Ex4-1

$$f(t+\Delta t)=f(t)+\Delta t\cdot\frac{df}{dt}+\frac{\Delta t^2}{2!}\frac{d^2 f}{dt^2}+\frac{\Delta t^3}{3!}\frac{d^3 f}{dt^3}+\sigma(\Delta t^4)$$

$$f(t+2\Delta t)=f(t)+2\Delta t\cdot\frac{df}{dt}+\frac{(2\Delta t)^2}{2!}\frac{d^2 f}{dt^2}+\frac{(2\Delta t)^3}{3!}\frac{d^3 f}{dt^3}+\sigma(\Delta t^4)$$

$$\therefore 4f(t+\Delta t)-f(t+2\Delta t)-3f(t)=2\Delta t\cdot\frac{df}{dt}+\sigma(\Delta t^3)$$

$$\therefore \frac{df}{dt}=\frac{4f(t+\Delta t)-f(t+2\Delta t)-3f(t)}{2\Delta t}+\sigma(\Delta t^2)$$

Ex4-2

Suppose $t_0=0$, then the six elements can be represented as

$$f(0), f(0.1), f(0.2), f(0.3), f(0.4), f(0.5)$$

$$\therefore \frac{df(0)}{dt}=\frac{f(0.1)-f(0)}{0.1}$$

$$\frac{df(0.1)}{dt}=\frac{f(0.2)-f(0)}{0.2}$$

$$\frac{df(0.2)}{dt}=\frac{f(0.3)-f(0.1)}{0.2}$$

$$\frac{df(0.3)}{dt}=\frac{f(0.4)-f(0.2)}{0.2}$$

$$\frac{df(0.4)}{dt}=\frac{f(0.5)-f(0.3)}{0.2}$$

$$\frac{df(0.5)}{dt}=\frac{f(0.5)-f(0.4)}{0.1}$$

$$\therefore \frac{d}{dt}\begin{bmatrix} f(0)\\ f(0.1)\\ f(0.2)\\ f(0.3)\\ f(0.4)\\ f(0.5)\end{bmatrix}=\begin{bmatrix} -10 & 10 & 0 & 0 & 0 & 0\\ -5 & 0 & 5 & 0 & 0 & 0\\ 0 & -5 & 0 & 5 & 0 & 0\\ 0 & 0 & -5 & 0 & 5 & 0\\ 0 & 0 & 0 & -5 & 0 & 5\\ 0 & 0 & 0 & 0 & -10 & 10\end{bmatrix}\begin{bmatrix} f(0)\\ f(0.1)\\ f(0.2)\\ f(0.3)\\ f(0.4)\\ f(0.5)\end{bmatrix}$$

$$\therefore D=\begin{bmatrix} -10 & 10 & 0 & 0 & 0 & 0\\ -5 & 0 & 5 & 0 & 0 & 0\\ 0 & -5 & 0 & 5 & 0 & 0\\ 0 & 0 & -5 & 0 & 5 & 0\\ 0 & 0 & 0 & -5 & 0 & 5\\ 0 & 0 & 0 & 0 & -10 & 10\end{bmatrix}$$

Ex4-3

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x$$

a) Forward Euler

$$\therefore x_{k+1} = (I + \Delta t \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}) x_k$$

$$= \begin{bmatrix} 1-\Delta t & 0 \\ 0 & 1-2\Delta t \end{bmatrix} x_k$$

The integration is stable when $\left| eigs(\begin{bmatrix} 1-\Delta t & 0 \\ 0 & 1-2\Delta t \end{bmatrix}) \right| < 1$

$$\therefore |1-\Delta t| < 1, |1-2\Delta t| < 1, \Delta t > 0$$

$$\Rightarrow 0 < \Delta t < 1$$

$\therefore$ the system is unstable when $\Delta t \geq 1$

b) Backward Euler

$$\therefore x_{k+1} = (I - \Delta t \cdot \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix})^{-1} x_k$$

$$= \begin{bmatrix} 1+\Delta t & 0 \\ 0 & 1+2\Delta t \end{bmatrix}^{-1} x_k$$

The integration is stable when $\left| eigs(\begin{bmatrix} 1+\Delta t & 0 \\ 0 & 1+2\Delta t \end{bmatrix}^{-1}) \right| < 1$

$$\therefore \left| \frac{1}{|1+\Delta t|} \right| < 1, \left| \frac{1}{|1+2\Delta t|} \right| < 1, \Delta t > 0$$

$$\Rightarrow \Delta t > 0$$

$\therefore$ the system is always stable

Ex4-4

$$\dot{x} = v$$

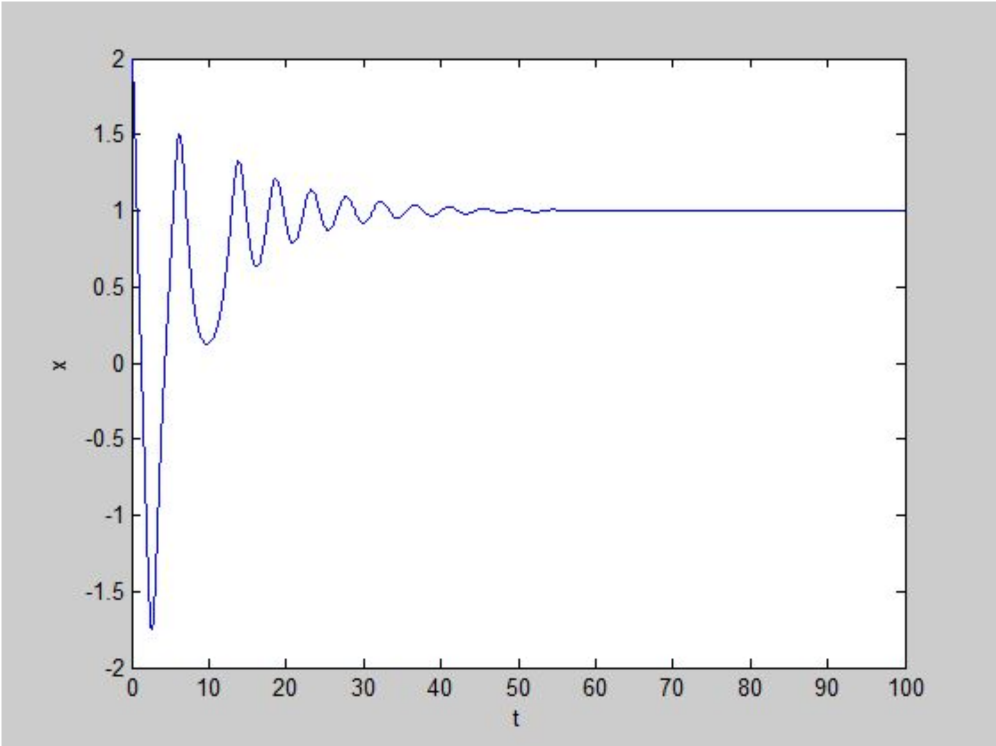
$$\dot{v} = -\delta v - \beta x - \alpha x^3 + \gamma \cos(\omega t)$$

$$\alpha = 1, \beta = -1, \delta = 0.2$$

$\therefore \dot{x} = v$

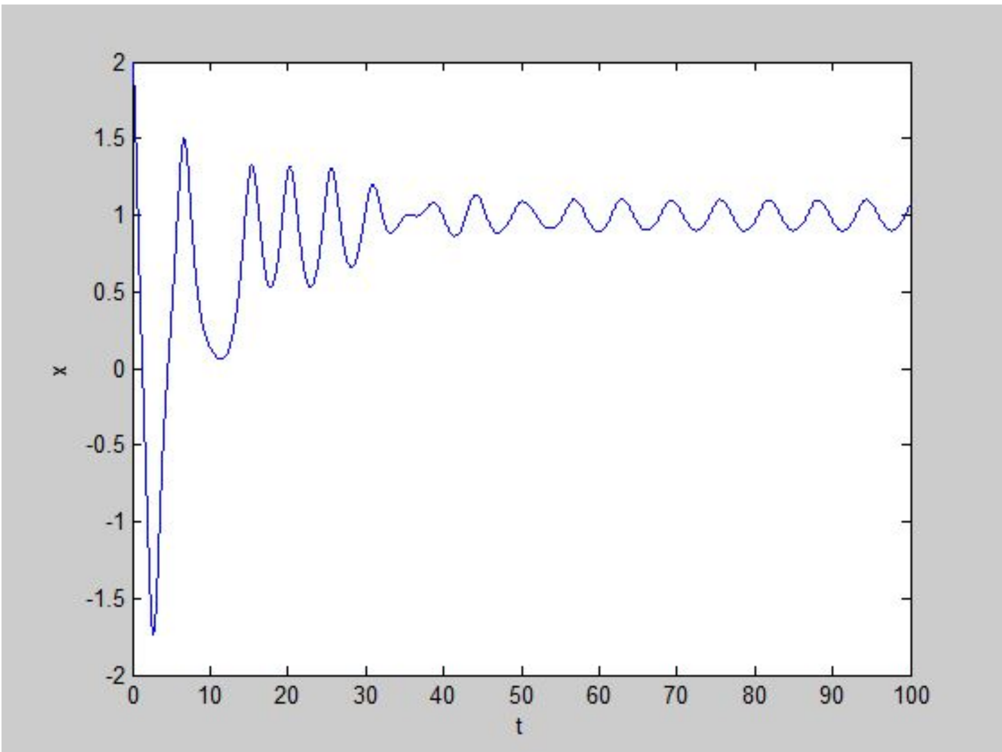
$\dot{v} = -0.2v + x - x^3 + \gamma\cos(\omega t)$

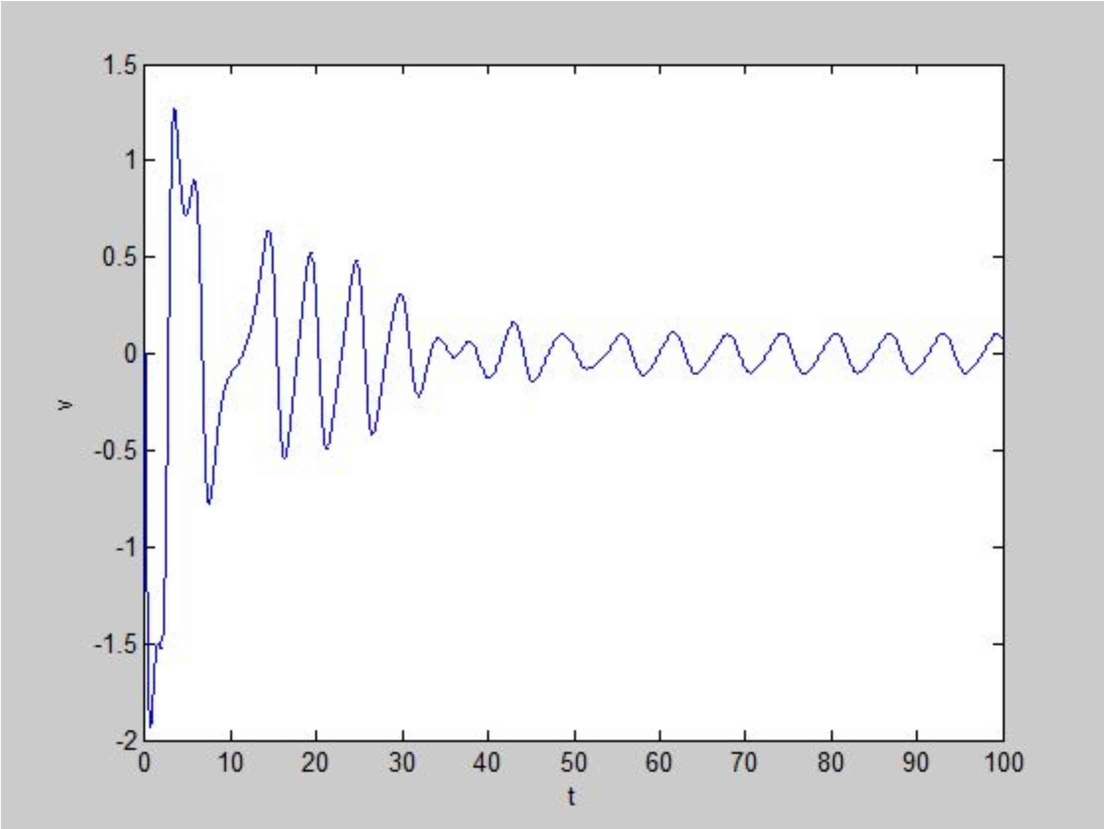
a)



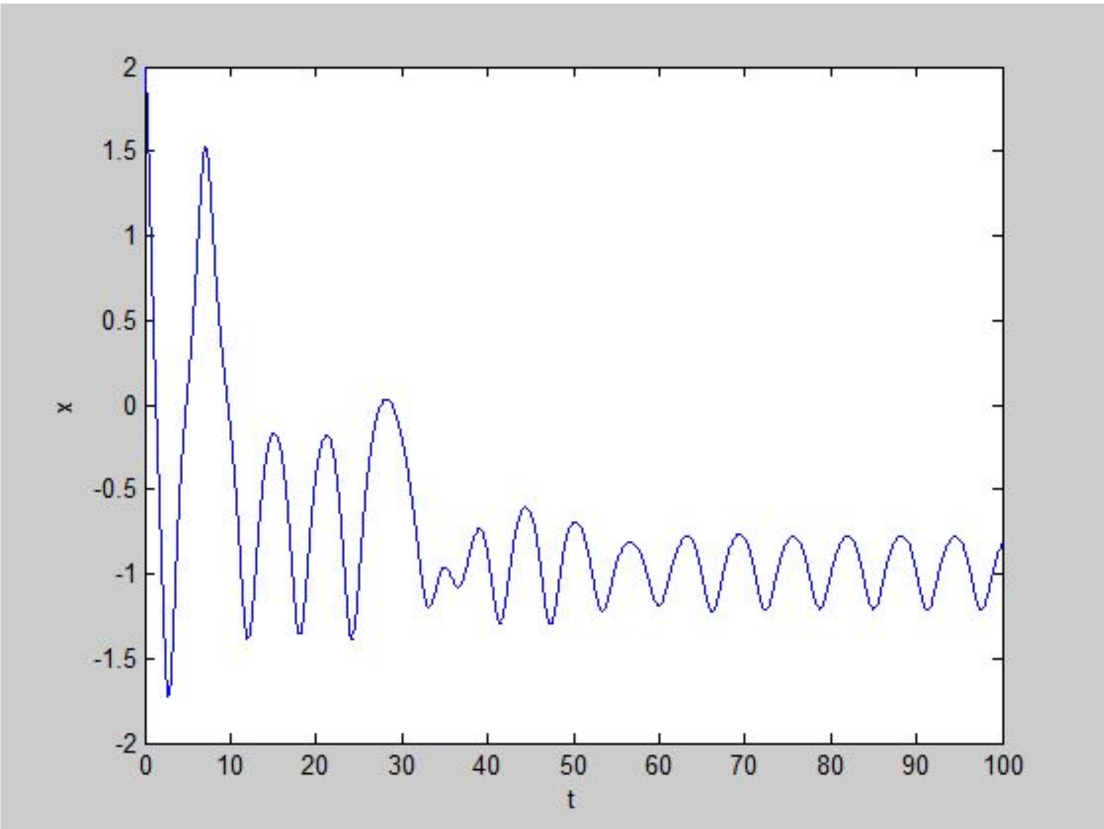
b)

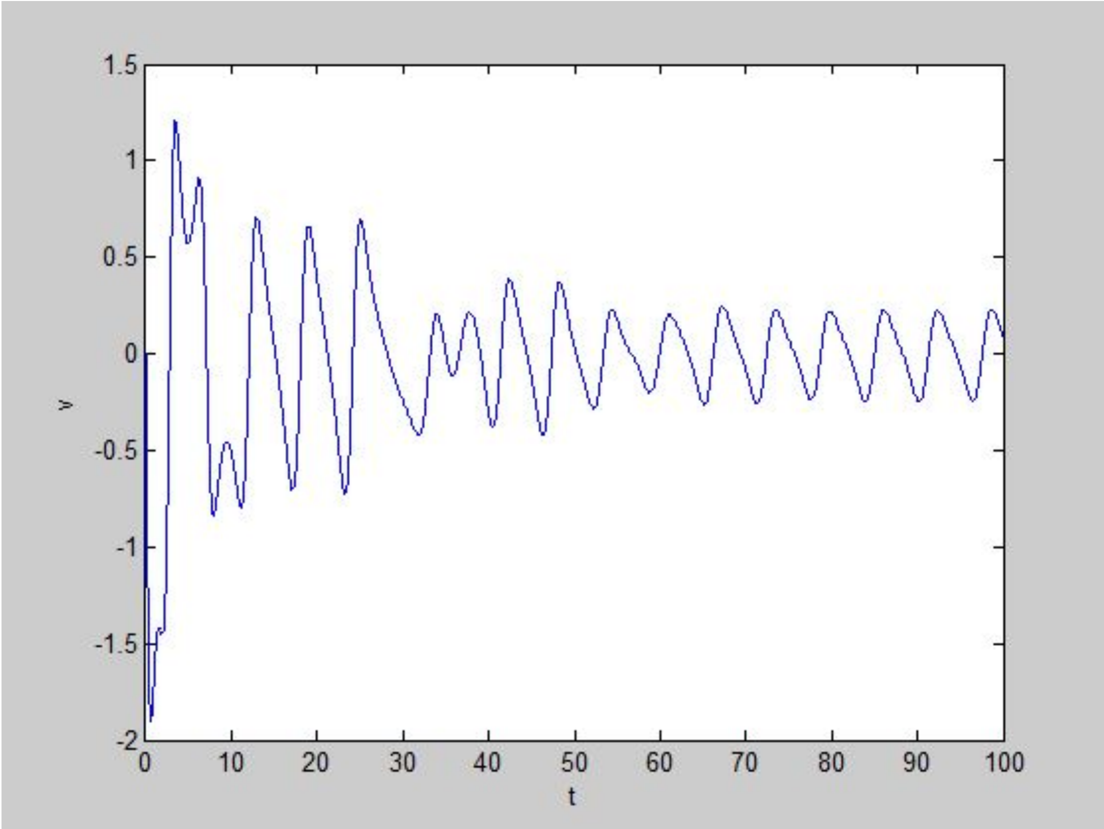
$\gamma = 0.1$



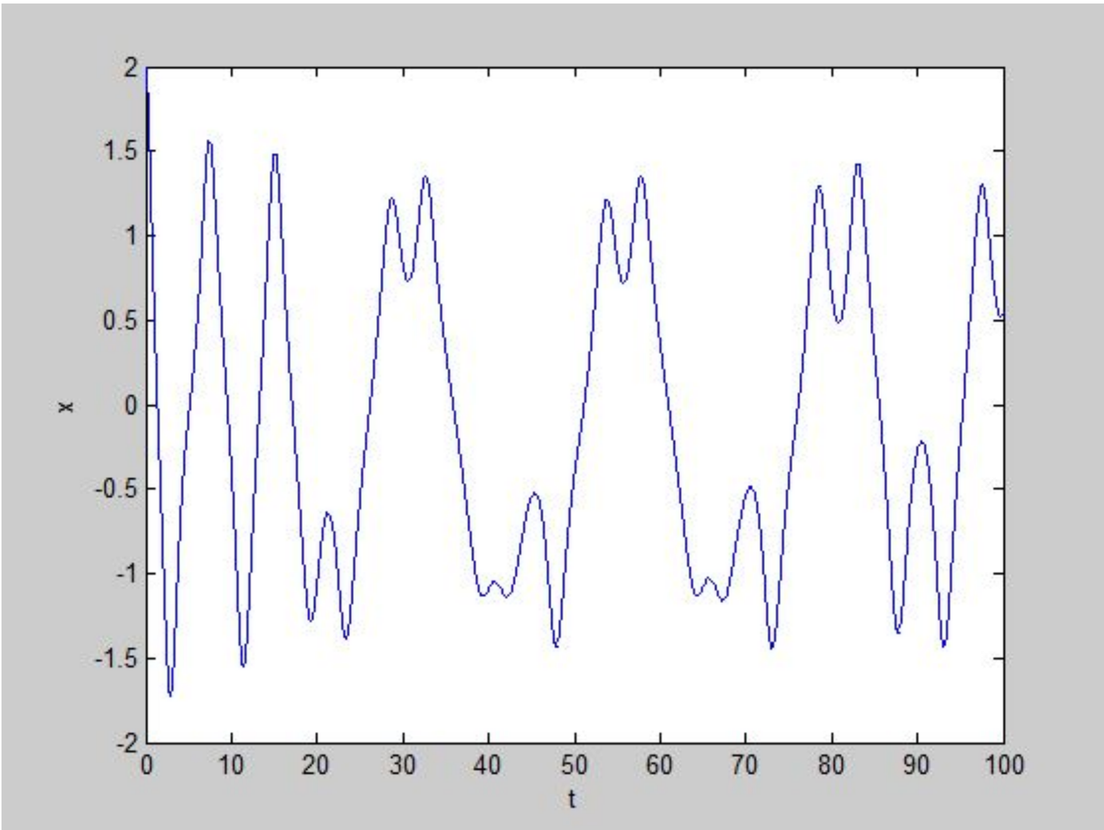


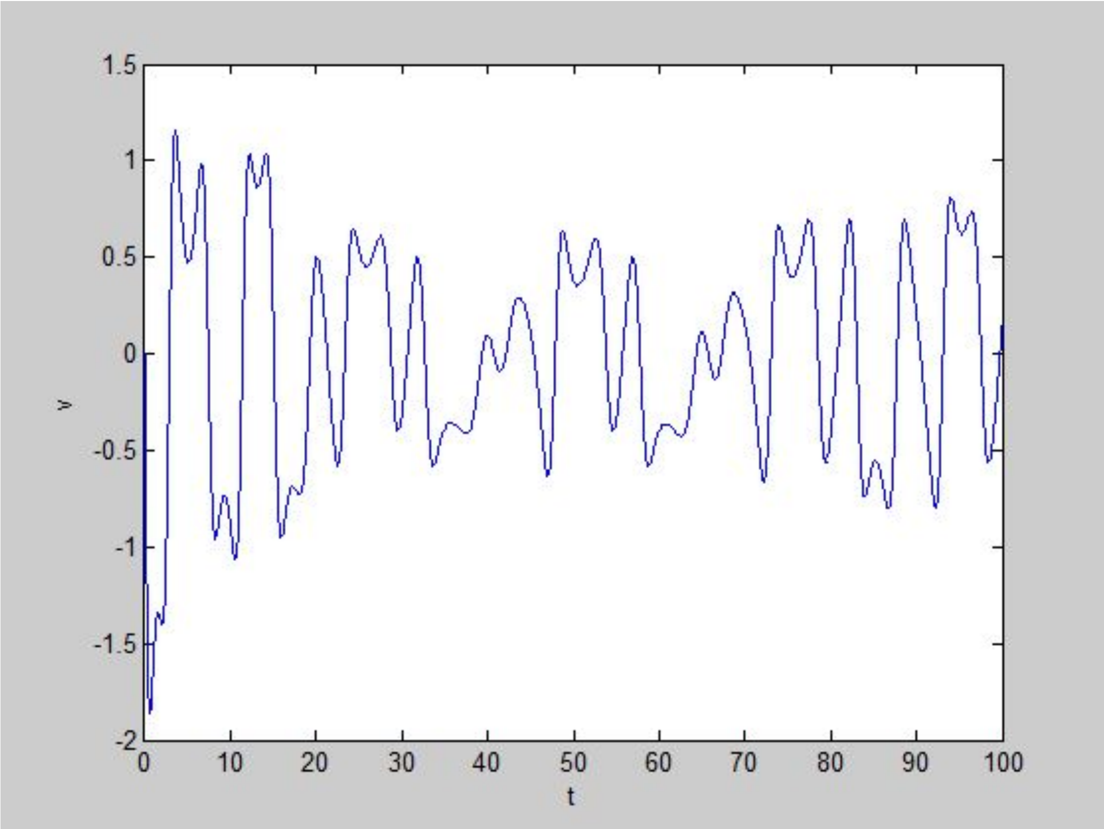
$\gamma = 0.2$




1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
0
10
20
30
40
50
60
70
80
90
100
t

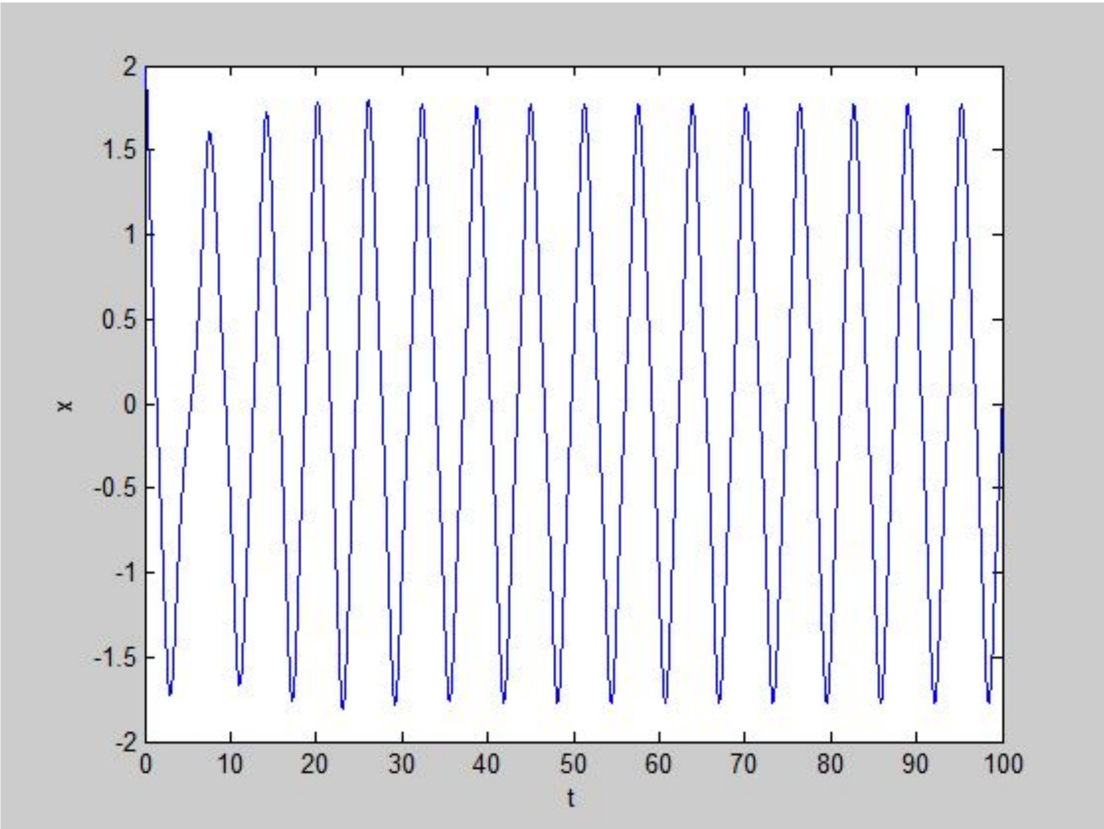
$\gamma = 0.3$


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
x
0
10
20
30
40
50
60
70
80
90
100
t

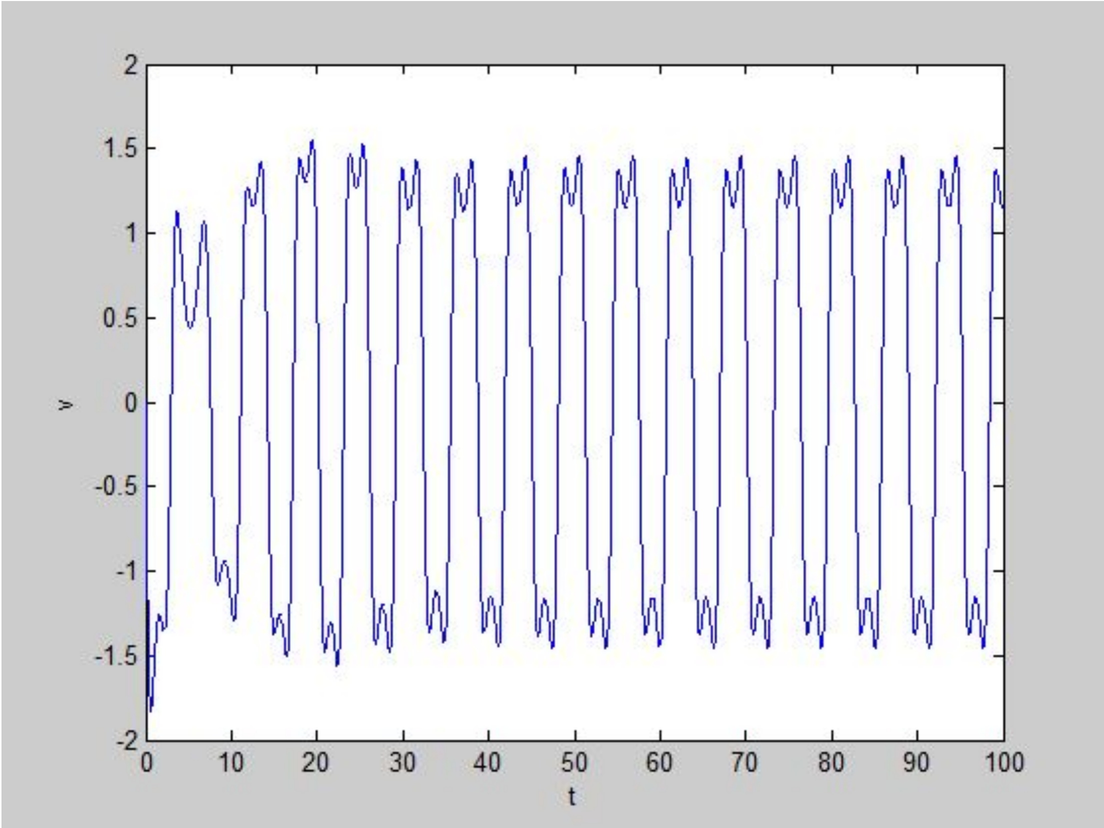


1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
0
10
20
30
40
50
60
70
80
90
100
t

$\gamma = 0.4$

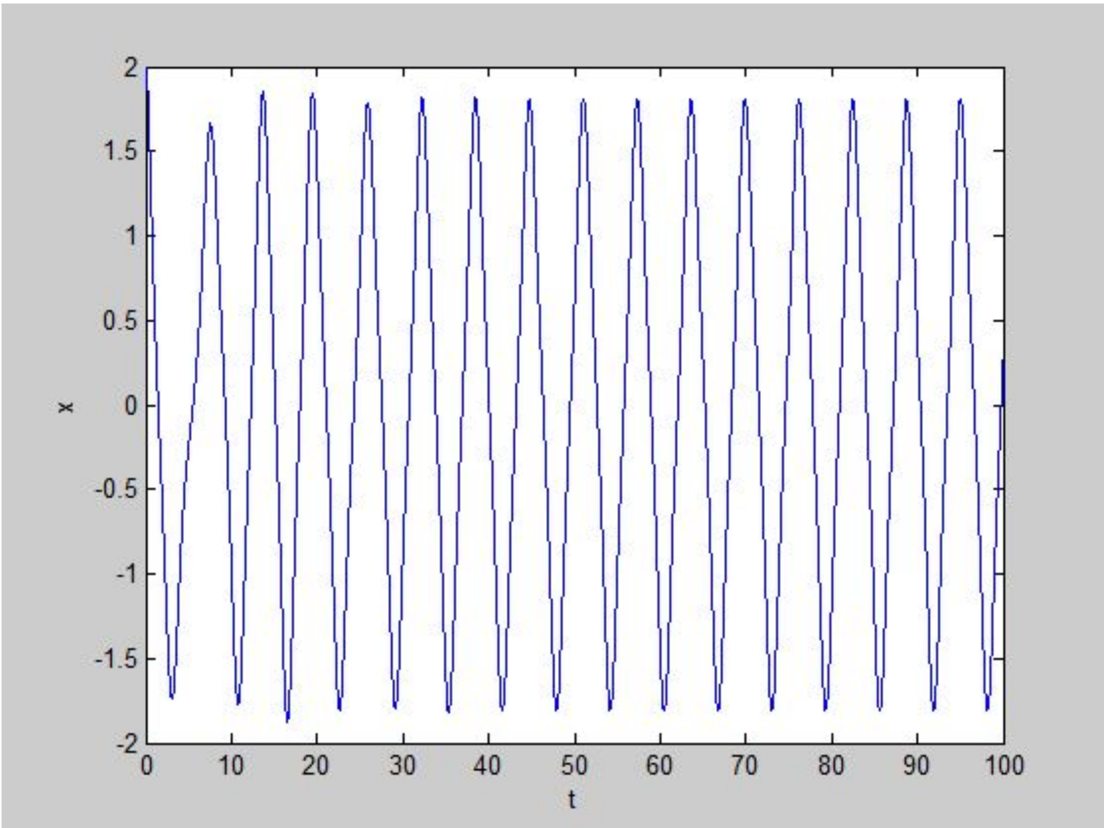


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
x
0
10
20
30
40
50
60
70
80
90
100
t

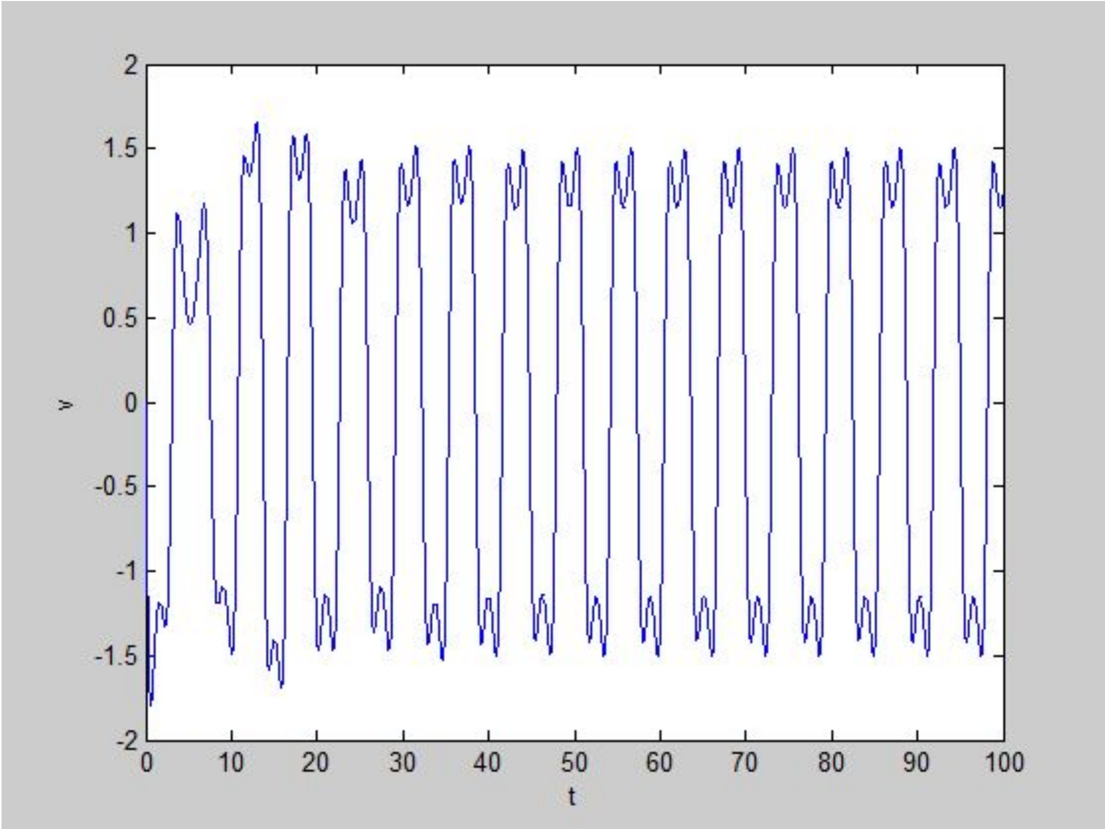


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
0
10
20
30
40
50
60
70
80
90
100
t

$\gamma = 0.5$



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
x
0
10
20
30
40
50
60
70
80
90
100
t



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
0
10
20
30
40
50
60
70
80
90
100
t