Vanderpool

TMath 403

Winter 2023

True/False: If the statement is false, give a counterexample.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] All subgroups in a ring R, are ideals.

2. [3] In $\mathbb{Z}_5[x]$, we have $x^5 - x = x(x-1)(x-2)(x-3)(x-4)$.

3. [3] The minimal polynomial of $\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$ is $x^3 - 1$.

4. [6] Construct algebraic objects for each rectangle below so that the object satisfies the conditions given in the row *and* the condition in the column, if possible. If not possible, briefly explain why.

	not Commutative	not an Integral Domain	not a PID
Ring with Unity/ Ring with Multiplicative Identity			
Field			

5. [4] Explain the connection/describe the theorem between field extensions and constructible numbers (with straight-edge and compass). Consider providing an example to clarify your explanation. No proofs are needed here.

6. [3] Let R be a commutative ring with unity. Outline at least one way to build a field with R. Consider providing an example to clarify your explanation. No proofs are needed here.

7. Recall $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}, i^2 = -1\}$. Define $\pi : \mathbb{Z}[i] \to \mathbb{Z}[i]/\langle 2 - i\rangle$ where $\pi(a + bi) = [a + bi]$.

   (a) [2] Identify three elements from $\mathbb{Z}[i]$ in the same coset.

   (b) [3] What do the distinct cosets look like?

8. [8] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
   No, doing both questions will not earn you extra credit.

   Theorem 1. *Let F be a field and let $f(x)$ be a polynomial in $F[x]$ that is reducible over F. Prove that $\langle f(x)\rangle$ is not a prime ideal in $F[x]$. (Recall a proper ideal P in a commutative ring R is prime if $ab \in P$ implies $a \in P$ or $b \in P$.)*

   Theorem 2. *Let R be an integral domain. Show that if the only ideals in R are $\{0\}$ and R itself, then R must be a field.*