ME564 - Autumn 2014 **Due: 4:00:00pm Monday November 24^{th}, 2014**
Homework 5

Exercise 5-1 Show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

Exercise 5-2 Show that $\nabla \cdot (\nabla \times \mathbf{f}) = 0$ for any vector field $\mathbf{f}$.

Exercise 5-3 Let W be the three-dimensional region under the graph (function) of $f(x, y) = \exp(x^2 + y^2)$ and over the region in the plane defined by $1 \leq x^2 + y^2 \leq 2$.

(a) Find the volume of W.

(b) Find the flux of the vector field $\mathbf{F} = (2x - xy)\mathbf{i} - y\mathbf{j} + yz\mathbf{k}$ out of the region W.

Exercise 5-4 A (real valued) inner product space is a vector space that has an inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ that satisfies the following three axioms:

- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$,
- $\langle a\mathbf{x}, \mathbf{y} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle$, for all real numbers $a \in \mathbb{R}$,
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, with equality only if $\mathbf{x} = 0$.

Consider the space of bounded functions on the interval $[0, 1]$ with the following inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Bounded just means that the magnitude of both f and g never exceed some fixed large number on the interval $[0, 1]$.

(a) Verify that this space of functions and inner product satisfy the following three axioms above.

(b) Show that the functions $\cos(\pi m x)$ and $\cos(\pi n x)$ for non-negative integers m and n are orthogonal (using the inner product above) for all $m \neq n$. You may find the following identify useful: $\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$.

(c) Plot $\cos(\pi m x)$ on the interval $[0, 1]$ with $dx = 0.01$ for $m = 0, 1, 2, 3, 4$, and 5. Verify numerically, using trapezoidal integration (i.e. `trapz`), that $\cos(\pi m x)$ and $\cos(\pi n x)$ are orthogonal for the following (m, n) pairs: $(1, 4)$, $(2, 6)$, and $(3, 15)$.

Note that we have an *infinite* set of orthogonal functions, which each represent a unique and orthogonal *vector direction* in the inner product space of bounded functions on $[0, 1]$. We are starting to build an infinite dimensional vector space (called a Hilbert space) for representing functions. These functions will be the solutions of PDEs in ME565. (Note that we will eventually need to include sine functions as well.)