ME564 - Autumn 2017 **Due: 4:00:00pm Friday October 20^{th}, 2017**
Homework 2

Exercise 2–1: Consider the following ODE:

$$\ddot{x} + 7\dot{x} + 10x = 0.$$

(a) Solve this ODE by substituting in $x(t) = e^{\lambda t}$ and finding roots of the characteristic polynomial.

(b) Now, write this second order ODE as a system of first order equations, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, by introducing a new variable $v = \dot{x}$. Solve the linear system by decomposing the matrix $\mathbf{A}$ into the eigenvalues and eigenvectors. Hint: computing the inverse of the matrix of eigenvectors is easier if they are not normalized (i.e., both of my eigenvectors have integer entries).

(c) For both of the solutions in parts (a) and (b), write down the specific solution for an initial condition $x(0) = 1$ and $\dot{x}(0) = 1$.

(d) What is the long-time behavior of this system?

Exercise 2–2: Write the following ODE as a system of first order ODEs:

$$\dddot{x} - 7\dot{x} - 6x = 0$$

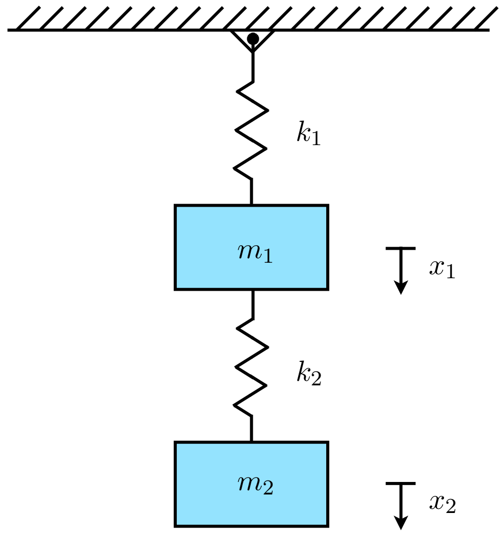
What are the eigenvalues of the system of ODEs? (it is OK to use Matlab).

What are the long-time behaviors of this system for the following two different initial conditions: $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$?

Please plot the response for `t=0:0.01:10` using `ode45` for each of these initial conditions. Do the Matlab plots agree with your calculations? Please briefly explain your observations in 1-2 sentences.

What will the response look like for $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix}$?

Exercise 2–3: Consider the double spring-mass system presented in class (equations below).



$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0$$
$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

Write these coupled second order linear differential equations as

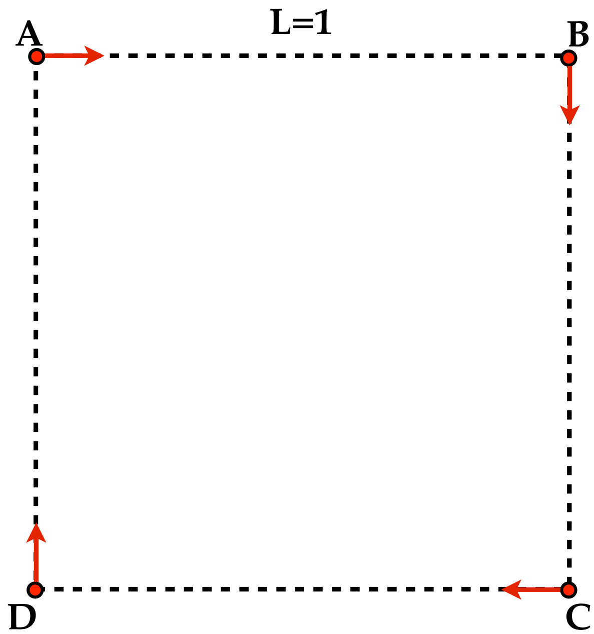
(a) A single fourth order ODE in x_1

(b) A single fourth order ODE in x_2

(c) A system of four coupled linear ODEs in terms of the positions and velocities of each mass. Please write this as a matrix ODE.

For the remaining parts, you may assume that $k_1 = k_2 = m_1 = m_2 = 1$. What are the eigenvalues of the matrix system of ODEs in part (c)? You can use Matlab to compute these. Use Matlab to simulate the system from $t = 0$ to $t = 15$ using `ode45` with a sufficiently small time-step, and plot the results; start with an initial condition $x_1(0) = x_2(0) = 1$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$. Do the simulations agree with your intuition based on the eigenvalues?

Exercise 2–4: There are four boats at four corners of a square. Each boat starts motoring towards the neighboring boat in the clockwise direction (so A pursues B, B pursues C, C pursues D and D pursues A). As the boats begin to move, they always move in a direction pointing towards the boat they are pursuing, so that they eventually meet in the middle. Each boat motors at a constant speed of 1 mile per hour in pursuit of its neighbor. The square has sides of length 1 mile.



- Write down a differential equation that describes the motion of a boat.
- Sketch the motion of the boats in time.
- How long until the boats meet in the middle?

Hint: Try to find any symmetry or convenient coordinates to use that make the problem simpler. The simplest solution may surprise you! You might also want to try and draw the situation and explain your reasoning.

Exercise 2–5: Consider the following nonlinear ODE:

$$\frac{dx}{dt} = x^2$$

(a) Find the solution $x(t)$ using any method you prefer.

(b) A major problem in engineering mathematics is to find a good coordinate system where complex system become *simple*. There is a change of coordinates $y(x)$ that makes this system linear, with dynamics

$$\frac{dy}{dt} = y.$$

Using the chain rule, we find that:

$$\frac{dy(x(t))}{dt} = \frac{\partial y}{\partial x}\dot{x} = y(x).$$

Plugging in $\dot{x} = x^2$, we get

$$\frac{\partial y}{\partial x}x^2 = y(x).$$

Solve for $y(x)$ using a series expansion. Show that a simple Taylor series will not work. Then try a series that includes negative powers of x: $y(x) = \cdots + c_{-2}x^{-2} + c_{-1}x^{-1} + c_0 + x_1 x + x_2 x^2 + \cdots$.