Overview of Topics

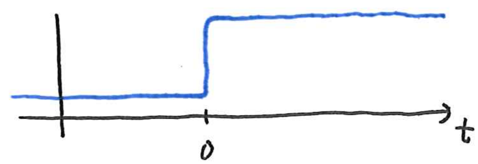
1. Laplace Transforms

The Fourier Transform is defined for functions that are well-behaved... $f(x) \to 0$ as $|x| \to \infty$.

However, lots of signals and functions we work with are not that well behaved.

Example: The Heaviside function

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$



Other Examples: $e^{\lambda t}$, $\cos(t)$, t^2 (unbounded, or blow up)

Solution: Multiply "badly-behaved" function by sufficiently large damping exponential, $e^{-\gamma t}$, to obtain a new well-defined function:

$$F(t) = \begin{cases} 0 & , t < 0 \\ f(t)\, e^{-\gamma t} & , t > 0 \end{cases}$$

Here, we assume $|f(t)| \leq K e^{\alpha t}$ for some $K > 0$, $\alpha > 0$ and $\gamma > \alpha$ (i.e. damping γ is stronger than growth α).

(Note: now $t \geq 0$, since $t < 0$ would defeat the purpose of multiplying by $e^{-\gamma t}$...)

Now, we can Fourier Transform nice function $F(t) = e^{-\gamma t} f(t)$.

$$\hat{F}(\omega) = \int_0^\infty F(t) e^{-i\omega t} dt$$

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \int_0^\infty F(\tau) e^{-i\omega \tau} d\tau \, d\omega$$

$$\Downarrow$$

$$\underbrace{e^{-\gamma t} f(t)}_{\times e^{+\gamma t}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{e^{i\omega t}}_{\times e^{+\gamma t}} \int_0^\infty f(\tau) e^{-(\gamma + i\omega)\tau} d\tau \, d\omega$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\gamma + i\omega)t} \int_0^\infty f(\tau) e^{-(\gamma + i\omega)\tau} d\tau \, d\omega$$

Let $s = \gamma + i\omega$ and $ds = i\,d\omega \Rightarrow d\omega = \frac{1}{i} ds$ ('s' is Laplace variable)

$$f(t) = \underbrace{\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \underbrace{\int_0^\infty f(\tau) e^{-s\tau} d\tau}_{\bar{f}(s) = \mathcal{L}(f(t))} ds}_{= \mathcal{L}^{-1}(\bar{f}(s))}$$

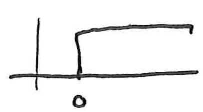
$$\boxed{\begin{aligned} \bar{f}(s) &= \int_0^\infty f(t) e^{-st} dt \\ f(t) &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(s) e^{st} ds \end{aligned}}$$

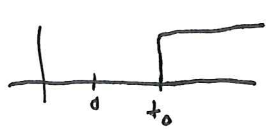
Laplace Transform Pair

Examples of Laplace Transforms:

Function of Time 't' $\xrightarrow{\mathcal{L}\{f\}}$ Function of Generalized Frequency 's'

$f(t)$		$\bar{f}(s)$
(A) $f(t) = 1$	$\int_0^\infty e^{-st}\,dt$	$\bar{f}(s) = \frac{1}{s}$ ← Question: why positive?
(B) $f(t) = e^{at}$	$\int_0^\infty e^{(a-s)t}\,dt$	$\bar{f}(s) = \frac{1}{s-a}$ ← Answer: Definite Integral
(C) $f(t) = \sin(at)$		$\bar{f}(s) = \frac{a}{s^2+a^2}$
(D) $f(t) = \cos(at)$		$\bar{f}(s) = \frac{s}{s^2+a^2}$
(E) $f(t) = t^n$		$\bar{f}(s) = \frac{n!}{s^{n+1}}$
(F) $f(t) = H(t)$	Same as (A)	$\bar{f}(s) = \frac{1}{s}$
(G) $f(t) = H(t-t_0)$	$\int_{t_0}^\infty e^{-st}\,dt$	$\bar{f}(s) = \frac{e^{-t_0 s}}{s}$





Extremely Important Properties of Laplace Transforms

Ⓐ **Derivative:** Let $\bar{f}(s) = \mathcal{L}(f(t))$

Then $\mathcal{L}\left(\frac{df}{dt}\right) = \int_0^\infty \underbrace{\frac{df}{dt}}_{dv} \underbrace{e^{-st}}_{u} dt$

$$= \left[f(t)e^{-st}\right]_0^\infty + \int_0^\infty s\, f(t) e^{-st} dt$$

$$= f(0) + s\mathcal{L}(f(t)) = f(0) + s\,\bar{f}(s)$$

Ⓑ **Higher Derivatives:** $\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).$

Ⓒ **Convolution:** $\left.\begin{array}{l} \mathcal{L}\{f(t)\} = \bar{f}(s) \\ \mathcal{L}\{g(t)\} = \bar{g}(s) \end{array}\right\}$ $\mathcal{L}\{f * g\} = \mathcal{L}\left\{\int_0^t f(t-\tau) g(\tau) d\tau\right\} = \bar{f}(s)\bar{g}(s)$