vanderpool

TMath 403

Key

Winter 2023

True/False: If the statement is false, give a counterexample.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] All subgroups in a ring R, are ideals.

(+.5) False, Consider $\mathbb{Z}$ in $\mathbb{R}$.
Note $\mathbb{Z}$ is a subgroup but $r\cdot\mathbb{Z} \not\subseteq \mathbb{Z}$.
For example, let $r = \frac{1}{2}$.

(+.5)
subgroup (+.5)
ideal (+.5)
counter ex (+1)

2. [3] In $\mathbb{Z}_5[x]$, we have $x^5 - x = x(x-1)(x-2)(x-3)(x-4)$.

(+.5) True. Note

x	0	1	2	3	4
x^5-x	0	0	0	0	0

The roots factor thm => then
$x-0, x-1, x-2, x-3, x-4$ are factors of x^5-x
So $x^5-x = (x-0)(x-1)(x-2)(x-3)(x-4)k$ for some $k \in \mathbb{Z}_5[x]$
Since x^5-x is monic $\Rightarrow k=1$, so $x^5-x = x(x-1)(x-2)(x-3)(x-4)$

(+1.5)

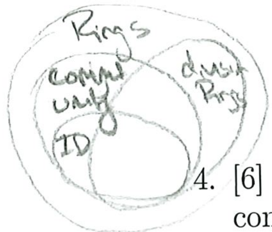
(Note: Fermat's little Thm is also a great way to do this)

(+.5)
Notation (+.5)

3. [3] The minimal polynomial of $\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$ is $x^3 - 1$. in $\mathbb{Q}[x]$.

(+.5) False. A minimal polynomial must be monic and irreducible.
Notice $(1)^3 - 1 = 0 \Rightarrow (x-1)$ is a factor of x^3-1
In fact $x^3 - 1 = (x-1)(x^2+x+1)$.
$\therefore$ x^3-1 is reducible $\Rightarrow$ x^3-1 is not minimal

(+.5)
min poly (+.5)
irreducible (+.5)
factoring (+1)



4. [6] Construct algebraic objects for each rectangle below so that the object satisfies the conditions given in the row *and* the condition in the column, if possible. If not possible, briefly explain why.

	not Commutative	not an Integral Domain	not a
Ring with Unity/ Ring with Multiplicative Identity	$\mathbb{H} = \{a+bi+cj+dk \mid a,b,c,d \in \mathbb{R}\}$ $Mat_{2\times 2}(\mathbb{R})$ (+1)	$\mathbb{Z}_{12}$ note $3 \cdot 4 = 0$ (+1)	$\mathbb{Z}[x]$ $\mathbb{R}[x,y]$
Field	Not possible Field def requires commutivity (+1)	Not possible Field def requires an Integral domain (+1)	Not poss Fields h 2 ideal where principal

5. [4] Explain the connection/describe the theorem between field extensions and constructible numbers (with straight-edge and compass). Consider providing an example to clarify your explanation. No proofs are needed here.

stat (+.5)

(+1) Constructible Numbers satisfy the properties of a field. Direct constructions allow us to build $\mathbb{Z}$, and then $\mathbb{Q}$. The field of Constructible Numbers is thus a field extention

(+1.5) It can be proved a number, α, is constructible if and only if the dimension of the field extension over $\mathbb{Q}$ (so $[\mathbb{Q}(\alpha), \mathbb{Q}] = 2^i$ for $i \in \mathbb{N}$),

sense (+.5) true things (+.5)

6. [3] Let R be a ~~commutative ring with unity~~ integral domain. Outline at least one way to build a field with R. Consider providing an example to clarify your explanation. No proofs are needed here.

stat (+.5)
construction (+1.5)
sense (+.5)
true things (+.5)

Let R be an integral domain. We can build the field of fractions, F_R.

Specifically

$F_R = \{(x,y) \mid x, y \in R\}/\sim$

where $(x,y) \sim (r,s)$ if $xs = yr$.

eg $R = \mathbb{Z}$ then $F_R = F_{\mathbb{Z}} = \mathbb{Q}$

Let R be an integral domain. Let M be a maximal ideal.

Then R/M would be a field.

eg. $R = \mathbb{Z}$
$M = \langle 5 \rangle = \{5s \mid s \in \mathbb{Z}\}$

then $R/M = \mathbb{Z}/\langle 5 \rangle \cong \mathbb{Z}_5$

7. Recall $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}, i^2 = -1\}$. Define $\pi : \mathbb{Z}[i] \to \mathbb{Z}[i]/\langle 2 - i\rangle$ where $\pi(a + bi) = [a + bi]$.

(a) [2] Identify three elements from $\mathbb{Z}[i]$ in the same coset.

def of ideal (+.5)
def of ideal used correctly (+.5)
two elements (+.5)
three elements (+.5)

Note $\langle 2-i\rangle = \{r(2-i) \mid r \in \mathbb{Z}[i]\} = \{(a+bi)(2-i) \mid a,b \in \mathbb{Z}\}$

So 0, $2-i$, $4-2i$, $i(2-i) = 1+2i$, and $(2+i)(2-i) = 4+1 = 5$ are all in the same coset as 0

(b) [3] What do the distinct cosets look like? ie - identify distinct elements $\mathbb{Z}[i]/_{\langle 2-i\rangle}$

start (+.5)
true (+.5)

(+1) Note $[2-i] = [0] \Rightarrow [2] = [i]$
Every element in $\mathbb{Z}[i]$ with i can be swapped w/ a 2.
So at most we have elements that look like $[\theta]$ where $\theta \in \mathbb{Z}$.

(+1) Note from (a), $[5] = [0] \Rightarrow [6] = [1]$ and $[n] = [n-5]$ $\forall n \in \mathbb{Z}$
So at most we have $[0], [1], [2], [3], [4]$

Will take a bit to prove $[i] \cap [j] = \emptyset$ for $i \neq j$ and $i, j \in \{0,1,2,3,4\}$

8. [8] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

Theorem 1. *Let F be a field and let $f(x)$ be a polynomial in $F[x]$ that is reducible over F. Prove that $\langle f(x)\rangle$ is not a prime ideal in $F[x]$. (Recall a proper ideal P in a commutative ring R is prime if $ab \in P$ implies $a \in P$ or $b \in P$.)*

Theorem 2. *Let R be an integral domain. Show that if the only ideals in R are $\{0\}$ and R itself, then R must be a field.*

Thm 1 Proof:
Let $f \in F[x]$ that is reducible over F.
Then $\exists\, g, h \in F[x]$ so that
$f(x) = g(x)h(x)$ where
$\deg(g(x)), \deg(h(x)) < \deg(f(x))$.
We want to show $\langle f(x)\rangle$ is not a prime ideal.
Consider $g(x)h(x)$ which equals $f(x)$ and thus $g(x)h(x) \in \langle f(x)\rangle$.
We claim $g(x), h(x) \notin \langle f(x)\rangle$ and thus $\langle f(x)\rangle$ is not prime.
To see $g(x), h(x) \notin \langle f(x)\rangle$, recall
$\langle f(x)\rangle = \{z f(x) \mid z \in F[x]\}$.
Since x has no mult. inverse, by construction $\deg(z f(x)) \geq \deg(f(x))$ $\forall z \in F[x]$.
However $\deg(g(x)), \deg(h(x)) < \deg(f(x))$
Thus $z f(x) \neq g(x)$ and $z f(x) \neq h(x)$ $\forall z \in F[x]$.
Thus $\langle f(x)\rangle$ is not prime. //

(use Weierstrass abac x2)

Thm 2 Proof:

Let R be an integral domain with the only ideals in R are $\{0\}$ & itself.
We will verify R is a field by verifying the conditions of a field:

1) R forms an abelian group with $+$
2) R has a 2nd binary operator, $\times$
3) Distribution is satisfied from the left and right.
4) R is a commutative ring with unity
5) R is an integral domain
6) R is a division ring

Notice the first 5 conditions are all satisfied with our assumptions, thus we need only verify R is a division ring. More specifically that $\forall x \in R$ with $x \neq 0$, that $\exists x^{-1} \in R \ni x x^{-1} = 1$.

Let $x \neq 0$ and $x \in R$. We will find $x^{-1} \in R$.
Consider $\langle x \rangle = \{ax \mid a \in R\}$. Since $x \neq 0$, $\langle x \rangle \neq \{0\}$.
Recall the only ideals in R are $\{0\}$ and itself, thus $\langle x \rangle = R$.
Since R has $1 \in R$, we know $1 \in \langle x \rangle$.
Thus $\exists a \in R \ni ax = 1$ in R. Notice then a is the mult. inverse to x which is what we were looking for.

Since this argument works $\forall x \neq 0$ in R, we have mult. inverses for all nonzero elements $\Rightarrow$ R is a division ring //

$\therefore$ R is a field.