Key

Vanderpool

TMath 403

Winter 2023

True/False: If the statement is false, give a counterexample.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] A subring of a field, is a field.

(.5) false $\mathbb{Z}$ is a subring of $\mathbb{R}$ but $\mathbb{Z}$ is not a field.

start (.5) In particular $2 \in \mathbb{Z}$ has no multiplicative inverse.

def of subring (.5) def of field (.5) Note if $x \in \mathbb{Z}$ then $2x$ is even $\forall x \in \mathbb{Z}$. So $2x \neq 1$

found counterex (1) for any x in $\mathbb{Z}$ $\Rightarrow$ 2 has no mult. inverse.

2. [3] The map $\phi : \mathbb{Z}_4 \to \mathbb{Z}_{12}$ defined by $\phi(x) = 3x$ is a ring homomorphism.

(.5) false Recall ring homomorphisms must maintain mult. structure

start (.5) def of ring homom (1) found counterex (1)

However $\phi(1 \cdot 3) = \phi(3) = 3 \cdot 3 = 9$

and $\phi(1)\phi(3) = (3 \cdot 1)(3 \cdot 3) = 3 \cdot 9 = 27 \equiv 3 \mod 12$

So $\phi(1 \cdot 3) \neq \phi(1)\phi(3)$

3. [3] The set $\{(x, y, z) \in \mathbb{R}^3 | x - 2y + 2z = 0\}$ is a (vector) subspace.

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(.5) true

start (.5) def of subspace/prop (.5) smsc (1.5)

We verify:

3) (If $(x,y,z) \in V$ and $\alpha \in \mathbb{R}$ then $\alpha(x,y,z) \in V$.)

Since $(x,y,z) \in V$ we know $x - 2y + 2z = 0$

Consider $\alpha(x,y,z)$ or $(\alpha x, \alpha y, \alpha z)$

Notice $\alpha x - 2(\alpha y) + 2(\alpha z)$
$= \alpha(x - 2y + 2z) = \alpha \cdot 0$

So $\alpha(x,y,z) \in V$.

1) (non empty) note $(0,0,0) \in V$

2) (If $(x,y,z), (a,b,c) \in V$ then $(x,y,z) - (a,b,c) \in V$.)

Since $(x,y,z), (a,b,c) \in V$ we know $x - 2y + 2z = 0$ and $a - 2b + 2c = 0$

Consider $(x,y,z) - (a,b,c)$ or $(x-a, y-b, z-c)$.

Notice $(x-a) - 2(y-b) + 2(z-c)$
$= (x - 2y + 2z) - (a - 2b + 2c) = 0$

So $(x,y,z) - (a,b,c) \in V$

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4. [5] Place the following algebraic objects into a square below so that the object satisfies the condition given in the row *and* the condition in the column. Note, there maybe more than one square that the object fits in, but you need only place each object once.

$\mathbb{Z}$ (+1) $\quad \mathbb{Z}_{12}$ (+1) $\quad M = \text{Mat}_2(\mathbb{R}) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c, d \in \mathbb{R}\}$ (+1) $\quad \mathbb{Z}_6 \times \mathbb{Z}_3$ (+1) $\quad \mathbb{Z}[i]$ (+1) ID not ED

	not Commutative	not a Field	not a PID
Ring	M	$\mathbb{Z}$ $\mathbb{Z}_6$ M $\mathbb{Z}_6 \times \mathbb{Z}_3$, $\mathbb{Z}[i]$	$\mathbb{Z}_6 \times \mathbb{Z}_3$ $\langle (1,0),(0,1) \rangle$ $\mathbb{Z}[i]$ $\langle i, 2 \rangle$
Integral Domain		$\mathbb{Z}$ $\quad \mathbb{Z}[i]$	$\mathbb{Z}[i]$

5. Find an example for each of the following in $\mathbb{Z}[x]$, if possible. No proofs are necessary. Be careful of notation!!

note there are lots of correct answers for these

(a) [1] a principal ideal

$\langle x \rangle = \{ x \cdot f(x) | f(x) \in \mathbb{Z}[x] \}$ so all polynomials w/ deg equal to or higher than 1

notation (.5) got one (.5)

(b) [2] an ideal that is not principal

$\langle x, 2 \rangle = \{ x f(x) + 2 \cdot g(x) | f(x), g(x) \in \mathbb{Z}[x] \}$

so all polynomials with an even constant term

notation (.5) got one (1) understood def (.5)

(c) [2] a maximal ideal

$\langle x, 2 \rangle$ will work

notation (.5) got one (1) understood def (.5)

(d) [2] a factor ring of ~~R~~ $\mathbb{Z}[x]$ that is also a field

$\mathbb{Z}[x] / \langle x, 2 \rangle$

(1) mod out by max ideal
(.5) got one
(.5) notation

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Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

6. [3] Given that 4 is a root, factor $x^4 + 4x^3 + x^2 + 2$ in $\mathbb{Z}_5[x]$ into irreducible elements.

(+.5) 4 is a root => (x-4) divides so " $(x-4)(x^3+3x^2+3x+2)$

$$\begin{array}{r} x^3+3x^2+3x+2 \\ x-4 \,\overline{)\, x^4+4x^3+x^2+2} \\ -(x^4-4x^3) \\ 3x^3+x^2+2 \\ -(3x^3-2x^2) \\ 3x^2+2 \\ -(3x^2-12x) \\ 2x+2 \\ -(2x-8) \;\; R0 \end{array}$$

long: (+1)

(+1) not irreducible? $\cancel{0}, \cancel{1}, \cancel{2}, 3, \cancel{4}$ — 3 is a root => x-3 divides

$$\begin{array}{r} x^2+x+1 \\ x-3 \,\overline{)\, x^3+3x^2+3x+2} \\ -(x^3-3x^2) \\ x^2+3x+2 \\ -(x^2-3x) \\ x+2 \\ -(x-3) \\ 5 \equiv 0 \end{array}$$

So (+.5)

$(x-4)(x-3)(x^2+x+1)$

note has no roots in $\mathbb{Z}_5$ so irreducible

$\cancel{0}, \cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}$

7. [8] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

Theorem 1. *Let ϕ be a ring homomorphism from a ring R to a ring S. Let B be an ideal of S. The set $\phi^{-1}(B)$ defined as $\{r \in R | \phi(r) \in B\}$ is an ideal of R.*

Theorem 2. *Let F be a field. Show that the field of quotients of F is ring-isomorphic to F.*

Thm 1: We need to verify:

1) $\phi^{-1}(B)$ is an abelian group.
 a) note ϕ is a group homomorphism so $\phi(0) = 0 \in S$. Since B is an ideal, B is an abelian group => $0 \in B$. Thus $0 \in \phi^{-1}(B)$ so $\neq \emptyset$.
 b) Let $x, y \in \phi^{-1}(B)$. Since R is an abelian group $-y \in R$
 Consider $\phi(x-y) = \phi(x) - \phi(y)$. Since B is an abelian group $\phi(x)-\phi(y) \in B$
 c) We inherit abelian property b/c R is an abelian group under +

2) $\phi^{-1}(B)$ is a ring
 a) let $x, y \in \phi^{-1}(B)$ Notice $\phi(xy) = \phi(x)\phi(y) \in B$ b/c B is closed under multiplication

3) $\phi^{-1}(B)$ "absorbs"
 Let $\alpha \in R$ and $r \in \phi^{-1}(B)$. Consider $\phi(\alpha r) = \phi(\alpha)\phi(r)$.
 Since $r \in \phi^{-1}(B)$ $\phi(r) \in B$ and B is an ideal so $s\phi(r) \in B$ $\forall s$ => $\phi(\alpha)\phi(r) \in B$. Thus $\alpha r \in \phi^{-1}(B)$.

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Thm 2: Let F be a field. Show that the field of quotients of F is ring isomorphic to F.

Pf: Let $\mathcal{F}$ be the field of quotients of F, so

$$\mathcal{F} = \{(x,y) \mid x,y \in F,\ y \neq 0\}/\sim \quad \text{where } (a,b)\sim(x,y) \text{ if } ay = bx.$$

Define $\psi : F \to \mathcal{F}$

$$x \longmapsto (x,1)$$

Claim: ψ is a ring homomorphism. Let $a, b \in F$ then

$$\psi(a+b) = (a+b, 1) = (a\cdot 1 + b\cdot 1, 1\cdot 1)$$
$$= (a,1) + (b,1) \quad \text{by def of addition in } \mathcal{F}$$
$$= \psi(a) + \psi(b)$$

$$\psi(a\cdot b) = (a\cdot b, 1) = (a\cdot b, 1\cdot 1)$$
$$= (a,1)\cdot(b,1) \quad \text{by def of mult. in } \mathcal{F}$$
$$= \psi(a)\psi(b)$$

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We now need to verify one-to-one and onto.

One-to-one: Let $a, b \in F \ni \psi(a) = \psi(b)$. We WTS $a = b$.

Since $\psi(a) = \psi(b) \Rightarrow (a,1) = (b,1)$
$\Rightarrow a\cdot 1 = b\cdot 1 \quad \therefore a = b$ ✓

Onto: Let $(\alpha,\beta) \in \mathcal{F}$, we need to construct $x \in F \ni \psi(x) = (\alpha,\beta)$.

Note $\beta \neq 0$ by def of $\mathcal{F}$. Since F is a field $\exists\, \beta^{-1} \in F$.

Consider $(\alpha\beta^{-1}, 1)$. Notice $\alpha\beta^{-1}\cdot\beta = \alpha$
$\therefore (\alpha\beta^{-1}, 1) = (\alpha,\beta)$

Since F is a field $\alpha\beta^{-1} \in F$. Notice $\psi(\alpha\beta^{-1}) = (\alpha\beta^{-1}, 1) = (\alpha,\beta)$ which is what we wanted to show.

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