Overview of Topics

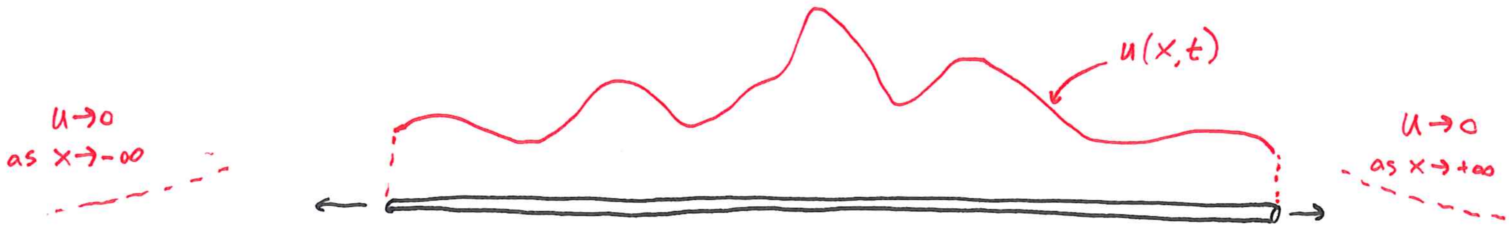
(1) Fourier transform solution to PDE on ∞ domain

1D Heat Eq$^{\underline{n}}$

(2) Convolution w/ a Gaussian

Fourier Transforms & Solutions to PDEs on Infinite Domains

Consider the 1D heat equation on $-\infty < x < \infty$:



$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad \left(\text{or } u_t = \alpha^2 u_{xx} \right)$$

with initial condition: $u(x,0) = f(x)$

(usually assume that $u(x,t) = 0$ for $x = \pm\infty$)

Two options:

① Separation of Variables

② Fourier Transform

(next page)

$$u_t = \alpha^2 u_{xx}$$

$$u(x,0) = f(x)$$

$|u|$ and $|u_x| \to 0$ as $|x| \to \infty$

Fourier Transform in 'x' to get a simple ODE in $\hat{u}$ ($\hat{u} = \mathcal{F}(u)$)

$$\mathcal{F}(u_x) = i\omega \mathcal{F}(u) = i\omega \hat{u}$$

$$\mathcal{F}(u_{xx}) = -\omega^2 \mathcal{F}(u) = -\omega^2 \hat{u}$$

$$\mathcal{F}(u_t) = \frac{\partial}{\partial t} \mathcal{F}(u) = \hat{u}_t$$

So $u_t = \alpha^2 u_{xx} \Longrightarrow \underbrace{\hat{u}_t = -\alpha^2 \omega^2 \hat{u}}_{\text{ODE!}}$

$$\frac{\partial \hat{u}}{\partial t} = \lambda \hat{u} \Longrightarrow \hat{u}(\omega,t) = \hat{u}(\omega,0)e^{\lambda t}$$

$$\lambda = -\alpha^2\omega^2 \Longrightarrow \boxed{\hat{u}(\omega,t) = \hat{u}(\omega,0)e^{-\alpha^2\omega^2 t}}$$

How do we get $\hat{u}(\omega,0)$?

Simple! Fourier transform initial conditions:

$$\hat{u}(\omega,0) = \mathcal{F}(u(x,0)) = \mathcal{F}(f(x)).$$

Solution to 1D Heat Equation (in Fourier "frequency" domain) is:

$$\hat{u}(\omega,t) = \left[\underbrace{\left[\hat{f}(\omega)\right]}_{\text{Initial Cond. in Fourier coords.}} \underbrace{e^{-\alpha^2\omega^2 t}}_{\text{All frequencies are damped... at different rates!}}\right]$$

To bring back to space (where we live...):

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\underbrace{\left[\int_{-\infty}^{\infty} f(y)e^{-iy\omega}\,dy\right]}_{\hat{f}(\omega)} e^{-\alpha^2\omega^2 t}\right] e^{i\omega x}\,d\omega$$

can switch order... no problem.

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega(x-y)-\alpha^2\omega^2 t} f(y)\,dy\,d\omega$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(x-y)-\alpha^2\omega^2 t}\,d\omega\right] f(y)\,dy$$

Lets look at inner integral:

$$[\cdot] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\xi} e^{-\alpha^2\omega^2 t}\,d\omega \quad \text{for } \xi = x-y$$

$$= \underbrace{\mathcal{F}^{-1}\left(e^{-\alpha^2\omega^2 t}\right)}_{\hat{g}(\omega,t)} = \underbrace{\frac{e^{-\frac{\xi^2}{4\alpha^2 t}}}{2\sqrt{\alpha^2\pi t}}}_{g(\xi,t)}$$

(Quite difficult to show this step!)

ASIDE

$$= \int_{-\infty}^{\infty} g(x-y,t)\,f(y)\,dy \quad \left(\text{convolution } \int\right).$$

$$= \underbrace{(g * f)}_{\text{convolution}}(x,t)$$

Take a very important step back:

$$u(x,t) = \int_{-\infty}^{\infty} \underbrace{\frac{e^{-(x-y)^2/(4\alpha^2 t)}}{2\alpha\sqrt{\pi t}}}_{g(x-y,t) = \mathcal{F}^{-1}(\hat{g}) = \mathcal{F}^{-1}\left(e^{-\alpha^2\omega^2 t}\right).} f(y)\,dy$$

$\hat{g} = e^{-\alpha^2\omega^2 t}$ is a Gaussian in ω

g is <u>also</u> a Gaussian, but in x.

So $u(x,t)$ = Convolution of $f(x)$ with a Gaussian $g(x,t)$.

The variance of $g(x,t)$ increases with time t, effectively spreading out solutions...