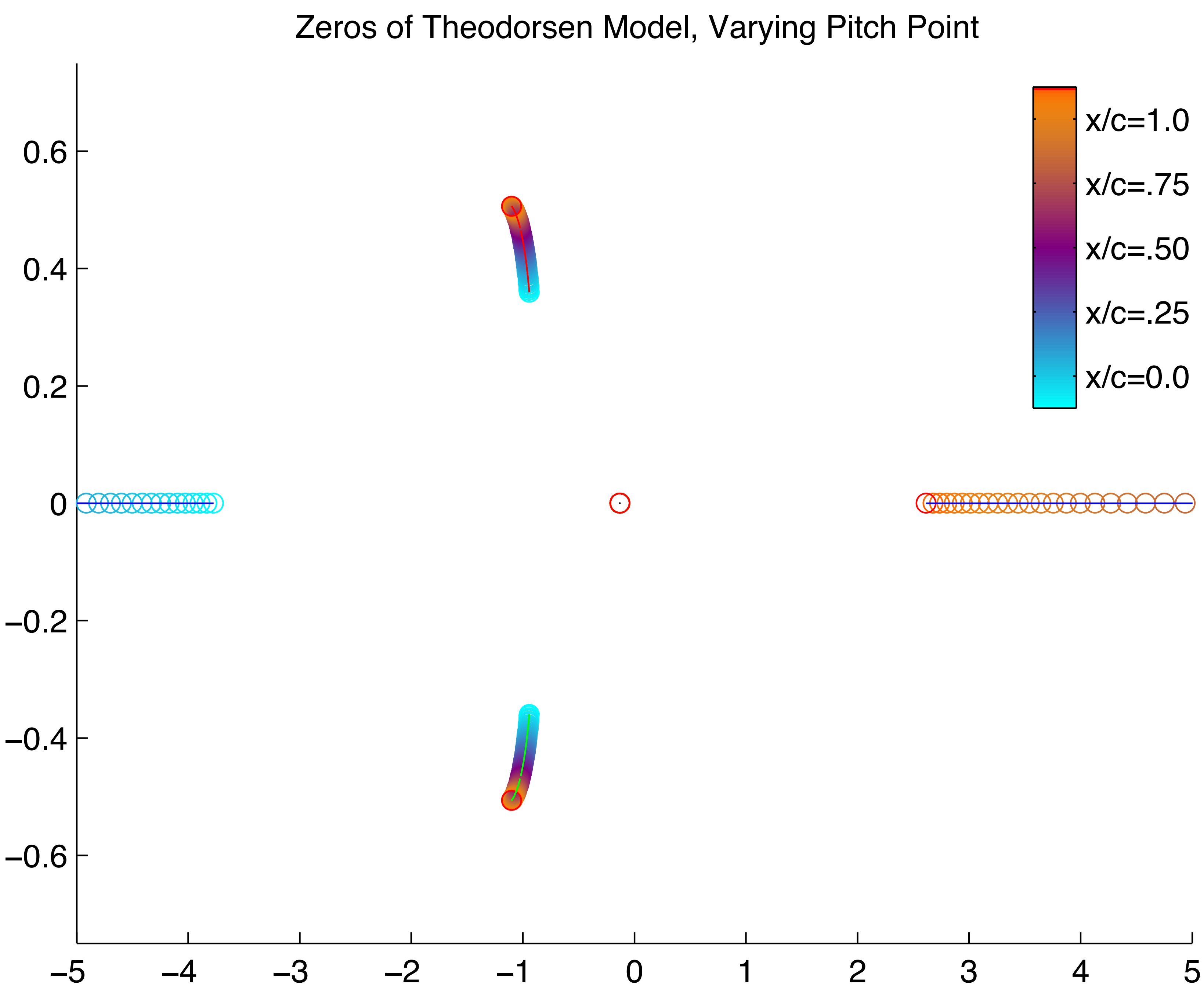
High frequencies dominated by added-mass forces

Crossover region determined by Theodorsen's function $\mathrm{C(k)}$



Brunton and Rowley, AIAA ASM 2011

Zeros of Theodorsen's Model



As pitch point moves aft of center, zero enters RHP at +infinity.

non-minimum phase response:

Given a step in angle of attack, lift initially moves in opposite direction (because of negative added-mass forces), before the circulatory lift forces have a change to catch up and system relaxes to a positive lift steady state.

Brunton and Rowley, AIAA ASM 2011

Indicial Response Models

Given an impulse in angle of attack, $\alpha = \delta(t)$, the time history of Lift is $C_L^\delta(t)$

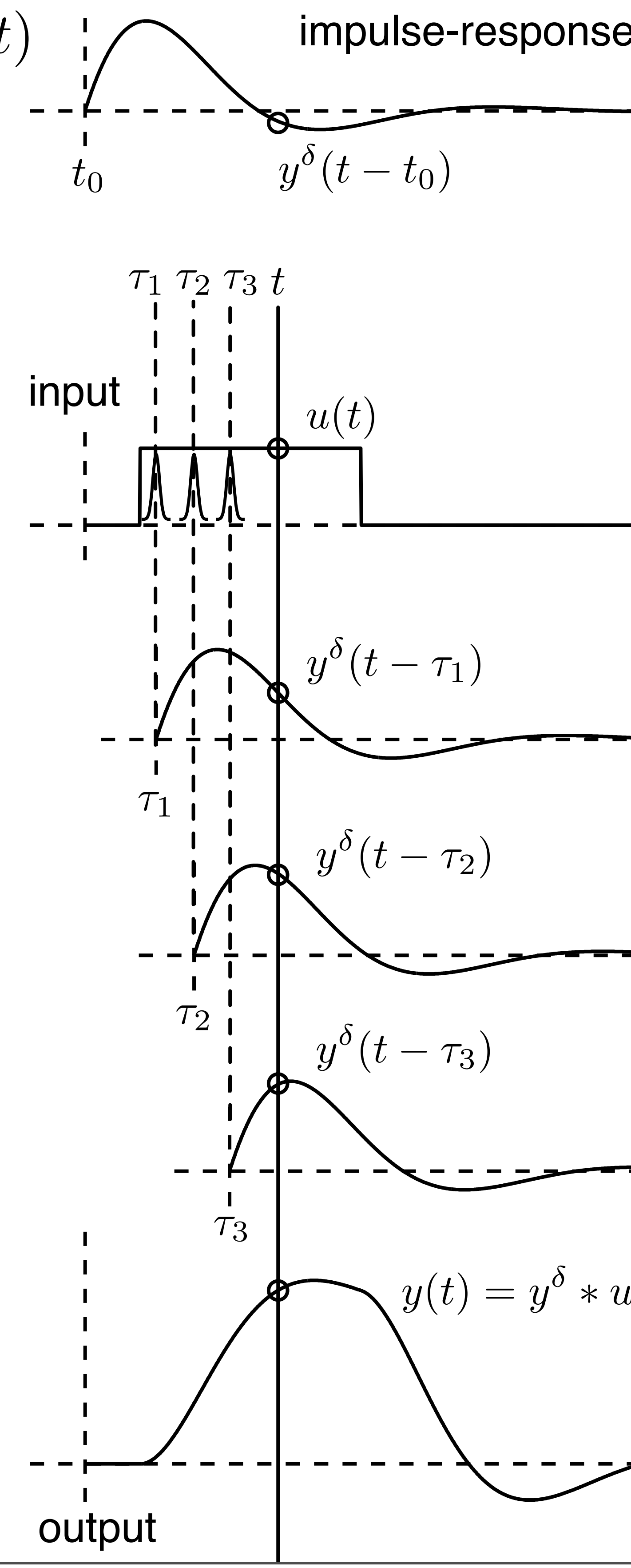
The response to an arbitrary input $\alpha(t)$ is given by linear superposition:

$$C_L(t) = \int_0^t C_L^\delta(t-\tau)\alpha(\tau)d\tau = \left(C_L^\delta * \alpha\right)(t)$$

Given a step in angle of attack, $\dot{\alpha} = \delta(t)$, the time history of Lift is $C_L^S(t)$

The response to an arbitrary input $\alpha(t)$ is given by:

$$C_L(t) = C_L^S(t)\alpha(0) + \int_0^t C_L^S(t-\tau)\dot{\alpha}(\tau)d\tau$$



Model Summary

Reconstructs Lift for arbitrary input

Linear time-invariant (LTI) models

Based on experiment, simulation or theory

Wagner developed indicial response analytically
using same approximations as Theodorsen

convolution integral inconvenient for feedback control design

Wagner, 1925.

Reisenthel, 1996.

Leishman, 2006.

Reduced Order Indicial Response

Stability derivatives plus fast dynamics

$$C_L(\alpha, \dot{\alpha}, \ddot{\alpha}, \mathbf{x}) = C_{L_\alpha}\alpha + C_{L_{\dot{\alpha}}}\dot{\alpha} + C_{L_{\ddot{\alpha}}}\ddot{\alpha} + C\mathbf{x}$$

Quasi-steady and added-mass

Fast dynamics

Transfer Function

$$Y(s) = \left[\frac{C_{L_\alpha}}{s^2} + \frac{C_{L_{\dot{\alpha}}}}{s} + C_{L_{\ddot{\alpha}}} + G(s)\right] s^2 U(s)$$

State-Space Form

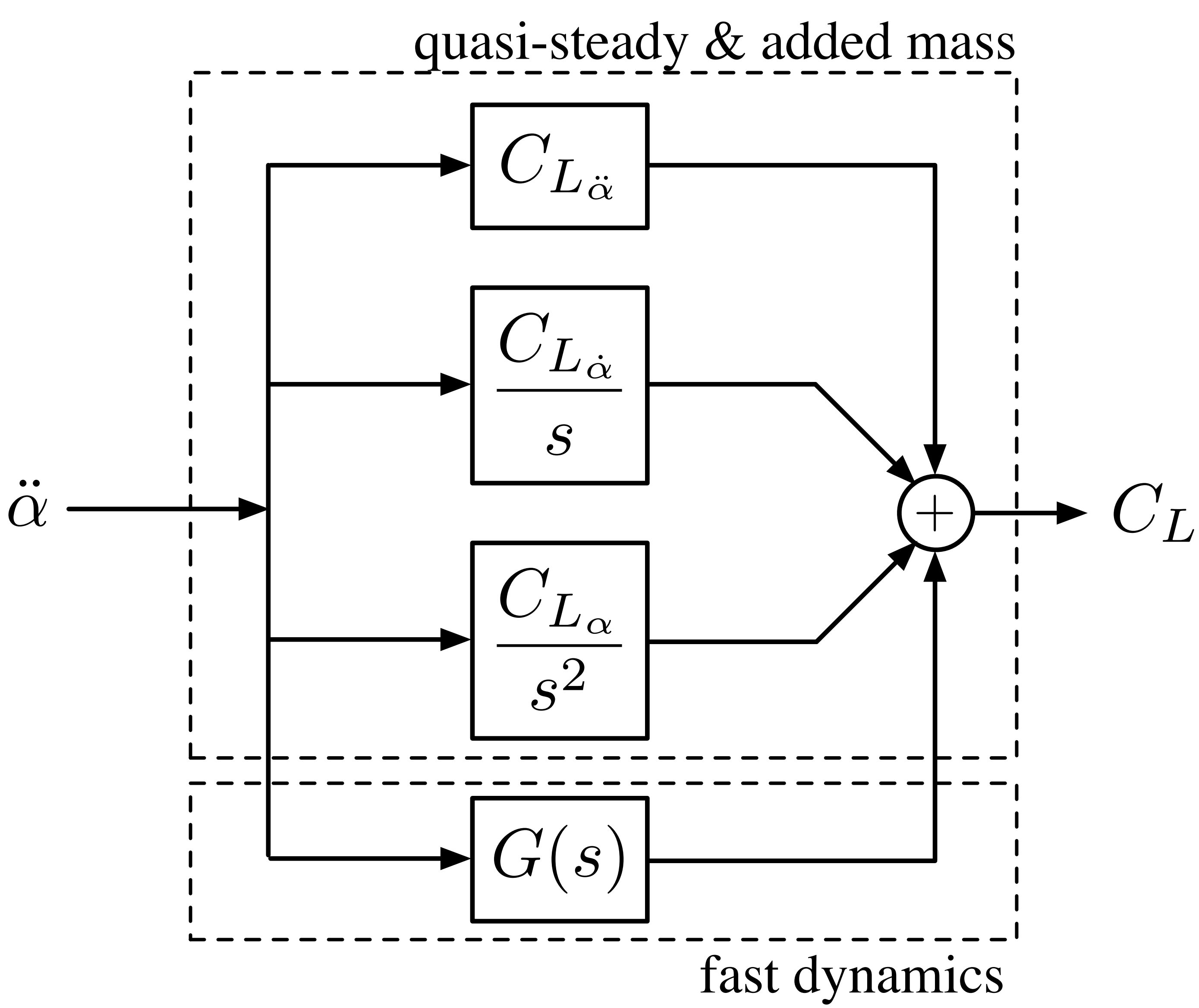
$$\frac{d}{dt}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} = \begin{bmatrix}A_r & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}B_r\\ 0\\ 1\end{bmatrix}\ddot{\alpha}$$

$$C_L = \begin{bmatrix}C_r & C_{L_\alpha} & C_{L_{\dot{\alpha}}}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} + C_{L_{\ddot{\alpha}}}\ddot{\alpha}$$

Brunton and Rowley, *in preparation.*

Reduced Order Indicial Response

$$C_L(t) = C_L^S(t)\alpha(0) + \int_0^t C_L^S(t-\tau)\dot{\alpha}(\tau)d\tau$$



fast dynamics

Reduced-order model

$$\frac{d}{dt}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} = \begin{bmatrix}A_r & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}B_r\\ 0\\ 1\end{bmatrix}\ddot{\alpha}$$

input

$$C_L = \begin{bmatrix}C_r & C_{L_\alpha} & C_{L_{\dot{\alpha}}}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \alpha\\ \dot{\alpha}\end{bmatrix} + C_{L_{\ddot{\alpha}}}\ddot{\alpha}$$

quasi-steady and added-mass

Model Summary

Linearized about $\alpha = 0$

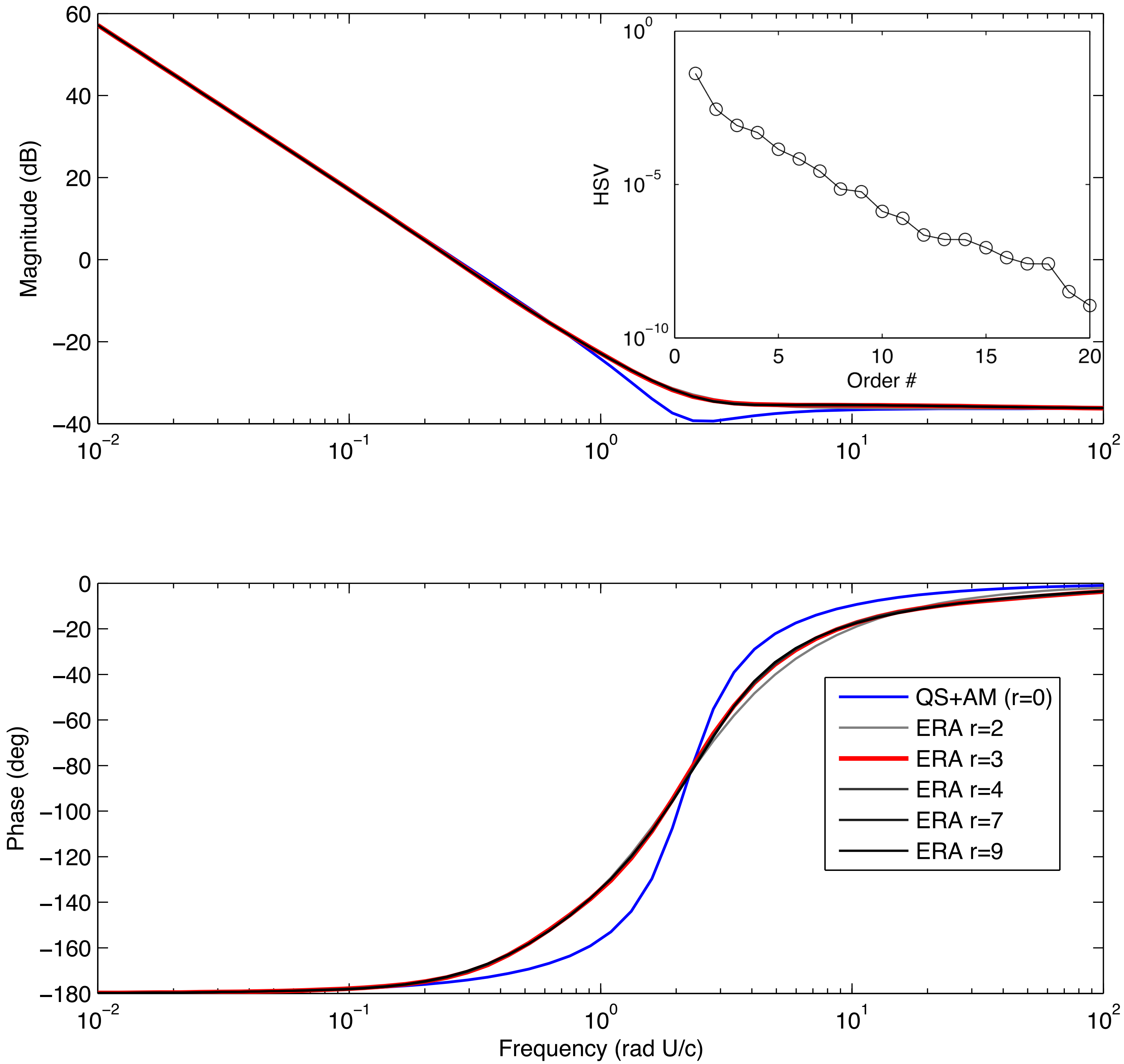
Based on experiment, simulation or theory

Recovers stability derivatives $C_{L_\alpha}, C_{L_{\dot{\alpha}}}, C_{L_{\ddot{\alpha}}}$ associated with quasi-steady and added-mass

ODE model ideal for control design

Brunton and Rowley, *in preparation.*

Bode Plot - Pitch (LE)



Frequency response

input is $\ddot{\alpha}$ (α is angle of attack)

output is lift coefficient C_L

Pitching at leading edge

Model without additional fast dynamics [QS+AM (r=0)] is inaccurate in crossover region

Models with fast dynamics of ERA model order >3 are converged

Punchline: additional fast dynamics (ERA model) are essential

Brunton and Rowley, *in preparation*.

Bode Plot - Pitch (QC)

Frequency response

input is $\ddot{\alpha}$ (α is angle of attack)

output is lift coefficient C_L

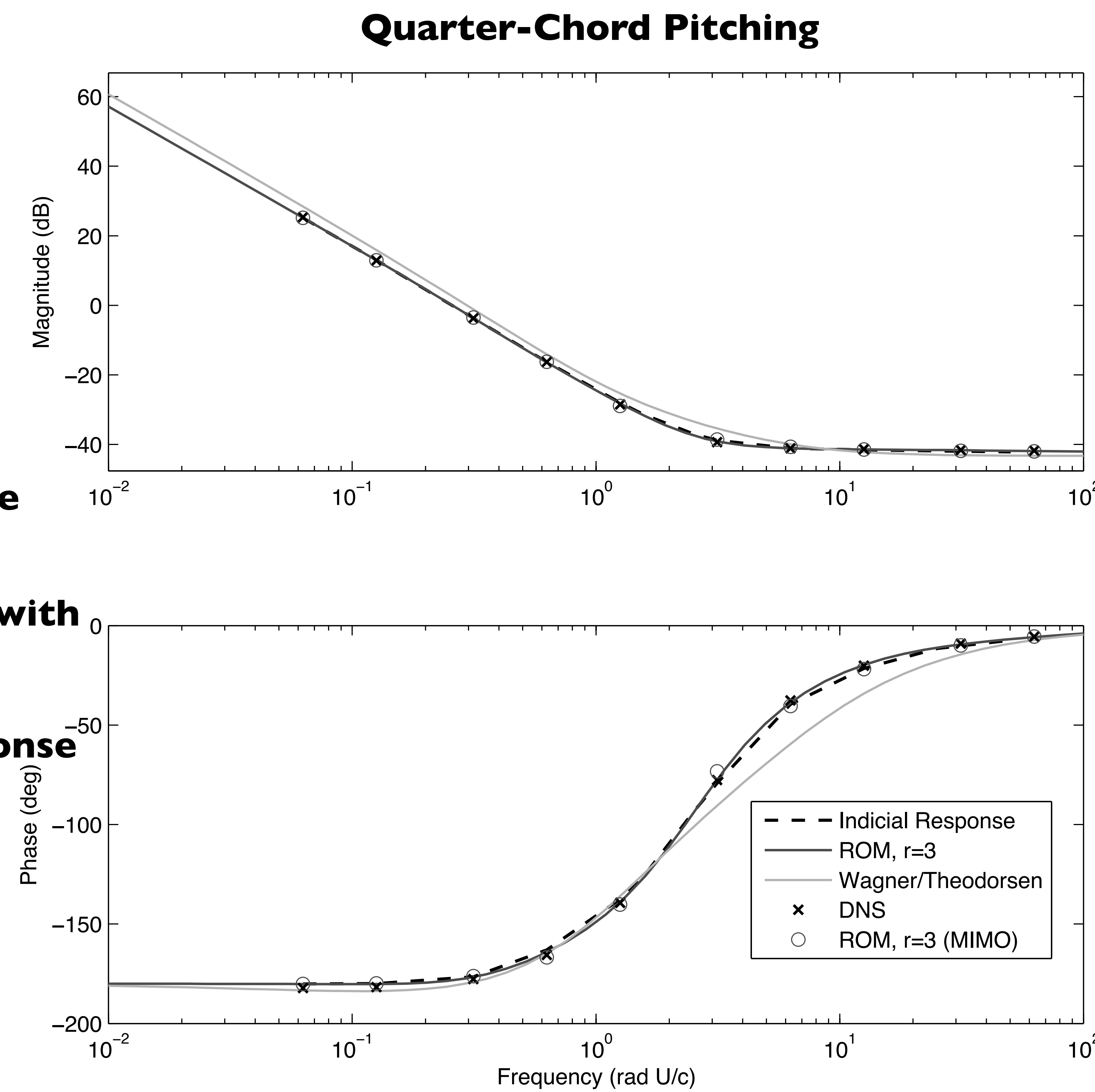
Pitching at quarter chord

Reduced order model with ERA r=3 accurately reproduces Indicial Response

Indicial Response and ROM agree better with DNS than Theodorsen's model.

Asymptotes are correct for Indicial Response because it is based on experiment

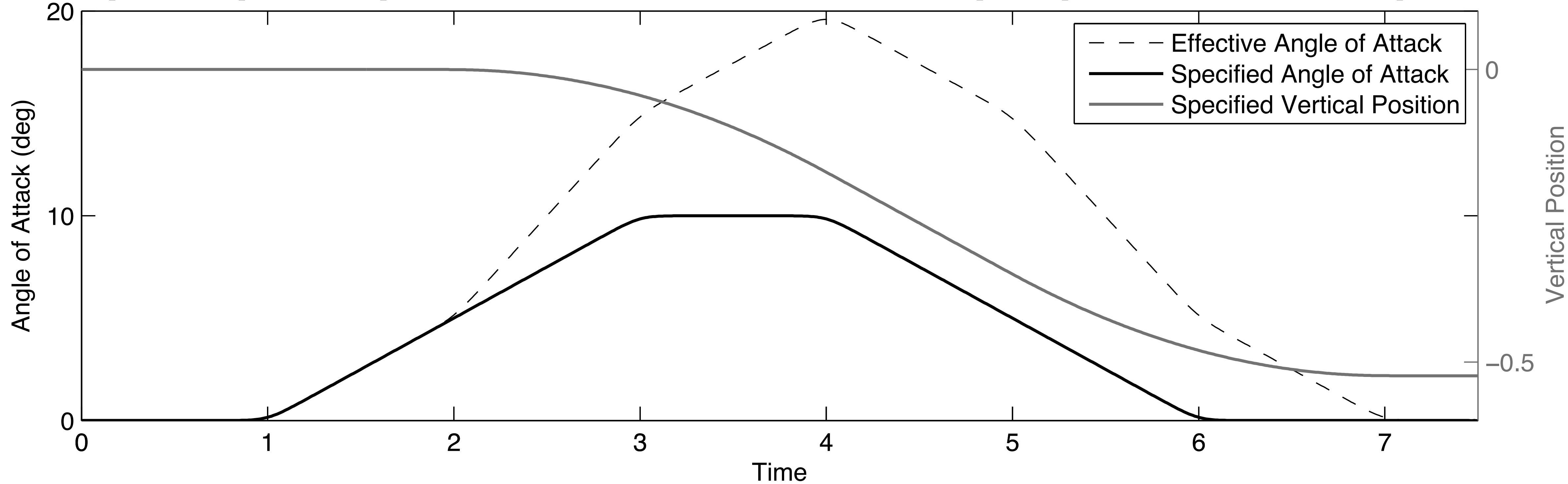
Model for pitch/plunge dynamics [ERA, r=3 (MIMO)] works as well, for the same order model

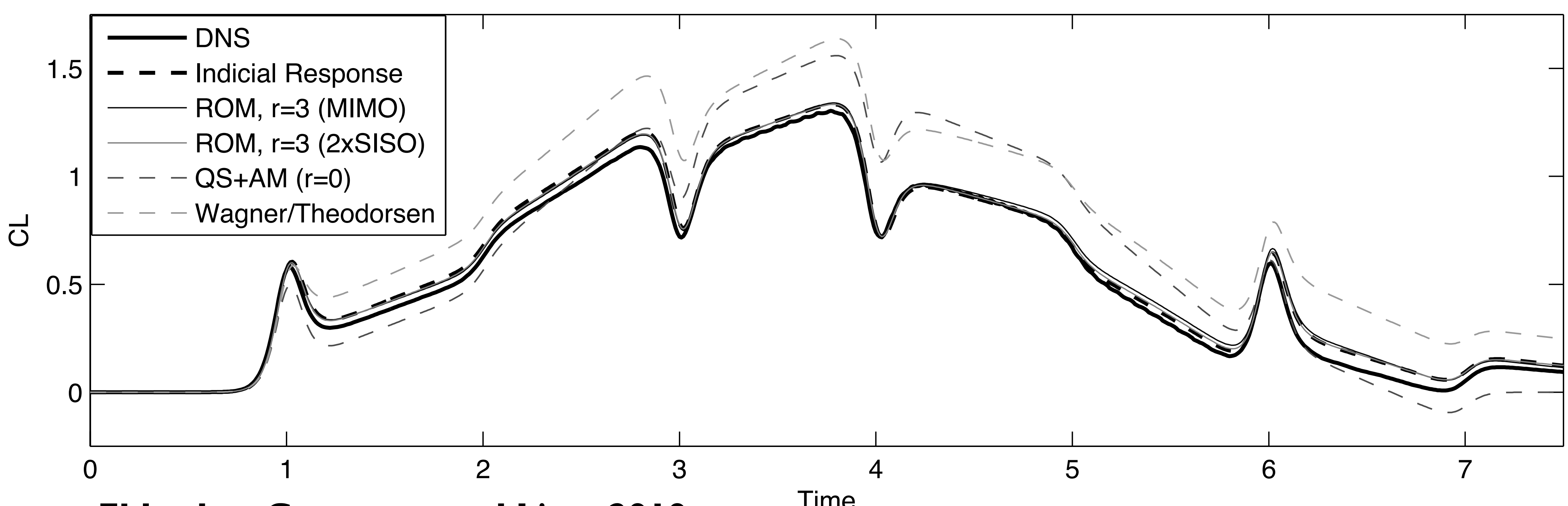


Brunton and Rowley, *in preparation.*

Pitch/Plunge Maneuver

Canonical pitch-up, hold, pitch-down maneuver, followed by step-down in vertical position



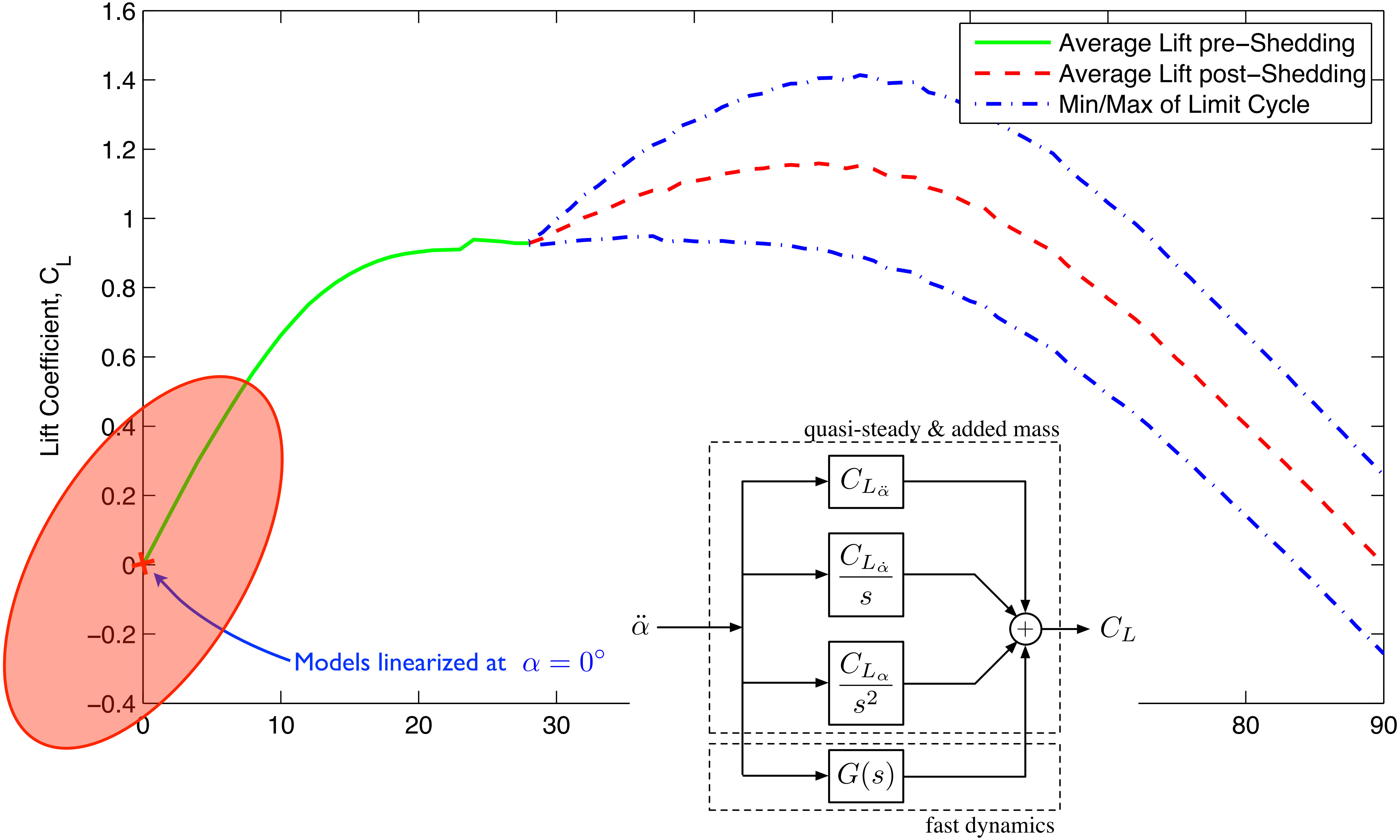


OL, Altman, Eldredge, Garmann, and Lian, 2010

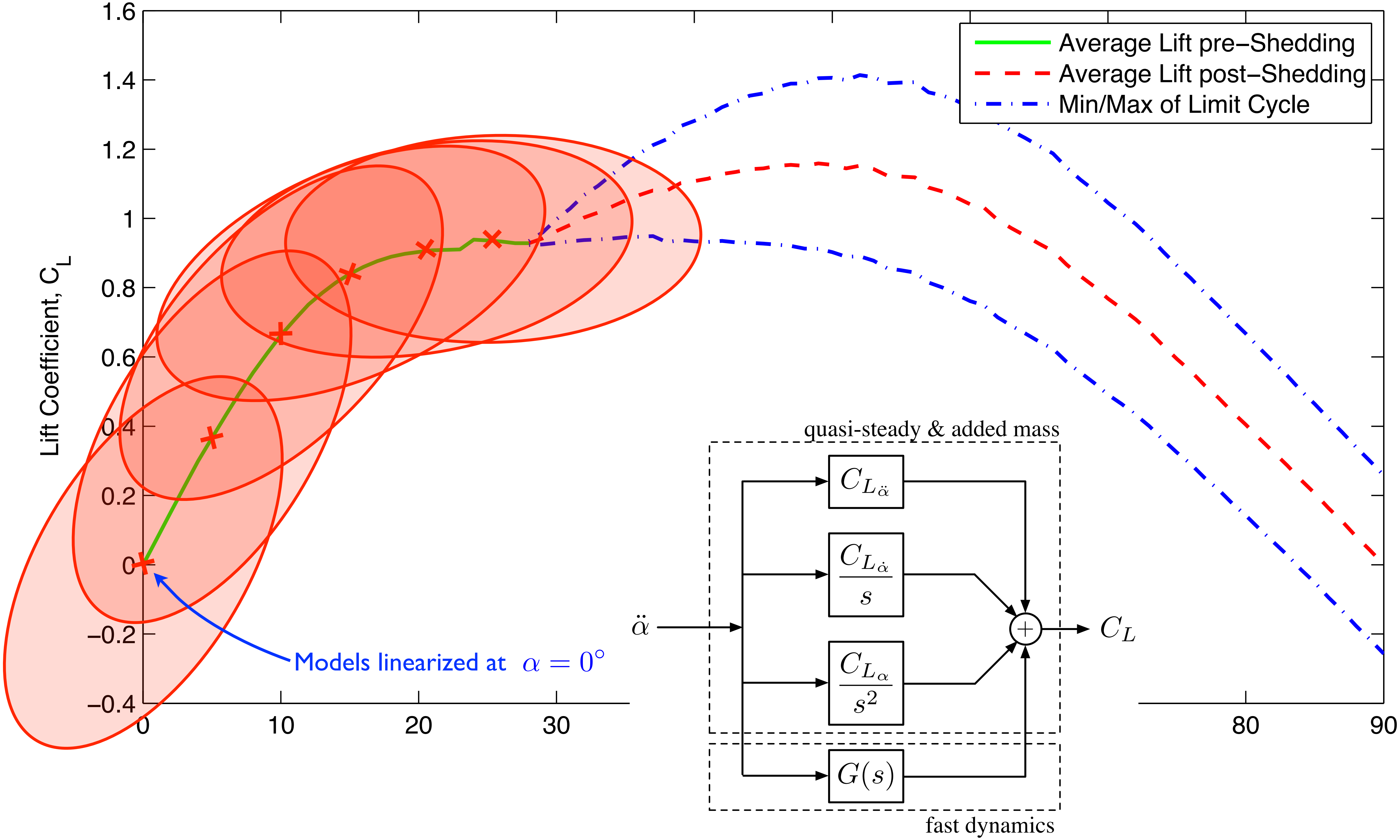
Brunton and Rowley, *in preparation*.

Reduced order model for indicial response accurately captures lift coefficient history from DNS

Lift vs. Angle of Attack



Lift vs. Angle of Attack



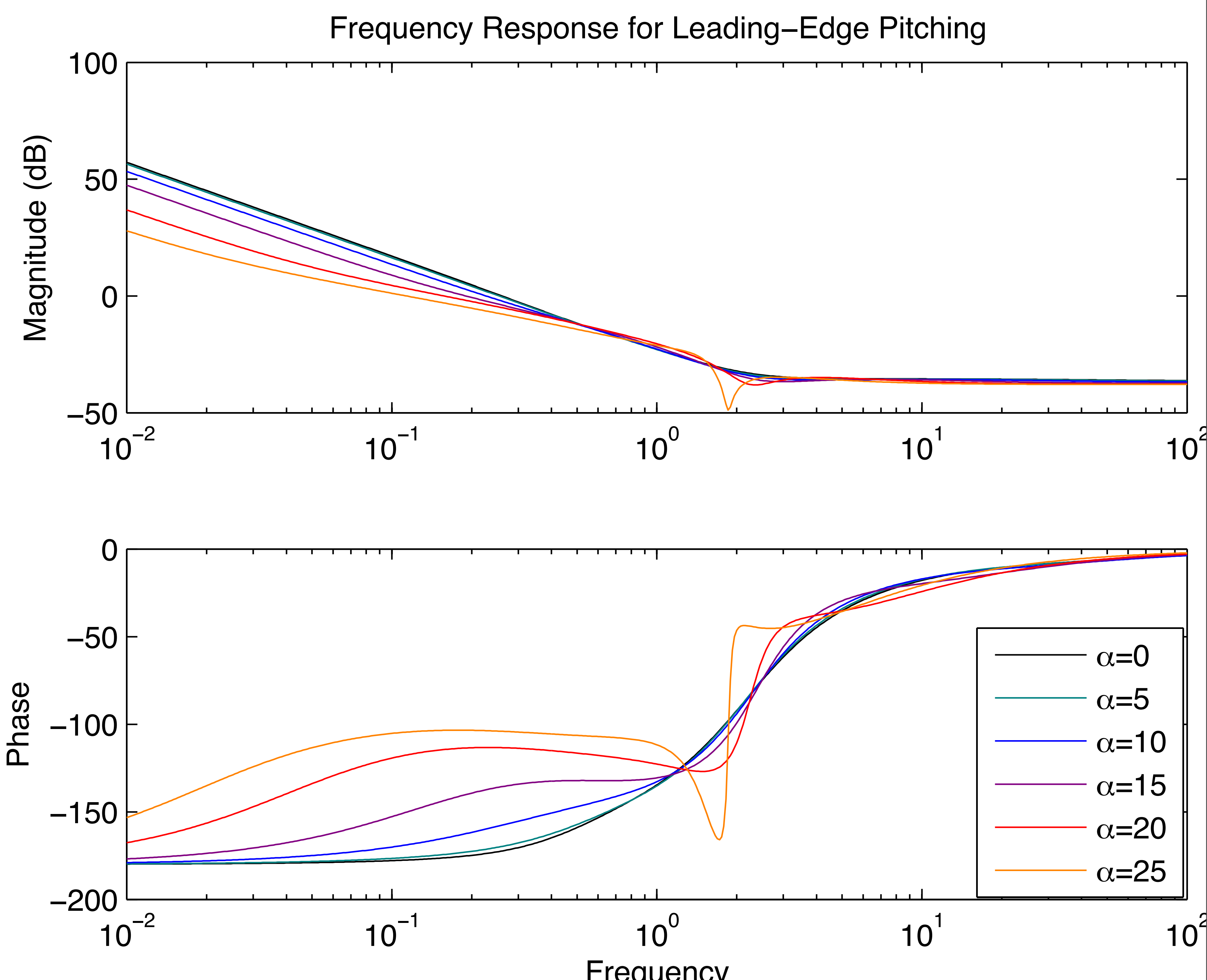
Bode Plot of ERA Models

Results

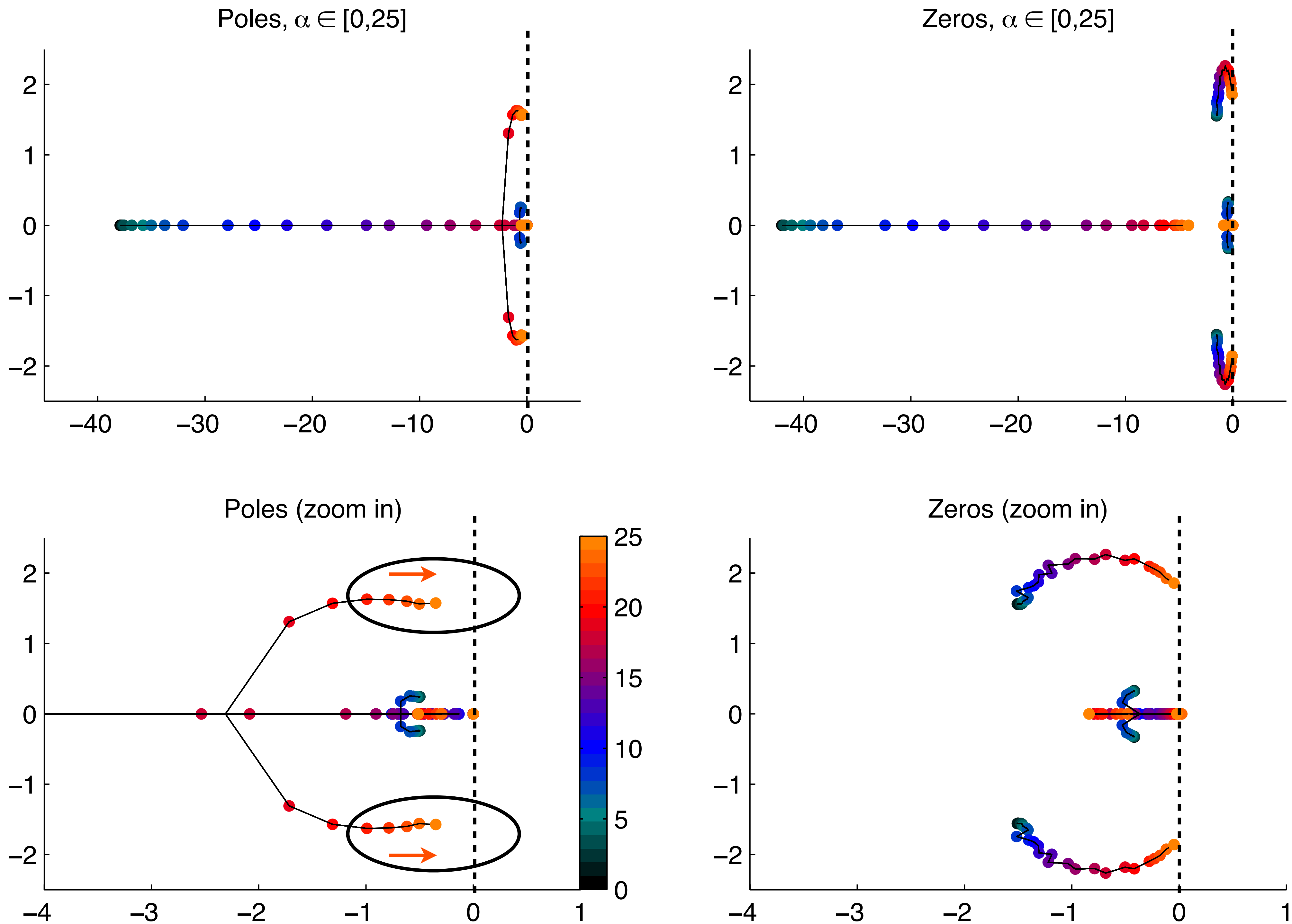
Lift slope decreases for increasing angle of attack, so magnitude of low frequency motions decreases for increasing angle of attack.

At larger angle of attack, phase converges to -180 at much lower frequencies. I.e., solutions take longer to reach equilibrium in time domain.

Consistent with fact that for large angle of attack, system is closer to Hopf instability, and a pair of eigenvalues are moving closer to imaginary axis.



Poles and Zeros of ERA Models

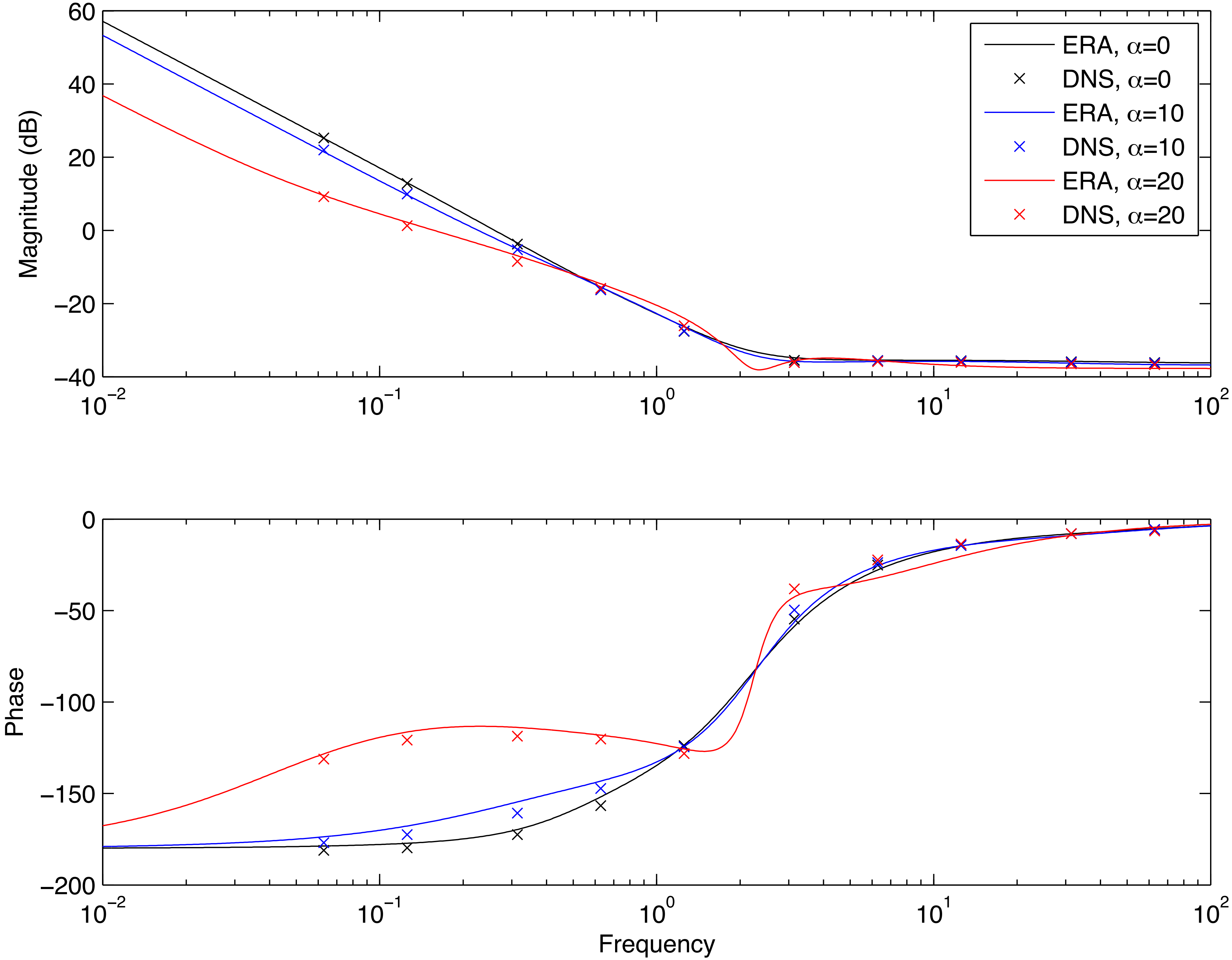


As angle of attack increases, pair of poles (and pair of zeros) march towards imaginary axis.

This is a good thing, because a Hopf bifurcation occurs at $\alpha_{\text{crit}} \approx 28^\circ$

Brunton and Rowley, AIAA ASM 2011

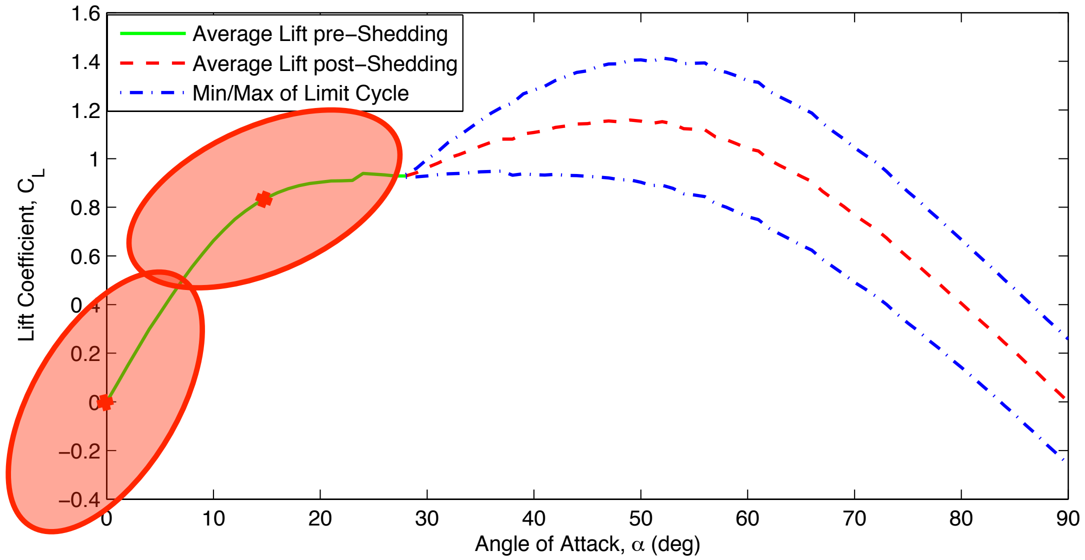
Bode Plot of Model (-) vs Data (x)

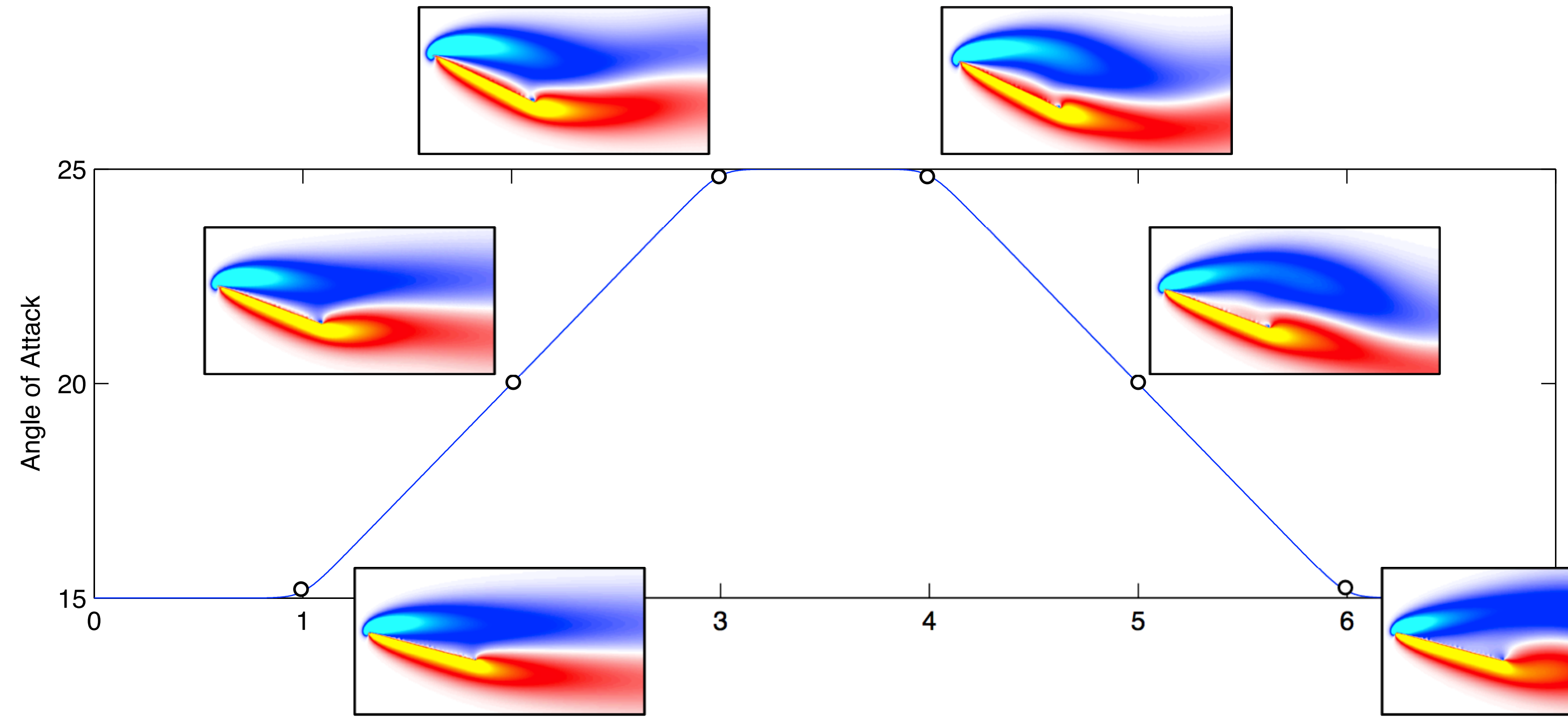


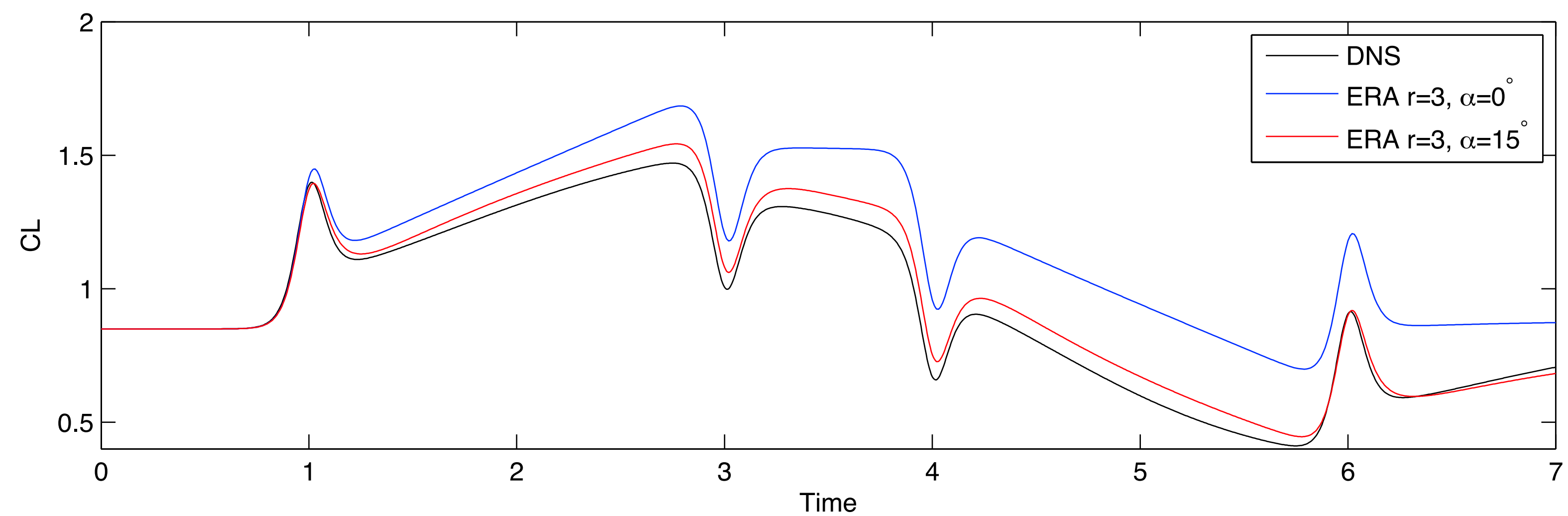
Direct numerical simulation confirms that local linearized models are accurate for small amplitude sinusoidal maneuvers

Brunton and Rowley, AIAA ASM 2011

Large Amplitude Maneuver







Compare models linearized at

$\alpha = 0^\circ$ **and** $\alpha = 15^\circ$

For pitching maneuver with

$$\alpha \in [15^\circ, 25^\circ]$$

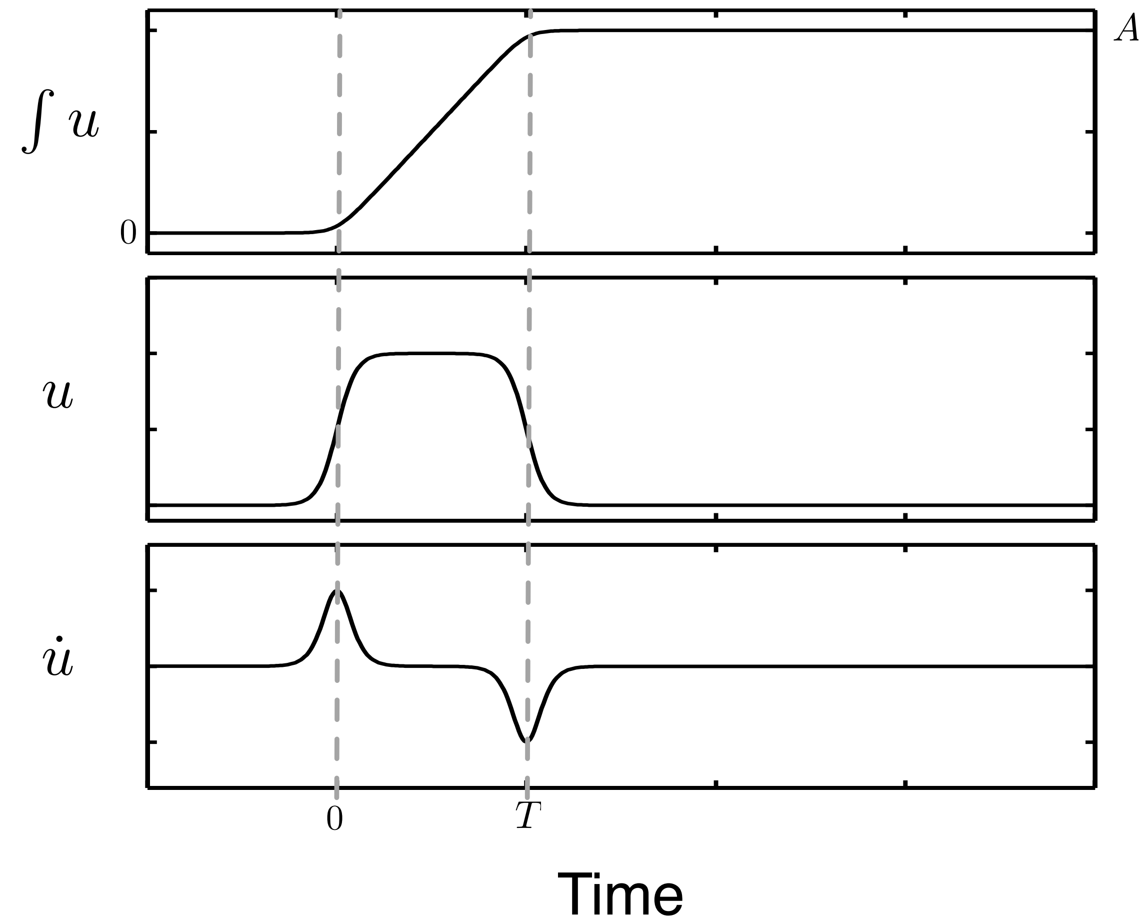
Model linearized at $\alpha = 15^\circ$

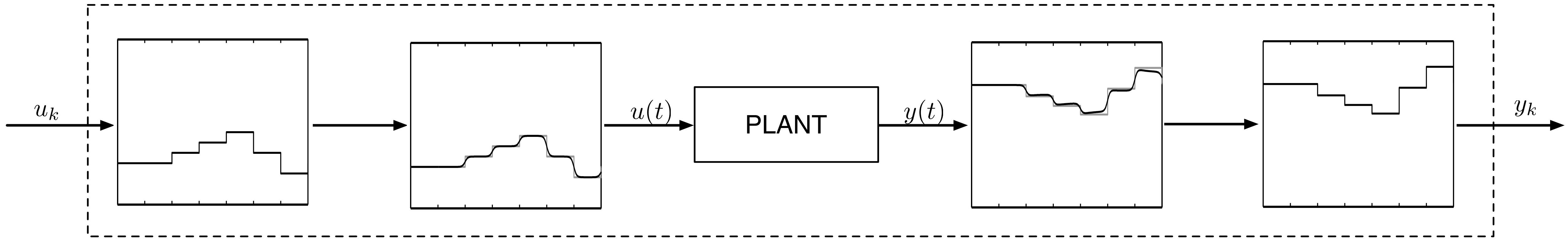
captures lift response more accurately

$$G(t) = \log \left[\frac{\cosh(a(t-t_1))\cosh(a(t-t_4))}{\cosh(a(t-t_2))\cosh(a(t-t_3))} \right] \qquad \alpha(t) = \alpha_0 + \alpha_{\max} \frac{G(t)}{\max(G(t))}$$

Brunton and Rowley, AIAA ASM 2011 **OL, Altman, Eldredge, Garmann, and Lian, 2010**

(Indicial) Step Response

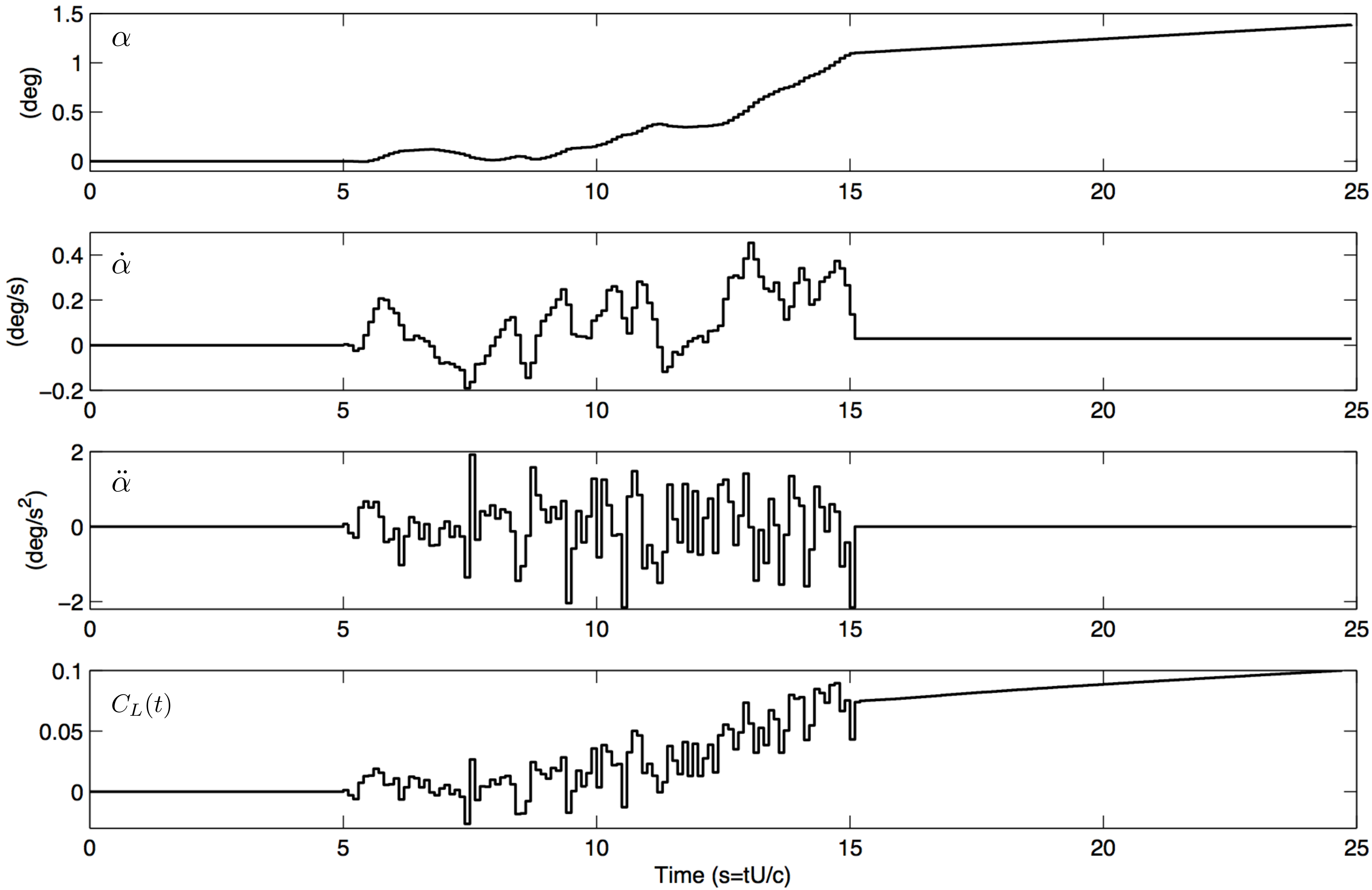




Previously, models are based on aerodynamic step response

Idea: Have pilot fly aircraft around for 5-10 minutes, back out the Markov parameters, and construct ERA model.

Random Input Maneuver



Idea: Have pilot fly aircraft around for 5-10 minutes, back out the Markov parameters, and construct ERA model.

Moving Base Flow

Base flow velocity:

$$u(x,y,t) = \|\mathbf{V}\| \cos(\alpha) - \dot{\theta}(y - y_C)$$

$$v(x,y,t) = \|\mathbf{V}\| \sin(\alpha) + \dot{\theta}(x - x_C)$$

Vorticity:

$$\nabla \times (u,v) = v_x - u_y = \dot{\theta} + \dot{\theta} = 2\dot{\theta}$$

where (x_C, y_C) is the center of mass.

Moving Base Flow

Faster simulations (Cholesky decomposition)

allows more aggressive maneuvers and gusts

subject of current research

Immersed Boundary Method

T. Colonius and K. Taira, 2008
A fast immersed boundary method using a nullspace approach and multi-domain far-field boundary conditions.

Conclusions

Reduced order model based on indicial response at non-zero angle of attack

- Based on eigensystem realization algorithm (ERA)
- Models appear to capture dynamics near Hopf bifurcation
- Locally linearized models outperform models linearized at $\alpha = 0^\circ$

Observer/Kalman Filter Identification for more realistic input/output data

- Efficient computation of reduced-order models
- Ideal for simulation or experimental data

Future Work:

- Combine models linearized at different angles of attack
- Add large amplitude effects such as LEV and vortex shedding
- Test modeling procedure in Prof. Williams' wind tunnel experiment

Wagner, 1925.

Theodorsen, 1935.

Leishman, 2006.

OL, Altman, Eldredge, Garmann, and Lian, 2010

Brunton and Rowley, AIAA ASM 2009-2011

Juang and Pappa, 1985.

Ma, Ahuja, Rowley, 2010.

Juang, Phan, Horta, Longman, 1991.