Assessment Committee

pg 3 #6b is a solid assessment problem

Key

Final TMath 402 Winter 2026

Recall you may use a two-sided 8.5" by 11" sheet of notes (or surface with equal area) with what ever you like written or typed on it. Additionally, a non-internet accessing calculator, or Desmos Test Mode can be used.

(note: pg 5 #8 was very challenging - they'd never seen anything like it. I'd run in final exam the class collaborated on it.)

Show your work for the following problems to earn full marks.

1. [3] Prove or provide a counterexample for the following statement: Let S_{10} be the permutation group on ten elements. For all $\sigma \in S_{10}$, $\sigma^{10} = ()$, where () is the identity permutation.

False (+1.5)

Counterexample:

Consider $\sigma = (123)(45678) \in S_{10}$

note that the order of σ is $3 \cdot 5 = 15$ so $\sigma^{10} \neq ()$

def of S_{10} (+1)

permutation + raised to 10 (+1)

logic/notation (+1.5)

2. [3] Prove or provide a counterexample for the following statement: We can define a ring structure on the set $R = \{a + b\sqrt{5} | a, b \in \mathbb{Z}\}$.

Proof: (+1.5)

- R has an abelian gp structure with $(a+b\sqrt{5}) + (\alpha+\beta\sqrt{5}) = (a+\alpha) + (b+\beta)\sqrt{5}$ (+1)
  Note the unit is $0+0\sqrt{5}$, inverses $-(a+b\sqrt{5}) = -a-b\sqrt{5}$
  Closed b/c $\mathbb{Z}$ is closed $\checkmark$ $(a+b\sqrt{5})+(\alpha+\beta\sqrt{5}) = (\alpha+\beta\sqrt{5})+(a+b\sqrt{5})$
- multiplicative structure distributes nicely with $(a+b\sqrt{5})\cdot(\alpha+\beta\sqrt{5}) = (a\alpha + b\beta\cdot 5) + (a\beta + b\alpha)\sqrt{5}$ (+1)

logic/notation (+1.5)

3. Consider $\theta : D_4 \to \mathbb{Z}_4$ defined by $\theta(r^k f) = k$.

   (a) [2] If θ is a homomorphism, find $\theta(f)$. Show your computations/justification.

   $\theta(f) = \theta(r^0 f) = 0$ (+1)

   $\theta(f) = \theta(rf\, r^2 f\, rf) = \theta(rf) + \theta(r^2 f) + \theta(rf) = 1 + 2 + 1 = 0 \bmod 4$

   notation (+1.5)

   (b) [2] Can θ be extended to define a homormophism? Justify your answer.

   No? (+1.5)

   We trace the elements to make sure structure is preserved.

   $e \longmapsto 0$ $\quad f \longmapsto 0$

   Scratch ✱ computations (+1)

4. Consider the partial subgroup lattice for $\mathbb{Z}_9 \times \mathbb{Z}_7$ shown to the right.

   (a) [1] Find a generating set for the subgroup denoted with K.

   (b) [2] Find a generating set for the subgroup denoted with H.

   (c) [1] What number should α be replaced with in the order column?

   (d) [1] One edge is missing. Add the missing edge to the subgroup lattice.

   (e) [2] Could $\mathbb{Z}_9 \times \mathbb{Z}_7$ form a field with component-wise addition and multiplication reduced by the appropriate modulo ? Why or why not?

5. Consider a homomorphism $\phi : \mathbb{Z}_{30} \to \mathbb{Z}_{45}$.

   (a) [3] What are the possible sizes for the image of ϕ? Clearly provide justification and reasoning.

   (b) [4] How many different homomorphisms can be defined for ϕ with $\ker(\phi) = \{0, 15\}$.

6. Let $X = \{1, 2, 3, 4, 5\}$ and let S be the set of all permutations on X.

(a) [5] Let $\alpha = (1, 2, 4)$ and $\beta = (1, 5)$. Find the following:

i. α as a product of transpositions

(+1) $\alpha = (124) = (14)(12)$ or $(12)(14)$

(there are several answers for this)

ii. α^{-1}

(+1) $= (421)$ or (241) or (142)

note $(124)(142) = (1)(2)(4) = (142)(124)$

iii. $\alpha \circ \beta$

$(124)(15) = (1245)$ (+1)

Composition in correct direction (+.5)

iv. Does α and β commute? Justify your answer.

$\beta\alpha = (15)(124) = (152)$ (+1) $\neq \alpha\beta$ so do not commute (+.5)

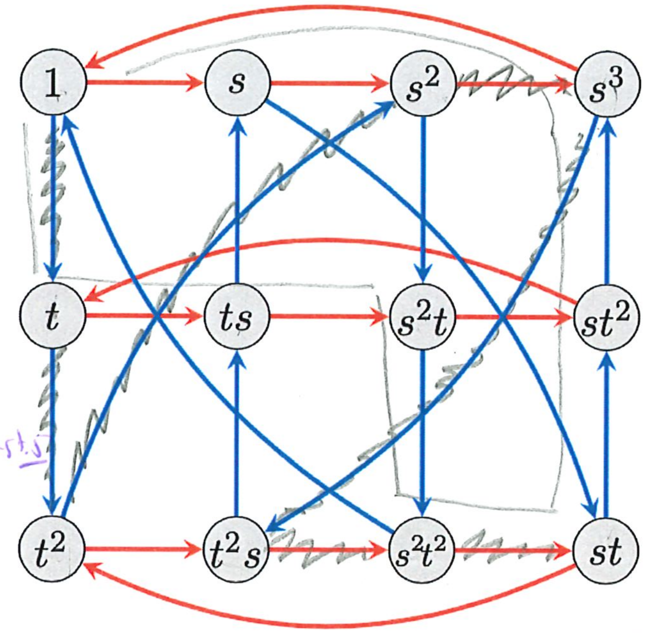
(b) [4] Define a subset $H = \{\sigma \in S | \sigma(3) = 3\}$. That is, all permutations that do not move 3 (or stabilize 3). For example, α from part (a) is in H since 3 is not moved. Provide a formal proof that H is a subgroup of S.

Proof: We will verify H is a group in its own right. Use $H \leq S_5$ so H inherits associativity. We will check for an identity, inverses, and closure.

- Identity: Note that $()$ does not move any elements so 3 is fixed by $()$ $\Rightarrow$ $() \in H$.
- Inverses: Let $\sigma \in H$ then σ keeps 3 in place. Since $\sigma^{-1}\sigma = ()$ and σ does not move 3 we know σ^{-1} must also leave 3 alone. Thus $\sigma^{-1} \in H$.
- Closure: Let $\sigma, \tau \in H$ then consider $\sigma\tau$. Consider $\sigma\tau$ acting on 3. Note σ does not move 3, and then τ does not either thus $\sigma\tau$ does not move 3. $\Rightarrow \sigma\tau \in H$

Proof correct (+4)

7. Use the provided Cayley graph of a group G to answer the following:



(a) [1] How many generators are used in the Cayley graph?

2

(b) [2] Simplify $ts^2tst^2st^3sts^2$ to a form on the Cayley graph.

(+1) follow the arrows   identify any identities/relations on the Cayley graph   (+1) st

(c) [2] Find the inverse of $ts^2tst^2st^3sts^2$

(+1) [we need a path back from st to 1

(+1) [tts works   note there are multiple answers ?

(d) [2] Find the left cosets of $\langle t\rangle$

$\langle t\rangle = \{t, t^2, s^2, s^2t, s^2t^2, 1\}$ (+.5)

$s\langle t\rangle = \{st, st^2, s^3, s^3t, s^3t^2, s\}$
or $\{st, st^2, s^3, t^2s, ts, s\}$] (+1)

(+.5) [note b/c $|G| = 12$ and $|\langle t\rangle| = 6$ there are only two / have all elements

(e) [2] Is there a subgroup H in G that is normal in G? Provide some justification for your answer.

(+.5) Yup $\langle t\rangle$ works

Note $[G:\langle t\rangle] = 2$ and we had a Written HW proof showing if the index is 2 then $\langle t\rangle \trianglelefteq G$.

definition of normal (+.5)
used def right (+1)

Also we could just see

$\langle t\rangle s = \{ts, t^2s, s^3, s^2ts, s^2t^2s, s\}$
$= \{ts, t^2s, s^3, st^2, st, s\} = s\langle t\rangle$ ↳ checking the

defined where variables see from (1)

abelian def (1,5)
def of cyclic (1)
in factor group (1)
center / Z(G) (1,5)

8. [8] Recall the center of G is a normal subgroup defined as $Z(G) = \{z \in G | zg = gz \forall g \in G\}$. Prove formally that if $G/Z(G)$ is cyclic, then G is abelian.

Proof (1+4) (work)

Pf: Let $a, b \in G$. We will show $a \cdot b = b \cdot a$,

Consider $aZ(G) \in G/Z(G)$ and $bZ(G) \in G/Z(G)$

b/c $G/Z(G)$ is cyclic $\exists\ x \in G \ni \langle xZ(G) \rangle = G/Z(G)$

That is $\exists\ n, m \in \mathbb{Z} \ni$

$aZ(G) = x^n Z(G)$ and $bZ(G) = x^m Z(G)$.

That is $\exists$ elements $\bar{\alpha}, \hat{\alpha}$ and $\bar{\beta}, \hat{\beta} \in Z(G) \ni$

$a\bar{\alpha} = x^n \hat{\alpha}$ and $b\bar{\beta} = x^m \hat{\beta}$

Since G is a group we can left compose with $\bar{\alpha}^{-1}$ and $\bar{\beta}^{-1}$ on the respective equations to get

$a = x^n \hat{\alpha}\bar{\alpha}^{-1}$ and $b = x^m \hat{\beta}\bar{\beta}^{-1}$.

Since $Z(G)$ is a subgroup $\hat{\alpha}\bar{\alpha}^{-1} = \alpha \in Z(G)$
and $\hat{\beta}\bar{\beta}^{-1} = \beta \in Z(G)$

So $a = x^n \alpha$ and $b = x^m \beta$. (*)

Consider $ab = (x^n \alpha)(x^m \beta)$ by (*)
$= x^n (\alpha x^m)\beta$ b/c associativity of G
$= x^n (x^m \alpha)\beta$ b/c $\alpha \in Z(G)$
$= x^{n+m} \alpha\beta$ b/c assoc + def of order
$= x^{m+n} \alpha\beta = x^m x^n \alpha\beta$ $\beta \in Z(G)$ + commutes
$= x^m x^n \beta\alpha$ $\alpha \in Z(G)$ + commutes
$= x^m \beta x^n \alpha$ b/c assoc + def of order
$= ba$ by (*)

Thus G is abelian.