A Comparison of Methods for Fast Computation of FTLE Fields

Steven L. Brunton and Clarence W. Rowley
Department of Mechanical and Aerospace Engineering, Princeton University, NJ 08540, USA

A number of efficient methods are developed for the computation of a time-series of FTLE fields. The efficiency and accuracy of the methods are compared with each other and with the standard computation. It is demonstrated that the unidirectional interpolated methods have the best balance of speed and accuracy.

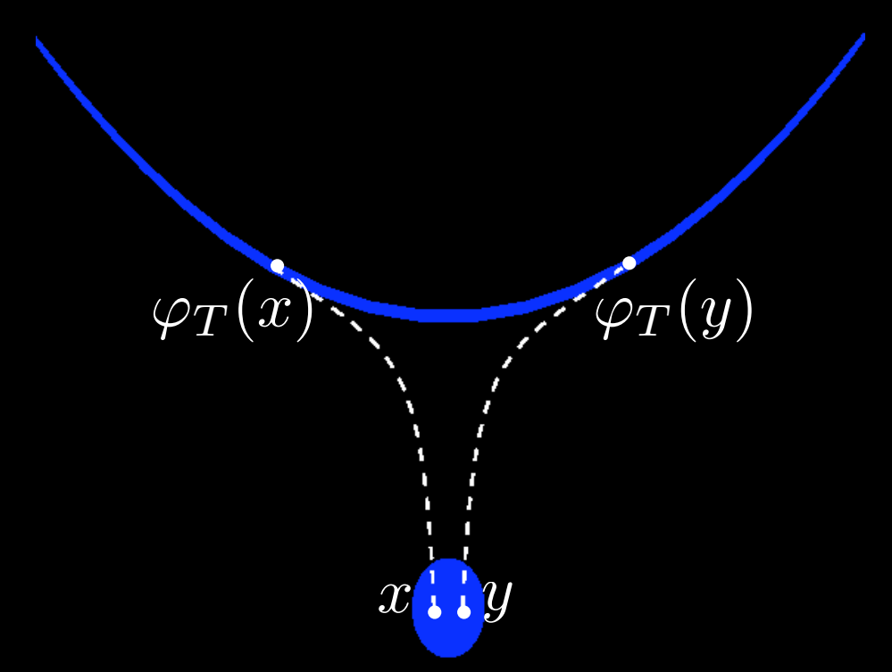
FTLE field illustrating the separated flow and wake structures associated with a flat plate plunging sinusoidally in a viscous fluid.

The goals of this work are:

Develop fast, accurate algorithms for computing a time-series of FTLE fields,

Utilize flow map at previous time to speed up flow map calculation at current time.

Finite Time Lyapunov Exponents (FTLE)



$$\|\delta x_T\| = \sqrt{\langle \delta x_0, \Delta \delta x_0 \rangle}$$

$$\Delta = \frac{d\varphi_T}{dx}^* \frac{d\varphi_T}{dx}$$ Cauchy-Green deformation tensor

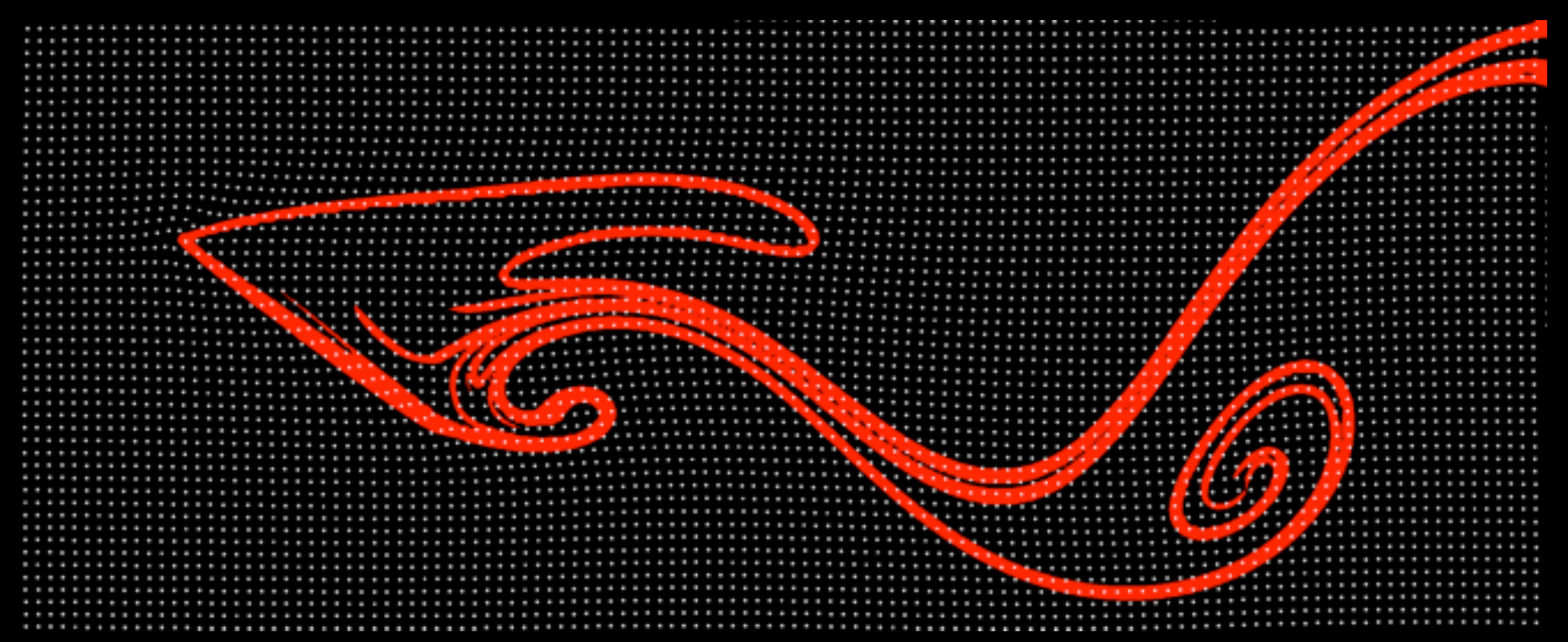
Maximal stretching occurs when δx_0 is aligned with the eigenvector corresponding to the maximum eigenvalue of Δ :

$$\delta x_0 \in \operatorname{span}\{\xi_{\max}\} \quad \text{where} \quad \Delta \xi_{\max} = \lambda_{\max}(\Delta)\xi_{\max}$$

$$\max_{\delta x_0} \|\delta x_T\| = \sqrt{\lambda_{\max}}\|\delta x_0\| = e^{\sigma_T(x)|T|}\|\delta x_0\|$$

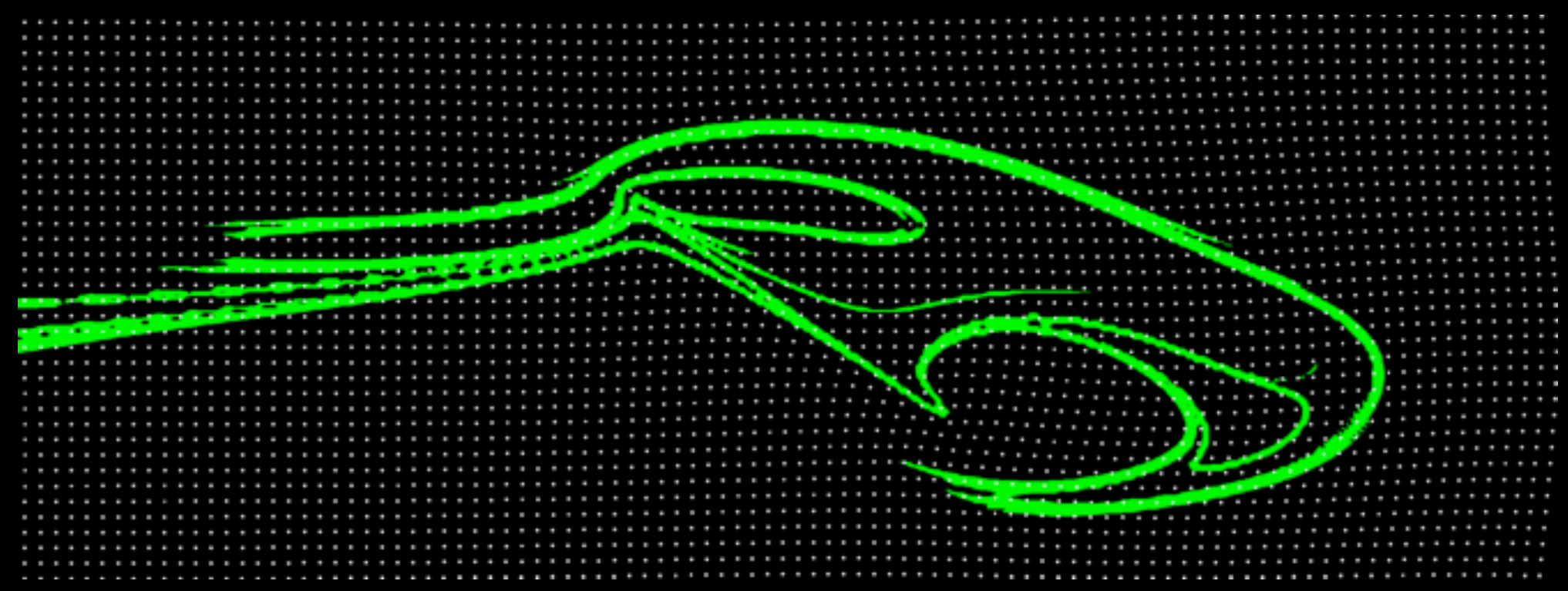
$$\boxed{\sigma_T(x) = \frac{1}{|T|} \log \sqrt{\lambda_{\max}(\Delta)}}$$

Finite Time Lyapunov Exponent
Shadden et al. (2005)



Flat plate at high angle-of-attack.

Forward-time attracting set



Backward-time attracting set

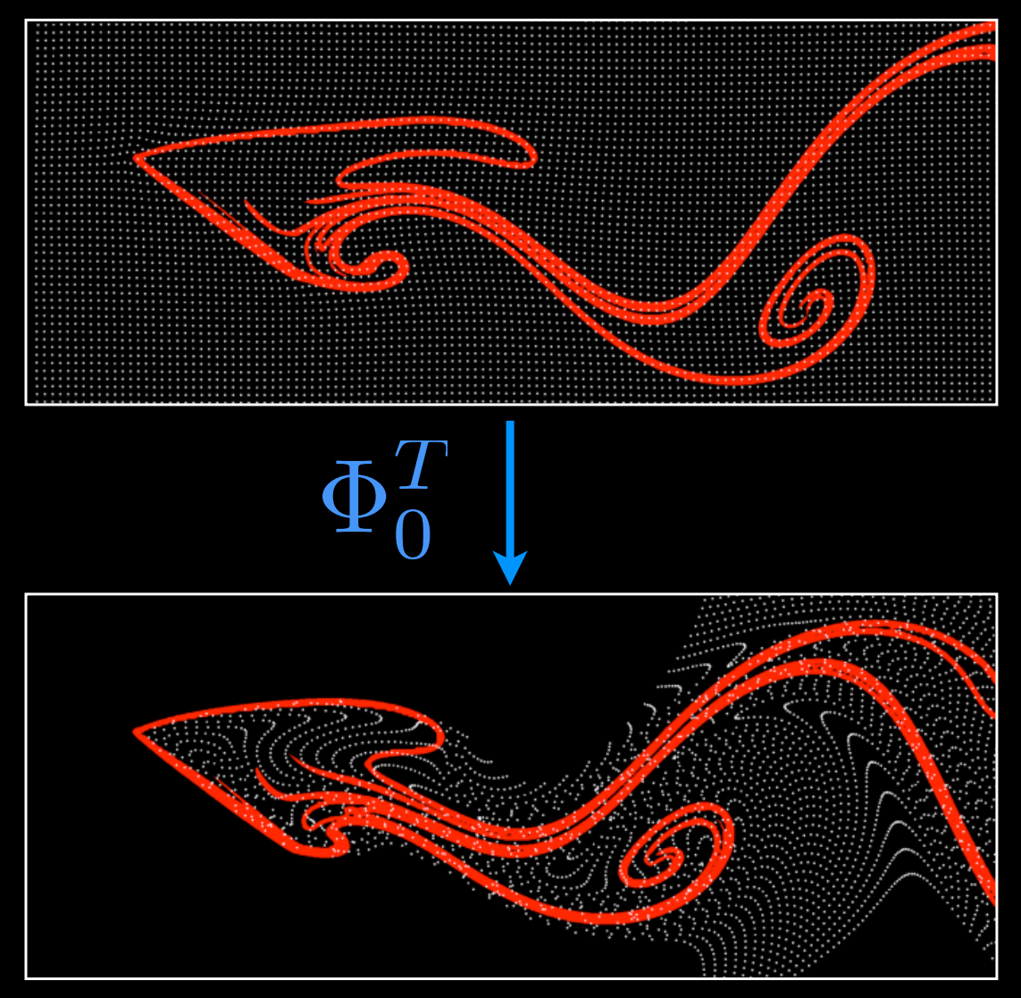
Re = 100, $\alpha = 35^\circ$

Standard Computational Method (Exact)

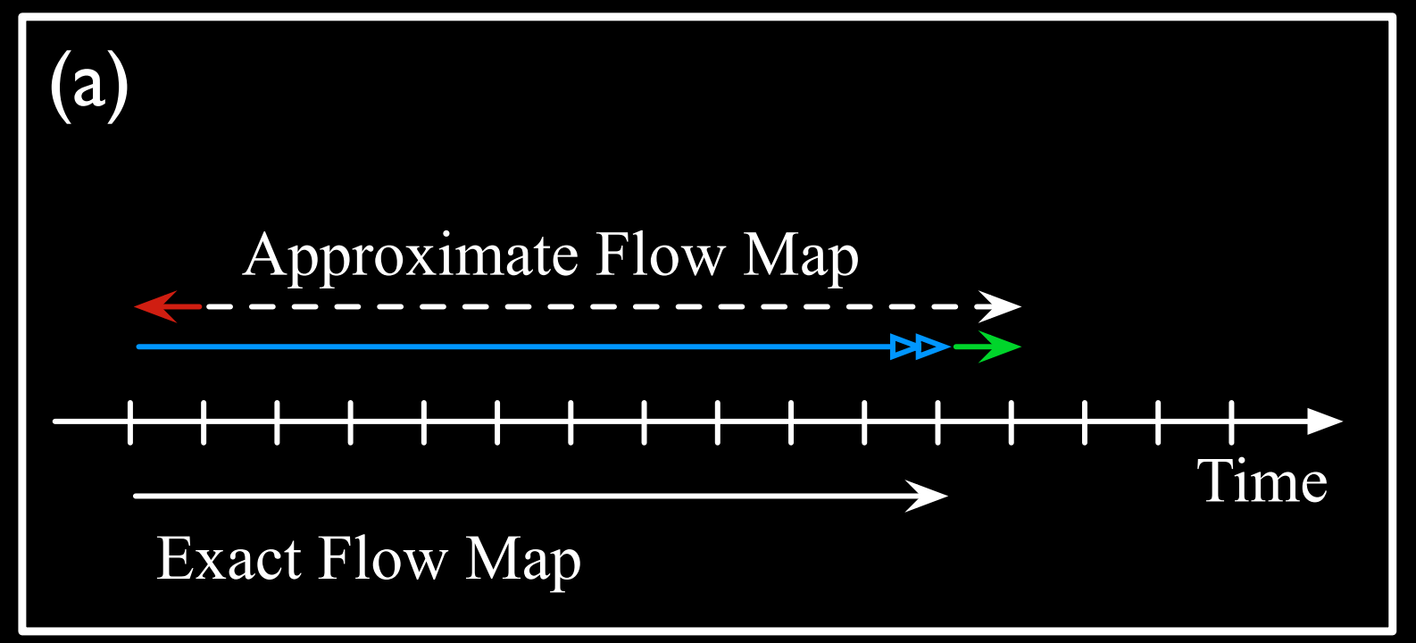
step 1: Initialize a grid of particles on domain

step 2: Advect particles along the flow Φ_0^T
(i.e., integrate particle through time-varying velocity field)

step 3: Extract finite time Lyapunov exponent according to the formula given.

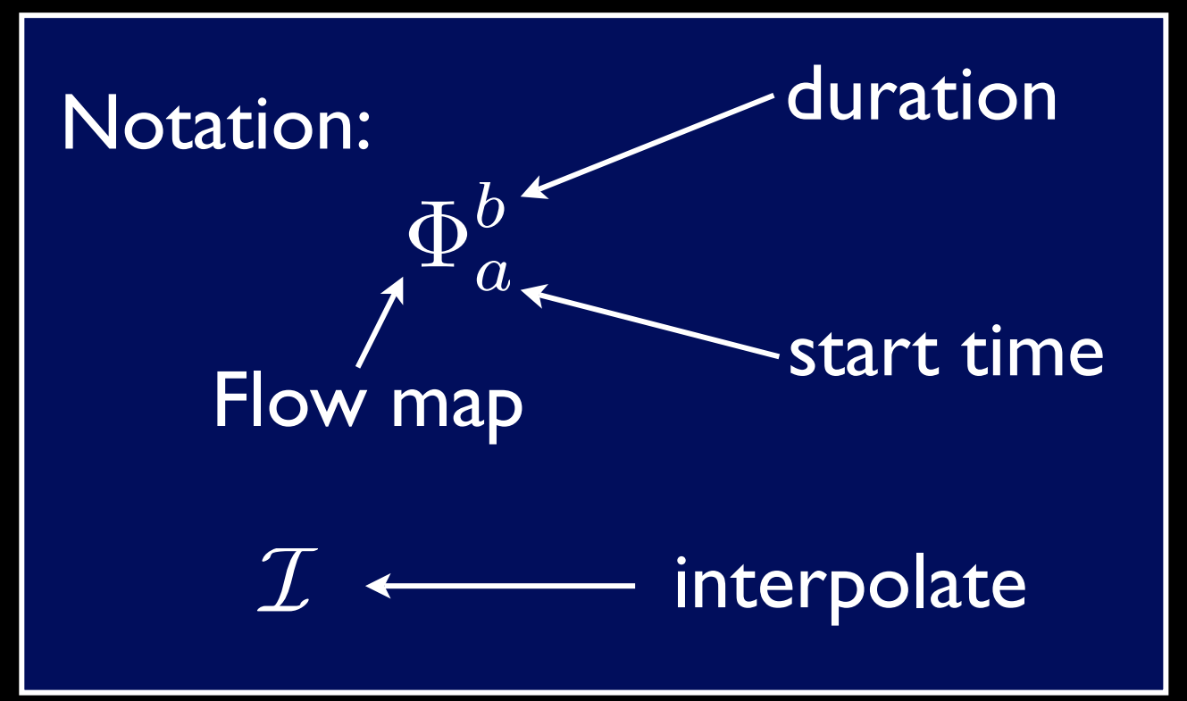


Bidirectional Interpolated Method

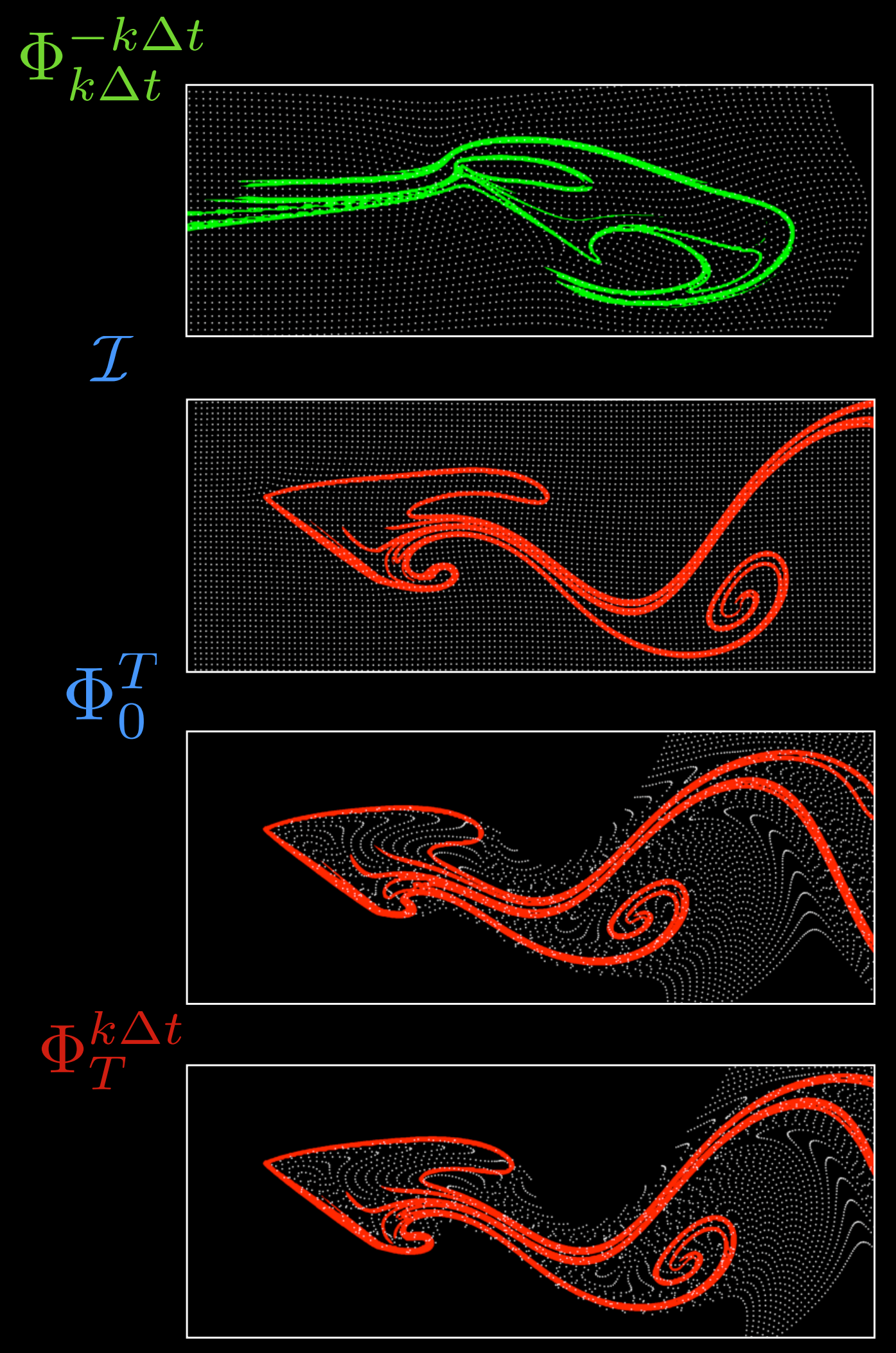


$$\tilde{\Phi}_{k\Delta t}^T = \Phi_T^{k\Delta t} \circ \Phi_0^T \circ \mathcal{I} \circ \Phi_{k\Delta t}^{-k\Delta t}$$

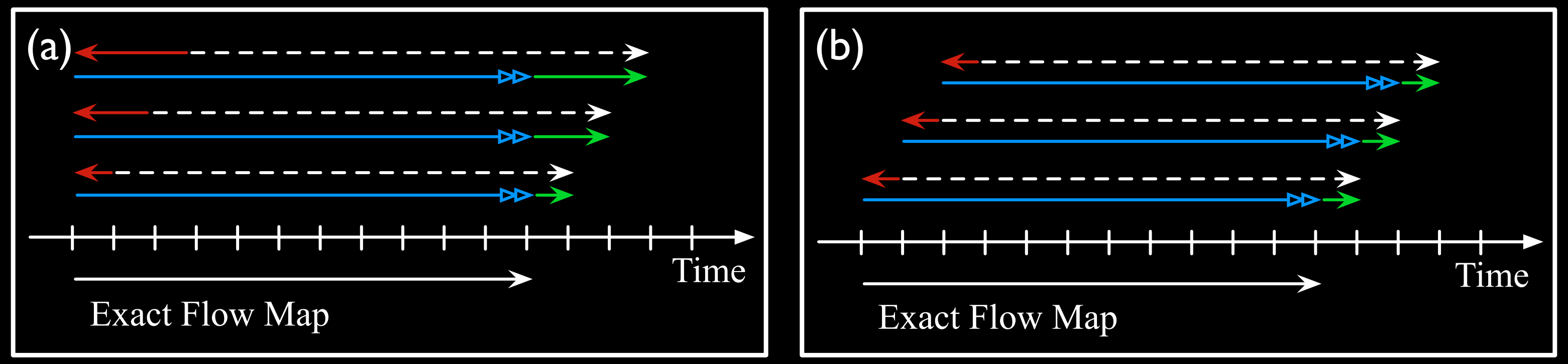
🟩	Integrate Backward
🟦	Interpolate Forward
🟥	Integrate Forward
----	Approximate Result



Notation:
Φ_a^b
duration
start time
Flow map
$\mathcal{I}$ ← interpolate



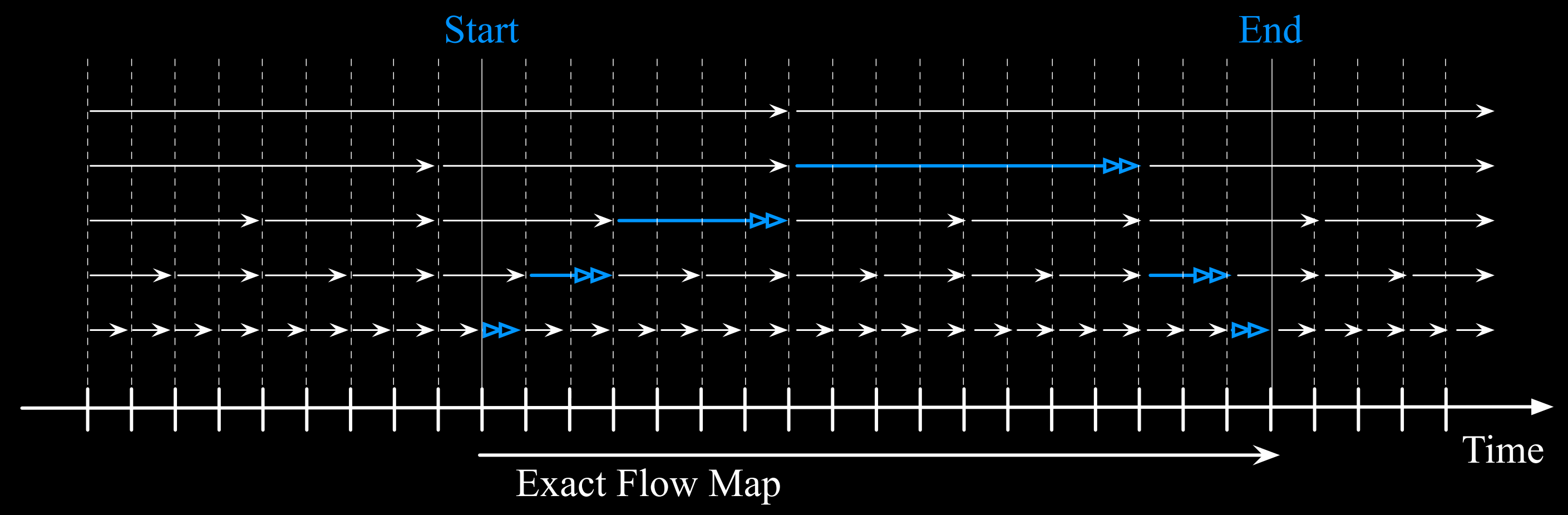
Bidirectional Method for Time-series



$$\tilde{\Phi}_{k\Delta t}^T = \Phi_T^{k\Delta t} \circ \Phi_0^T \circ \mathcal{I} \circ \Phi_{k\Delta t}^{-k\Delta t}$$

$$\tilde{\Phi}_{k\Delta t}^T = \Phi^{\Delta t} \circ \Phi^T \circ \mathcal{I} \circ \Phi_{k\Delta t}^{-\Delta t}$$

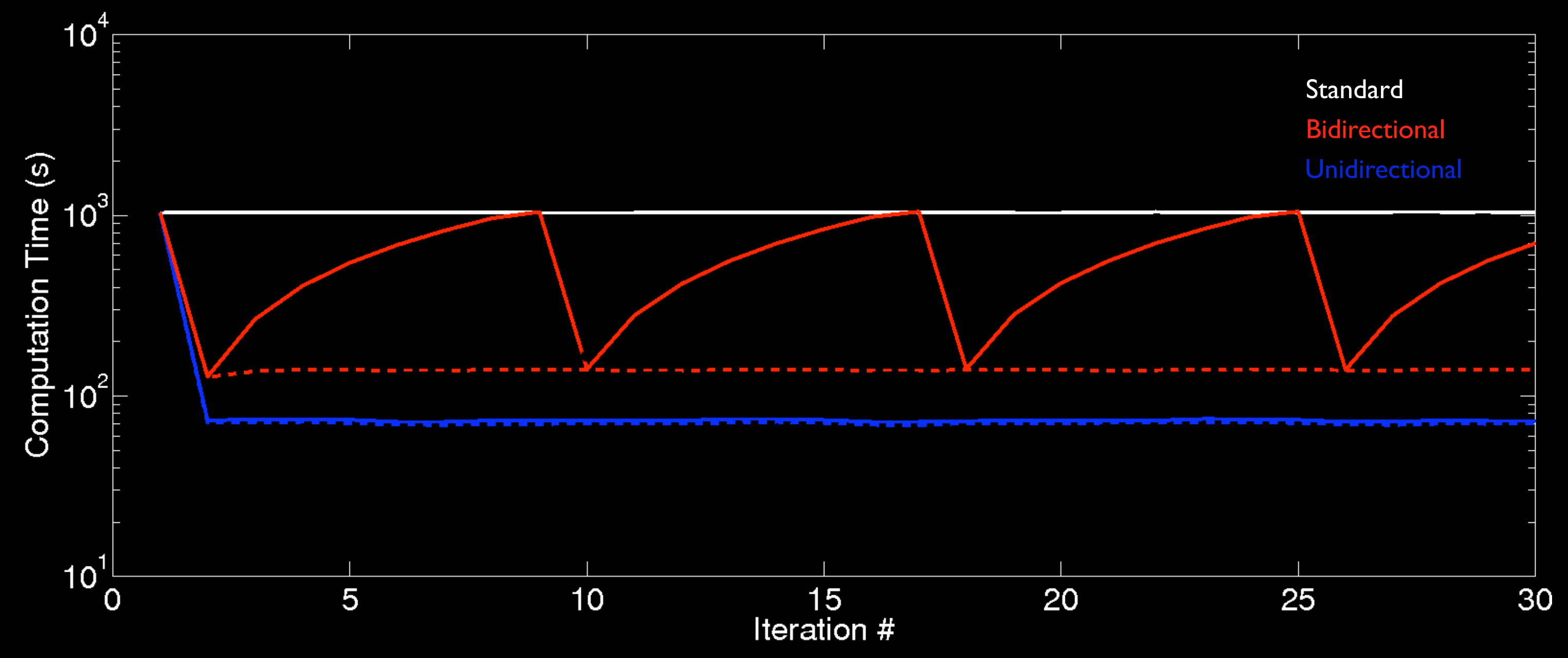
Unidirectional Interpolated Method



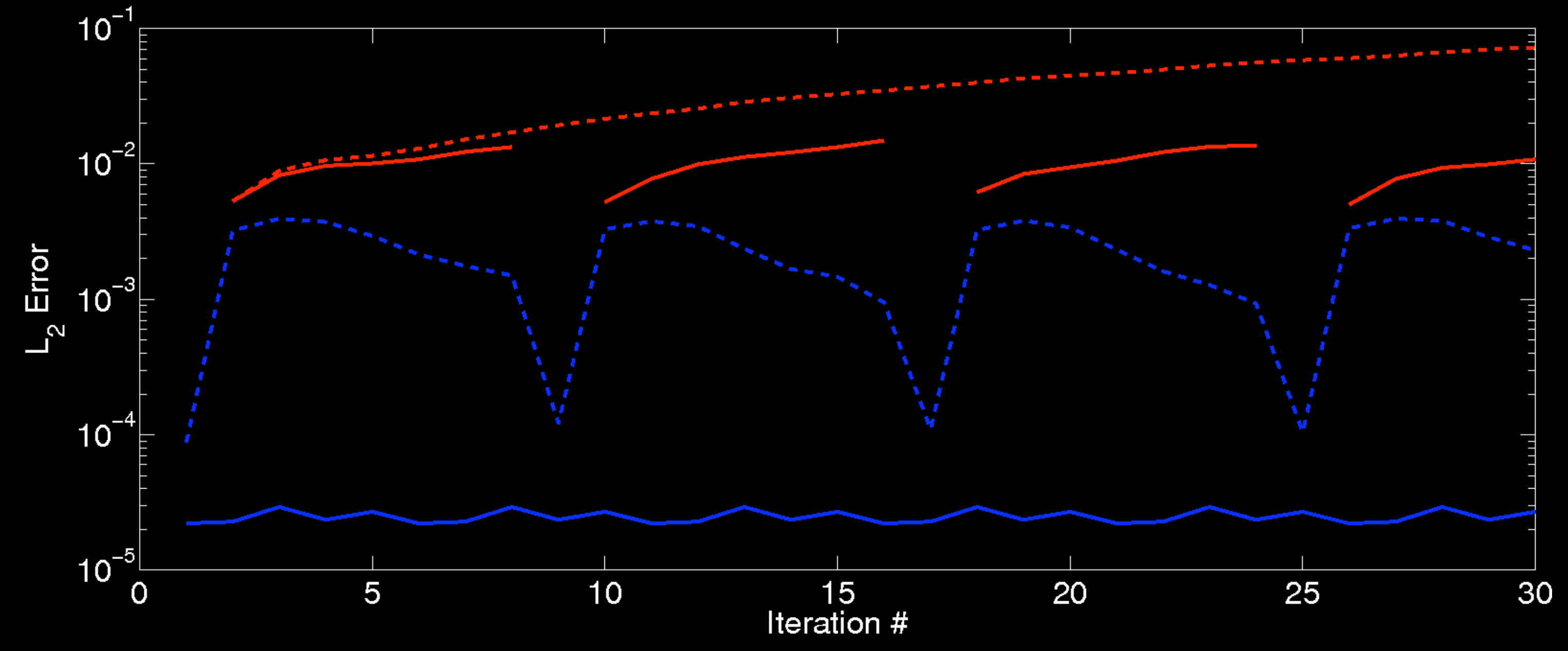
$$\Phi_0^{kh} = \Phi_{(k-1)h}^{kh} \circ \ldots \circ \Phi_h^{2h} \circ \Phi_0^h$$

Comparison of Methods

Computation Time

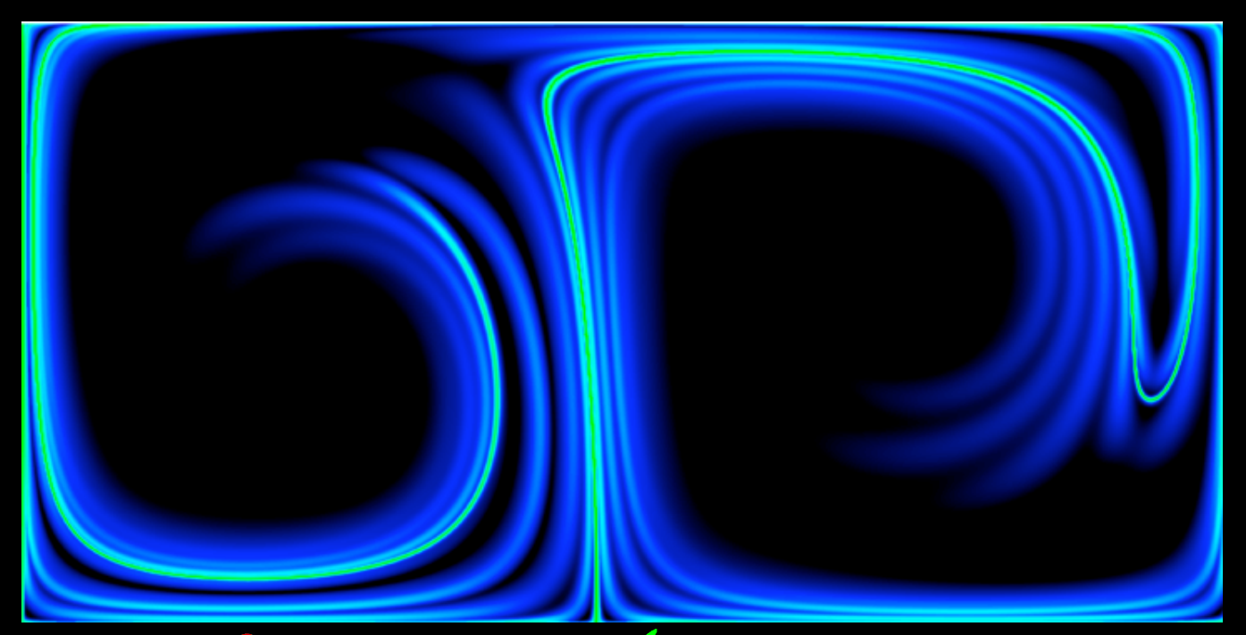


Error



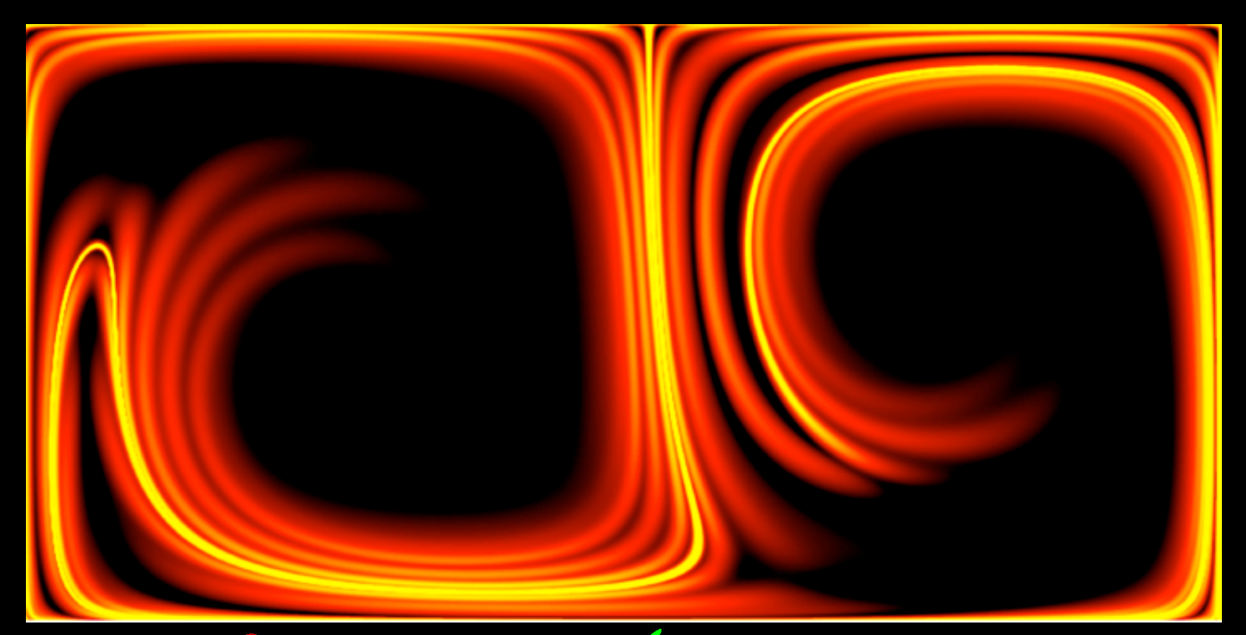
Graphical Comparison

Exact forward-time FTLE



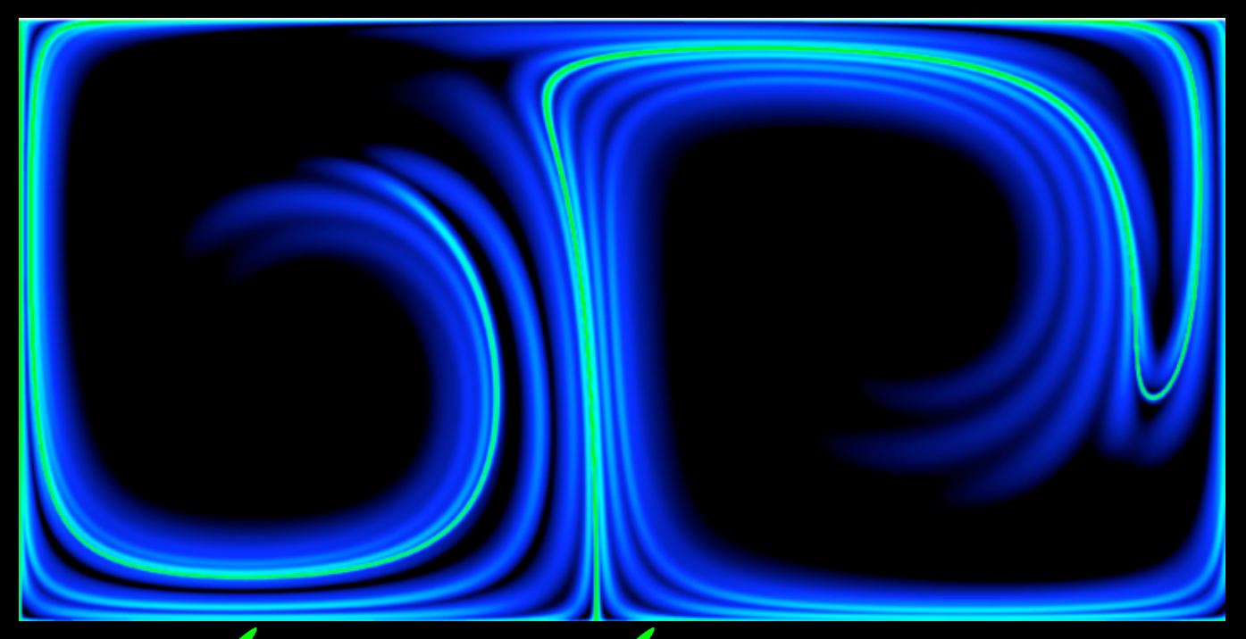
Fast: ☒ Accurate: ☑

Exact backward-time FTLE



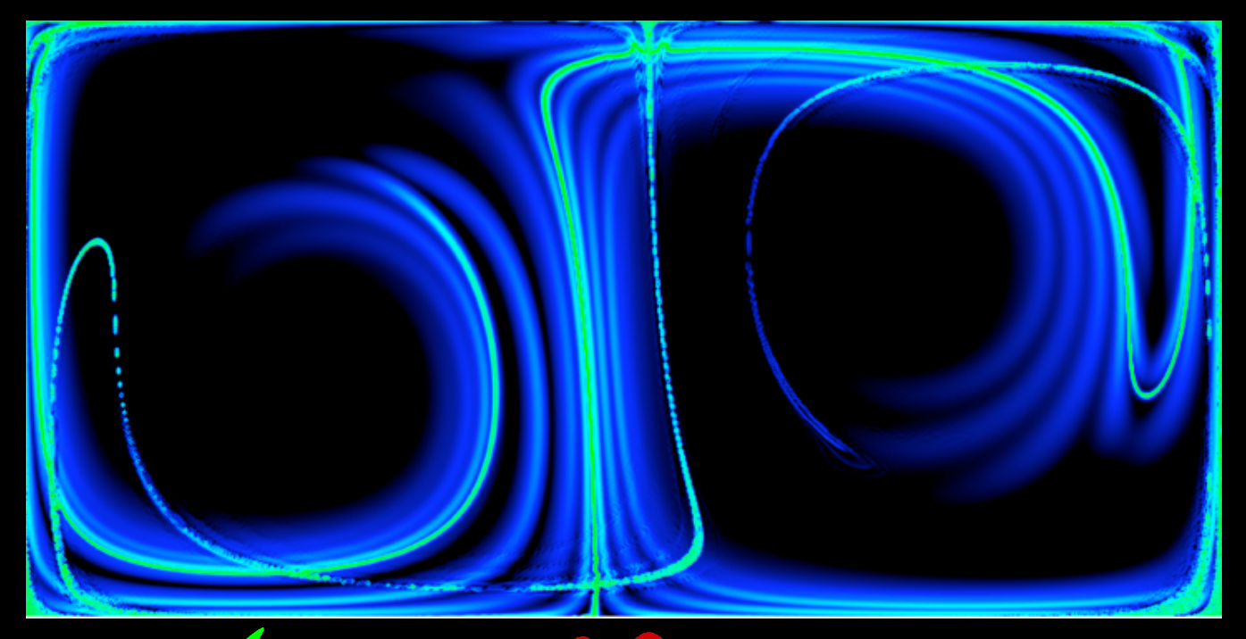
Fast: ☒ Accurate: ☑

Unidirectional



Fast: ☑ Accurate: ☑

Bidirectional



Fast: ☑ Accurate: ☒

Results

The unidirectional interpolated method is the only algorithm which is both fast and accurate, achieving an order of magnitude speed-up. The bidirectional method is fast, but significant error is introduced when particles are integrated backward in time and then interpolated forward through a reference flow map. The bidirectional error is large where the opposite-time FTLE field has large magnitude.

References

Haller, G., "Lagrangian coherent structures from approximate velocity data." *Physics of Fluids*, **14**:6, 2002.

Shadden, S.C., Lekien, F., and Marsden, J.E., "Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimenional aperiodic flows." *Physica D*, **212**:34, 2005.

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