note: vector spaces should have shown up

Key

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TMath 403

Practice

This exam is intended for practice. The actual exam may look nothing like it.

True/False: If the statement is false, give a counterexample.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. A subring of a field, is a field. False $\mathbb{Z}$ in $\mathbb{R}$

note: $1 \in \mathbb{Z}$ so $\mathbb{Z} \neq \emptyset$

- $\mathbb{Z}$ is an abelian group under $+$
- $\mathbb{Z}$ is closed under multiplication

$\therefore$ $\mathbb{Z}$ is a subring

However, $\mathbb{Z}$ is not a field. In particular, 2 has no mult. inverse.
Notice $\forall x \in \mathbb{Z}$, $2x$ is even but 1 is odd. $\therefore \nexists x \in \mathbb{Z} \ni x2 = 1$.

2. Let R be a ring and $r \in R$. The subgroup generated by r is the same as the ideal $\langle r \rangle$.

False: Consider $R = \mathbb{Z}[x]$,

Notice $1 \in \mathbb{Z}[x]$ and as a subgroup 1 generates $\{a_0 \mid a_0 \in \mathbb{Z}\}$
since 1 generates $\mathbb{Z}$ under addition.

However $\langle 1 \rangle$ as an ideal is all of $R = \mathbb{Z}[x]$.
Recall as ideals $\langle 1 \rangle = \{1 \cdot x \mid x \in \mathbb{Z}[x]\}$.
Since 1 is the mult. identity $\langle 1 \rangle = \mathbb{Z}[x]$ as an ideal.

3. $\mathrm{GL}_2(\mathbb{R}) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc \neq 0\}$ is a division ring.

~~True~~ False

Addition defined by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{bmatrix}$

Inherits it's associativity and commutativity from $\mathbb{R}$.

Note addition remains ~~in $GL_2(\mathbb{R})$~~ b/c

$(a+\alpha)(d+\delta) - (b+\beta)(c+\gamma) = (ad + a\delta + \alpha d + \alpha\delta) - (bc + b\gamma + \beta c + \beta\gamma)$

$= (ad - bc) + (\alpha\delta - \beta\gamma) + a\delta + \alpha d - b\gamma + \beta c$

not closed under addition?

eg $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin GL_2(\mathbb{R})$

in $GL_2(\mathbb{R})$ in $GL_2(\mathbb{R})$

Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

4. For each square below provide an example that satisfies the condition given in the row *and* the condition in the column, if possible. If not possible, briefly explain why.

	not a Commutative Ring	not a Field	not a PID
Commutative Ring	not possible the row & col are in direct contradiction with each other.	$\mathbb{Z}_{12}$	$\mathbb{Z}[x]$ with $\langle x, 2\rangle$
Integral Domain	not possible b/c ID's definition requires a Commutative Ring	$\mathbb{Z}$	$\mathbb{Z}[x]$ with $\langle x, 2\rangle$

5. Find a unit in $\mathbb{Z}_6[x]$. Justify your answer. find $x \in \mathbb{Z}_6[x] \ni \exists y \in \mathbb{Z}_6[x] \ni xy = 1$

$(2x+1)(3x+1)$

$\cancel{6x^2} + 2x + 3x + 1$

$5x+1$

want to find $(ax+1)(3x+1) = 1$

$\Rightarrow 3a \equiv 0 \bmod 6 \Rightarrow 2 \mid a$

$\Rightarrow 3+a \equiv 0 \bmod 6 \Rightarrow a = 3$

trying odd element

$(ax+1)(3x^2+2x+1)$

x^3: $a \cdot 3 \equiv 0 \bmod 6 \Rightarrow 2 \mid a$

x^2: $3 + 2a \equiv 0 \bmod 6 \Rightarrow a$ is odd

trying odd element

$(ax+5)(3x^2+3x+1)$

x^3: $3a \equiv 0 \bmod 6$

x^2: $3+a \equiv 0 \bmod 6$

x: $3+a \equiv 0 \bmod 6$

$5 \cdot 5 = 1$

6. Given that 3 is a root, factor $5x^4 + 2x^2 - 3$ in $\mathbb{Z}_7[x]$ into irreducible elements.

3 is a root $\Rightarrow$ $x-3$ divides $5x^4 + 2x^2 - 3$, so $5x^4 + 2x^2 - 3 = (x-3)(5x^3 + x^2 + 5x + 1)$

$$\begin{array}{r} 5x^3 + x^2 + 5x + 1 \\ x-3 \,\big)\, 5x^4 + 0x^3 + 2x^2 + 0x - 3 \\ -(5x^4 - x^3) \\ x^3 + 2x^2 \\ -(x^3 - 3x^2) \\ 5x^2 + 0x \\ -(5x^2 - 1x) \\ x - 3 \end{array}$$

there are more roots/factors?

roots of $5x^3 + x^2 + 5x + 1$ $\quad x = 0, 1, 2, 3, 4, 5, 6$

$$\begin{array}{r} 5x^2 + 5 \\ x-4 \,\big)\, 5x^3 + x^2 + 5x + 1 \\ -(5x^3 - 20x^2) \\ 21x^2 \;\; 15x + 1 \\ -(5x - 20) \\ 21 \Rightarrow 0 \bmod 7 \end{array}$$

$(x-3)(x-4)(5x^2+5)$

$x = 0, 1, 2, 3, 4, 5, 6$

$5(x-3)(x-4)(x^2+1)$

7. Let $D = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$. Let F_D be the field of fractions of D.

(a) Let I be the ideal generated by $\sqrt{2}$. Identify the ring D/I up to isomorphism and prove the isomorphism.

$I = \langle\sqrt{2}\rangle = \{(\sqrt{2})x \mid x \in \mathbb{Z}[\sqrt{2}]\}$ notice $\sqrt{2}(x + y\sqrt{2}) = x\sqrt{2} + 2y$

$= \{2y + x\sqrt{2} \mid x, y \in \mathbb{Z}\}$

Think this is $\cong \mathbb{Z}_2$

$D \xrightarrow{\psi} \mathbb{Z}_2$

$a + b\sqrt{2} \longmapsto a \bmod 2$

$\ker \psi = \{a + b\sqrt{2} \mid a \equiv 0 \bmod 2; b \in \mathbb{Z}\}$

$= \{2c + b\sqrt{2} \mid c, b \in \mathbb{Z}\} = I$

$D/\ker\psi = D/I$

1st isomorphism $\Rightarrow$ we have an isomophism between D/I & $\mathbb{Z}_2$

(b) Identify two representatives of the same element in F_D.

$[1 + 0\sqrt{2}, 1] = [2 + 0\sqrt{2}, 2]$ b/c $(1 + 0\sqrt{2})2 = 1 \cdot (2 + 0\sqrt{2})$

thinking of $\left(\frac{1 + 0\sqrt{2}}{1} = \frac{2 + 0\sqrt{2}}{2} \right)$

(c) Show addition is well defined in F_D

let $a, b, \alpha, \beta \in D \ni [a, b] = [\alpha, \beta]$ $(*)$

and $c, d, \gamma, \delta \in D \ni [c, d] = [\gamma, \delta]$. $(☺)$

We need to verify $[a, b] + [c, d] = [\alpha, \beta] + [\gamma, \delta]$.

By def of addition it suffices to show $[ad + bc, bd] = [\alpha\delta + \beta\gamma, \beta\delta]$

By def of the equiv. relation it suffices to show $(ad + bc)\beta\delta = bd(\alpha\delta + \beta\gamma)$

or $ad\beta\delta + bc\beta\delta = bd\alpha\delta + bd\beta\gamma$.

Note $(*)$ implies $a\beta = b\alpha$ and $(☺)$ implies $c\delta = d\gamma$. So

$ad\beta\delta + bc\beta\delta = (a\beta)(d\delta) + (c\delta)(b\beta)$ b/c D is commutative & assoc

$= b\alpha d\delta + d\gamma b\beta$ by $(*)$ & $(☺)$

$= bd\alpha\delta + bd\beta\gamma$ by commutivity which is what we wanted to show

(d) Build a ring homomorphism between D and F_D.

$\varphi : \mathbb{Z}[\sqrt{2}] \longrightarrow F_D$

$a + b\sqrt{2} \longmapsto [a + b\sqrt{2}, 1]$

note φ preserves addition b/c $\varphi((a + b\sqrt{2}) + (\alpha + \beta\sqrt{2})) = \varphi((a + \alpha) + (b + \beta)\sqrt{2})$

$= [(a + \alpha) + (b + \beta)\sqrt{2}, 1]$

$= [a + b\sqrt{2}, 1] + [\alpha + \beta\sqrt{2}, 1]$

$= \varphi(a + b\sqrt{2})\ \varphi(\alpha + \beta\sqrt{2})$

note φ preserves mult. b/c $\varphi((a + b\sqrt{2})(\alpha + \beta\sqrt{2})) = \varphi((a\alpha + 2b\beta) + (b\alpha + a\beta)\sqrt{2})$

need more room

$= \varphi(a + b\sqrt{2})\ \varphi(\alpha + \beta\sqrt{2})$

8. [8] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

Theorem 1. *Let $f : R \to S$ be a ring homomorphism between commutative rings. If f is surjective, and I is an ideal of R, show that $f(I)$ is an ideal of S.*

Theorem 2. *Let R be a ring with a multiplicative identity. If x is a zero divisor then x is not a unit.*

Thm 1: Let $f: R \to S$ be a ring homomorphism. Recall a ring homomorphism is also a group homomorphism on $+$ and since I is a subgroup, $f(I)$ is also a subgroup (from MTH 410).

Recall also that the image of a ring is also a ring as f preserves multiplication.

It remains to show $s \cdot f(I) \in f(I)$ $\forall s \in S$.

Note that f is surjective so $\exists r \in R \ni f(r) = s$. Thus

$s \cdot f(I) = f(r) f(I)$

$= f(rI)$ since f preserves multiplication

$= f(I)$ since I is an ideal in R.

Thus $s \cdot f(I) \in f(I)$ //

Thm 2:

Let $x \in R$ be a zero divisor. Then $x \neq 0$ and $\exists y \neq 0$ in R so that $x \cdot y = 0$. (*)

Assume towards contradiction that x is a unit in R. Then $\exists x^{-1} \in R$ so that $x^{-1} \cdot x = 1$.

Return to (*): $x \cdot y = 0$

Apply x^{-1} to both sides: $x^{-1} x y = x^{-1} 0$

Reassociate & simplify $(1) y = 0$

$\Rightarrow y = 0$

which is a contradiction.

$\therefore$ x is not a unit. //