ME564 - Autumn 2017 **Due: 11:59pm Monday December 4^{th}, 2017**
Homework 5

Exercise 5-1 Solve the following

- Compute the gradient of the function $f(x, y, z) = x^2 \sin(y) + z$.
- Compute the curl of the new vector field $\mathbf{V} = \nabla f$.
- Show that $\nabla \times (\nabla f) = 0$ for any scalar function f. That is to say, all gradient fields are irrotational.

Exercise 5-2 A (real valued) inner product space is a vector space that has an inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ that satisfies the following three axioms:

- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$,
- $\langle a\mathbf{x}, \mathbf{y} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle$, for all real numbers $a \in \mathbb{R}$,
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, with equality only if $\mathbf{x} = 0$.

Consider the space of bounded functions on the interval $[0, 1]$ with the following inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Bounded just means that the magnitude of both f and g never exceed some fixed large number for x on the interval $[0, 1]$.

(a) Verify that this space of functions and inner product satisfy the following three axioms above.

(b) Show that the functions $\cos(\pi mx)$ and $\cos(\pi nx)$ for non-negative integers m and n are orthogonal (using the inner product above) for all $m \neq n$. You may find the following identify useful: $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$.

(c) Plot $\cos(\pi mx)$ on the interval $[0, 1]$ with $dx = 0.01$ for $m = 0, 1, 2, 3, 4$, and 5. Verify numerically, using trapezoidal integration (i.e. `trapz`), that $\cos(\pi mx)$ and $\cos(\pi nx)$ are orthogonal for the following (m, n) pairs: $(1, 4)$, $(2, 6)$, and $(3, 15)$.

Note that we have an *infinite* set of orthogonal functions, which each represent a unique and orthogonal *vector direction* in the inner product space of bounded functions on $[0, 1]$. We are starting to build an infinite dimensional vector space (called a Hilbert space) for representing functions. These functions will be the solutions of PDEs in ME565. (Note that we will eventually need to include sine functions as well.)