403 Midterm Take home

1) The subgroup lattice of $\mathbb{Z}_6 \times \mathbb{Z}_3$ is shown below.

[4] a) Identify what kind of algebraic object $\mathbb{Z}_6 \times \mathbb{Z}_3$ is

Abelian Group? Yes? Recall the group structure is defined coordinate-wise

So $(0,0)$ is the additive identity (since $0 \in \mathbb{Z}_6$ and $0 \in \mathbb{Z}_3$ are the add. identities)

For $(x,s) \in \mathbb{Z}_6 \times \mathbb{Z}_3$, $-x$ & $-y$ are the additive inverses in $\mathbb{Z}_6$ & $\mathbb{Z}_3$ respectively

$\Rightarrow (-x,-s)$ is the additive inverse in $\mathbb{Z}_6 \times \mathbb{Z}_3$

Closure is inherited from $\mathbb{Z}_6$ and $\mathbb{Z}_3$ coordinate-wise

Commutativity is similarly inherited from $\mathbb{Z}_6$ and $\mathbb{Z}_3$

Ring? Yes? From DHW2 Q§16.7 #36 we get a ring structure if we use coordinate wise multiplication... infact that problem also could have been cited for abelian group structure...

Commutative Ring? Yes? Multiplication is commutative in $\mathbb{Z}_6$ and $\mathbb{Z}_3$ so $\forall\, x,y \in \mathbb{Z}_6$ and $s,t \in \mathbb{Z}_3$, $(x,s)\cdot(y,t) = (xy, st) = (yx, ts)$
$= (y,t)(x,s)$

Ring with identity? Yes? Note $1 \in \mathbb{Z}_6$ and $1 \in \mathbb{Z}_3$ are identities in their respective rings, so $\forall (x,s) \in \mathbb{Z}_6 \times \mathbb{Z}_3$ $(x,s)(1,1) = (x,s)$ $\therefore (1,1)$ is id

Division ring? No. Note $(2,0) \neq (0,0)$ and we claim $\nexists\, (x,s) \in \mathbb{Z}_6 \times \mathbb{Z}_3$ $\ni (2,0)\cdot(x,s) = (1,1)$.

Note if $(2,0)\cdot(x,s) = (1,1) \Rightarrow 2x \equiv 1 \bmod 6$ and $0 \cdot s \equiv 1 \bmod 3$

Neither of these equations have solutions. Examine the mult. table in $\mathbb{Z}_6$ & $\mathbb{Z}_3$

x	0	1	2	3	4	5
2	0	2	4	0	2	4

x	0	1	2
0	0	0	0

Integral Domain? No. Note $(1,0)\cdot(0,1) = (0,0)$ but neither $(0,1)$ nor $(1,0)$ are $(0,0)$. Thus we have zero divisors.

UFD? No. recall UFD's are integral domains, since $\mathbb{Z}_6 \times \mathbb{Z}_3$ is not an Integral domain, $\mathbb{Z}_6 \times \mathbb{Z}_3$ is not a UFD.

PID? No. recall PID are integral domains, Since $\mathbb{Z}_6 \times \mathbb{Z}_3$ is not an Integral domain, $\mathbb{Z}_6 \times \mathbb{Z}_3$ is not a PID.

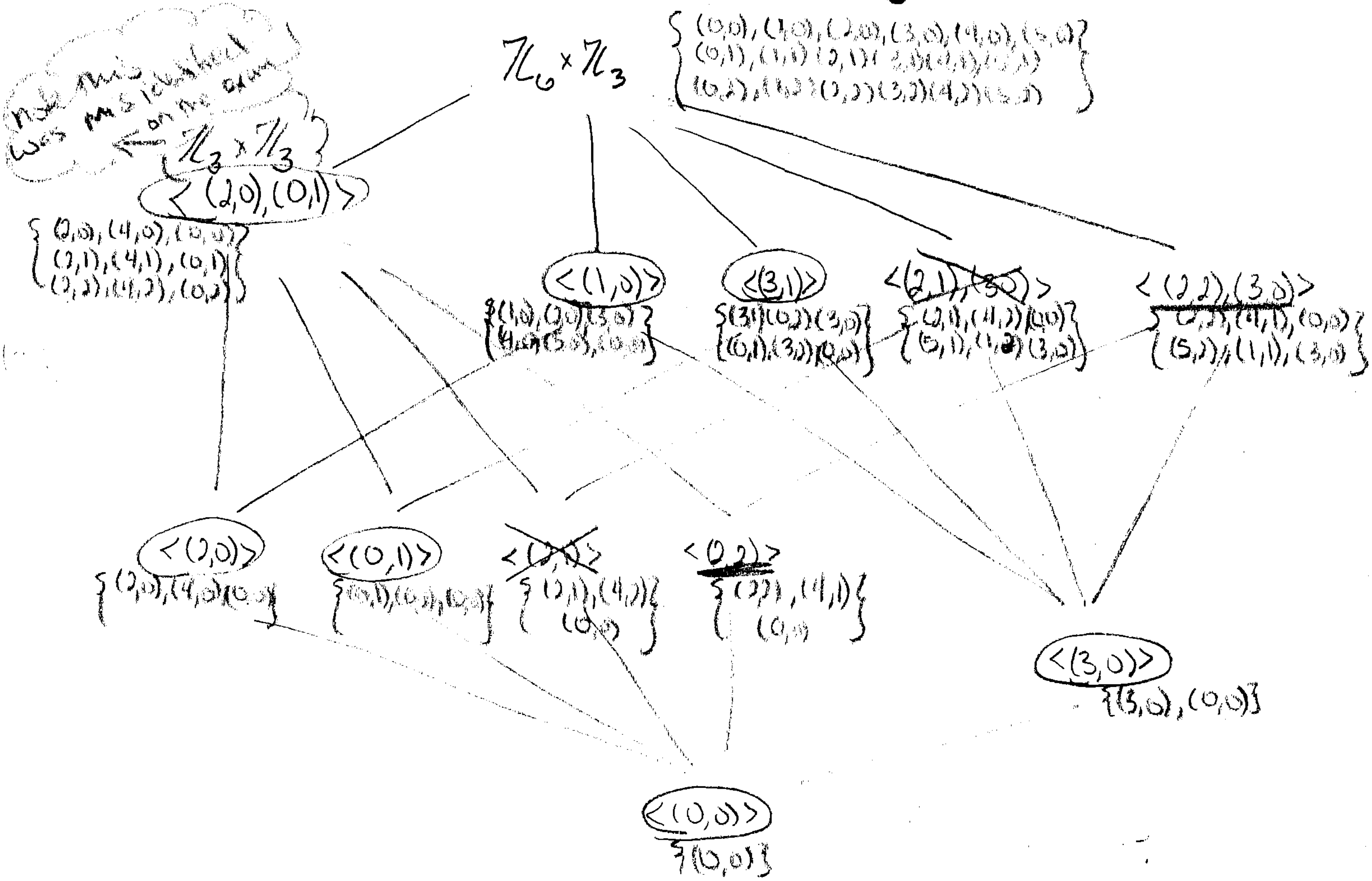
ED? No. Same argument made for UFD + PID's

Field? No. Same argument made for UFD + PID's.

[2]b) Circle any subgroups that are also ideals

[2]c) Underline any subgroups that are subrings but not ideals

[2]d) Cross out any subgroups that are not subrings



[5]e) Identify a Quotient Ring of $\mathbb{Z}_6 \times \mathbb{Z}_3$ that is a field.
Write down the elements of the quotient ring + give the addition + mult. tables

There are many correct answers here – we need to make sure we use a maximal ideal. So we could choose any one of: $\langle(1,0)\rangle$, $\langle(3,1)\rangle$ or $\langle(2,0),(0,1)\rangle$

$\mathbb{Z}_6 \times \mathbb{Z}_3 / \langle(1,0)\rangle \cong \mathbb{Z}_3$ or $\mathbb{Z}_6 \times \mathbb{Z}_3 / \langle(2,0),(0,1)\rangle \cong \mathbb{Z}_2$

$[a,b] \longmapsto b$

representatives/elements

$(0,0) + \langle(1,0)\rangle$

$(0,1) + \langle(1,0)\rangle$

$(0,2) + \langle(1,0)\rangle$

addition & mult tables look like $\mathbb{Z}_3$

representatives/elements

$(0,0) + I$

$(1,1) + I$

where $I = \langle(2,0),(0,1)\rangle$

+	(0,0)+I	(1,1)+I
(0,0)+I	(0,0)+I	(1,1)+I
(1,1)+I	(1,1)+I	(0,0)+I

×	(0,0)+I	(1,1)+I
(0,0)+I	(0,0)+I	(0,0)+I
(1,1)+I	(0,0)+I	(1,1)+I