ME564 - Autumn 2018 **Due: 5:00:00pm Friday November 30^{th}, 2018**
Homework 5

Exercise 5–1: Lets say that we are integrating the following stable ODE:

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \mathbf{x}$$

(a) If we are using a forward Euler scheme, for what $\Delta t > 0$ will the resulting simulation become unstable?

(b) If we use the backward Euler scheme, for what $\Delta t > 0$ will the simulation become unstable?

Exercise 5–2: A (real valued) inner product space is a vector space that has an inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ that satisfies the following three axioms:

- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$,
- $\langle a\mathbf{x}, \mathbf{y} \rangle = a \langle \mathbf{x}, \mathbf{y} \rangle$, for all real numbers $a \in \mathbb{R}$,
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, with equality only if $\mathbf{x} = 0$.

Consider the space of bounded functions on the interval $[0, 1]$ with the following inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx.$$

Bounded just means that the magnitude of both f and g never exceed some fixed large number on the interval $[0, 1]$. You can approximate this inner product in Matlab by defining the two vectors f and g on a discrete grid from 0 to 1.

(a) Verify that this space of functions and inner product satisfy the following three axioms above.

(b) Show that the functions $\cos(\pi m x)$ and $\cos(\pi n x)$ for non-negative integers m and n are orthogonal (using the inner product above) for all $m \neq n$. You may find the following identify useful: $\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$.

(c) Plot $\cos(\pi m x)$ on the interval $[0, 1]$ with $dx = 0.01$ for $m = 0, 1, 2, 3, 4$, and 5. Verify numerically, using trapezoidal integration (i.e. `trapz`), that $\cos(\pi m x)$ and $\cos(\pi n x)$ are orthogonal for the following (m, n) pairs: $(1, 4)$, $(2, 6)$, and $(3, 15)$.

Note that we have an *infinite* set of orthogonal functions, which each represent a unique and orthogonal *vector direction* in the inner product space of bounded functions on $[0, 1]$. We are starting to build an infinite dimensional vector space (called a Hilbert space) for representing functions. These functions will be the solutions of PDEs in ME565. (Note that we will eventually need to include sine functions as well.)
