ME564- MATHS

100

HOMEWORK # 2

Exercise 2.1

(a) $f(t) = 5e^{(-1+i)t}$

Euler's formula: $e^{ix} = \cos x + i\sin x$

$\therefore f(t) = 5e^{(-1+i)t}$

$f(t) = 5e^{-t+it}$

$f(t) = 5e^{-t} \cdot e^{it}$

$f(t) = 5e^{-t} \cdot [\cos(t) + i\sin(t)]$ [Using Euler's formula]

$\boxed{f(t) = 5e^{-t} \cdot \cos(t) + i5e^{-t}\sin(t)}$

$\therefore$ Real part of $f(t) = 5e^{-t} \cdot \cos(t)$

Imaginary part of $f(t) = 5e^{-t}\sin(t)$

(b) $f(t) = e^{(1-i)t}$

$\therefore f(t) = e^{(1-i)t}$

$= e^{t-it}$

$= e^{t} \cdot e^{-it}$

Using Euler's formula,

$\therefore\ f(t) = e^t\,[\cos(-t) + i\sin(-t)]$

But, $\cos(-t) = \cos t$

$\sin(-t) = -\sin t$

$\therefore\ f(t) = e^t\,[\cos(t) - i\sin(t)]$

$$\boxed{f(t) = e^t.\cos(t) - i\,e^t.\sin(t)}$$

$\therefore$ Real part of $f(t) = e^t.\cos(t)$

$\therefore$ Imaginary part of $f(t) = -e^t.\sin(t)$

Problem 1:

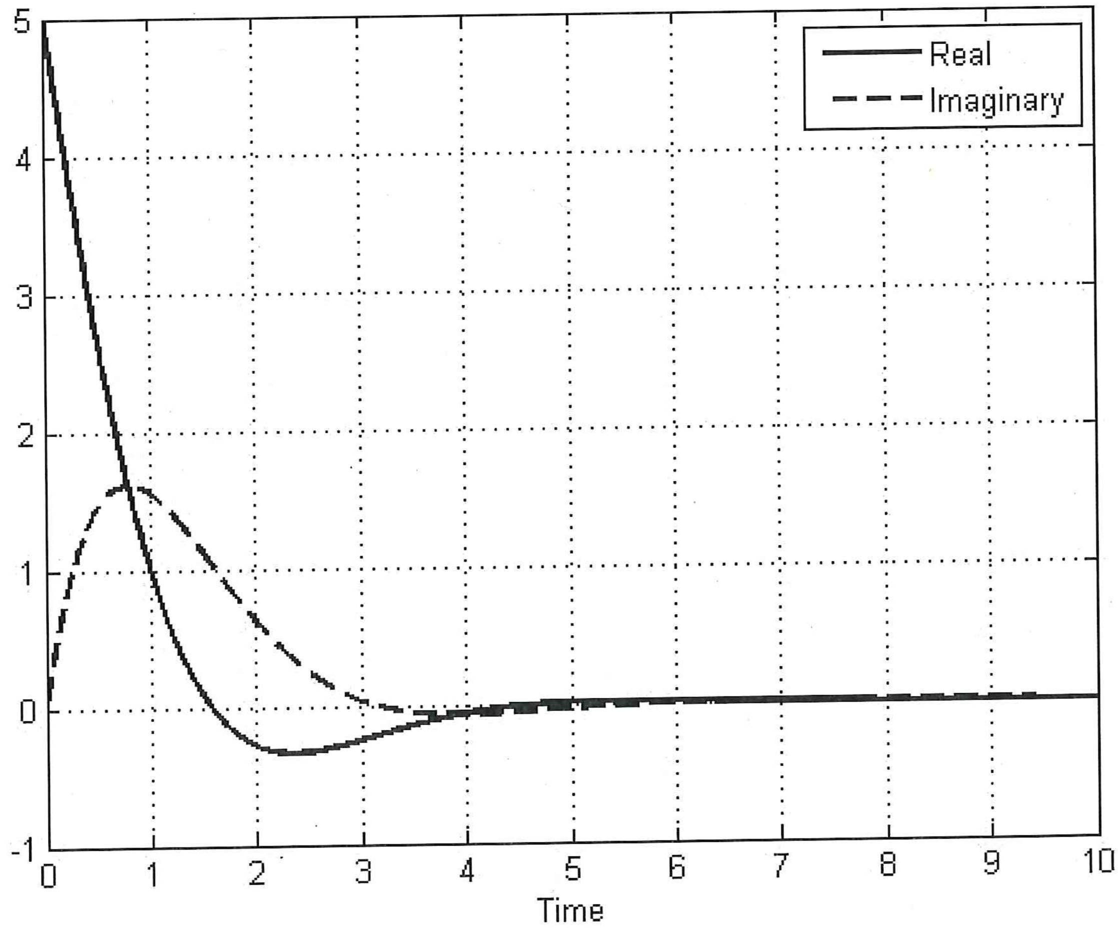
MATLAB Code:

```
t=0:0.01:10;
R1=exp(t).*cos(t);
I1=-1.*exp(t).*sin(t);
plot(t,R1,'k','linewidth',1.5)
hold on;
grid on;
plot(t,I1,'k--','linewidth',1.5)
legend('Real','Imaginary')
xlabel('Time')
```

Problem 1: A

The Real and imaginary parts are plotted for the function: $f(t) = 5e^{(-1+i)t}$

$$R[f(t)] = 5e^{-t}\cos(t)$$
$$I[f(t)] = 5e^{-t}\sin(t)$$

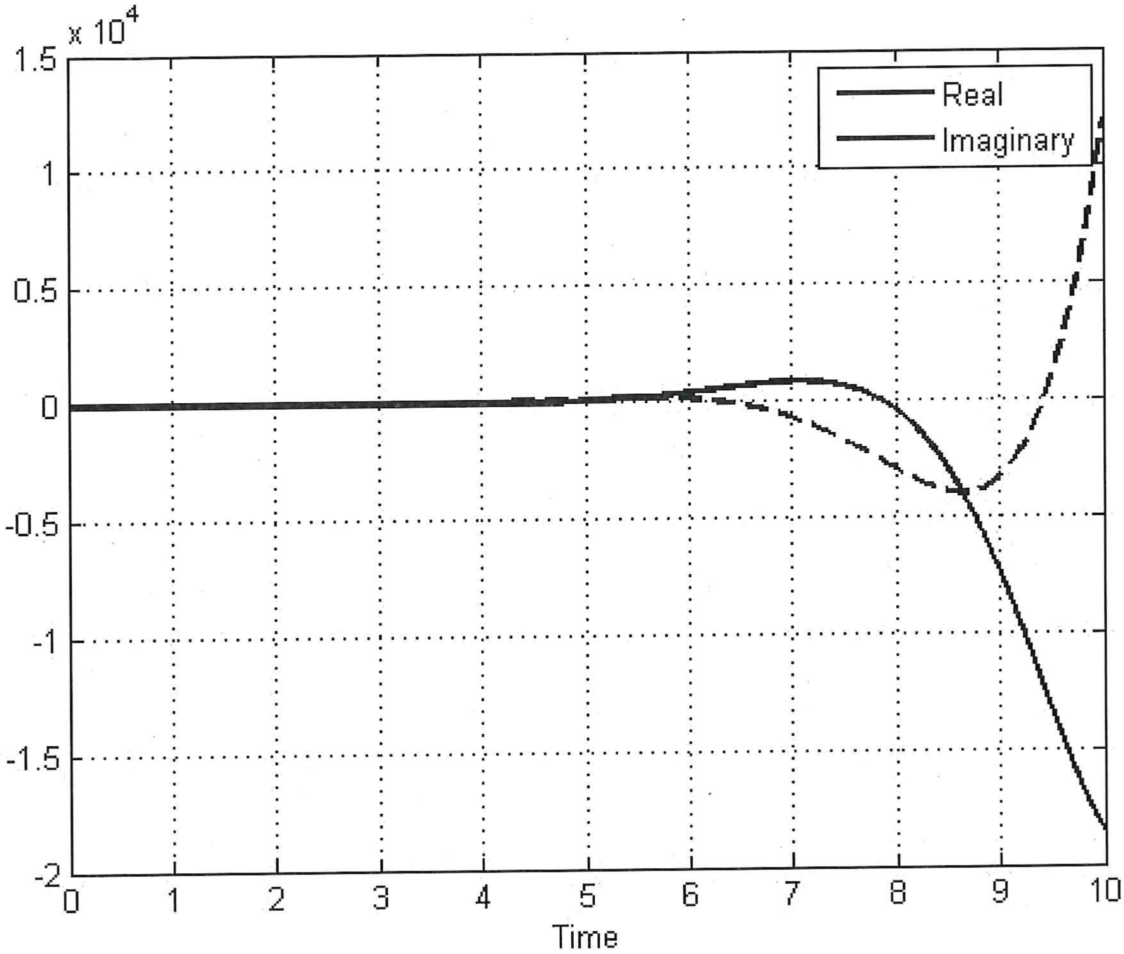


Problem 1: B

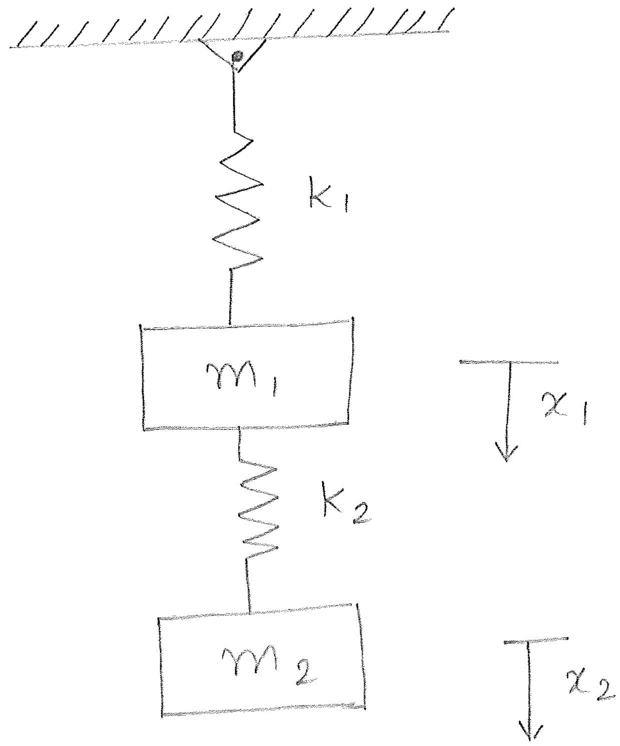
The Real and imaginary parts are plotted for the function: $f(t) = e^{(1-i)t}$

$R[f(t)] = e^t \cdot \cos(t)$

$I[f(t)] = -e^t \sin(t)$



Excercise 2-2



$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = 0 \quad \text{——— (1)}$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \quad \text{——— (2)}$$

(a) A single fourth order ODE in x_1

consider eq^n (1)

$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = 0$$

$$\therefore \quad m_1\ddot{x}_1 + k_1 x_1 + k_2 x_1 - k_2 x_2 = 0$$

$$\therefore \quad k_2 x_2 = m_1\ddot{x}_1 + k_1 x_1 + k_2 x_1$$

$$\therefore \quad x_2 = \frac{m_1\ddot{x}_1 + k_1 x_1 + k_2 x_1}{k_2}$$

$$\therefore \quad x_2 = \frac{m_1}{k_2}\ddot{x}_1 + \frac{k_1}{k_2}x_1 + x_1 \quad \text{——— (3)}$$

Taking derivative of eq^n (3) w.r.t time,

$$\dot{x}_2 = \frac{m_1}{K_2}\dddot{x}_1 + \frac{K_1}{K_2}\dot{x}_1 + \dot{x}_1$$

Taking derivative once again,

$$\ddot{x}_2 = \frac{m_1}{K_2}\ddddot{x}_1 + \frac{k_1}{K_2}\ddot{x}_1 + \ddot{x}_1 \quad \text{——— (4)}$$

Now, eq^n (2) is given by,

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \quad \text{——— (2)}$$

Substitute values of $\ddot{x}_2$ & x_2 from (3) & (4) respectively into eq^n (4)

$$0 = m_2\left[\frac{m_1}{K_2}\ddddot{x}_1 + \frac{K_1}{K_2}\ddot{x}_1 + \ddot{x}_1\right] + k_2\left[\frac{m_1}{K_2}\ddot{x}_1 + \frac{K_1}{K_2}x_1 + x_1 - x_1\right]$$

$$\frac{m_1 m_2}{K_2}\ddddot{x}_1 + \frac{m_2 K_1}{K_2}\ddot{x}_1 + m_2\ddot{x}_1 + m_1\ddot{x}_1 + K_1 x_1 = 0$$

$$\left(\frac{m_1 m_2}{K_2}\right)\ddddot{x}_1 + \left(\frac{m_2 k_1 + m_2 k_2 + m_1 k_2}{K_2}\right)\ddot{x}_1 + k_1 x_1 = 0$$

Multiply by k_2 throughout

$$m_1 m_2 \ddddot{x}_1 + (m_2 k_1 + m_1 k_2 + m_2 k_2)\ddot{x}_1 + k_1 k_2 x_1 = 0$$

$$\Rightarrow \boxed{m_1 m_2 \ddddot{x}_1 + \left[m_2 k_1 + (m_1 + m_2) k_2\right]\ddot{x}_1 + k_1 k_2 x_1 = 0}$$

(b) single 4^{th} order ODE in x_2

consider eq^n (2)

$$m_2 \ddot{x}_2 + k_2(x_2 - x_1) = 0$$

$$\therefore \quad m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$

$$\therefore \quad m_2 \ddot{x}_2 + k_2 x_2 = k_2 x_1$$

$$\therefore \quad x_1 = \frac{m_2 \ddot{x}_2 + k_2 x_2}{k_2}$$

$$x_1 = \frac{m_2}{k_2} \ddot{x}_2 + x_2 \quad \text{——— (5)}$$

Take derivative of x_1 w.r.t time

$$\dot{x}_1 = \frac{m_2}{k_2} \dddot{x}_2 + \dot{x}_2$$

Take derivative once more,

$$\ddot{x}_1 = \frac{m_2}{k_2} \ddddot{x}_2 + \ddot{x}_2 \quad \text{——— (6)}$$

Now eq^n (1) is given by,

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = 0 \quad \text{——— (1)}$$

Substitute values of (5) & (6) in eq^n (1)

$$m_1 \left[\frac{m_2}{k_2} \ddddot{x}_2 + \ddot{x}_2 \right] + \left(\frac{m_2}{k_2} \ddot{x}_2 + x_2 \right) k_1 + k_2 \left(\frac{m_2}{k_2} \ddot{x}_2 + x_2 - x_2 \right) =$$

$$m_1\left[\frac{m_2}{K_2}\ddddot{x}_2 + \ddot{x}_2\right] + \left(\frac{m_2}{K_2}\ddot{x}_2 + x_2\right).K_1 + K_2\left(\frac{m_2}{K_2}\ddot{x}_2\right) = 0$$

$$\frac{m_1 m_2}{K_2}\ddddot{x}_2 + m_1\ddot{x}_2 + \frac{K_1 m_2}{K_2}\ddot{x}_2 + K_1 x_2 + m_2\ddot{x}_2 = 0$$

Multipy throughout by 'k_2'

$$m_1 m_2 \ddddot{x}_2 + m_1 k_2 \ddot{x}_2 + k_1 m_2 \ddot{x}_2 + k_1 k_2 x_2 + k_2 m_2 \ddot{x}_2 = 0$$

$$m_1 m_2 \ddddot{x}_2 + (m_1 k_2 + m_2 k_2 + m_2 k_1)\ddot{x}_2 + k_1 k_2 x_2 = 0$$

$$\Rightarrow \boxed{m_1 m_2 \ddddot{x}_2 + \left[m_2 k_1 + k_2(m_1 + m_2)\right]\ddot{x}_2 + k_1 k_2 x_2 = 0}$$

(c) A system of 4 coupled linear ODE in terms of positions and velocities of each mass.

$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = 0 \quad \text{———} \quad (1)$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \quad \text{———} \quad (2)$$

Let $\dot{x}_1 = V_1$

$\therefore \ddot{x}_1 = \dot{V}_1$

Let $\dot{x}_2 = V_2$

$\therefore \ddot{x}_2 = \dot{V}_2$

Substituting new variables in eq^n (1), (2)

Equation (1) becomes,

$$m_1\dot{V}_1 + k_1 x_1 + k_2(x_1 - x_2) = 0$$

$$\therefore \quad \dot{v}_1 = \frac{-k_1 x_1 - k_2(x_1 - x_2)}{m_1}$$

$$\dot{v}_1 = -\frac{k_1}{m_1} x_1 - \frac{k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2$$

$$\dot{v}_1 = \left(\frac{-k_1 - k_2}{m_1}\right) x_1 + \left(\frac{k_2}{m_1}\right) x_2 \quad \text{——— (7)}$$

$$\& \quad \dot{x}_1 = v_1$$

Eq^n (2) after substituting new variables become

$$m_2 \dot{v}_2 + k_2(x_2 - x_1) = 0$$

$$\therefore \quad m_2 \dot{v}_2 = -k_2(x_2 - x_1)$$

$$\dot{v}_2 = -\frac{k_2}{m_2}(x_2 - x_1)$$

$$\dot{v}_2 = +\frac{k_2}{m_2} x_1 - \frac{k_2}{m_2} x_2 \quad \text{——— (8)}$$

$$\& \quad \dot{x}_2 = v_2$$

In matrix form we can write the eq^n as,

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1 - k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}$$

(d) Eigen values of Matrix in part (c)

The matrix in part (c) is,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1 - k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix}$$

Assume, $k_1 = k_2 = m_1 = m_2 = 1$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

Using MATLAB, the eigen values are,

$\lambda_1 = 1.6180i$

$\lambda_2 = -1.6180i$

$\lambda_3 = 0.6180i$

$\lambda_4 = -0.6180i$

Simulate the system from $t=0$ to $t=15$ using ODE45 with sufficiently small step size.

Initial conditions,

$x_1(0) = 1$ $\qquad$ $\dot{x}_1(0) = V_1 = 0$

$x_2(0) = 1$ $\qquad$ $\dot{x}_2(0) = V_2 = 0$

$$\therefore \quad y_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \text{Initial vector,}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

Matlab code,

```
t = 0:0.01:15
yo = [1; 0; 1; 0]
A = [0  1  0  0
    -2  0  1  0
     0  0  0  1
     1  0 -1  0]
[t, y] = ode45(@(t,y) A*y, t, yo);
plot(t, y(:,1), 'r-');
```

Do the simulations agree with your intutions based on the eigen values?

The eigen values of the system are complex and conjugates of each other

$\lambda_1 = 1.6180i$ $\lambda_3 = 0.6180i$

$\lambda_2 = -1.6180i$ $\lambda_4 = -0.6180i$

As the eigen values lie on the complex axis of the Real-Complex plane, the system must be "marginally stable", and "oscillating".

Also the final solution, $x(t)$, $\dot{x}(t)$ etc must be functions of $\cos(t)$ and $\sin(t)$.

From the graphs of function we can see that the system shows oscillatory behaviour. Further more the graphs look like they are result of superposition of cos and sin terms.

Therefore, the simulations agree with the intution based on eigen values.

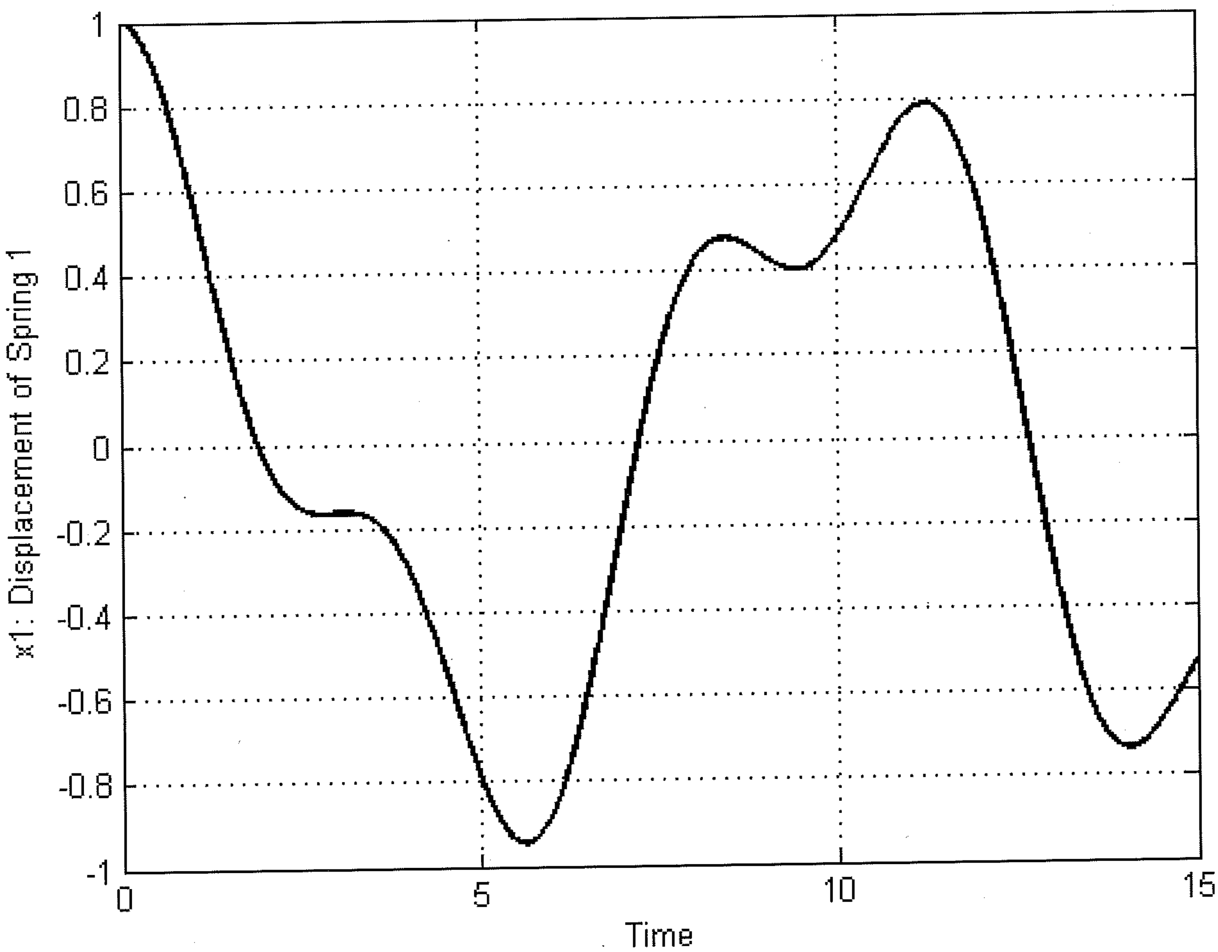
good

Problem 2:

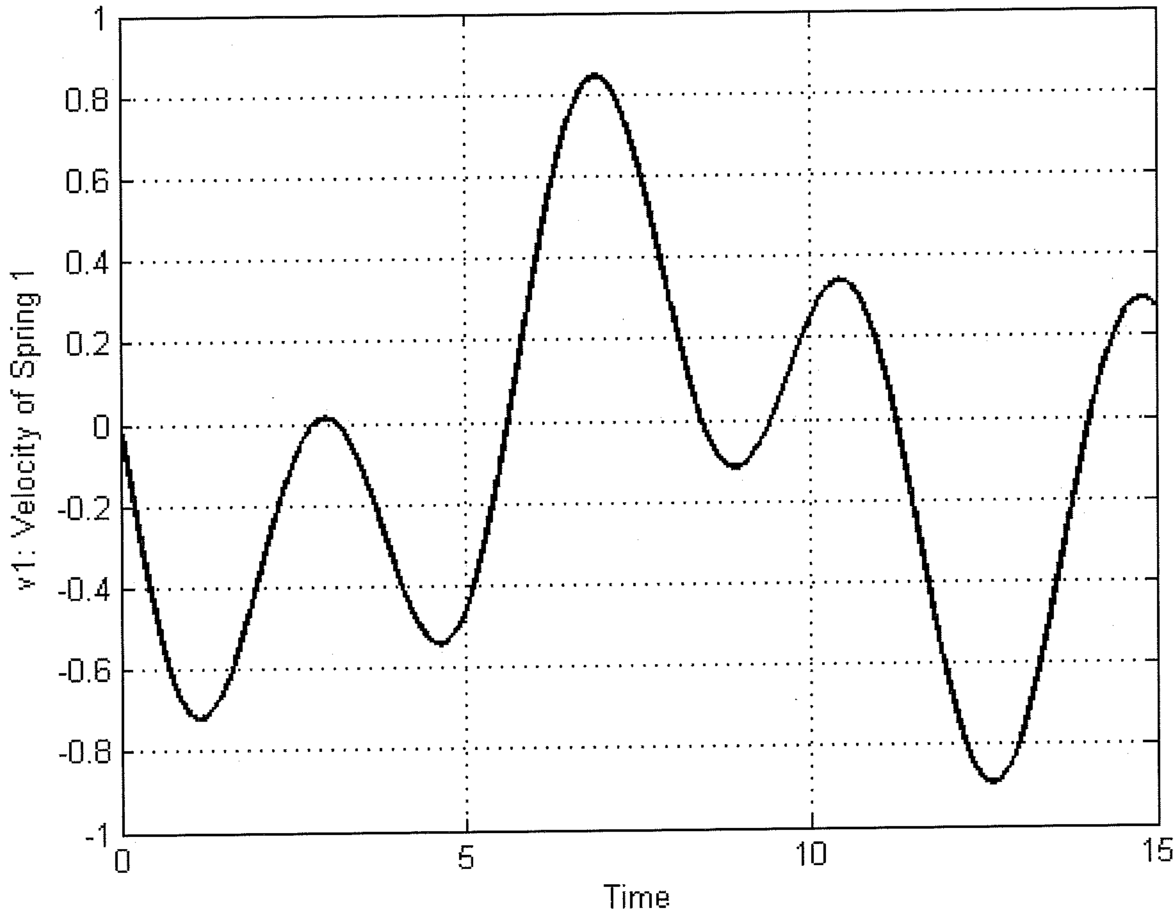
MATLAB CODE:

```
clc
clear
t=0:0.01:15;
y0=[1;0;1;0];
A=[0 1 0 0;
-2 0 1 0;
0 0 0 1;
1 0 -1 0];
[t,y]=ode45(@(t,y)A*y,t,y0);
plot(t,y(:,4),'r','linewidth',2)
grid on;
xlabel('Time')
ylabel('v2: Velocity of Spring 2')
```

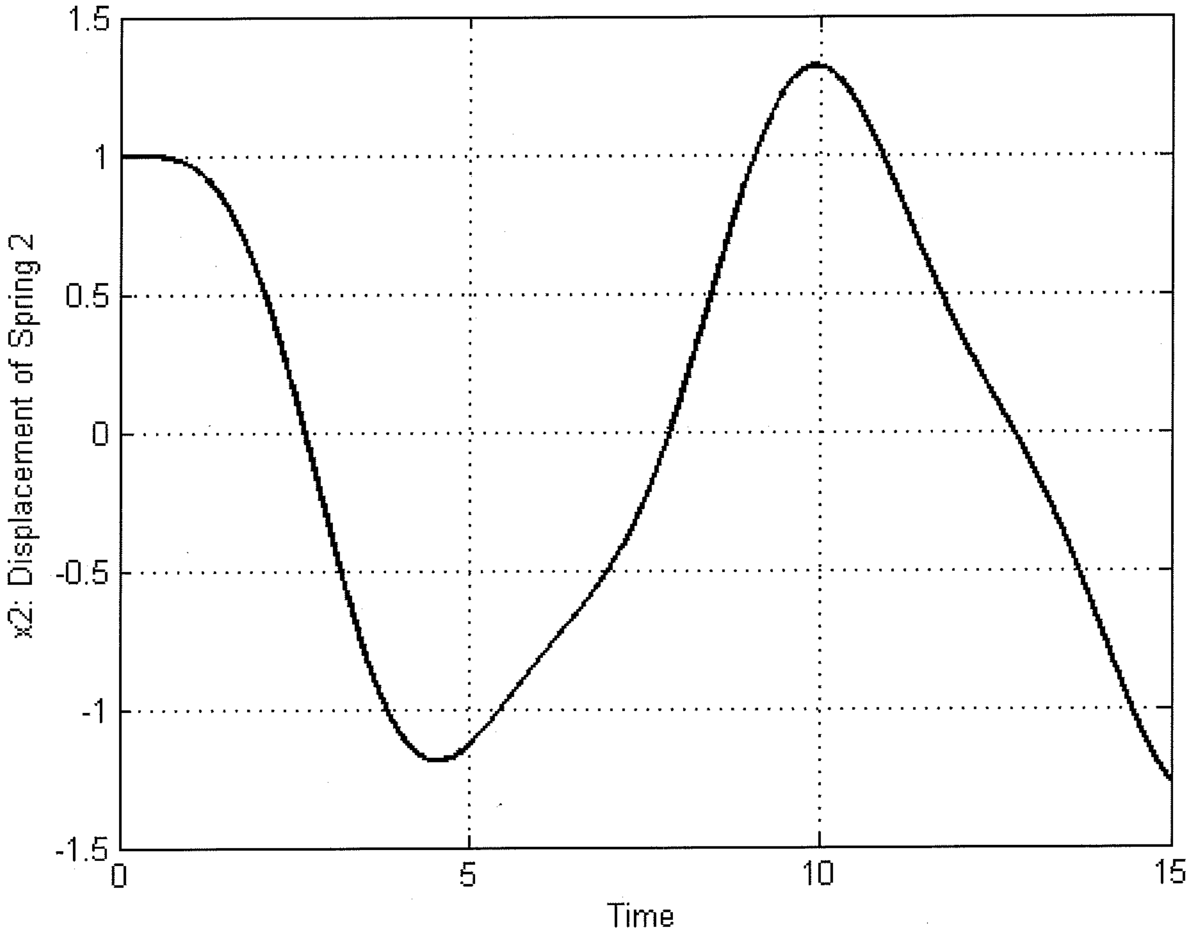
Displacement of Spring 1 ‘x1’ vs time ‘t’



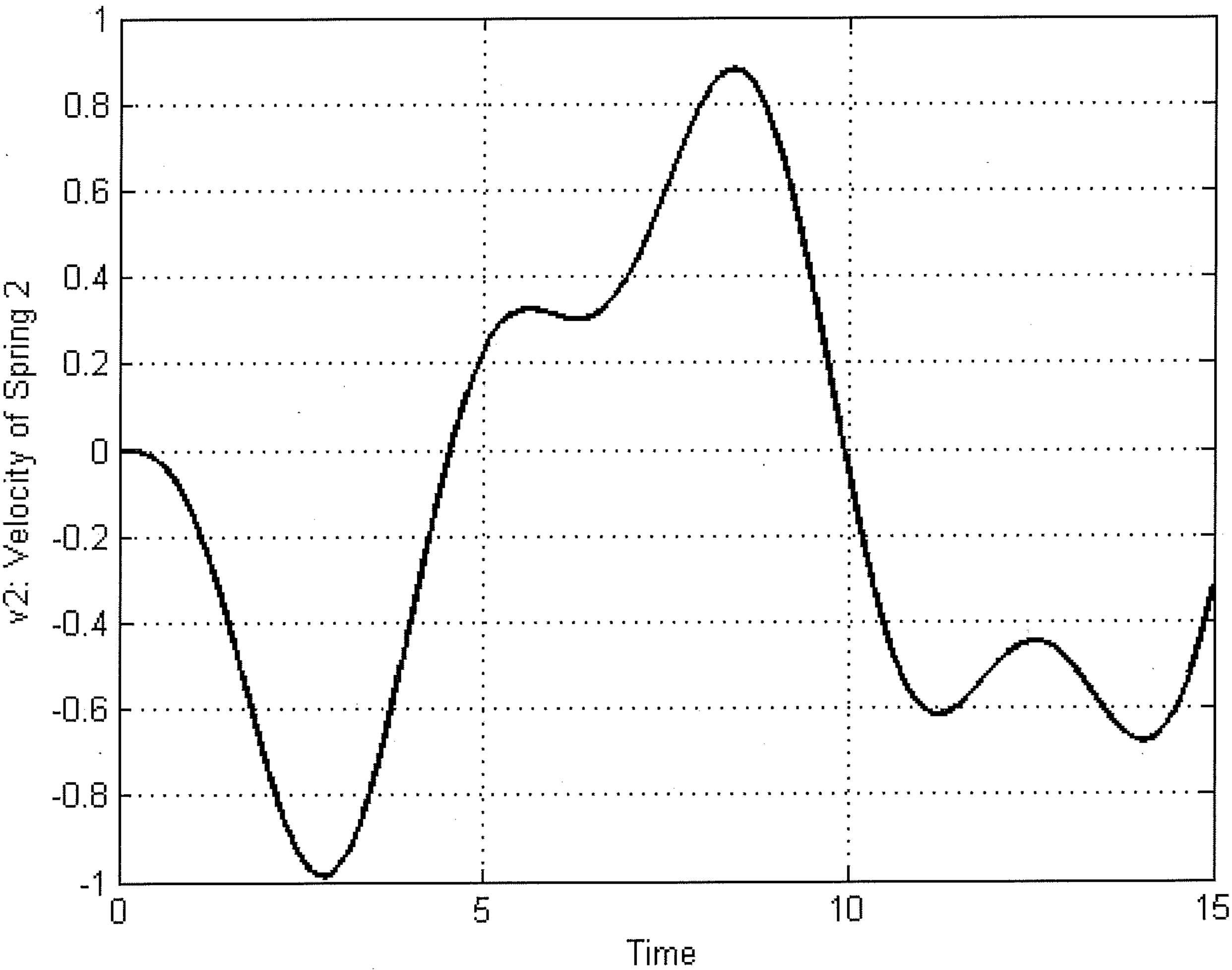
Velocity of Spring 1 'v1' vs time 't'



Displacement of Spring 2 ‘x2’ vs time ‘t’



Velocity of Spring 2 'v2' vs time 't'



Excercise 2-3

$$\ddot{x} + 3\dot{x} + 2x = 0$$

Let $\dot{x} = v$

$\ddot{x} = \dot{v}$

$\dot{v} = -3\dot{x} - 2x = -3v - 2x$

$$\therefore \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\therefore \boxed{\frac{d}{dt} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}}$$

$$\therefore \dot{y} = A y$$

where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

To find eigen values:

$$\det(A - \lambda I) = 0$$

$$\therefore \left| \begin{matrix} 0 & 1 \\ -2 & -3 \end{matrix} \right| - \left| \begin{matrix} \lambda & 0 \\ 0 & \lambda \end{matrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda(3+\lambda) + 2 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda^2 + 3\lambda + 2$$

$$\therefore \quad \lambda^2 + 2\lambda + \lambda + 2 = 0$$

$$(\lambda+2)(\lambda+1) = 0$$

$$\therefore \quad \lambda_1 = -2$$

and

$$\lambda_2 = -1$$

$\therefore$ Eigen values: $\lambda_1 = -2$ and $\lambda_2 = -1$

<u>Eigen vectors</u>,

Case (i) $\lambda_1 = -2$

$$[A - \lambda I]\,\underline{x} = \underline{0}$$

$$\therefore \quad \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = [A - \lambda I]$$

$$\therefore \quad \begin{bmatrix} +2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad 2x_1 + x_2 = 0 \quad \text{——— (1)}$$

&

$$-2x_1 - x_2 = 0 \quad \text{——— (2)}$$

From (1) and (2) we get,

$$x_2 = -2x_1$$

$$\therefore \quad \xi_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Case (2) $\lambda = -1$

$$[A - \lambda I]\,\underline{x} = \underline{0}$$

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = [A - \lambda I]$$

$$\therefore \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0 \quad \text{———— (1)}$$

$$-2x_1 \mp 2x_2 = 0 \quad \text{———— (2)}$$

from (1) & (2)

$$x_1 = -x_2$$

$$\therefore \quad \xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigen vectors : $\xi_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\quad \xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\therefore \quad T = \begin{bmatrix} | & | \\ \xi_1 & \xi_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^{-1} = \frac{1}{|T|}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$|T| = \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = -1-(-2) = 1$$

$$T^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$\underline{y}(t) = T e^{\underline{D}t} T^{-1} \underline{y}(0)$$

$$\Rightarrow \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -e^{-2t} & -e^{-2t} \\ 2e^{-t} & e^{-t} \end{bmatrix}\begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -e^{-2t}+2e^{-t} & -e^{-2t}+e^{-t} \\ +2e^{-2t}-2e^{-t} & +2e^{-2t}-e^{-t} \end{bmatrix}\begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

$$\therefore \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ 2e^{-2t}-2e^{-t} & 2e^{-2t}-e^{-t} \end{bmatrix}\begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x(t) &= (2e^{-t}-e^{-2t})\,x(0) + (e^{-t}-e^{-2t})\,v(0) \\ v(t) &= 2(e^{-2t}-e^{-t})\,x(0) + (2e^{-2t}-e^{-t})\,\dot{v}(0) \end{aligned}$$

Excercise 2-4

$$\ddddot{x} + 5\dddot{x} + 5\ddot{x} - 5\dot{x} - 6x = 0$$

Let
$$x = a$$
$$\dot{x} = \dot{a} = b \quad \text{—— (1)}$$
$$\ddot{x} = \dot{b} = c \quad \text{—— (2)}$$
$$\dddot{x} = \dot{c} = d \quad \text{—— (3)}$$
$$\ddddot{x} = \dot{d} = -5\dddot{x} - 5\ddot{x} + 5\dot{x} + 6x$$

$$\therefore \quad \dot{d} = -5d - 5c + 5b + 6a \quad \text{—— (4)}$$

From eq^n (1) — (4) we can write the Matrix of ODE,

$$\therefore \begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 5 & -5 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The system is in form: $\dot{y} = Ay$

$$\therefore \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 5 & -5 & -5 \end{bmatrix}$$

Using MATLAB to find eigen values of A,

$$\lambda_1 = -3$$
$$\lambda_2 = 1$$
$$\lambda_3 = -2$$
$$\lambda_4 = -1$$

Using MatLab we can find T, D for the Matrix A

$$T = \begin{bmatrix} 0.0349 & 0.5 & 0.1085 & 0.5 \\ -0.1048 & 0.5 & -0.2169 & -0.5 \\ 0.3143 & 0.5 & 0.4339 & 0.5 \\ -0.9429 & 0.5 & -0.8677 & -0.5 \end{bmatrix}$$

$$D = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Using Matlab we can compute T^{-1}

$$T^{-1} = \begin{bmatrix} 7.1589 & 3.5795 & -7.1589 & -3.5795 \\ 0.5 & 0.9167 & 0.5 & 0.0833 \\ -9.2195 & -3.0732 & 9.2195 & 3.0732 \\ 3.0 & -0.5 & -2.0 & -0.5 \end{bmatrix}$$

$$\dot{y}(t) = \begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dddot{x} \\ \ddddot{x} \end{bmatrix}$$

The solution is given by,

$$y(t) = T e^{Dt} T^{-1} y(0)$$

Case (1) $y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\therefore \quad y(t) = T\, e^{Dt}\, T^{-1}\, y(0)$$

$$y(t) = T\, e^{\underline{D}t}\, T^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y(t) \doteq \begin{bmatrix} 0.0349 & 0.5 & 0.1085 & 0.5 \\ -0.1048 & 0.5 & -0.2169 & -0.5 \\ 0.3143 & 0.5 & 0.4339 & 0.5 \\ -0.9429 & 0.5 & -0.8677 & -0.5 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-2t} \\ 0 & 0 & 0 & e \end{bmatrix}$$

$$\cdot \begin{bmatrix} 7.1589 & 3.5795 & -7.1589 & -3.5795 \\ 0.5 & 0.9167 & 0.5 & 0.0833 \\ -9.2195 & -3.0732 & 9.2195 & 3.0732 \\ 3.0 & -0.5 & -2.0 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y(t) = e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$ (Solved using MATLAB]

$$\therefore \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

For initial condition $y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$y(t) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \\ \dddot{x} \end{bmatrix} = \begin{bmatrix} e^t \\ e^t \\ e^t \\ e^t \end{bmatrix}$$

$\therefore \boxed{\begin{matrix} x(t) = e^t \\ \dot{x}(t) = v(t) = e^t \end{matrix}}$ for $y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

&

Case (2) $y(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$$y(t) = T e^{\underline{D} t} T^{-1} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Using matlab to compute matrix multiplications,

$$y(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\therefore y(t) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \\ \dddot{x} \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -e^{-t} \\ e^{-t} \\ -e^{-t} \end{bmatrix}$$

$$\therefore \boxed{\begin{aligned} x(t) &= e^{-t} \\ \dot{x}(t) = v(t) &= -e^{-t} \end{aligned}} \quad \text{for } y(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

<u>Long term behaviours</u>

case (1) $y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow x(t) = e^{t}$

For this initial condition, $x(t)$ will increase exponentially with time.

At $t \to \infty$, $x(t) \longrightarrow \infty$

$\therefore$ At $t \to \infty$, the function goes to infinity

case (2) $y(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow x(t) = e^{-t}$

For this initial condition, $x(t)$ will decrease exponentially with time

As $t \to \infty$, $x(t) = 1/e^{t} \to 0$

$\therefore$ At $t \to \infty$, the function will reach '0'.

Observations

- From the graphs of $x(t)$ for initial condition $y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ we can see that,

the function goes on increasing exponentially as we go on increasing time.
This behaviour matches exactly with the calculated value $x(t) = e^t$

- From the graphs of $x(t)$ for initial condition $y(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ we can see that

the function goes on decreasing exponentialy as we increase time
This behaviour matches with the calculated value $x(t) = -e^t$

- Also, another observation is that the behaviour of a system depends on the initial condition.
For the same system we get 2 entirely different behaviours for 2 separate initial conditions.

Problem 4:

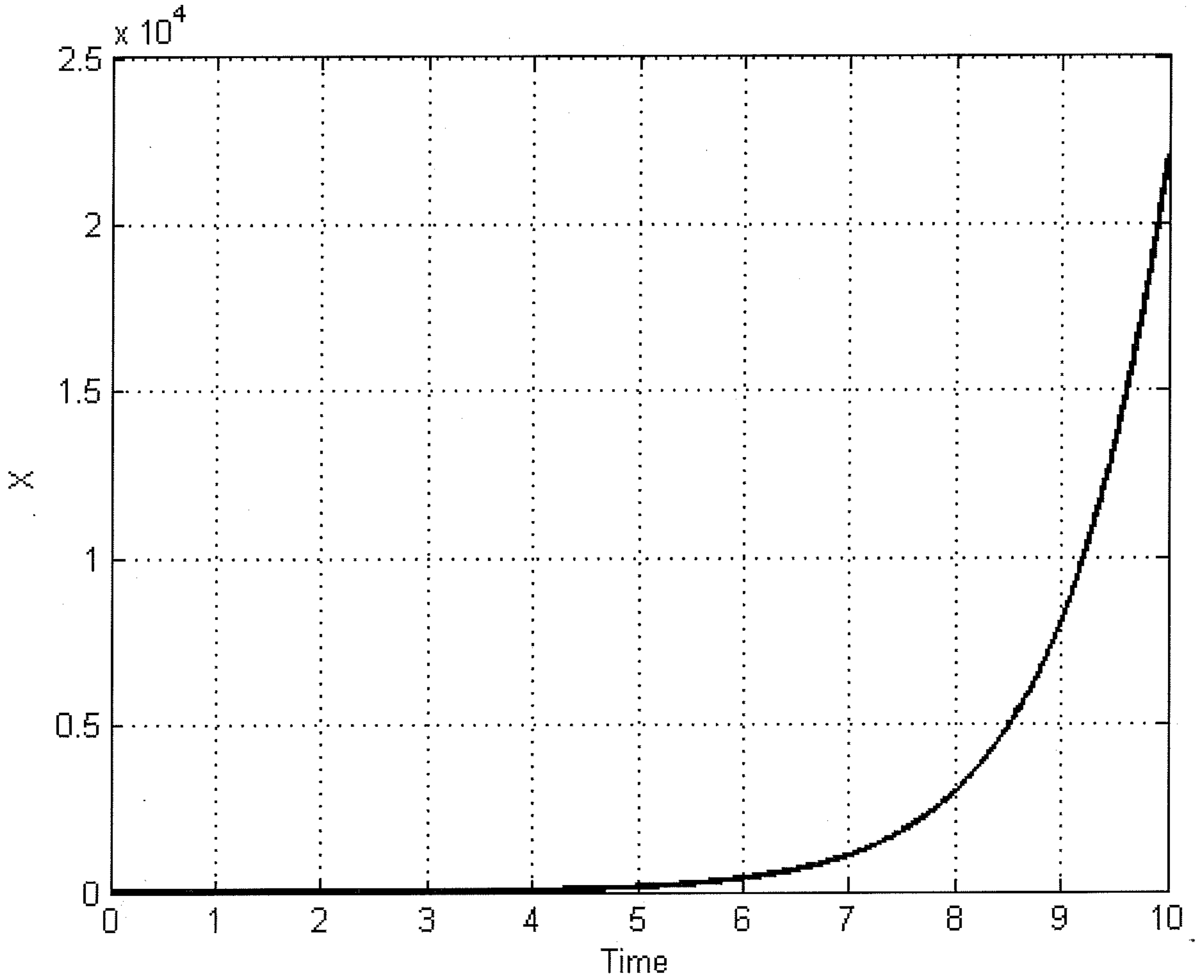
Matlab Code:

```
clc

clear

t=0:0.01:10;

y0=[1;-1;1;-1];

A=[0 1 0 0;

0 0 1 0;

0 0 0 1;

6 5 -5 -5];

[t,y]=ode45(@(t,y)A*y,t,y0);

plot(t,y(:,1),'k','linewidth',2)

grid on;

xlabel('Time')

ylabel('X')
```

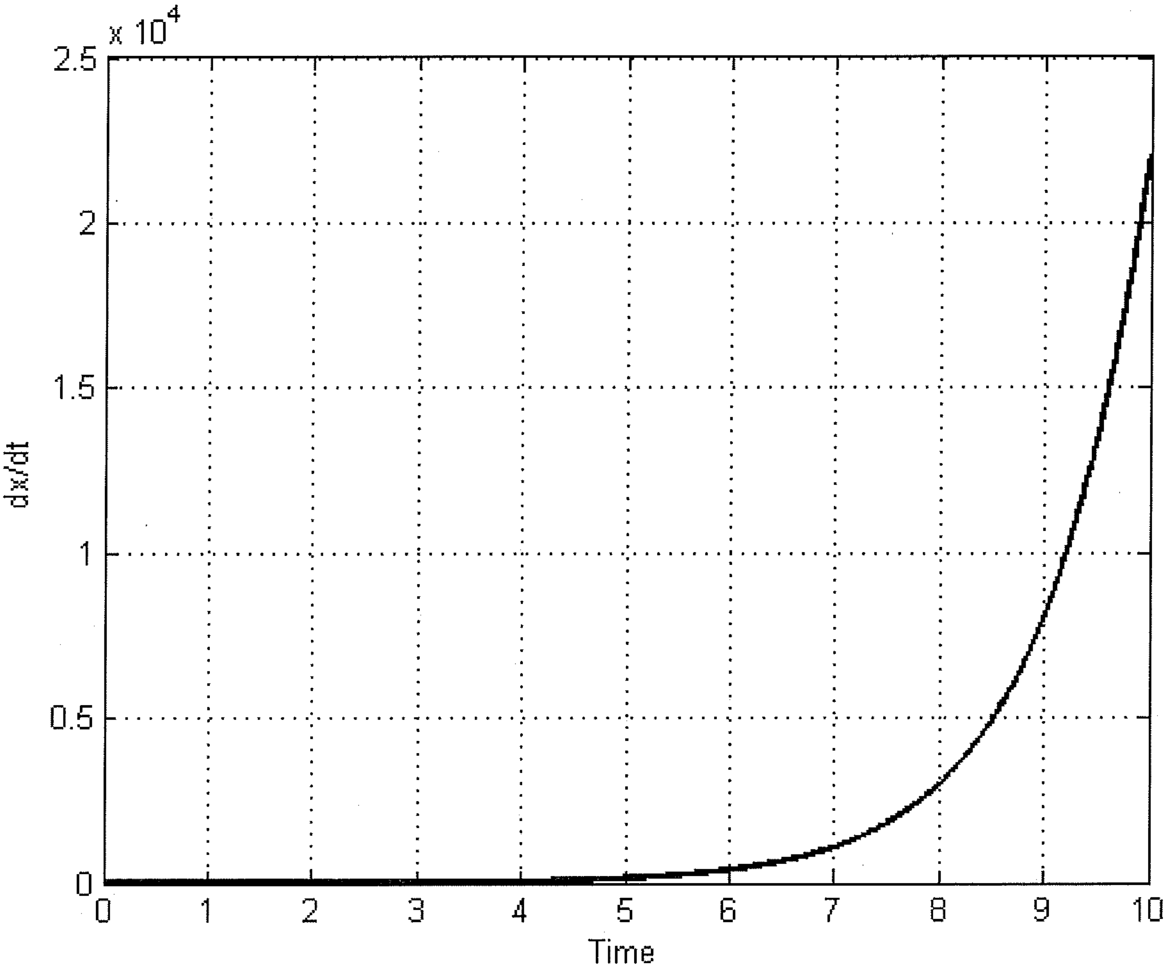
Graphs of x and dx/dt for different initial conditions

Case 1:

Displacement 'x' Vs Time 't'

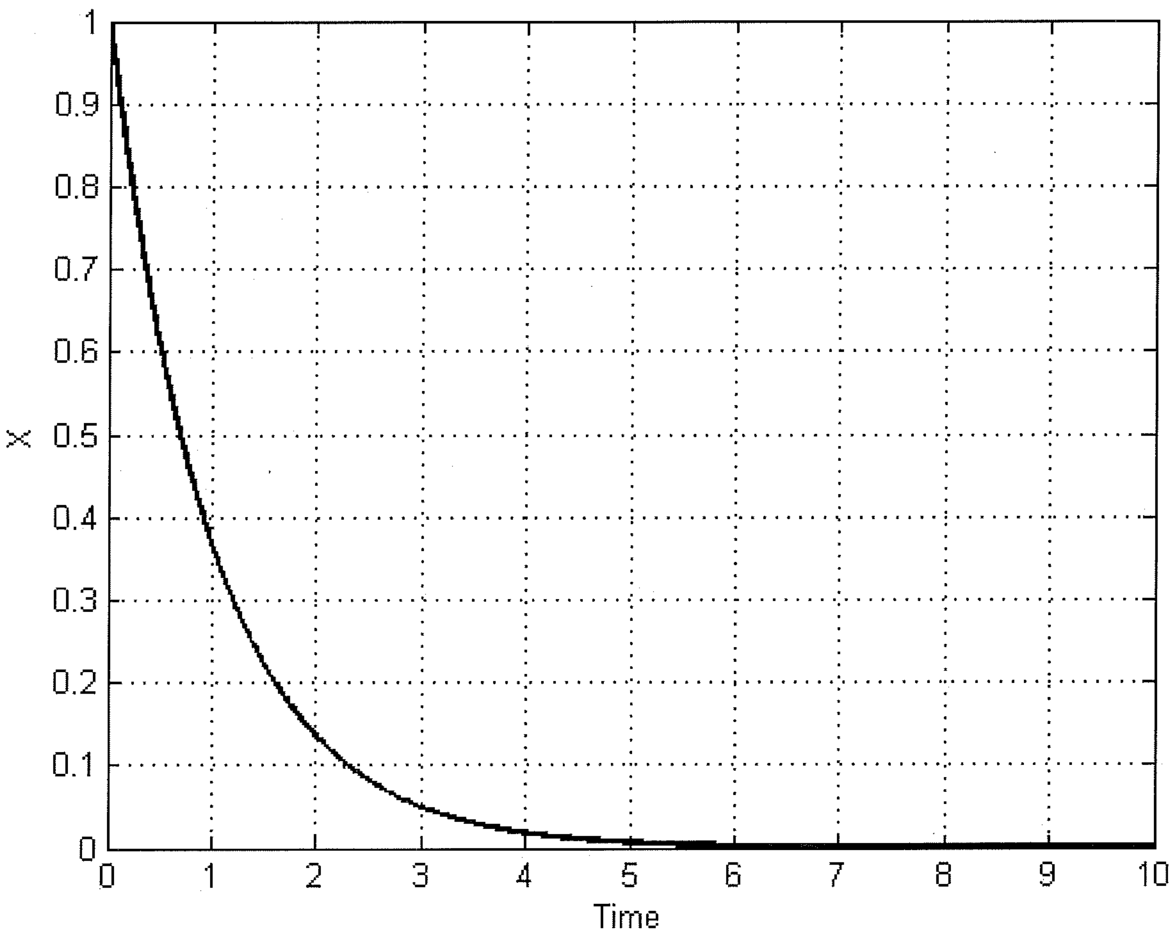


Velocity 'dx/dt' vs Time 't'

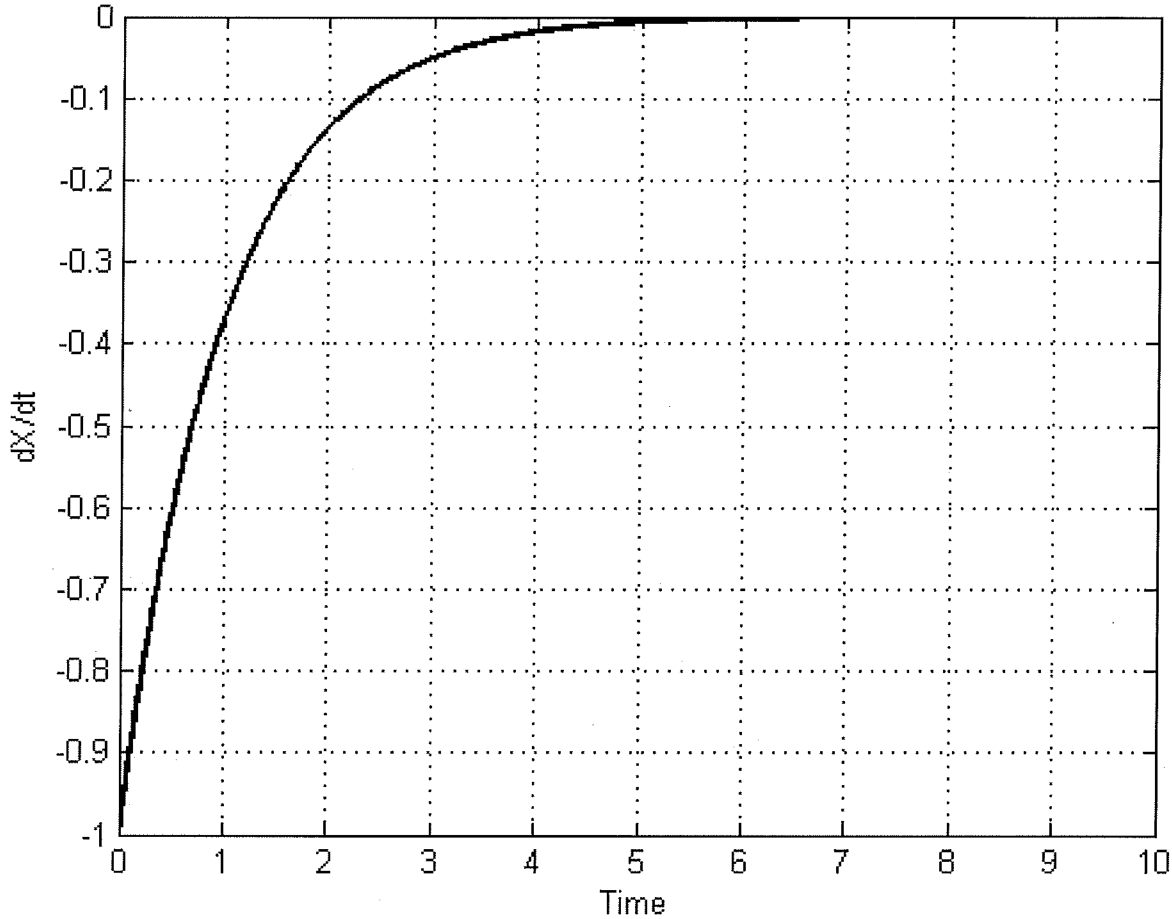


Case 2:

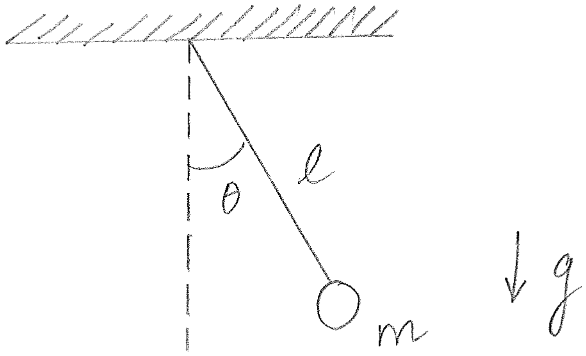
Displacement ‘x’ Vs Time ‘t’



Velocity 'dx/dt' vs Time 't'



Excercise 2.5



Kinetic energy $= T = \frac{1}{2} ml^2 \dot{\theta}^2$

Potential energy $= V = -gml\cos\theta$

$L = T - V = \frac{1}{2} ml^2 \dot{\theta}^2 - (-gml\cos\theta) = \frac{1}{2} ml^2 \dot{\theta}^2 + gml\cos\theta$

The equations of motion of pendulum are,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{\theta}}(T-V)\right] - \frac{\partial}{\partial \theta}(T-V) = 0$$

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2} ml^2 \dot{\theta}^2 + gml\cos\theta\right)\right] - \frac{\partial}{\partial \theta}\left(\frac{1}{2} ml^2 \dot{\theta}^2 + gml\cos\theta\right) = 0$$

$$\frac{d}{dt}\left[\frac{2ml^2\dot{\theta}}{2} + gml\frac{\partial \cos\theta}{\partial \dot{\theta}}\right] - \frac{1}{2} ml^2 \frac{\partial \dot{\theta}^2}{\partial \theta} - gml(-\sin\theta) = 0$$

$$\therefore \frac{d}{dt}\left[ml^2\dot{\theta} + 0\right] - 0 + gml\sin\theta = 0$$

$$\therefore ml^2 \frac{d\dot{\theta}}{dt} + gml\sin\theta = 0$$

$$ml^2\ddot{\theta} + gml\sin\theta = 0$$

$$ml^2\ddot{\theta} = -gml\sin\theta$$

$$\ddot{\theta} = -\frac{gml\sin\theta}{ml^2}$$

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

Let $\dot{\theta} = \omega$

$$\therefore \quad \dot{\omega} = \ddot{\theta} = -\frac{g}{l}\sin\theta$$

$\therefore$ We can write this in form $\dot{x} = f(x)$

$$\therefore \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{l}\sin\theta \end{bmatrix}$$

$$f = \begin{bmatrix} \omega \\ -\frac{g}{l}\sin\theta \end{bmatrix}$$

Assuming $g = l = m = 1$

$$f = \begin{bmatrix} \omega \\ -\sin\theta \end{bmatrix}$$

$$\therefore \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = f = \begin{bmatrix} \omega \\ -\sin\theta \end{bmatrix}$$

Fixed points,

$$f(\bar{x}) = \begin{bmatrix} \omega \\ -\sin\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad \omega = 0 \quad \&$

$-\sin\theta = 0$

$\therefore \sin\theta = 0$

$\therefore \theta = 0, \pi, 2\pi, n\pi$

$\theta = n\pi \quad$ where $\quad n = 0, 1, 2, 3 \ldots$

$\therefore \quad n = \text{even} = 0, 2, 4$

$\therefore \quad n = \text{odd} = 1, 3, 5$

Case (1) $\quad \omega = 0 \;\&\; \theta = n\pi \; (n = \text{even})$ (Fixed pt 1)

Let us linearize the ODE according to form,

$$\dot{x} - \dot{\bar{x}} = \frac{d}{dt}(\Delta x) = \left.\frac{\Delta f}{\Delta x}\right|_{\bar{x}} \Delta x$$

$$\frac{\Delta f}{\Delta x} = \begin{bmatrix} \partial\omega/\partial\theta & \partial\omega/\partial\omega \\ \partial(-\sin\theta)/\partial\theta & \partial(-\sin\theta)/\partial\omega \end{bmatrix}$$

$$\frac{\Delta f}{\Delta x} = \begin{bmatrix} 0 & 1 \\ -\cos\theta & 0 \end{bmatrix}$$

$$\therefore \frac{d}{dt}\begin{bmatrix} \Delta\theta \\ \Delta\omega \end{bmatrix} = \left.\frac{\Delta f}{\Delta x}\right|_{\substack{\omega = 0,\ \theta = n\pi \\ n\,(\text{even})}} \begin{bmatrix} \Delta\theta \\ \Delta\omega \end{bmatrix}$$

$$\therefore \frac{d}{dt}\begin{bmatrix}\Delta\theta \\ \Delta\omega\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\underset{(n\,\text{even})}{\cos(n\pi)} & 0\end{bmatrix}\begin{bmatrix}\Delta\theta \\ \Delta\omega\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} - \frac{d}{dt}\begin{bmatrix}\bar{\theta} \\ \bar{\omega}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix}\Delta\theta \\ \Delta\omega\end{bmatrix}$$

$$\therefore \boxed{\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix}\Delta\theta \\ \Delta\omega\end{bmatrix}} \rightarrow \text{Linearized ODE}$$

$$\therefore \boxed{\begin{aligned}\dot{\theta} &= \Delta\omega \\ \dot{\omega} &= \ddot{\theta} = -\Delta\theta\end{aligned}} \rightarrow \text{Linearized Solution}$$

stability,

The linearized ODE is,

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix}\Delta\theta \\ \Delta\omega\end{bmatrix}$$

$$x;\ \dot{x} = A\,x$$

The A matrix is,

$$A = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}$$

Eigen values of A are,

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\therefore \lambda^2 = 1$$

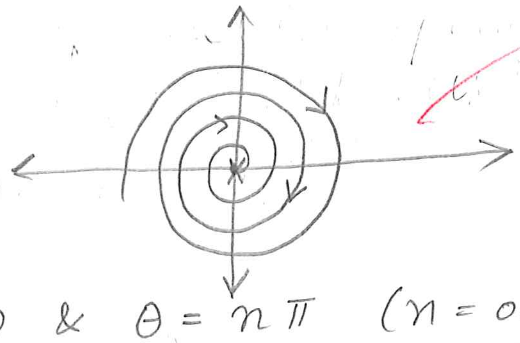
$$\boxed{\lambda = \pm i}$$

Eigen values of A are,

$$\lambda_1 = i$$

$$\lambda_2 = -i$$

Stability :
Since the system has imaginary complex conjugate eigen values $\pm i$, the system is "marginally stable". The system is a stable spiral centered at the fixed point



Case (2) $\omega = 0$ & $\theta = n\pi$ (n = odd) (Fixed pt 2)

Using procedure similar to case (1)

The linearized ODE is,

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos(n\pi) & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \omega \end{bmatrix}$$
n = odd

As $\cos(n\pi) = -1$

$$\therefore \frac{d}{dt}\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

Linearized ode

$$\boxed{\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}\theta \\ \omega\end{bmatrix}}$$

Eigen values.

$$A = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix}-\lambda & 1 \\ 1 & -\lambda\end{vmatrix} = 0$$

$$\therefore \quad \lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$$\boxed{\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}}$$

Stability : Saddle point

The fixed point is the saddle point.
So, in the vicinity of the point, one direction is stable and the other direction is unstable